



Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt



The ternary Goldbach problem with arithmetic weights attached to two of the variables

D.I. Tolev¹

Faculty of Mathematics and Informatics, Sofia University "St. Kl. Ohridsky", 5 J. Bourchier, 1164 Sofia, Bulgaria

ARTICLE INFO

Article history:

Received 22 April 2009

Available online 16 September 2009

Communicated by Robert C. Vaughan

Keywords:

Goldbach problems

Primes in arithmetic progressions

Sieve methods

ABSTRACT

We consider the ternary Goldbach problem with two prime variables of the form $k^2 + m^2 + 1$ and find an asymptotic formula for the number of its solutions.

© 2009 Elsevier Inc. All rights reserved.

Notations. By Greek letters we denote real numbers and by small Latin letters – integers. However, the letter p , with or without subscripts, is reserved for primes. By ε we denote an arbitrarily small positive number, not the same in different appearances. N is a sufficiently large odd integer and $\mathcal{L} = \log N$. We denote by $\mathcal{J}$ the set of all subintervals of the interval $[1, N]$ and if $J_1, J_2 \in \mathcal{J}$ then $\mathbf{J} = \langle J_1, J_2 \rangle$ is the corresponding ordered pair. Respectively $\mathbf{k} = \langle k_1, k_2 \rangle$ is two-dimensional vector with integer components k_1, k_2 and, in particular, $\mathbf{1} = \langle 1, 1 \rangle$. We write $(m_1, \dots, m_k)$ for the greatest common factor of $m_1, \dots, m_k$. As usual $\tau(k)$ is the number of positive divisors of k ; $r(k)$ is the number of solutions of the equation $m_1^2 + m_2^2 = k$ in integers m_j ; $\varphi(k)$ is the Euler function; $\Omega(k)$ is the number of the prime factors of k , counted with the multiplicity; $\chi(k)$ is the non-principal character modulo 4 and $L(s, \chi)$ is the corresponding Dirichlet's L -function. We mark by $\square$ an end of a proof, or its absence.

1. Introduction and statement of the result

In 1937 Vinogradov [13] considered the sum

$$I^{(3)}(N) = \sum_{p_1 + p_2 + p_3 = N} (\log p_1)(\log p_2)(\log p_3)$$

E-mail address: dtolev@fmi.uni-sofia.bg.

¹ Supported by Sofia University Grant 028/2009.

and proved that

$$I^{(3)}(N) = \frac{1}{2} N^2 \mathfrak{S}^{(3)}(N) + O(N^2 \mathcal{L}^{-A}), \quad (1)$$

where $A > 0$ is an arbitrarily large constant and

$$\mathfrak{S}^{(3)}(N) = \prod_{p \nmid N} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p|N} \left(1 - \frac{1}{(p-1)^2}\right). \quad (2)$$

It is expected that a similar formula holds true for the sum

$$I^{(2)}(N) = \sum_{p_1+p_2=N} (\log p_1)(\log p_2),$$

but this has not been proved so far. However, using Vinogradov's method, one may establish that $I^{(2)}(n)$ is close to $n\mathfrak{S}^{(2)}(n)$ for almost all $n \leq N$. Here $\mathfrak{S}^{(2)}(n)$ is given by

$$\mathfrak{S}^{(2)}(n) = \begin{cases} c_0 \lambda(n) & \text{for } 2 \mid n, \\ 0 & \text{for } 2 \nmid n, \end{cases}$$

where

$$c_0 = 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right), \quad \lambda(k) = \prod_{\substack{p|k \\ p>2}} \frac{p-1}{p-2}. \quad (3)$$

More precisely (see, for example, Vaughan [12, Chapter 2]), for any constant $A > 0$ we have

$$\sum_{n \leq N} |I^{(2)}(n) - n\mathfrak{S}^{(2)}(n)| \ll N^2 \mathcal{L}^{-A}. \quad (4)$$

Another classical achievement in prime number theory is the solution of the Hardy–Littlewood problem, concerning the representation of large integers as a sum of two squares and a prime. It was solved by Linnik [6] and related problems have been studied by Linnik, Hooley and other mathematicians (see Hooley [4, Chapter 5]). In particular, one can show that

$$\sum_{p \leq N} r(p-1) = \pi N \mathcal{L}^{-1} \prod_{p>2} \left(1 + \frac{\chi(p)}{p(p-1)}\right) + O(N \mathcal{L}^{-1-\theta_0} (\log \mathcal{L})^5), \quad (5)$$

where

$$\theta_0 = \frac{1}{2} - \frac{1}{4} e \log 2 = 0.0029 \dots \quad (6)$$

A sharper estimate for the remainder term in (5) was established by Bredihin [1].

In the present paper we prove a theorem which, in some sense, is a combination of (1) and (5). Define

$$R(N) = \sum_{p_1+p_2+p_3=N} r(p_1-1)r(p_2-1)(\log p_1)(\log p_2)(\log p_3) \quad (7)$$

and

$$\begin{aligned} \mathfrak{S}_R(N) &= \pi^2 \mathfrak{S}^{(3)}(N) \prod_{p \nmid N(N-1)(N-2)} \left(1 + \chi(p) \frac{2p^2 + p\chi(p) - 6p + 3\chi(p)}{p^2(p^2 - 3p + 3)} \right) \\ &\times \prod_{p|N} \left(1 + \chi(p) \frac{2p^2 + p\chi(p) - 4p + 2\chi(p)}{p^2(p-1)(p-2)} \right) \\ &\times \prod_{p|N-1} \left(1 + \chi(p) \frac{4p^2 - p\chi(p) - 6p + 3\chi(p)}{p^2(p^2 - 3p + 3)} \right) \\ &\times \prod_{p|N-2} \left(1 + \chi(p) \frac{2p^2 - p^2\chi(p) + p\chi(p) - 6p + 3\chi(p)}{p^2(p^2 - 3p + 3)} \right), \end{aligned} \quad (8)$$

where $\mathfrak{S}^{(3)}(N)$ is given by (2).

Theorem 1. We have the following asymptotic formula

$$R(N) = \frac{1}{2} N^2 \mathfrak{S}_R(N) + O(N^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^7), \quad (9)$$

where θ_0 is the constant defined by (6).

It is clear that if $2 \nmid N$ then $1 \ll \mathfrak{S}_R(N) \ll 1$, so the main term in (9) dominates the remainder term provided that N is a sufficiently large odd integer.

Theorem 1 is related to a recent result of the author, which may be considered as a combination of (4) and (5). In the paper [11] the sum

$$\sum_{p_1+p_2=n} r(p_1-1)(\log p_1)(\log p_2)$$

was studied and it was proved that the expected asymptotic formula for it holds true for almost all even integers $n \leq N$. A similar problem was earlier considered by Matomäki [7].

