ARTICLE IN PRESS YJNTH:5756

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Eichler–Shimura isomorphism and group cohomology on arithmetic groups ☆,☆☆

Santiago Molina

CRM, Centre de Recerca Matemàtica, Barcelona, Spain

ARTICLE INFO

Article history:
Received 20 January 2017
Received in revised form 18 April 2017
Accepted 20 April 2017
Available online xxxx
Communicated by D. Goss

Keywords:
Eichler–Shimura
Automorphic forms
Cohomology of arithmetic groups
Automorphic representations

ABSTRACT

In this article, we give a group cohomological interpretation to the Eichler–Shimura isomorphism. For any quaternion algebra A over a totally real field with multiplicative group G, we interpret a weight $(k_1, k_2, \cdots, k_d)$-automorphic form of G as a $G(F)$-invariant homomorphism of $(\mathcal{G}_\infty, K_\infty)$-modules. Then the Eichler–Shimura isomorphism is given by the connection morphism provided by the natural exact sequences defining the $(\mathcal{G}_\infty, K_\infty)$-module of discrete series of weight $(k_1, k_2, \cdots, k_d)$.

© 2017 Elsevier Inc. All rights reserved.

0. Introduction

The Eichler–Shimura isomorphism establishes a bijection between the space of modular forms and certain cohomology groups with coefficients in a space of polynomials. More precisely, let $k \geq 2$ be an integer and let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup, then we have the following isomorphism of Hecke modules

☆ The author is supported in part by DGICYT Grant MTM2015-63829-P.

☆☆ The author/authors have been partially funded by the CERCA Programme of the Generalitat de Catalunya.

E-mail address: smolina@crm.cat.

http://dx.doi.org/10.1016/j.jnt.2017.04.007
0022-314X/© 2017 Elsevier Inc. All rights reserved.

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007

$$M_k(\Gamma,\mathbb{C}) \oplus S_k(\Gamma,\mathbb{C}) \simeq H^1(\Gamma, V(k)^\vee), \tag{0.1}$$

where $V(k)^\vee$ is the dual of the $\mathbb{C}$-vector space of homogeneous polynomials of degree $k-2$, $M_k(\Gamma,\mathbb{C})$ is the space of modular forms of weight k and $S_k(\Gamma,\mathbb{C}) \subset M_k(\Gamma,\mathbb{C})$ is the subspace of cuspidal modular forms (see [6, Thm. 8.4] and [4, Thm. 6.3.4]).

This isomorphism can be interpreted in geometric terms. Indeed, a modular form of weight k can be interpreted as a section of certain sheaf of differential forms on the open modular curve attached to Γ. With this in mind, the Eichler–Shimura isomorphism can be obtained comparing deRham and singular cohomology, noticing that the singular cohomology of the open modular curve is given by the group cohomology $H^\bullet(\Gamma, V(k)^\vee)$. The aim of this paper is to omit this geometric interpretation and to provide a new group cohomological interpretation.

Let us remark that the identification (0.1) provides an integral and rational structure to the space of modular forms, since the space of polynomials $V(k)$ has integral and rational models, namely, the space of polynomials with integer and rational coefficients.

The restriction of the Eichler–Shimura isomorphism to the spaces of cuspidal modular forms is given by the morphisms

$$\partial^\pm : S_k(\Gamma,\mathbb{C}) \longrightarrow H^1(\Gamma, V(k)^\vee),$$

where

$$\partial^\pm(f)(\gamma)(P) = \int_{z_0}^{\gamma z_0} P(1,-\tau)f(\tau)d\tau \pm \int_{z_0}^{\gamma z_0} P(1,\bar\tau)f(-\bar\tau)d(-\bar\tau),$$

for any z_0 in $\mathcal{H}$ the Poincaré hyperplane, $\gamma \in \Gamma$ and $P \in V(k)$. In fact, the morphism defining the cuspidal part of (0.1) is given by

$$\begin{array}{ccc} S_k(\Gamma,\mathbb{C}) \oplus \overline{S_k(\Gamma,\mathbb{C})} & \longrightarrow & H^1(\Gamma, V(k)^\vee) \\ (f_1, \bar f_2) & \longmapsto & (\partial^+ + \partial^-)(f_1) + (\partial^+ - \partial^-)(f_2). \end{array}$$

The image of $\partial^\pm$ lies in the subspaces $H^1(\Gamma, V(k)^\vee)^\pm \subset H^1(\Gamma, V(k)^\vee)$ where the natural action of $\omega = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizing Γ acts by ± 1.

In a general setting, F is a totally real number field of degree d, G is the multiplicative group of a quaternion algebra A over F, and ϕ is a weight $\underline{k} = (k_1, \cdots, k_d)$ cuspidal automorphic form of G with level $\mathcal{U} \subset G(\mathbb{A}^\infty)$ and central character $\psi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$. We assume that $\psi_{\sigma_i}(x) = \mathrm{sign}(x)^{k_i}|x|^{\mu_i}$, for any archimedean place $\sigma_i : F \hookrightarrow \mathbb{R}$. In this scenario, by an automorphic form we mean a function on $\mathcal{H}^r \times G(\mathbb{A}^\infty)/\mathcal{U}$, where $\mathcal{H}$ is the Poincaré upperplane and r is the cardinal of the set Σ of archimedean places where A splits, with values in $\bigotimes_{\sigma_i \in \infty\setminus\Sigma} V_{\mu_i}(k_i)^\vee$ (see §1.2 for a precise definition of $V_{\mu_i}(k_i)$), that satisfies the usual transformation laws with respect to the weight-k_i-actions of $G(F)$. The interesting cohomology subgroups to consider are:

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007

$$H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}}) = \bigoplus_{i=1}^{n} H^r(\Gamma_{g_i}, V_\psi(\underline{k})^\vee), \qquad \Gamma_{g_i} = G(F)^+ \cap g_i \mathcal{U} g_i^{-1},$$

where $V_\psi(\underline{k})$ is the tensor product of the polynomial spaces $V_{\mu_i}(k_j)$ $(j = 1, \cdots, d)$, $G(F)^+ \subseteq G(F)$ is the subgroup of totally positive elements, $\{g_i\}_{i=1,\cdots,n} \subset G(\mathbb{A}^\infty)$ is a set of representatives of the double coset space $G(F)^+ \backslash G(\mathbb{A}^\infty)/\mathcal{U}$, and $\mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} = C(G(\mathbb{A}^\infty)/\mathcal{U}, V_\psi(\underline{k})^\vee)$. Similarly as in the classical case, for any character $\varepsilon : G(F)/G(F)^+ \longrightarrow \pm 1$ we can define a morphism

$$\partial^\varepsilon : S_{\underline{k}}(\mathcal{U}, \psi) \longrightarrow H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}})(\varepsilon),$$

from the set $S_{\underline{k}}(\mathcal{U}, \psi)$ of automorphic cuspforms of weight $\underline{k}$, level $\mathcal{U}$ and central character ψ, to the ε-isotypical component of $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}})$. Such a map is given by:

$$\partial^\epsilon \phi = \sum_{\gamma \in G(F)/G(F)^+} \epsilon(\gamma) \partial \phi^\gamma,$$

where $\partial\phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ is the class of the cocycle

$$(G(F)^+)^r \ni (g_1, g_2, \cdots, g_r) \longmapsto \int_{\tau_1}^{g_1\tau_1} \cdots \int_{g_1\cdots g_{r-1}\tau_r}^{g_1\cdots g_r\tau_r} P_\Sigma(1, -\underline{z}) \langle \phi(\underline{z}, g), P^\Sigma \rangle d\underline{z},$$

with $P_\Sigma \in \bigotimes_{j=1}^r V_{\mu_j}(k_j)$, $P^\Sigma \in \bigotimes_{j=r+1}^d V_{\mu_j}(k_j)$ and $\underline{z} = (z_1, \cdots, z_r), (\tau_1, \cdots, \tau_r) \in \mathcal{H}^r$. Our result will provide a group cohomological interpretation to the morphisms ∂^ε, for any character ε.