The method used for the proof of Theorem 1 can also be applied for finding asymptotic formulas for the sums

$$\sum_{p_1+p_2+p_3=N} r(p_1-1)\tau(p_2-1)(\log p_1)(\log p_2)(\log p_3)$$

and

$$\sum_{p_1+p_2+p_3=N} \tau(p_1-1)\tau(p_2-1)(\log p_1)(\log p_2)(\log p_3).$$

It would be interesting to consider the ternary Goldbach equation with weights of the above type attached to all of the variables. We would be in a position to attack this problem if we had more information about the number of solutions of the ternary equation with all prime variables lying in independent arithmetic progressions with large moduli. However, the best result of this type available in the literature at present, which is due to the author [10] and improves a theorem of K. Halupczok [3], is not strong enough for our aims.

2. Some lemmas

First we consider the Goldbach binary problem with one prime variable lying in a given interval and belonging to an arithmetic progression. Suppose that $n \leq N$, let k and l be integers with $(k, l) = 1$ and let $J \in \mathcal{J}$. We denote

$$I_{k,l}^{(2)}(n, J) = \sum_{\substack{p_1 + p_2 = n \\ p_1 \equiv l \pmod{k} \\ p_1 \in J}} (\log p_1)(\log p_2); \quad (10)$$

$$\mathfrak{S}_{k,l}^{(2)}(n) = \begin{cases} c_0 \lambda(nk) & \text{if } (k, n-l) = 1 \text{ and } 2 \mid n, \\ 0 & \text{otherwise;} \end{cases} \quad (11)$$

$$\Phi^{(2)}(n, J) = \sum_{\substack{m_1 + m_2 = n \\ m_1 \in J}} 1; \quad (12)$$

$$\Delta_{k,l}^{(2)}(n, J) = I_{k,l}^{(2)}(n, J) - \frac{\mathfrak{S}_{k,l}^{(2)}(n)}{\varphi(k)} \Phi^{(2)}(n, J). \quad (13)$$

If $J = [1, N]$ then we write for simplicity $I_{k,l}^{(2)}(n)$, $\Phi^{(2)}(n)$ ($= n - 1$) and $\Delta_{k,l}^{(2)}(n)$.

Our first lemma is a generalization of (4) and states that $\Delta_{k,l}^{(2)}(n, J)$ is small on average with respect to k and n and uniformly for l and J . More precisely, we have

Lemma 1. *For any constant $A > 0$ there exists $B = B(A) > 0$ such that*

$$\sum_{k \leq \sqrt{N} \mathcal{L}^{-B}} \max_{(l,k)=1} \max_{J \in \mathcal{J}} \sum_{n \leq N} |\Delta_{k,l}^{(2)}(n, J)| \ll N^2 \mathcal{L}^{-A}.$$

This lemma is very similar to results of Mikawa [8] and Laporta [5]. These authors study the equation $p_1 - p_2 = n$ and without the condition $p_1 \in J$. However, inspecting the arguments presented in [5], the reader will readily see that the proof of Lemma 1 can be obtained in the same manner. $\square$

Next we consider Goldbach's ternary problem with two primes from arithmetic progressions and belonging to given intervals. Suppose that $\mathbf{k} = \langle k_1, k_2 \rangle$ and $\mathbf{l} = \langle l_1, l_2 \rangle$ are two-dimensional vectors with integer components and let $\mathbf{J} = \langle J_1, J_2 \rangle$ be a pair of intervals $J_1, J_2 \in \mathcal{J}$. We denote

$$I_{\mathbf{k},\mathbf{l}}^{(3)}(N, \mathbf{J}) = \sum_{\substack{p_1 + p_2 + p_3 = N \\ p_i \equiv l_i \pmod{k_i} \\ p_i \in J_i, i=1,2}} (\log p_1)(\log p_2)(\log p_3),$$

$$\Phi^{(3)}(N, \mathbf{J}) = \sum_{\substack{m_1 + m_2 + m_3 = N \\ m_i \in J_i, i=1,2}} 1. \quad (14)$$

Using the notations of K. Halupczok [3], we define $\mathfrak{S}_{\mathbf{k}, \mathbf{l}}^{(3)}(N)$ in the following way. Consider the sets of primes

$$\mathcal{A} = \{p: p \nmid k_1 k_2, p \mid N\};$$

$$\mathcal{B} = \{p: p \nmid k_1 k_2 N\};$$

$$\mathcal{C} = \{p: p \mid k_1, p \nmid k_2, p \mid N - l_1\} \cup \{p: p \mid k_2, p \nmid k_1, p \mid N - l_2\};$$

$$\mathcal{D} = \{p: p \mid k_1, p \nmid k_2, p \nmid N - l_1\} \cup \{p: p \mid k_2, p \nmid k_1, p \nmid N - l_2\};$$

$$\mathcal{E} = \{p: p \mid k_1, p \mid k_2, p \mid N - l_1 - l_2\};$$

$$\mathcal{F} = \{p: p \mid k_1, p \mid k_2, p \nmid N - l_1 - l_2\}.$$

If $\mathcal{E} \neq \emptyset$ then we assume that

$$\mathfrak{S}_{\mathbf{k}, \mathbf{l}}^{(3)}(N) = 0. \quad (15)$$

If $\mathcal{E} = \emptyset$ then we put

$$\mathfrak{S}_{\mathbf{k}, \mathbf{l}}^{(3)}(N) = \prod_{p \in \mathcal{A} \cup \mathcal{D}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \in \mathcal{B}} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \in \mathcal{C} \cup \mathcal{F}} \left(1 + \frac{1}{p-1}\right). \quad (16)$$

We also define

$$\Delta_{\mathbf{k}, \mathbf{l}}^{(3)}(N, \mathbf{J}) = I_{\mathbf{k}, \mathbf{l}}^{(3)}(N, \mathbf{J}) - \frac{\mathfrak{S}_{\mathbf{k}, \mathbf{l}}^{(3)}(N) \Phi^{(3)}(N, \mathbf{J})}{\varphi(k_1) \varphi(k_2)}. \quad (17)$$

If $J_1 = J_2 = [1, N]$ then we write for simplicity $I_{\mathbf{k}, \mathbf{l}}^{(3)}(N)$, $\Phi^{(3)}(N) (= N^2/2 + O(N))$ and $\Delta_{\mathbf{k}, \mathbf{l}}^{(3)}(N)$.

The next lemma is analogous to Lemma 1 and states that $\Delta_{\mathbf{k}, \mathbf{l}}^{(3)}(N, \mathbf{J})$ is small on average with respect to $\mathbf{k}$ and uniformly for $\mathbf{l}$ and $\mathbf{J}$. More precisely, we have

Lemma 2. For any constant $A > 0$ there exists $C = C(A) > 0$ such that

$$\sum_{k_1 \leq \sqrt{N} \mathcal{L}^{-C}} \sum_{k_2 \leq \sqrt{N} \mathcal{L}^{-C}} \max_{\substack{(l_i, k_i)=1 \\ i=1,2}} \max_{\substack{J_i \in \mathcal{J} \\ i=1,2}} |\Delta_{\mathbf{k}, \mathbf{l}}^{(3)}(N, \mathbf{J})| \ll N^2 \mathcal{L}^{-A}.$$

This statement is slightly more general than a theorem from author's recent paper [10], which improves a result of K. Halupczok [3]. There are no conditions $p_i \in J_i$ in the theorems of [10] and [3], but the reader can easily verify that the methods developed in these articles imply also the validity of Lemma 2. $\square$

In several occasions we will need the following simple

Lemma 3. Suppose that $j \in \{1, -1\}$ and let m, k, l, n be natural numbers. Then the quantities $\mathfrak{S}_{4m, 1+jm}^{(2)}(n)$ and $\mathfrak{S}_{(k, 4m), (l, 1+jm)}^{(3)}(n)$ do not depend on j .

The proof follows directly from the definitions of $\mathfrak{S}_{k,l}^{(2)}(n)$ and $\mathfrak{S}_{k,l}^{(3)}(n)$. We leave the easy verification to the reader. $\square$

The next two lemmas are due to C. Hooley and play an essential role in the proof of (5), as well as in the solutions of other related problems.

Lemma 4. For any constant $\omega > 0$ we have

$$\sum_{p \leq N} \left| \sum_{\substack{d|p-1 \\ \sqrt{N}\mathcal{L}^{-\omega} < d < \sqrt{N}\mathcal{L}^{\omega}}} \chi(d) \right| \ll N\mathcal{L}^{-1-\theta_0}(\log \mathcal{L})^5,$$

where θ_0 is defined by (6). The constant in the Vinogradov symbol depends only on ω .

Lemma 5. For any constant $\omega > 0$ we have

$$\sum_{p \leq N} \left| \sum_{\substack{d|p-1 \\ \sqrt{N}\mathcal{L}^{-\omega} < d < \sqrt{N}\mathcal{L}^{\omega}}} \chi(d) \right|^2 \ll N\mathcal{L}^{-1}(\log \mathcal{L})^7,$$

where the constant in the Vinogradov symbol depends on ω .

The proofs of very similar results (with $\omega = 48$ and with the condition $d | N - p$ rather than $d | p - 1$) are available in [4, Chapter 5]. The reader will easily see that the methods used there yield also the validity of Lemmas 4 and 5. $\square$

The next lemma is analogous to another result of Hooley from [4, Chapter 5].

Lemma 6. Let n be an integer satisfying $1 \leq n \leq N$. Suppose that $\omega > 0$ is a constant and let $\mathcal{P} = \mathcal{P}_{\omega}(N)$ be the set of primes $p \leq N$ such that $p - 1$ has a divisor lying between $\sqrt{N}\mathcal{L}^{-\omega}$ and $\sqrt{N}\mathcal{L}^{\omega}$. Then we have

$$\sum_{\substack{p_1+p_2=n \\ p_1 \in \mathcal{P}}} 1 \ll N\mathcal{L}^{-2-2\theta_0}(\log \mathcal{L})^6, \quad (18)$$

where θ_0 is defined by (6). The constant in the Vinogradov symbol depends only on ω .

Proof. We proceed as in [4, Chapter 5, Section 7]. Denote the sum in the left side of (18) by Σ and let $D_1 = \sqrt{N}\mathcal{L}^{-\omega}$, $D_2 = \sqrt{N}\mathcal{L}^{\omega}$. Suppose that α is a real number satisfying $1 < \alpha < 3/2$. We have

$$\Sigma \leq \sum_{\substack{p_1+p_2=n \\ \Omega(p_1-1) \leq \alpha \log \mathcal{L} \\ p_1 \in \mathcal{P}}} 1 + \sum_{\substack{p_1+p_2=n \\ \Omega(p_1-1) > \alpha \log \mathcal{L}}} 1 = \Sigma_1 + \Sigma_2, \quad (19)$$

say.

Consider first Σ_1 . We have

$$\Sigma_1 \leq \sum_{\substack{p_1+p_2=n \\ \Omega(p_1-1) \leq \alpha \log \mathcal{L}}} \sum_{\substack{d|p_1-1 \\ D_1 < d < D_2}} 1 = \sum_{p_1+p_2=n} \sum_{\substack{md=p_1-1 \\ D_1 < d < D_2 \\ \Omega(md) \leq \alpha \log \mathcal{L}}} 1.$$

Obviously, the inequality $\Omega(md) \leq \beta$ implies the validity of at least one of the inequalities $\Omega(m) \leq \frac{1}{2}\beta$ and $\Omega(d) \leq \frac{1}{2}\beta$. Hence

$$\Sigma_1 \leq \sum_{p_1+p_2=n} \sum_{\substack{md=p_1-1 \\ D_1 < d < D_2 \\ \Omega(d) \leq \frac{1}{2}\alpha \log \mathcal{L}}} 1 + \sum_{p_1+p_2=n} \sum_{\substack{md=p_1-1 \\ D_1 < d < D_2 \\ \Omega(m) \leq \frac{1}{2}\alpha \log \mathcal{L}}} 1 = \Sigma_1^{(1)} + \Sigma_1^{(2)}, \quad (20)$$

say. Consider $\Sigma_1^{(2)}$. The conditions imposed in its definition imply $m < D_2$ and clearly

$$\Sigma_1^{(2)} \ll \sum_{p_1+p_2=n} \sum_{\substack{md=p_1-1 \\ D_1 \mathcal{L}^{-10} < m < D_2 \\ \Omega(m) \leq \frac{1}{2}\alpha \log \mathcal{L}}} 1 + N\mathcal{L}^{-10} = \Sigma_1^{(3)} + N\mathcal{L}^{-10}, \quad (21)$$

say. Obviously $\Sigma_1^{(1)} \leq \Sigma_1^{(3)}$ and from this inequality, (20) and (21) we find

$$\Sigma_1 \ll \Sigma_1^{(3)} + N\mathcal{L}^{-10}. \quad (22)$$

To estimate $\Sigma_1^{(3)}$ we change the order of summation and find

$$\Sigma_1^{(3)} = \sum_{\substack{D_1 \mathcal{L}^{-10} < m < D_2 \\ \Omega(m) \leq \frac{1}{2}\alpha \log \mathcal{L}}} \lambda_m(n), \quad (23)$$

where $\lambda_m(n)$ is the number of primes $p < n$ satisfying $p \equiv 1 \pmod{m}$ and such that $n-p$ is a prime too. We apply Theorem 2.4 of Halberstam and Richert [2] (with $x = y = n$, $k = m$, $l = 1$, $g = 1$, $a_1 = -1$, $b_1 = n$) and find

$$\lambda_m(n) \ll \prod_{p|mn} \left(1 - \frac{1}{p}\right)^{-1} \cdot \frac{m}{\varphi(m)} \cdot \frac{n/m}{\log^2(n/m)} \ll \frac{N}{m} \mathcal{L}^{-2} (\log \mathcal{L})^2.$$