Let $F_\infty \simeq \mathbb{R}^d$ be the product of the archimedean completions of F, let $\mathcal{G}_\infty$ be the Lie algebra of $G(F_\infty)$ and let $K_\infty \subseteq G(F_\infty)$ be a maximal compact subgroup. Then the $(\mathcal{G}_\infty, K_\infty)$-module generated by ϕ is isomorphic to $D_\psi(\underline{k})$, the tensor product of discrete series of weight k_j at archimedean places in Σ and polynomial spaces $V_{\mu_j}(k_j)$ at archimedean places not in Σ. This implies that any $\phi \in S_{\underline{k}}(\mathcal{U}, \psi)$ provides an element

$$\phi \in H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}}); \qquad \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} := \mathrm{Hom}_{(\mathcal{G}_\infty, K_\infty)}(D_\psi(\underline{k}), \mathcal{A}^{\mathcal{U}}),$$

where $\mathcal{A}^{\mathcal{U}}$ is the $(\mathcal{G}_\infty, K_\infty)$-module of smooth admissible functions $f : G(\mathbb{A})/\mathcal{U} \to \mathbb{C}$. Our main result (Theorem 2.4) can be rewritten as follows:

Theorem 0.1. *There exists an exact sequence of $G(F)$-modules*

$$0 \to \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})^{\mathcal{U}} \to \mathcal{A}(I_1^\varepsilon(\underline{k}), \mathbb{C})^{\mathcal{U}} \to \mathcal{A}(I_2^\varepsilon(\underline{k}), \mathbb{C})^{\mathcal{U}} \to \cdots \to \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})^{\mathcal{U}} \to 0,$$

such that, up to an explicit constant, the morphism ∂^ϵ is given by the corresponding connection morphism

$$H^0(G(F),\mathcal{A}(D_\psi(\underline{k}),\mathbb{C})^{\mathcal{U}}) \longrightarrow H^r(G(F),\mathcal{A}(V_\psi(\underline{k})(\varepsilon),\mathbb{C})^{\mathcal{U}})$$
$$\simeq H^r(G(F)^+,\mathcal{A}(V_\psi(\underline{k}),\mathbb{C})^{\mathcal{U}})(\varepsilon).$$

We obtain the above exact sequence from extensions of the $(\mathcal{G}_{\sigma_i},K_{\sigma_i})$-modules of discrete series $D_{\mu_i}(k_i)$ at every place $\sigma_i\in\Sigma$. The archimedean local nature of these connection morphisms ∂^ε implies that the $G(\mathbb{A}^\infty)$-representation generated by $\partial^\varepsilon\phi$ coincides with the restriction to $G(\mathbb{A}^\infty)$ of π_ϕ, the automorphic representation attached to ϕ.

The image of a cuspidal automorphic representation π_ϕ through the morphisms ∂^ε is used in many papers to give a group cohomological construction of cyclotomic and anti-cyclotomic p-adic L-functions and Stickelberger elements attached to quadratic extensions of a totally real number field (see for instance [7,5,1]). The explicit form of ∂^ε given in Theorem 2.4 provides the interpolation properties of these objects.

Another application is the construction of Stark–Heegner points. By means of the connection morphisms $\partial^{\varepsilon^{\pm 1}}$, where $(\varepsilon^+,\varepsilon^-)$ is a well chosen pair of characters, one can construct a complex torus $\mathbb{C}^{[L:\mathbb{Q}]}/\Lambda$ attached to a weight 2 automorphic representation π_ϕ with field of coefficients L. It is conjectured that such complex torus coincides with the abelian variety of GL_2-type attached to π_ϕ. In [3], we use the cohomological description of $\partial^{\varepsilon^{\pm 1}}$ to construct Stark–Heegner points in the complex torus, that we conjecture to be global points in the corresponding abelian variety. Such points are conjecturally defined over class fields of quadratic extensions of F and satisfy explicit reciprocity laws.

Notation. Throughout this paper, we will denote by $\int_{S^1}d\theta=\int_{\mathrm{SO}(2)}d\theta$ the Haar measure of $S^1=\mathrm{SO}(2)$ such that $\mathrm{vol}(S^1)=\pi$.

Let F be a number field. For any place v of F, we denote by F_v its completion at v. Given a finite set of places S of F, we denote by F_S the product of completions at every place in S. We denote by F_∞ the product of completions at every archimedean place. Similarly, for any subset Σ of archimedean places, $F_{\infty\setminus\Sigma}$ will be the product of completions at every archimedean place not in Σ. We denote by $\mathbb{A}$ the ring of adeles of F. For any set S of places of F, we write $\mathbb{A}^S$ for the ring adeles outside S. Consistent with this notation, we denote by $\mathbb{A}^\infty$ the ring of finite adeles of F.

1. Discrete series

1.1. Finite dimensional representations

Let A be a quaternion algebra defined over a local field F. Let K/F be an extension where A splits. For any natural number $k\in\mathbb{N}$, let $\mathcal{P}_{k-2}^K\simeq\mathrm{Sym}^{k-2}(K^2)$ be the finite K-vector space of homogeneous polynomials of degree $k-2$. We have a well defined action of $\mathrm{GL}_2(K)$ on $\mathcal{P}_{k-2}^K$ given by

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} * P\right)(x,y) := P\left((x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = P\left(ax+cy, bx+dy\right). \tag{1.2}$$

If we fix an embedding

$$\iota : A \longrightarrow A \otimes_F K \simeq \mathrm{M}_2(K),$$

then $\mathcal{P}_{k-2}^K$ is equipped with an action of $A^\times$.

We denote by $\det : A \to F$ the reduced norm of A, and let us consider

$$A^+ = \{a \in A : \det(a) \in F^2\}.$$

We write $V(k)_K = \mathcal{P}_{k-2}^K \otimes \det^{\frac{2-k}{2}}$ with the natural action of A^+. It is clear that the centre of $A^\times$ acts trivially on $V(k)_K$. Notice that, if k is even, the action of A^+ on $V(k)_K$ extends to a natural action of A.