We substitute this upper bound for $\lambda_m(n)$ in (23) and then proceed precisely as in [4, Chapter 5, Section 7] to find

$$\Sigma_1^{(3)} \ll N\mathcal{L}^{-2} (\log \mathcal{L})^2 \sum_{\substack{D_1 \mathcal{L}^{-10} < m < D_2 \\ \Omega(m) \leq \frac{\alpha}{2} \log \mathcal{L}}} \frac{1}{m} \ll N\mathcal{L}^{\gamma(\alpha/2)-3} (\log \mathcal{L})^3, \quad (24)$$

where

$$\gamma(c) = c - c \log c. \quad (25)$$

From (22) and (24) we get

$$\Sigma_1 \ll N\mathcal{L}^{\gamma(\alpha/2)-3} (\log \mathcal{L})^3. \quad (26)$$

Consider now Σ_2 . We have

$$\Sigma_2 \leq \sum_{\substack{p_1+p_2=n \\ \alpha \log \mathcal{L} < \Omega(p_1-1) \leq 10 \log \mathcal{L}}} 1 + \sum_{\substack{m \leq N \\ \Omega(m) > 10 \log \mathcal{L}}} 1 = \Sigma_2^{(1)} + \Sigma_2^{(2)}, \quad (27)$$

say. It is shown in [4, Chapter 5, Section 7] that

$$\Sigma_2^{(2)} \ll N \mathcal{L}^{-4}. \quad (28)$$

Consider $\Sigma_2^{(1)}$. Denote by $\mathcal{R}_N$ the set of integers $m \leq N$ composed only of primes $\leq N^{\frac{1}{20 \log \mathcal{L}}}$. Applying the method of [4, Chapter 5, Section 7] we get

$$\begin{aligned} \Sigma_2^{(1)} &\leq \sum_{\substack{p_1+p_2=n \\ \alpha \log \mathcal{L} < \Omega(p_1-1) \\ p_1-1 \notin \mathcal{R}_N}} 1 + \sum_{\substack{p_1+p_2=n \\ \Omega(p_1-1) \leq 10 \log \mathcal{L} \\ p_1-1 \in \mathcal{R}_N}} 1 \\ &= \sum_{\substack{p_1+p_2=n \\ \alpha \log \mathcal{L} < \Omega(p_1-1) \\ p_1-1 \notin \mathcal{R}_N}} 1 + O(\sqrt{N}) \\ &= \Sigma_2^{(3)} + O(\sqrt{N}), \end{aligned} \quad (29)$$

say. If $p_1 - 1 \notin \mathcal{R}_N$ then there exists a prime $q > N^{\frac{1}{20 \log \mathcal{L}}}$ such that $p_1 - 1 = qr$ for some positive integer $r < N^{1 - \frac{1}{20 \log \mathcal{L}}}$. On the other hand, from the condition $\Omega(p_1 - 1) > \alpha \log \mathcal{L}$ it follows that $\Omega(r) > \alpha \log \mathcal{L} - 1$. Therefore

$$\Sigma_2^{(3)} \leq \sum_{\substack{r < N^{1 - \frac{1}{20 \log \mathcal{L}}} \\ \Omega(r) > \alpha \log \mathcal{L} - 1}} \kappa_r(n), \quad (30)$$

where $\kappa_r(n)$ is the number of primes $q \leq N/r$ such that $rq + 1$ and $-rq + n - 1$ are primes too. We apply again Theorem 2.4 of [2] (with $x = y = N/r$, $k = 1$, $g = 2$, $a_1 = r$, $a_2 = -r$, $b_1 = 1$, $b_2 = n - 1$, $E = -r^3 n$) to find

$$\kappa_r(n) \ll \prod_{p|E} \left(1 - \frac{1}{p}\right)^{\rho(p)-2} \cdot \prod_{p|n-1} \left(1 - \frac{1}{p}\right)^{-1} \cdot \frac{N/r}{\log^3(N/r)}, \quad (31)$$

where $\rho(p)$ is the number of solutions of the congruence $(a_1 m + b_1)(a_2 m + b_2) \equiv 0 \pmod{p}$. It is easy to verify that

$$\rho(p) = \begin{cases} p & \text{for } p \mid r, \ p \mid n - 1, \\ 0 & \text{for } p \mid r, \ p \nmid n - 1, \\ 1 & \text{for } p \nmid r, \ p \mid n - 2, \\ 2 & \text{for } p \nmid r, \ p \nmid n - 2. \end{cases} \quad (32)$$

From (31) and (32) it follows that

$$\kappa_r(n) \ll \frac{N}{r \log^3(N/r)} (\log \mathcal{L})^3.$$

We substitute this upper bound for $\kappa_r(n)$ in (30) and then we notice that the inequality $r < N^{1-\frac{1}{20\log \mathcal{L}}}$ implies $\log(N/r) \gg \mathcal{L}(\log \mathcal{L})^{-1}$. We find

$$\Sigma_2^{(3)} \ll N \mathcal{L}^{-3} (\log \mathcal{L})^6 \sum_{\substack{r < N \\ \Omega(r) > \alpha \log \mathcal{L} - 1}} \frac{1}{r}.$$

Now we estimate the sum over r in the way proposed in [4, Chapter 5, Section 7] to get

$$\Sigma_2^{(3)} \ll N \mathcal{L}^{\gamma(\alpha)-3} (\log \mathcal{L})^6, \quad (33)$$

where $\gamma(c)$ is defined by (25). Using (27), (28), (29) and (33) we obtain

$$\Sigma_2 \ll N \mathcal{L}^{\gamma(\alpha)-3} (\log \mathcal{L})^6. \quad (34)$$

From (19), (26) and (34) it follows that

$$\Sigma \ll N (\mathcal{L}^{\gamma(\alpha/2)-3} + \mathcal{L}^{\gamma(\alpha)-3}) (\log \mathcal{L})^6.$$

We choose α from the condition $\gamma(\alpha/2) = \gamma(\alpha)$, which gives $\alpha = e/2$. This completes the proof of the lemma. $\square$

3. Proof of Theorem 1

Beginning. We put

$$D = \sqrt{N} \mathcal{L}^{-B(10)-C(10)-1}, \quad (35)$$

where $B(A)$ and $C(A)$ are specified respectively in Lemma 1 and Lemma 2. Obviously

$$r(m) = 4 \sum_{d|m} \chi(d) = 4(r_1(m) + r_2(m) + r_3(m)), \quad (36)$$

where

$$r_1(m) = \sum_{\substack{d|m \\ d \leq D}} \chi(d), \quad r_2(m) = \sum_{\substack{d|m \\ D < d < N/D}} \chi(d), \quad r_3(m) = \sum_{\substack{d|m \\ d \geq N/D}} \chi(d). \quad (37)$$

Hence using (7) and (36) we get

$$R(N) = 16 \sum_{1 \leq i, j \leq 3} R_{i,j}(N), \quad (38)$$

where

$$R_{i,j}(N) = \sum_{p_1+p_2+p_3=N} r_i(p_1-1)r_j(p_2-1)(\log p_1)(\log p_2)(\log p_3). \quad (39)$$

We shall prove that the main term in (9) comes from $R_{1,1}(N)$ and the other sums $R_{i,j}(N)$ contribute only to the remainder term. Because of the symmetry we have to consider only the expressions $R_{i,j}(N)$ with $i \leq j$.