1.2. *Discrete series and exact sequences*

Assume that $F = \mathbb{R}$ and $A = \mathrm{M}_2(\mathbb{R})$. Let $\mathcal{G}L_2(\mathbb{R})$ be the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$. For any $k \in \mathbb{Z}$ and $\mu \in \mathbb{C}$, we define $I_\mu(k)$ as the $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-module of smooth admissible vectors in

$$\left\{ f : \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C} : \ f\left(\begin{pmatrix} t_1 & x \\ & t_2 \end{pmatrix} g\right) = \mathrm{sign}(t_1)^k (t_1 t_2)^{\frac{\mu}{2}} \left(\frac{t_1}{t_2}\right)^{\frac{k}{2}} f(g) \right\}.$$

Write $V_\mu(k) = V(k)_{\mathbb{C}} \otimes \det^{\frac{\mu}{2}}$. If we assume that $k \geq 2$, we have the well defined morphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules

$$\iota : V_\mu(k) \longrightarrow I_\mu(2-k); \qquad \iota(P)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{\frac{2-k+\mu}{2}} P(c,d).$$

Moreover, we have a $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-invariant pairing (see [2, §2])

$$\langle\ ,\ \rangle_I : I_\mu(k) \times I_{-\mu}(2-k) \longrightarrow \mathbb{C}; \qquad (f,g) \longmapsto \int_{\mathrm{SO}(2)} f(\theta) g(\theta) d\theta,$$

providing the morphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules

$$\bar\varphi : I_\mu(k) \longrightarrow V_{-\mu}(k)^\vee; \qquad \langle \bar\varphi(f), P\rangle_V := \langle f, \iota(P)\rangle_I.$$

Composing with the natural $\mathrm{GL}_2(\mathbb{R})^+$-morphism

$$V_{-\mu}(k)^\vee \longrightarrow V_\mu(k); \qquad F \longmapsto P_F(x,y) = \langle F, (Yx - Xy)^{k-2}\rangle_{V_{(X,Y)}}, \tag{1.3}$$

we obtain a map

$$\begin{aligned}&\varphi : I_\mu(k) \longrightarrow V_\mu(k);\\ &\varphi(f)(x,y) \;=\; \langle \bar\varphi(f), (Yx - Xy)^{k-2}\rangle_{V(X,Y)} = \int_{S^1} f(\theta)(y\sin\theta + x\cos\theta)^{k-2} d\theta.\end{aligned}$$

Remark 1.1. Notice that we have the symmetry

$$\begin{aligned}\langle F, P_G\rangle_V &= \langle F, \langle G, (Yx - Xy)^{k-2}\rangle_{V(X,Y)}\rangle_{V(x,y)}\\ &= (-1)^k\langle G, \langle F, (Xy - Yx)^{k-2}\rangle_{V(x,y)}\rangle_{V(X,Y)} = (-1)^k\langle G, P_F\rangle_V,\end{aligned}$$

for all $F, G \in V_\mu(k)^\vee$.

The kernel of φ is $D_\mu(k)$ the *Discrete Series* $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module of weight k and central character $x \mapsto \mathrm{sign}(x)^k|x|^\mu$. This definition implies that $D_\mu(k)$ lies in the following exact sequence of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules:

$$0 \longrightarrow D_\mu(k) \longrightarrow I_\mu(k) \xrightarrow{\varphi} V_\mu(k) \longrightarrow 0. \tag{1.4}$$

Since any $g \in \mathrm{GL}_2(\mathbb{R})^+$ can be written uniquely as $g = u \cdot \tau(x,y) \cdot \kappa(\theta)$, where

$$u \in \mathbb{R}^+,\ \tau(x,y) = \begin{pmatrix} y^{1/2} & xy^{-1/2}\\ & y^{-1/2}\end{pmatrix} \in B,\ \kappa(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} \in \mathrm{SO}(2),$$

we have that

$$I_\mu(k) = \bigoplus_{t\equiv k \pmod 2} \mathbb{C} f_t; \qquad f_t(u\cdot\tau(x,y)\cdot\kappa(\theta)) = u^\mu y^{\frac{k}{2}} e^{ti\theta}.$$

The $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-module structure of $I_s(k)$ can be described as follows: Let $L, R \in \mathcal{G}L_2(\mathbb{R})$ be the *Maass differential operators* defined in [2, §2.2]

$$L = e^{-2i\theta}\left(-iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial\theta}\right), \quad R = e^{2i\theta}\left(iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial\theta}\right).$$

Then, the $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-module $I_s(k)$ is characterized by the relations:

$$Rf_t = \left(\frac{k+t}{2}\right) f_{t+2}; \qquad Lf_t = \left(\frac{k-t}{2}\right) f_{t-2}; \tag{1.5}$$

$$\kappa(\theta)f_t = e^{ti\theta} f_t; \qquad uf_t = u^\mu f_t, \tag{1.6}$$

for any $\kappa(\theta) \in \mathrm{SO}(2)$ and $u \in \mathbb{R}^+ \subset \mathrm{GL}_2(\mathbb{R})^+$.

Write $z = x + iy$ and $\bar{z} = x - iy$. For $n \in \{0, 1, \cdots, k-2\}$, let us consider the elements $P_n \in V_\mu(k)$, $P_n(x,y) = z^n \bar{z}^{k-2-n}$. It is clear that $\{P_n\}_{n=0,\cdots,k-2}$ is a basis for the $\mathbb{C}$-vector space $V_\mu(k)$. We compute that

$$\begin{aligned}\varphi(f)(x,y) &= \int_{S^1} f(\theta)((2i^{-1})(z-\bar{z})\sin\theta + 2^{-1}(z+\bar{z})\cos\theta)^{k-2} d\theta \\ &= \int_{S^1} 2^{2-k} f(\theta)(ze^{-i\theta} + \bar{z}e^{i\theta})^{k-2} d\theta \\ &= 2^{2-k} \sum_{n=0}^{k-2} \binom{k-2}{n} P_n(x,y) \int_{S^1} f(\theta) e^{-i(2n-k+2)\theta} d\theta\end{aligned}$$

By orthogonality, we deduce that $\varphi(f_{2n-k+2}) = 2^{2-k}\pi\binom{k-2}{n} P_n(x,y)$.

Since $\kappa(\theta)P_n = e^{(2n-k+2)i\theta} P_n$, the morphism of $\mathbb{C}$-vector spaces

$$s : V_\mu(k) \longrightarrow I_\mu(k); \qquad s(P_n) = \frac{2^{k-2}}{\pi}\binom{k-2}{n}^{-1} f_{2n-k+2}, \tag{1.7}$$

defines a section of φ as $\mathrm{SO}(2)\mathbb{R}^+$-modules.

Remark 1.2. Since (1.3) is an isomorphism, we can define a non-degenerate $\mathrm{GL}_2(\mathbb{R})^+$-invariant bilinear pairing $V_\mu(k) \times V_{-\mu}(k) \to \mathbb{C}$

$$\langle P_F, Q \rangle = \langle F, Q \rangle_V, \qquad F \in V_{-\mu}(k)^\vee_{\mathbb{C}}, \quad Q \in V_{-\mu}(k),$$

which is symmetric or antisymmetric depending on the parity of k, by Remark 1.1. Moreover, by the definition of (1.3),

$$P(s,t) = \langle P, Q_{s,t} \rangle, \qquad Q_{s,t}(x,y) = (ys - xt)^{k-2}.$$

In particular,

$$P(-1,i) = i^{2-k}\langle P, P_0 \rangle = i^{k-2}\langle P_0, P \rangle. \tag{1.8}$$

Since s is a section of φ, we compute

$$\langle P, Q \rangle = \langle \bar{\varphi}(s(P)), Q \rangle_V = \langle s(P), \iota(Q) \rangle_I, \tag{1.9}$$

for all $P, Q \in V_\mu(k)$.

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007

1.3. The $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module of discrete series

We want to give structure of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module to $I_\mu(k)$. Hence, we have to define the action of $\omega = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in O(2) \setminus \mathrm{SO}(2)$. That is to say, we have to define $\omega \in \mathrm{End}(I_\mu(k))$ such that

$$(i) \quad \omega f_t \in \mathbb{C} f_{-t}; \qquad (ii) \quad \omega^2 = 1; \qquad (iii) \quad \omega R = L\omega.$$

If we write $\omega f_t = \lambda(t) f_{-t}$, condition (ii) implies that $\lambda(t)\lambda(-t) = 1$. Moreover, condition (iii) implies that $\lambda(t) = \lambda(t+2)$. We obtain two possible $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module structures for $I_\mu(k)$: Letting $\lambda(t) = 1$ for all $t \equiv k \bmod 2$, or letting $\lambda(t) = -1$ for all $t \equiv k \bmod 2$. Write $I_\mu(k)^\pm$ for the $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module such that $\omega f_t = \pm f_{-t}$, respectively.

By abuse of notation, write also $V_\mu(k)$ and $V_\mu(k)_\mathbb{R}$ (in case $\mu \in \mathbb{R}$) for the $\mathrm{GL}_2(\mathbb{R})$-representations

$$V_\mu(k) = V_\mu(k)_\mathbb{C} = \mathcal{P}^\mathbb{C}_{k-2} \otimes |\det|^{\frac{2-k+\mu}{2}}, \qquad V_\mu(k)_\mathbb{R} = \mathcal{P}^\mathbb{R}_{k-2} \otimes |\det|^{\frac{2-k+\mu}{2}}.$$

Remark 1.3. With this $\mathrm{GL}_2(\mathbb{R})$-module structure, the pairing $\langle \cdot, \cdot \rangle$ introduced in Remark 1.2 is not a $\mathrm{GL}_2(\mathbb{R})$-invariant in general. In fact one can show that

$$\langle gP, Q \rangle = \mathrm{sign}(\det g)^k \langle P, g^{-1} Q \rangle.$$

Note that, for any $f \in I_\mu(k)^\pm$, we have that $\omega f(\theta) = \pm f(-\theta)$, hence we compute that,

$$\begin{aligned} \varphi(\omega f)(x, y) &= \int_{S^1} \omega f(\theta)(x\cos\theta + y\sin\theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(-\theta)(x\cos\theta + y\sin\theta)^{k-2} d\theta \\ &= \pm \int_{S^1} f(\theta)(x\cos\theta - y\sin\theta)^{k-2} d\theta = \pm \omega(\varphi(f))(x, y) \end{aligned}$$

This implies that the exact sequence of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules (1.4) provides the exact sequences of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules

$$0 \longrightarrow D_\mu(k) \xrightarrow{\iota} I_\mu(k)^+ \longrightarrow V_\mu(k) \longrightarrow 0, \tag{1.10}$$

$$0 \longrightarrow D_\mu(k) \xrightarrow{\iota \circ I} I_\mu(k)^- \longrightarrow V_\mu(k)(\varepsilon) \longrightarrow 0, \tag{1.11}$$

where $\varepsilon : \mathrm{GL}_2(\mathbb{R}) \to \pm 1$ is the character given by $\varepsilon(g) = \operatorname{sign}\det(g)$, $D_\mu(k)$ is the $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module with fixed action of ω given by $\omega f(\theta) = f(-\theta)$, and I is the automorphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules

$$I : D_\mu(k) \longrightarrow D_\mu(k) : \quad I(f_t) = \mathrm{sign}(t) f_t.$$

Note that $\iota \circ I$ is a monomorphism of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules because $I(\omega f) = -\omega(I(f))$.

1.4. Matrix coefficients

Let us consider $A(\mathbb{C})$ the $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-module of admissible C^∞ functions $f : \mathrm{GL}_2(\mathbb{R})^+ \to \mathbb{C}$. For any $f_0 \in I_{-\mu}(2-k)$, I claim that

$$\varphi_{f_0} : I_\mu(k) \longrightarrow A(\mathbb{C}), \qquad \varphi_{f_0}(f)(g_\infty) = \langle g_\infty f, f_0 \rangle_I, \qquad g_\infty \in \mathrm{GL}_2(\mathbb{R})^+,$$

provides a well defines morphism of $(\mathcal{G}L_2(\mathbb{R}), \mathrm{SO}(2))$-modules. Indeed, for any element of the Lie algebra $G \in \mathcal{G}L_2(\mathbb{R})$,

$$\begin{aligned} G\varphi_{f_0}(f)(g_\infty) &= \frac{d}{dt}(\varphi_{f_0}(g_\infty \exp(tG))) \mid_{t=0} = \frac{d}{dt}(\langle g_\infty \exp(tG) f, f_0 \rangle_I) \mid_{t=0} \\ &= \varphi_{f_0}(Gf)(g_\infty). \end{aligned}$$

1.5. $\mathbb{R}$-structures of discrete series

As we can see in [2, §2.2], R and L are not in $\mathcal{G}L_2(\mathbb{R})$, they are *Caley transformations* in $\mathcal{G}L_2(\mathbb{C})$ of elements in $\mathcal{G}L_2(\mathbb{R})$. In fact, $\mathcal{G}L_2(\mathbb{R})$ is generated by

$$\begin{aligned} R + L &= -2y\sin(2\theta)\frac{\partial}{\partial x} + 2y\cos(2\theta)\frac{\partial}{\partial y} + \sin(2\theta)\frac{\partial}{\partial \theta}; \qquad u\frac{\partial}{\partial u}; \\ i(R-L) &= -2y\cos(2\theta)\frac{\partial}{\partial x} - 2y\sin(2\theta)\frac{\partial}{\partial y} + \cos(2\theta)\frac{\partial}{\partial \theta}; \quad \text{and} \quad \frac{\partial}{\partial \theta}. \end{aligned}$$

If we define $h_t := f_t + f_{-t} \in I_\mu(k)^\pm$ and $g_t := i(f_t - f_{-t}) \in I_\mu(k)^\pm$, it is easy to compute that

$$\begin{aligned} (R+L)h_t &= \left(\frac{k+t}{2}\right) h_{t+2} + \left(\frac{k-t}{2}\right) h_{t-2}, \qquad \frac{\partial}{\partial \theta} h_t = -t g_t, \\ i(R-L)h_t &= \left(\frac{k+t}{2}\right) g_{t+2} - \left(\frac{k-t}{2}\right) g_{t-2}, \qquad \omega h_t = \pm h_t \\ \kappa(\theta) h_t &= \cos(t\theta) h_t - \sin(t\theta) g_t, \qquad \omega g_t = \mp g_t. \end{aligned}$$

Hence the $\mathbb{R}$-vector space $I_\mu(k)^\pm_\mathbb{R} \subset I_\mu(k)^\pm$ generated by h_t and g_t defines a $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module over $\mathbb{R}$.