The evaluation of $R_{1,1}(N)$. Using (14), (17), (37) and (39) we get

$$R_{1,1}(N) = \sum_{d_1, d_2 \leq D} \chi(d_1)\chi(d_2)I_{\mathbf{d},1}^{(3)}(N) = R'_{1,1}(N) + R^*_{1,1}(N), \quad (40)$$

where

$$R'_{1,1}(N) = \sum_{d_1, d_2 \leq D} \frac{\chi(d_1)\chi(d_2)}{\varphi(d_1)\varphi(d_2)} \mathfrak{S}_{\mathbf{d},1}^{(3)}(N) \Phi^{(3)}(N),$$

$$R^*_{1,1}(N) = \sum_{d_1, d_2 \leq D} \chi(d_1)\chi(d_2) \Delta_{\mathbf{d},1}^{(3)}(N).$$

From (35) and Lemma 2 it follows that

$$R^*_{1,1}(N) \ll N^2 \mathcal{L}^{-1}. \quad (41)$$

Consider $R'_{1,1}(N)$. It is clear that

$$R'_{1,1}(N) = \frac{1}{2} N^2 \Gamma(N) + O(N^{1+\varepsilon}), \quad (42)$$

where

$$\Gamma(N) = \sum_{d_1, d_2 \leq D} \frac{\chi(d_1)\chi(d_2)}{\varphi(d_1)\varphi(d_2)} \mathfrak{S}_{\mathbf{d},1}^{(3)}(N). \quad (43)$$

It remains to establish an asymptotic formula for $\Gamma(N)$. The calculations are long and complicated, but rather routine and straightforward. We point out only the main steps and leave the details to the reader.

Using (15), (16) and (43) we find

$$\Gamma(N) = \sum_{d \leq D} \frac{\chi(d)}{\varphi(d)} \psi_N(d) \sum_{\substack{t \leq D \\ (t, d, N-2)=1}} f_{N,d}(t), \quad (44)$$

where

$$\psi_N(d) = \prod_{p \nmid dN} \left(1 + \frac{1}{(p-1)^3}\right) \prod_{p \in \mathcal{U}_1 \cup \mathcal{U}_2} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \mid N-1 \\ p \nmid d}} \left(1 + \frac{1}{p-1}\right), \quad (45)$$

$$\begin{aligned}
f_{N,d}(t) &= \frac{\chi(t)}{\varphi(t)} \prod_{\substack{p \nmid dN \\ p|t}} \left(1 + \frac{1}{(p-1)^3}\right)^{-1} \prod_{\substack{p \in \mathcal{U}_1 \cup \mathcal{U}_2 \\ p|t}} \left(1 - \frac{1}{(p-1)^2}\right)^{-1} \prod_{p|(d,t,N-1)} \left(1 + \frac{1}{p-1}\right)^{-1} \\
&\times \prod_{\substack{p \nmid d(N-1) \\ p|t}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{\substack{p \in \mathcal{U}_3 \cup \mathcal{U}_4 \\ p|t}} \left(1 + \frac{1}{p-1}\right)
\end{aligned} \quad (46)$$

and

$$\begin{aligned}
\mathcal{U}_1 &= \{p: p \mid N, p \nmid d\}, & \mathcal{U}_2 &= \{p: p \nmid N-1, p \mid d\}, \\
\mathcal{U}_3 &= \{p: p \mid N-1, p \nmid d\}, & \mathcal{U}_4 &= \{p: p \nmid N-2, p \mid d\}.
\end{aligned}$$

First we evaluate the sum over t in (44). From (46) it follows that

$$f_{N,d}(t) \ll (\log \mathcal{L})^2 t^{-1} \log \log(10t) \quad (47)$$

with absolute constant in the Vinogradov symbol. Hence the corresponding Dirichlet series

$$F_{N,d}(s) = \sum_{\substack{t=1 \\ (t,d,N-2)=1}}^{\infty} f_{N,d}(t) t^{-s}$$

is absolutely convergent in $\operatorname{Re}(s) > 0$. Clearly $f_{N,d}(t)$ is multiplicative with respect to t and applying Euler's identity we find

$$F_{N,d}(s) = \prod_{p \nmid (d,N-2)} T_{N,d}(p, s), \quad T_{N,d}(p, s) = 1 + \sum_{l=1}^{\infty} f_{N,d}(p^l) p^{-ls}.$$

From (46) we establish that

$$T_{N,d}(p, s) = (1 - \chi(p) p^{-s-1})^{-1} (1 + \chi(p) p^{-s-1} Y_{N,d}(p)),$$

where

$$Y_{N,d}(p) = \begin{cases} (p-3)(p^2-3p+3)^{-1} & \text{if } p \nmid dN(N-1)(N-2), \\ 2(p-2)^{-1} & \text{if } p \mid d, p \nmid N(N-1)(N-2), \\ (p-1)^{-1} & \text{if } p \nmid d, p \mid N, \\ 2(p-2)^{-1} & \text{if } p \mid d, p \mid N, \\ (2p-3)(p^2-3p+3)^{-1} & \text{if } p \nmid d, p \mid N-1, \\ (p-1)^{-1} & \text{if } p \mid d, p \mid N-1, \\ (p-3)(p^2-3p+3)^{-1} & \text{if } p \nmid d, p \mid N-2. \end{cases} \quad (48)$$

Hence we get

$$F_{N,d}(s) = L(s+1, \chi) \prod_{p|(d,N-2)} (1 - \chi(p) p^{-s-1}) \prod_{p \nmid (d,N-2)} (1 + \chi(p) p^{-s-1} Y_{N,d}(p)). \quad (49)$$

From this formula it follows that $F_{N,d}(s)$ has an analytic continuation to $\operatorname{Re}(s) > -1$. Using (48) and the simplest bound for $L(s+1, \chi)$ we get

$$F_{N,d}(s) \ll N^\varepsilon T^{1/6} \quad \text{for } \operatorname{Re}(s) \geq -\frac{1}{2}, \quad |\operatorname{Im}(s)| \leq T. \quad (50)$$

We apply the version of Perron's formula given at Tenenbaum [9, Chapter II.2] and also (47) to find

$$\sum_{\substack{t \leq D \\ (t,d,N-2)=1}} f_{N,d}(t) = \frac{1}{2\pi i} \int_{\kappa-iT}^{\kappa+iT} F_{N,d}(s) \frac{D^s}{s} ds + O\left(\sum_{t=1}^{\infty} \frac{N^\varepsilon D^\kappa \log \log(10t)}{t^{1+\kappa} (1+T|\log \frac{D}{t}|)}\right),$$

where $\kappa = 1/100$, $T = N^{3/4}$. It is easy to verify that the remainder term above is $O(N^{-1/100})$ and applying the residue theorem we see that the main term is equal to

$$F_{N,d}(0) + \frac{1}{2\pi i} \left(\int_{\kappa-iT}^{-1/2-iT} + \int_{-1/2-iT}^{-1/2+iT} + \int_{-1/2+iT}^{\kappa+iT} \right) F_{N,d}(s) \frac{D^s}{s} ds.$$