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007

We check that the morphisms $\varphi : I_\mu(k)^+ \to V_\mu(k)_\mathbb{C}$ and $\varphi : I_\mu(k)^- \to V_\mu(k)(\varepsilon)_\mathbb{C}$ descend to morphisms of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules over $\mathbb{R}$

$$\varphi^+ : I_\mu(k)^+_\mathbb{R} \longrightarrow V_\mu(k)_\mathbb{R}, \qquad \varphi^- : I_\mu(k)^-_\mathbb{R} \longrightarrow V_\mu(k)_\mathbb{R}(\varepsilon).$$

Hence the kernel $D_\mu(k)_\mathbb{R} \subset D_\mu(k)$ of φ^+ defines a $(\mathcal{G}L_2(\mathbb{R}), O(2))$-module over $\mathbb{R}$, generated by h_k, such that $D_\mu(k)_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = D_\mu(k)$. Nevertheless, the automorphism of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules $I : D_\mu(k) \to D_\mu(k)$ does not descend to an automorphism of $(\mathcal{G}L_2(\mathbb{R}), SO(2))$-modules over $\mathbb{R}$ since $I(h_t) = -\mathrm{sign}(t) i g_t$. In fact,

$$I(D_\mu(k)_\mathbb{R}) = i D_\mu(k)_\mathbb{R} \subset D_\mu(k).$$

We obtain the exact sequences of $(\mathcal{G}L_2(\mathbb{R}), O(2))$-modules over $\mathbb{R}$

$$0 \longrightarrow D_\mu(k)_\mathbb{R} \xrightarrow{\iota} I_\mu(k)^+_\mathbb{R} \longrightarrow V_\mu(k)_\mathbb{R} \longrightarrow 0, \tag{1.12}$$

$$0 \longrightarrow D_\mu(k)_\mathbb{R} \xrightarrow{\iota \circ I} i I_\mu(k)^-_\mathbb{R} \longrightarrow i V_\mu(k)_\mathbb{R}(\varepsilon) \longrightarrow 0. \tag{1.13}$$

2. Connection morphisms

In this section, we assume that G is the multiplicative group of a quaternion algebra that splits at the set of archimedean places Σ. Write $r = \#\Sigma$. Let us consider the $\mathbb{C}$-vector space $\mathcal{A}(\mathbb{C})$ of functions $f : G(\mathbb{A}) \longrightarrow \mathbb{C}$ such that:

- There exists an open compact subgroup $U \subseteq G(\mathbb{A}^\infty)$ such that $f(gU) = f(g)$, for all $g \in G(\mathbb{A})$.
- Under a fixed identification $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$, $f \mid_{G(F_\Sigma)} \in C^\infty(\mathrm{GL}_2(\mathbb{R})^r, \mathbb{C})$.
- Fixing K_Σ, a maximal compact subgroup of $G(F_\Sigma)$ isomorphic to $O(2)^r$, we assume that any $f \in \mathcal{A}(\mathbb{C})$ is K_Σ-finite, namely, its right translates by elements of K_Σ span a finite-dimensional vector space.
- We assume that any $f \in \mathcal{A}(\mathbb{C})$ is $\mathcal{Z}$-finite, where $\mathcal{Z}$ is the centre of the universal enveloping algebra of $G(F_\Sigma)$.

Write ρ for the action of $G(\mathbb{A})$ given by right translation, then $(\mathcal{A}(\mathbb{C}), \rho)$ defines a smooth $G(\mathbb{A}^\infty)$-representation and a $(\mathcal{G}_\infty, K_\infty)$-module, where $\mathcal{G}_\infty$ is the Lie algebra of $G(F_\Sigma)$ and $K_\infty = K_\Sigma \times G(F_{\infty\setminus\Sigma})$. Moreover, $\mathcal{A}(\mathbb{C})$ is also equipped with the $G(F)$-action:

$$(h \cdot f)(g) = f(h^{-1} g), \quad h \in G(F),$$

where $g \in G(\mathbb{A})$, $f \in \mathcal{A}(\mathbb{C})$. Let us fix an isomorphism $G(F_\Sigma) \simeq \mathrm{GL}_2(\mathbb{R})^r$ that maps K_Σ to $O(2)^r$ and let V be a $(\mathcal{G}_\infty, K_\infty)$-module. We define

$$\mathcal{A}(V, \mathbb{C}) := \mathrm{Hom}_{(\mathcal{G}_\infty, K_\infty)}(V, \mathcal{A}(\mathbb{C})),$$

endowed with the natural $G(F)$- and $G(\mathbb{A}^\infty)$-actions.

Remark 2.1. Note that if the $(\mathcal{G}_\infty, K_\infty)$-module V comes from a finite dimensional $G(F_\infty)$-representation V,

$$\mathcal{A}(V, \mathbb{C}) \simeq C(G(\mathbb{A}^\infty), \mathrm{Hom}(V, \mathbb{C})) = C(G(\mathbb{A}^\infty), V^\vee),$$

where $V^\vee$ is seen as a $G(F)$-module by means of the usual injection $G(F) \hookrightarrow G(F_\infty)$, and the action of $G(F)$ on $C(G(\mathbb{A}^\infty), V^\vee)$ is given by $(h * f)(g) = h(f(h^{-1}g))$.

Fix $\sigma \in \Sigma$, $\mu \in \mathbb{C}$ and let us consider $D_\mu(k)$, $V_\mu(k)$ and $V_\mu(k)(\varepsilon)$ as $(\mathcal{G}_\infty, K_\infty)$-modules by means of the projection $G(F_\infty) \to G(F_\sigma)$. The exact sequences (1.10) and (1.11) provide the connection morphisms

$$\partial_\sigma^{\epsilon_\sigma} : H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})) \longrightarrow H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\epsilon_\sigma), \mathbb{C})),$$

for any of the two characters $\epsilon_\sigma : G(F_\sigma)/G(F_\sigma)^+ \to \pm 1$.

Let $V_\mathbb{R}$ be a $(\mathcal{G}_\infty, K_\infty)$-representation over $\mathbb{R}$ such that $V = V_\mathbb{R} \otimes \mathbb{C}$. This implies that we have a well defined complex conjugation on V by conjugating on the second factor. Thus, we have a complex conjugation on $\mathcal{A}(V, \mathbb{C})$, given by

$$\mathcal{A}(V, \mathbb{C}) \ni \phi \longmapsto \bar{\phi} \in \mathcal{A}(V, \mathbb{C}); \qquad \bar{\phi}(v) = \overline{\phi(\bar{v})}.$$

Lemma 2.2. *Assume that $V = V_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ for some $(\mathcal{G}_\infty, K_\infty)$-module $V_\mathbb{R}$ over $\mathbb{R}$, and let $\mu \in \mathbb{R}$. Then, for any $\phi \in H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C}))$, we have*

$$\overline{\partial_\sigma^{\epsilon_\sigma}(\phi)} = \epsilon_\sigma(c) \cdot \partial_\sigma^{\epsilon_\sigma}(\bar{\phi})$$

for any $c \in G(F_\sigma) \setminus G(F_\sigma)^+$.

Proof. We denote by $\mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \subset \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$ the subspaces where complex conjugation acts by ± 1, respectively. Since exact sequences (1.10) and (1.11) descend to exact sequences (1.12) and (1.13), we obtain

$$0 \longrightarrow \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm\varepsilon_\sigma} \longrightarrow \mathcal{A}(V \otimes I_\mu(k)^{\varepsilon_\sigma}, \mathbb{C})^{\pm\varepsilon_\sigma} \longrightarrow \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1} \longrightarrow 0.$$

Hence the connection morphism satisfies

$$\delta_\sigma^{\epsilon_\sigma}\left(H^i(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})^{\pm 1})\right) \subseteq H^{i+1}(G(F), \mathcal{A}(V \otimes V_\mu(k)(\varepsilon_\sigma), \mathbb{C})^{\pm\varepsilon_\sigma})$$

and the result follows. □

2.1. Explicit computation of the connection morphisms

Let us consider the section $s : V_\mu(k)_{\mathbb{C}} \to I_\mu(k)$ of (1.7).

We compute that

$$\begin{aligned}\langle f_m, \iota P_n\rangle_I &= \int_{S^1} e^{mi\theta} P_n(-\sin\theta, \cos\theta) d\theta \\ &= \int_{S^1} i^{2n-k+2} e^{mi\theta} e^{ni\theta} e^{(n-k+2)i\theta} d\theta = \pi i^{2n-k+2} \delta(2n-k+2+m),\end{aligned}$$

where $\delta(n)$ is the Dirac delta. Thus, $g_\infty^{-1} P = \sum_{n=0}^{k-2} \alpha_n(g_\infty) P_n$, where

$$\alpha_n(g_\infty) = \frac{i^{k-2-2n}}{\pi} \langle f_{k-2-2n}, \iota(g_\infty^{-1} P)\rangle_I = \frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2-2n}, \iota(P)\rangle_I.$$

Since $V_\mu(k)$ is generated by $\{P_0, \cdots, P_{k-2}\}$, we deduce that $\alpha_n = 0$ unless $n \in \{0, \cdots, k-2\}$. Since matrix coefficient morphism are $(\mathcal{G}_\sigma, K_\sigma)$-module morphisms by §1.4, we can compute on the one side

$$\begin{aligned}R\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty(R f_{k-2-2n}), \iota(P)\rangle_I \\ &= (k-n-1)\frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n}, \iota(P)\rangle_I = (n+1-k)\alpha_{n-1}(g_\infty), \\ L\alpha_n(g_\infty) &= \frac{i^{k-2-2n}}{\pi} \langle g_\infty(L f_{k-2-2n}), \iota(P)\rangle_I \\ &= (n+1)\frac{i^{k-2-2n}}{\pi} \langle g_\infty f_{k-2n-4}, \iota(P)\rangle_I = -(n+1)\alpha_{n+1}(g_\infty).\end{aligned}$$

On the other side, we have that $s(P_n) = \frac{2^{k-2}}{\pi}\binom{k-2}{n}^{-1} f_{2n-k+2}$. Hence,

$$(n < k-2)\ Rs(P_n) = \frac{2^{k-2}}{\pi}\binom{k-2}{n}^{-1}(n+1) f_{2n-k+4} = (k-2-n)s(P_{n+1}),$$

$$(n > 0)\ \ Ls(P_n) = \frac{2^{k-2}}{\pi}\binom{k-2}{n}^{-1}(k-n-1) f_{2n-k} = ns(P_{n-1}).$$

Assume that $\tilde\phi \in \mathcal{A}(V \otimes I_\mu(k), \mathbb{C})$ and the action of $(\mathcal{G}_\sigma, K_\sigma)$ on V is trivial. For any $f \in I_\mu(k)$ and $v \in V$, we will usually denote by $\tilde\phi_v(f)$ the expression $\tilde\phi(v \otimes f)$. We aim to compute

$$h_P(g_\infty) := \tilde\phi_v(s(g_\infty^{-1}P))(g_\infty, g), \qquad g_\infty \in G(F_\sigma)^+ \simeq \mathrm{GL}_2(\mathbb{R})^+,$$

for all $g \in G(\mathbb{A}^\sigma)$, $P \in V_\mu(k)$, and $v \in V$. Since $h_P(g_\infty) = \sum_{n=0}^{k-2} \alpha_n(g_\infty)\tilde\phi_v(s(P_n))(g_\infty, g)$, we compute

$$\begin{aligned}
Rh_P &= \sum_{n=0}^{k-2}\left((R\alpha_n)\tilde{\phi}_v(s(P_n)) + \alpha_n\tilde{\phi}_v(Rs(P_n))\right) = \\
&= \sum_{n=0}^{k-2}(n+1-k)\alpha_{n-1}\tilde{\phi}_v(s(P_n)) + \frac{k-1}{2^{2-k}\pi}\alpha_{k-2}\tilde{\phi}_v(f_k) \\
&\quad + \sum_{n=0}^{k-3}(k-2-n)\alpha_n\tilde{\phi}_v(s(P_{n+1})) \\
&= \frac{k-1}{\pi}2^{k-2}\left(\alpha_{k-2}\tilde{\phi}_v(f_k) - \alpha_{-1}\tilde{\phi}_v(f_{2-k})\right) = \frac{k-1}{\pi}2^{k-2}\alpha_{k-2}\tilde{\phi}_v(f_k), \\
Lh_P &= \sum_{n=0}^{k-2}\left((L\alpha_n)\tilde{\phi}_v(s(P_n)) + \alpha_n\tilde{\phi}_v(Ls(P_n))\right) = \\
&= \sum_{n=0}^{k-2}(-n-1)\alpha_{n+1}\tilde{\phi}_v(s(P_n)) + \frac{k-1}{2^{2-k}\pi}\alpha_0\tilde{\phi}_v(f_{-k}) + \sum_{n=1}^{k-2}n\alpha_n\tilde{\phi}_v(s(P_{n-1})) \\
&= \frac{k-1}{\pi}2^{k-2}\left(\alpha_0\tilde{\phi}_v(f_{-k}) - \alpha_{k-1}\tilde{\phi}_v(f_{k-2})\right) = \frac{k-1}{\pi}2^{k-2}\alpha_0\tilde{\phi}_v(f_{-k}).
\end{aligned}$$

Notice that $R = e^{2i\theta}2iy(\frac{\partial}{\partial\tau} - \frac{1}{4y}\frac{\partial}{\partial\theta})$ and $L = -e^{-2i\theta}2iy(\frac{\partial}{\partial\bar{\tau}} - \frac{1}{4y}\frac{\partial}{\partial\theta})$ with $\tau = x+iy$. Since s is a morphism of $\mathrm{SO}(2)\mathbb{R}^+$-modules, h_P is a function of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+ \simeq \mathcal{H}$, thus h_P is a function on τ and $\bar{\tau}$. Let us compute $\frac{\partial h_P}{\partial\tau}$ and $\frac{\partial h_P}{\partial\bar{\tau}}$: By (1.9) and (1.8),

$$\begin{aligned}
\frac{\partial h_P}{\partial\tau}(\tau,\bar{\tau}) &= \frac{y^{-1}e^{-2i\theta}}{2i}R(h_P) = \frac{(k-1)}{2\pi iye^{2i\theta}}i^{2-k}\langle g_\infty P_0, P\rangle\tilde{\phi}_v(f_k) \\
&= \frac{(k-1)}{2\pi iye^{2i\theta}}(g_\infty^{-1}P)(1,-i)\tilde{\phi}_v(f_k) = \frac{(k-1)}{2\pi i}P(1,-\tau)\frac{\tilde{\phi}_v(f_k)}{f_k}(\tau,\bar{\tau},g), \\
\frac{\partial h_P}{\partial\bar{\tau}}(\tau,\bar{\tau}) &= \frac{-y^{-1}e^{2i\theta}}{2i}L(h_P) = \frac{(1-k)}{2\pi iye^{-2i\theta}}i^{k-2}\langle g_\infty\omega P_0, P\rangle\tilde{\phi}_v(f_{-k}) \\
&= \frac{(1-k)}{2\pi iye^{-2i\theta}}(g_\infty^{-1}P)(1,i)\tilde{\phi}_v(f_{-k}) = \frac{(1-k)}{2\pi i}P(1,-\bar{\tau})\frac{\tilde{\phi}_v(f_{-k})}{f_{-k}}(\tau,\bar{\tau},g),
\end{aligned}$$

by Remark 1.3, where $\tilde{\phi}_v(f_k)f_k^{-1}$ and $\tilde{\phi}_v(f_{-k})f_{-k}^{-1}$ are seen as functions of $\mathrm{GL}_2(\mathbb{R})^+/\mathrm{SO}(2)\mathbb{R}^+ \simeq \mathcal{H}$. A similar (and classical) calculation shows that $\tilde{\phi}_v(f_k)f_k^{-1}$ and $\tilde{\phi}_v(f_{-k})f_{-k}^{-1}$ are holomorphic and anti-holomorphic, respectively.