From (50) it follows that the contribution from the above integrals is $O(N^{-1/100})$. Hence

$$\sum_{\substack{t \leq D \\ (t,d,N-2)=1}} f_{N,d}(t) = F_{N,d}(0) + O(N^{-1/100}). \quad (51)$$

Obviously, using (49) we get

$$F_{N,d}(0) = \frac{\pi}{4} \prod_{p|(d,N-2)} \left(1 - \frac{\chi(p)}{p}\right) \prod_{p \nmid (d,N-2)} \left(1 + \frac{\chi(p)}{p} \gamma_{N,d}(p)\right). \quad (52)$$

We use (44), (45), (48), (51) and (52) to find a new expression for $\Gamma(N)$ and after some calculations we obtain

$$\Gamma(N) = \frac{\pi}{4} \mathfrak{S}^{(3)}(N) \mathfrak{E}(N) \sum_{d \leq D} g_N(d) + O(N^{\varepsilon-1/100}), \quad (53)$$

where $\mathfrak{S}^{(3)}(N)$ is defined by (2),

$$\begin{aligned} \mathfrak{E}(N) &= \prod_{p \nmid N(N-1)} \left(1 + \frac{\chi(p)(p-3)}{p(p^2-3p+3)}\right) \prod_{p|N} \left(1 + \frac{\chi(p)}{p(p-1)}\right) \\ &\quad \times \prod_{p|N-1} \left(1 + \frac{\chi(p)(2p-3)}{p(p^2-3p+3)}\right), \end{aligned} \quad (54)$$

and

$$\begin{aligned}
 g_N(d) &= \frac{\chi(d)}{\varphi(d)} \prod_{\substack{p|d \\ p \nmid N(N-1)(N-2)}} \frac{1 + \frac{2\chi(p)}{p(p-2)}}{1 + \frac{\chi(p)(p-3)}{p(p^2-3p+3)}} \prod_{p|(d,N)} \frac{1 + \frac{2\chi(p)}{p(p-2)}}{(1 - \frac{1}{(p-1)^2})(1 + \frac{\chi(p)}{p(p-1)})} \\
 &\times \prod_{\substack{p|d \\ p \nmid N}} \left(1 + \frac{1}{(p-1)^3}\right)^{-1} \prod_{p|(d,N-1)} \frac{(1 + \frac{1}{p-1})(1 + \frac{\chi(p)}{p(p-1)})}{1 + \frac{\chi(p)(2p-3)}{p(p^2-3p+3)}} \\
 &\times \prod_{\substack{p|d \\ p \nmid N-1}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p|(d,N-2)} \frac{1 - \frac{\chi(p)}{p}}{1 + \frac{\chi(p)(p-3)}{p(p^2-3p+3)}}.
 \end{aligned} \tag{55}$$

It is clear that $g_N(d)$ is multiplicative with respect to d and satisfies

$$g_N(d) \ll (\log \log(10d))^3 d^{-1},$$

where the constant in Vinogradov's symbol is absolute. Hence the Dirichlet series

$$G_N(s) = \sum_{d=1}^{\infty} g_N(d) d^{-s}$$

is absolutely convergent in $\operatorname{Re}(s) > 0$ and applying the Euler identity we get

$$G_N(s) = \prod_p H_N(p, s), \quad H_N(p, s) = 1 + \sum_{l=1}^{\infty} g_N(p^l) p^{-ls}. \tag{56}$$

From (55) and (56) we find

$$H_N(p, s) = (1 - \chi(p)p^{-s-1})^{-1} (1 + \chi(p)p^{-s-1} K_N(p)),$$

where

$$K_N(p) = \begin{cases} \frac{p^2 + p\chi(p) - 3p + 3\chi(p)}{p^3 - 3p^2 + 3p + p\chi(p) - 3\chi(p)} & \text{if } p \nmid N(N-1)(N-2), \\ \frac{p^2 + p\chi(p) - 2p + 2\chi(p)}{p^3 - 3p^2 + p\chi(p) + 2p - 2\chi(p)} & \text{if } p \mid N, \\ \frac{2p^2 - 3p - p\chi(p) + 3\chi(p)}{p^3 - 3p^2 + 3p + 2p\chi(p) - 3\chi(p)} & \text{if } p \mid N-1, \\ \frac{p^2 - p^2\chi(p) - 3p + p\chi(p) + 3\chi(p)}{p^3 - 3p^2 + 3p + p\chi(p) - 3\chi(p)} & \text{if } p \mid N-2. \end{cases} \tag{57}$$

This gives

$$G_N(s) = L(s+1, \chi) \prod_p (1 + \chi(p)p^{-s-1} K_N(p)).$$

We see that $G_N(s)$ has an analytic continuation to $\operatorname{Re}(s) > -1$ and

$$G_N(s) \ll N^\varepsilon T^{1/6} \quad \text{for } \operatorname{Re}(s) \geq -\frac{1}{2}, \quad |\operatorname{Im}(s)| \leq T.$$

Applying Perron's formula and proceeding as above we find

$$\sum_{d \leq D} G_N(d) = G_N(0) + O(N^{-1/100}) = \frac{\pi}{4} \prod_p \left(1 + \frac{\chi(p)}{p} K_N(p) \right) + O(N^{-1/100}). \quad (58)$$

Using (53), (54), (57) and (58) we find

$$\Gamma(N) = \frac{1}{16} \mathfrak{S}_R(N) + O(N^{\varepsilon-1/100}), \quad (59)$$

where $\mathfrak{S}_R(N)$ is defined by (8). We leave the calculations to the reader.

From (40), (41), (42) and (59) we get

$$R_{1,1}(N) = \frac{1}{32} N^2 \mathfrak{S}_R(N) + O(N^2 \mathcal{L}^{-1}). \quad (60)$$

The estimation of $R_{1,2}(N)$. Using (10), (13), (37) and (39) we write

$$R_{1,2}(N) = \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{d \leq D} \chi(d) I_{d,1}^{(2)}(N-p) = R'_{1,2}(N) + R^*_{1,2}(N), \quad (61)$$

where

$$R'_{1,2}(N) = \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{d \leq D} \frac{\chi(d)}{\varphi(d)} \mathfrak{S}_{d,1}^{(2)}(N-p)(N-p-1), \quad (62)$$

$$R^*_{1,2}(N) = \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{d \leq D} \chi(d) \Delta_{d,1}^{(2)}(N-p). \quad (63)$$

From (37), (63) and Cauchy's inequality we find

$$\begin{aligned} |R^*_{1,2}(N)| &\ll \mathcal{L} \sum_{2 < p < N} \tau(p-1) \sum_{d \leq D} |\Delta_{d,1}^{(2)}(N-p)| \\ &\ll \mathcal{L} \sum_{n \leq N} \tau(n) \sum_{d \leq D} |\Delta_{d,1}^{(2)}(n)| \\ &\ll \mathcal{L} \left(\sum_{n \leq N} \sum_{d \leq D} \tau^2(n) |\Delta_{d,1}^{(2)}(n)| \right)^{1/2} \left(\sum_{n \leq N} \sum_{d \leq D} |\Delta_{d,1}^{(2)}(n)| \right)^{1/2} \\ &= \mathcal{L} U^{1/2} V^{1/2}, \end{aligned} \quad (64)$$

say. We use the trivial bound $\Delta_{d,1}^{(2)}(n) \ll \mathcal{L}^2 N d^{-1}$ and the well-known elementary inequality $\sum_{n \leq x} \tau^2(n) \ll x \log^3 x$ and we find

$$U \ll N^2 \mathcal{L}^6. \quad (65)$$

To estimate V we apply (35) and Lemma 1 and we get

$$V \ll N^2 \mathcal{L}^{-10}. \quad (66)$$

From (64), (65) and (66) it follows that

$$R_{1,2}^*(N) \ll N^2 \mathcal{L}^{-1}. \quad (67)$$

Consider now $R'_{1,2}(N)$. Using (3), (11), (12) and (62) we write it in the form

$$R'_{1,2}(N) = c_0 \sum_{2 < p < N} (\log p) r_2(p-1) (N-p-1) \lambda(N-p) \sum_{\substack{d \leq D \\ (d, N-p-1)=1}} \frac{\chi(d) \lambda(d)}{\varphi(d) \lambda((d, N-p))}.$$

It is not difficult to find an asymptotic formula for the sum over d . However such a formula is already established in Section 3.2 of [11] and it implies that $\sum_d \ll \log \mathcal{L}$. Therefore, using also (3), we find

$$R'_{1,2}(N) \ll N \mathcal{L} (\log \mathcal{L})^2 \sum_{p \leq N} |r_2(p-1)|.$$

It remains to apply (37) and Lemma 4 and we get

$$R'_{1,2}(N) \ll N^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^7. \quad (68)$$

From (61), (67) and (68) we obtain

$$R_{1,2}(N) \ll N^2 \mathcal{L}^{-\theta_0} (\log \mathcal{L})^7. \quad (69)$$

The estimation of $R_{1,3}(N)$. We use (14), (37) and (39) to write

$$\begin{aligned} R_{1,3}(N) &= \sum_{p_1+p_2+p_3=N} (\log p_1)(\log p_2)(\log p_3) \sum_{\substack{d|p_1-1 \\ d \leq D}} \chi(d) \sum_{\substack{m|p_2-1 \\ \frac{p_2-1}{m} \geq N/D}} \chi\left(\frac{p_2-1}{m}\right) \\ &= \sum_{\substack{d \leq D \\ m \leq D \\ 2|m}} \chi(d) \sum_{j=\pm 1} \chi(j) I_{(d, 4m), (1, 1+jm)}^{(3)}(N, [1, N], J_m), \end{aligned}$$

where $J_m = [1 + mN/D, N]$. From (17) we get

$$R_{1,3}(N) = R'_{1,3}(N) + R_{1,3}^*(N), \quad (70)$$

where

$$R'_{1,3}(N) = \sum_{\substack{d \leq D \\ m \leq D \\ 2|m}} \frac{\chi(d) \Phi^{(3)}(N, \langle [1, N], J_m \rangle)}{\varphi(d) \varphi(4m)} \sum_{j=\pm 1} \chi(j) \mathfrak{S}_{(d, 4m), \langle 1, 1+jm \rangle}^{(3)}(N), \quad (71)$$

$$R^*_{1,3}(N) = \sum_{\substack{d \leq D \\ m \leq D \\ 2|m}} \chi(d) \sum_{j=\pm 1} \chi(j) \Delta_{(d, 4m), \langle 1, 1+jm \rangle}^{(3)}(N, \langle [1, N], J_m \rangle).$$

From (35) and Lemma 2 we find

$$R^*_{1,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (72)$$

Consider $R'_{1,3}(N)$. According to Lemma 3 the expression $\mathfrak{S}_{(d, 4m), \langle 1, 1+jm \rangle}^{(3)}(N)$ does not depend on j . Therefore from (71) it follows that

$$R'_{1,3}(N) = 0. \quad (73)$$

Using (70), (72) and (73) we obtain

$$R_{1,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (74)$$

The estimation of $R_{2,2}(N)$. Let $\mathcal{P}$ be the set of primes, specified in Lemma 6 (with $\omega = B(10) + C(10) + 1$). Using (35), (37), (39) and the inequality $uv \leq u^2 + v^2$ we get

$$R_{2,2}(N) \ll \mathcal{L}^3 \sum_{\substack{p_1+p_2+p_3=N \\ p_2 \in \mathcal{P}}} \left| \sum_{\substack{d|p_1-1 \\ D < d < N/D}} \chi(d) \right|^2 = \mathcal{L}^3 \sum_{p_1 < N} \left| \sum_{\substack{d|p_1-1 \\ D < d < N/D}} \chi(d) \right|^2 \sum_{\substack{p_2+p_3=N-p_1 \\ p_2 \in \mathcal{P}}} 1.$$

We estimate the sum over p_2, p_3 using Lemma 6 and we find

$$R_{2,2}(N) \ll N \mathcal{L}^{1-2\theta_0} (\log \mathcal{L})^6 \sum_{p < N} \left| \sum_{\substack{d|p-1 \\ D < d < N/D}} \chi(d) \right|^2.$$

Then we apply Lemma 5 and obtain

$$R_{2,2}(N) \ll N^2 \mathcal{L}^{-2\theta_0} (\log \mathcal{L})^{13}. \quad (75)$$

The estimation of $R_{2,3}(N)$. Using (10), (37) and (39) we write $R_{2,3}(N)$ in the form

$$\begin{aligned} R_{2,3}(N) &= \sum_{p_1+p_2+p_3=N} (\log p_1)(\log p_2)(\log p_3) r_2(p_1-1) \sum_{\substack{m|p_2-1 \\ \frac{p_2-1}{m} \geq N/D}} \chi\left(\frac{p_2-1}{m}\right) \\ &= \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \chi(j) I_{4m, 1+jm}^{(2)}(N-p, J_m), \end{aligned}$$

where $J_m = [1 + mN/D, N]$. Having in mind (13) we write

$$R_{2,3}(N) = R'_{2,3}(N) + R^*_{2,3}(N), \quad (76)$$

where

$$R'_{2,3}(N) = \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{\substack{m \leq D \\ 2|m}} \frac{\Phi^{(2)}(N-p, J_m)}{\varphi(4m)} \sum_{j=\pm 1} \chi(j) \mathfrak{S}_{4m, 1+jm}^{(2)}(N-p), \quad (77)$$

$$R^*_{2,3}(N) = \sum_{2 < p < N} (\log p) r_2(p-1) \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} \chi(j) \Delta_{4m, 1+jm}^{(2)}(N-p, J_m). \quad (78)$$