For any $P \in V_\mu(k)$, $g \in G(\mathbb{A}^\sigma)$, $v \in V$ and $\phi \in \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})$ the expressions

$$\omega_\phi(P,v,g)(\tau) := P(1,-\tau)\frac{\phi(v\otimes f_k)}{2\pi i f_k}(\tau,g)d\tau, \tag{2.14}$$

$$\bar{\omega}_\phi(P,v,g)(\bar{\tau}) := P(1,-\bar{\tau})\frac{\phi(v\otimes f_{-k})}{2\pi i f_{-k}}(\bar{\tau},g)d\bar{\tau}, \tag{2.15}$$

define holomorphic and anti-holomorphic forms in $\mathcal{H}$, respectively. Moreover, it is easy to check that

$$\omega_\phi(P,v,g)(\gamma^{-1}\tau)=\omega_{\gamma\phi}(\gamma P,v,\gamma g)(\tau),\qquad \bar\omega_\phi(P,v,g)(\gamma^{-1}\bar\tau)=\bar\omega_{\gamma\phi}(\gamma P,v,\gamma g)(\bar\tau),$$

for any $\gamma\in G(F)\cap G(F_\sigma)^+$. Assume that $c_\phi\in H^n(G(F),\mathcal{A}(V\otimes D_\mu(k),\mathbb{C}))$ is represented by the n-cocycle $\phi: G(F)^n\to\mathcal{A}(V\otimes D_\mu(k),\mathbb{C})$. Then $\partial_\sigma^{\epsilon_\sigma}(c_\phi)$ is represented by the $(n+1)$-cocycle $d^n\tilde\phi$, where $\tilde\phi(\underline{\gamma})\in\mathcal{A}(V\otimes I_\mu(k)_{\mathbb{C}},\mathbb{C})$ is any preimage of $\phi(\underline{\gamma})$ for all $\underline{\gamma}\in G(F)^n$. We consider the $(n+1)$-cocycle $\partial_\sigma^{\varepsilon_\sigma}(\phi)=d^n\tilde\phi-d^nb$, where $b(\underline{\gamma})(g)(v\otimes P)=\tilde\phi(\underline{\gamma})(v\otimes s(P))(1,g)$. We compute, for all $P\in V_\mu(k)$, $v\in V$, $\underline{\gamma}=(\gamma_1,\cdots,\gamma_n)\in G(F)^n$, $\alpha\in G(F)$ and $g\in G(\mathbb{A}^\sigma)$,

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P)=\alpha\left((\tilde\phi-b)(\underline{\gamma})\right)(v\otimes s(P))(1,g)+&\\ +\sum_{i=1}^{n+1}(-1)^i(\tilde\phi-b)(\alpha\underline{\gamma}_i)(v\otimes s(P))(1,g),&\end{aligned}$$

where $\alpha\underline{\gamma}_i=(\alpha,\gamma_1,\cdots,\gamma_{i-1}\gamma_i,\cdots,\gamma_n)$ for $i=1,\cdots,n$, and $\alpha\underline{\gamma}_{n+1}=(\alpha,\gamma_1,\cdots,\gamma_{n-1})$. Since $(\tilde\phi-b)(\underline{\gamma})(v\otimes s(P))=0$ by construction, we obtain

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P)&=\alpha\left(\tilde\phi(\underline{\gamma})\right)(v\otimes s(P))(1,g)-\alpha\left(b(\underline{\gamma})\right)(v\otimes P)(g)\\&=\tilde\phi(\underline{\gamma})(v\otimes s(P))(\alpha^{-1},\alpha^{-1}g)-\tilde\phi(\underline{\gamma})(v\otimes s(\alpha^{-1}P))(1,\alpha^{-1}g).\end{aligned}$$

Since $\tilde\phi(\underline{\gamma})(v\otimes f_k)=\phi(\underline{\gamma})(v\otimes f_k)$ and $\tilde\phi(\underline{\gamma})(v\otimes f_{-k})=\epsilon_\sigma(c)\phi(\underline{\gamma})(v\otimes f_{-k})$, we deduce from the above computations that, for any $\alpha\in G(F)\cap G(F_\sigma)^+$,

$$\begin{aligned}\partial_\sigma^{\epsilon_\sigma}\phi(\alpha,\underline{\gamma})(g)(v\otimes P)&=(k-1)\int_i^{\alpha^{-1}i}\omega_{\phi(\underline{\gamma})}(\alpha^{-1}P,v,\alpha^{-1}g)-\epsilon_\sigma(c)\bar\omega_{\phi(\underline{\gamma})}(\alpha^{-1}P,v,\alpha^{-1}g)\\&=(1-k)\int_i^{\alpha i}\omega_{\alpha\phi(\underline{\gamma})}(P,v,g)-\epsilon_\sigma(c)\bar\omega_{\alpha\phi(\underline{\gamma})}(P,v,g).\end{aligned}$$

Remark 2.3. We have a well defined action of $G(F)/G(F)^+$ on $H^r(G(F)^+,M)$, for any $G(F)$-module M, given by

$$c^\gamma(\alpha_1,\cdots,\alpha_r)=\gamma\left(c(\gamma^{-1}\alpha_1\gamma,\cdots,\gamma^{-1}\alpha_r\gamma)\right),$$

for $\gamma\in G(F)$, and $\alpha_i\in G(F)^+$. The image of the restriction map

$$H^r(G(F),M)\longrightarrow H^r(G(F)^+,M)$$

lies in $H^0(G(F)/G(F)^+,H^r(G(F)^+,M))$.