Consider first $R'_{2,3}(N)$. From Lemma 3 we know that $\mathfrak{S}_{4m, 1+jm}^{(2)}(N-p)$ does not depend on j . Hence using (77) we get

$$R'_{2,3}(N) = 0. \quad (79)$$

Consider now $R^*_{2,3}(N)$. From (37), (78) and Cauchy's inequality we find

$$\begin{aligned} R^*_{2,3}(N) &\ll \mathcal{L} \sum_{2 < p < N} \tau(p-1) \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} |\Delta_{4m, 1+jm}^{(2)}(N-p, J_m)| \\ &\ll \mathcal{L} \sum_{n \leq N} \tau(n) \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} |\Delta_{4m, 1+jm}^{(2)}(n, J_m)| \\ &\ll \mathcal{L} U_1^{1/2} V_1^{1/2}, \end{aligned} \quad (80)$$

where

$$U_1 = \sum_{n \leq N} \tau^2(n) \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} |\Delta_{4m, 1+jm}^{(2)}(n, J_m)|, \quad V_1 = \sum_{n \leq N} \sum_{\substack{m \leq D \\ 2|m}} \sum_{j=\pm 1} |\Delta_{4m, 1+jm}^{(2)}(n, J_m)|.$$

We use the trivial bound $\Delta^{(2)} \ll \mathcal{L}^2 N m^{-1}$ and the inequality $\sum_{n \leq x} \tau^2(n) \ll x \log^3 x$ to find

$$U_1 \ll N^2 \mathcal{L}^6. \quad (81)$$

We estimate V_1 using (35) and Lemma 1 and we get

$$V_1 \ll N^2 \mathcal{L}^{-10}. \quad (82)$$

Using (80), (81) and (82) we obtain

$$R^*_{2,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (83)$$

Now taking into account (76), (79) and (83) we find

$$R_{2,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (84)$$

The estimation of $R_{3,3}(N)$. We use (14), (37) and (39) to write

$$\begin{aligned} R_{3,3}(N) &= \sum_{p_1+p_2+p_3=N} (\log p_1)(\log p_2)(\log p_3) \sum_{\substack{m_1|p_1-1 \\ \frac{p_1-1}{m_1} \geq N/D}} \chi\left(\frac{p_1-1}{m_1}\right) \sum_{\substack{m_2|p_2-1 \\ \frac{p_2-1}{m_2} \geq N/D}} \chi\left(\frac{p_2-1}{m_2}\right) \\ &= \sum_{\substack{m_1, m_2 \leq D \\ 2|m_1, 2|m_2}} \sum_{\substack{j_1=\pm 1 \\ j_2=\pm 1}} \chi(j_1)\chi(j_2) I_{\langle 4m_1, 4m_2 \rangle, \langle 1+j_1m_1, 1+j_2m_2 \rangle}^{(3)}(N, \mathbf{J}_m), \end{aligned}$$

where $\mathbf{J}_m = \langle J_{m_1}, J_{m_2} \rangle$; $J_{m_v} = [1 + m_v N/D, N]$, $v = 1, 2$. We write

$$R_{3,3}(N) = R'_{3,3}(N) + R^*_{3,3}(N), \quad (85)$$

where

$$\begin{aligned} R'_{3,3}(N) &= \sum_{\substack{m_1, m_2 \leq D \\ 2|m_1, 2|m_2}} \frac{\Phi^{(3)}(N, \mathbf{J}_m)}{\varphi(4m_1)\varphi(4m_2)} \sum_{\substack{j_1=\pm 1 \\ j_2=\pm 1}} \chi(j_1)\chi(j_2) \mathfrak{S}_{\langle 4m_1, 4m_2 \rangle, \langle 1+j_1m_1, 1+j_2m_2 \rangle}^{(3)}(N), \\ R^*_{3,3}(N) &= \sum_{\substack{m_1, m_2 \leq D \\ 2|m_1, 2|m_2}} \sum_{\substack{j_1=\pm 1 \\ j_2=\pm 1}} \chi(j_1)\chi(j_2) \Delta_{\langle 4m_1, 4m_2 \rangle, \langle 1+j_1m_1, 1+j_2m_2 \rangle}^{(3)}(N, \mathbf{J}_m). \end{aligned} \quad (86)$$

According to Lemma 3, the expression $\mathfrak{S}^{(3)}$ in (86) does not depend of j_2 and therefore

$$R'_{3,3}(N) = 0. \quad (87)$$

On the other hand, using (35) and Lemma 2 we get

$$R^*_{3,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (88)$$

From (85), (87) and (88) it follows that

$$R_{3,3}(N) \ll N^2 \mathcal{L}^{-1}. \quad (89)$$

The end of the proof. The asymptotic formula (9) is a consequence of (38), (39), (60), (69), (74), (75), (84) and (89). The theorem is proved. $\square$

References

- [1] B. Bredihin, The dispersion method and binary additive problems, Russian Math. Surveys 20 (1965) 85–125.
- [2] H. Halberstam, H.-E. Richert, Sieve Methods, Academic Press, 1974.
- [3] K. Halupczok, On the ternary Goldbach problem with primes in independent arithmetic progressions, Acta Math. Hungar. 120 (4) (2008) 315–349.
- [4] C. Hooley, Applications of Sieve Methods to the Theory of Numbers, Cambridge Univ. Press, 1976.
- [5] M. Laporta, A short intervals result for $2n$ -twin primes in arithmetic progressions, Tsukuba J. Math. 23 (2) (1999) 201–214.
- [6] Y. Linnik, The Dispersion Method in Binary Additive Problems, Izdat. Leningrad Gos. Univ., Leningrad, 1961 (in Russian).

- [7] K. Matomäki, The binary Goldbach problem with one prime of the form $p = k^2 + l^2 + 1$, *J. Number Theory* 128 (2008) 1195–1210.
- [8] H. Mikawa, On prime twins in arithmetic progressions, *Tsukuba J. Math.* 16 (2) (1992) 377–387.
- [9] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press, 1995.
- [10] D. Tolev, The ternary Goldbach problem with primes from arithmetic progressions, arXiv:0902.2669v1, 2009; *Q. J. Math.*, in press.
- [11] D. Tolev, The binary Goldbach problem with arithmetic weights attached to one of the variables, arXiv:0903.3128v1, 2009; *Acta Arith.*, in press.
- [12] R. Vaughan, *The Hardy–Littlewood Method*, Cambridge Univ. Press, 1997.
- [13] I.M. Vinogradov, Representation of an odd number as a sum of three primes, *Dokl. Akad. Nauk SSSR* 15 (1937) 169–172 (in Russian).