Let $\psi : \mathbb{A}^\times/F^\times \to \mathbb{C}^\times$ be a Hecke character such that, for any archimedean place $\sigma_i : F \hookrightarrow \mathbb{R}$, $\psi_{\sigma_i}(x) = \mathrm{sign}(x)^{k_i}|x|^{\mu_i}$. Let $D_\psi(\underline{k})$ be the $(\mathcal{G}_\infty, K_\infty)$-module obtained by making the tensor product of $D_{\mu_i}(k_i)$ at the place σ_i, if $\sigma_i \in \Sigma$, and $V_{\mu_j}(k_j)$ at the place σ_j, if $\sigma_j \notin \Sigma$. An element of $D_\psi(\underline{k})$ is $f_{\underline{k}} \otimes P^\Sigma$, where $f_{\underline{k}} = \bigotimes_{\sigma_i\in\Sigma} f_{k_i}$, $f_{k_i} \in D_{\mu_i}(k_i)$ are the elements defined above, and $P^\Sigma \in \bigotimes_{\sigma_i\notin\Sigma} V_{\mu_i}(k_i)$. Let $V_\psi(\underline{k})$ be the $(\mathcal{G}_\infty, K_\infty)$-module obtained by making the tensor product of $V_{\mu_i}(k_i)$ at all the places σ_i. For any character $\epsilon : G(F_\infty)/G(F_\infty)^+ \simeq G(F)/G(F)^+ \to \pm 1$, we denote by $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon)$ the ε-isotypical component, namely, the subspace of $H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ such that the action of $G(F)/G(F)^+$ is given by the character. By the above remark, the restriction map provides an isomorphism

$$H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\varepsilon), \mathbb{C})) \simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon).$$

Using the above computations, we aim to give an explicit formula for the connection morphism:

Theorem 2.4. *Let ϕ be a weight $\underline{k}$ automorphic form of $G(\mathbb{A})$ with central character ψ. Then ϕ defines an element of $\phi \in H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C}))$. For a choice of signs at the places at infinity*

$$\epsilon : G(F_\infty)/G(F_\infty)^+ \simeq G(F)/G(F)^+ \to \pm 1,$$

the composition of the connection morphisms δ_{ϵ_σ}, for $\sigma \in \Sigma$,

$$\begin{aligned}\partial_\epsilon : H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})) &\longrightarrow H^r(G(F), \mathcal{A}(V_\psi(\underline{k})(\epsilon), \mathbb{C}))\\ &\simeq H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))(\epsilon),\end{aligned}$$

can be computed as follows:

$$\partial_\epsilon \phi = \prod_{j+1}^{r}(1-k_j) \sum_{\gamma\in G(F)/G(F)^+} \epsilon(\gamma)\partial\phi^\gamma,$$

where $\partial\phi \in H^r(G(F)^+, \mathcal{A}(V_\psi(\underline{k}), \mathbb{C}))$ is the class of the cocycle

$$(G(F)^+)^r \ni (g_1, g_2, \cdots, g_r) \longmapsto \int_{\tau_1}^{g_1\tau_1} \cdots \int_{g_1\cdots g_{r-1}\tau_r}^{g_1\cdots g_r\tau_r} P_\Sigma(1, -\underline{z}) \frac{\phi(f_{\underline{k}} \otimes P^\Sigma)}{(2\pi i)^r f_{\underline{k}}}(\underline{z}, 1, g) d\underline{z},$$

for any $\underline{P} = P^\Sigma \otimes P_\Sigma \in V_{\underline{\mu}}(\underline{k})$, $\underline{z} = (z_1, \cdots, z_r), (\tau_1, \cdots, \tau_r) \in \mathcal{H}^r$.

Proof. Let $S \subset \Sigma$ be a subset of archimedean places such that $\#S = s < r$. Assume that $\sigma = \sigma_j \in \Sigma \setminus S$ and let k be its corresponding weight and $\mu = \mu_j$. Let

$V = \bigotimes_{\sigma_i \in S'} D_{\mu_i}(k_i) \otimes \bigotimes_{\sigma_i \in \infty \setminus (\Sigma \setminus S)} V_{\mu_i}(k_i)(\varepsilon_{\sigma_i})$, where $S' = \Sigma \setminus (S \cup \{\sigma\})$. Thus, the composition of the connection morphisms corresponding to $\sigma_i \in S$, provides a morphism

$$\delta_S : H^0(G(F), \mathcal{A}(D_\psi(\underline{k}), \mathbb{C})) \longrightarrow H^s(G(F), \mathcal{A}(V \otimes D_\mu(k), \mathbb{C})).$$

By the previous computations, if $P \in V_\mu(k)$, $g \in G(\mathbb{A}^\sigma)$, $v \in V$,

$$\partial_\sigma^{\epsilon_\sigma} \partial_S \phi(\alpha, \underline{\gamma})(g)(v \otimes P) = (1-k) \int_i^{\alpha i} \omega_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g) - \epsilon_\sigma(c) \bar{\omega}_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g).$$

Notice that, letting $c \in (G(F) \cap G(F_S)^+) \setminus (G(F) \cap G(F_\sigma)^+)$, by means of the change of variables $\tau = g_\infty i \mapsto z = c\bar{\tau} = cg_\infty \omega i \in \mathcal{H}$, where $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$, we obtain that

$$\begin{aligned}
\int_i^{\alpha i} \bar{\omega}_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g) &= \int_i^{\alpha i} P(1, -\bar{\tau}) \frac{\alpha \partial_S \phi(\underline{\gamma})(v \otimes f_{-k})(g_\infty, g)}{f_{-k}(g_\infty)} d\bar{\tau} \\
&= - \int_{c(-i)}^{c\alpha(-i)} (c * P)(1, -z) \frac{c\alpha \partial_S \phi(\underline{\gamma})(v \otimes f_k)(cg_\infty \omega, cg)}{f_k(cg_\infty \omega)} dz \\
&= - \int_{\tau_0}^{c\alpha c^{-1} \tau_0} \omega_{c\alpha \partial_S \phi(\underline{\gamma})}(c * P, v, cg),
\end{aligned}$$

where $\tau_0 = c(-i)$, but in fact, this last expression does not depend on the choice of τ_0 because $\mathcal{H}$ is simply connected. Since $c\partial_S \phi(\underline{\gamma}) = \partial_S \phi(c\underline{\gamma}c^{-1})$ because $c \in G(F) \cap G(F_S)^+$, we obtain that

$$\partial_\sigma^{\epsilon_\sigma} \partial_S \phi(\alpha, \underline{\gamma}) = (1-k)(r(\alpha, \underline{\gamma}) + \varepsilon(c) c^{-1} r(c\alpha c^{-1}, c\underline{\gamma}c^{-1})),$$

where r is the cocycle

$$r(\alpha, \underline{\gamma})(g)(v \otimes P) = \int_i^{\alpha i} \omega_{\alpha \partial_S \phi(\underline{\gamma})}(P, v, g).$$

Applying a simple induction on S we obtain the desired result. $\square$

References

[1] F. Bergunde, L. Gehrmann, On the order of vanishing of stickelberger elements of hilbert modular forms, submitted for publication, arXiv:1506.04638.

[2] D. Bump, Automorphic Forms and Representations, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.

[3] X. Guitart, M. Masdeu, S. Molina, Automorphic darmon points, preprint.

[4] H. Hida, Elementary Theory of L-Functions and Eisenstein Series, 2 edition, London Math. Soc. Student Texts, vol. 85, Cambridge University Press, 1993.

[5] S. Molina, Anticyclotomic p-adic l-functions and the exceptional zero phenomenon, submitted for publication, arXiv:1509.08617.

[6] G. Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, Iwanami Shoten and Princeton University Press, 1971.

[7] M. Spieß, On special zeros of p-adic L-functions of Hilbert modular forms, Invent. Math. 196 (1) (2014) 69–138.

Please cite this article in press as: S. Molina, Eichler–Shimura isomorphism and group cohomology on arithmetic groups, J. Number Theory (2017), http://dx.doi.org/10.1016/j.jnt.2017.04.007