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Algebraic independence of the values of Mahler functions associated with a certain continued fraction expansion

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Abstract

It is proved that the function $\Theta(z) = \sum_{k \geqslant 0} \frac{z^{R_0+R_1+\cdots+R_k}}{(1-z^{R_0})(1-z^{R_1})\cdots(1-z^{R_k})}$, which can be expressed as a certain continued fraction, takes algebraically independent values at any distinct nonzero algebraic numbers inside the unit circle if the sequence $\{R_k\}_{k\geqslant 0}$ is the generalized Fibonacci numbers.

MSC: 11J81

Keywords: Algebraic independence; Mahler functions; Fibonacci numbers; Continued fractions

1. Introduction

Let $\{F_k\}_{k\geqslant 0}$ be the sequence of the Fibonacci numbers defined by

$$F_0 = 1, \quad F_1 = 2, \quad F_{k+2} = F_{k+1} + F_k \quad (k \geqslant 0).$$

Beresin, Levine, and Lubell [1] proved that if

$$\prod_{k\geqslant 0} (1 - z^{F_k}) = \sum_{k\geqslant 0} \varepsilon(k) z^k,$$

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then $\varepsilon(k) = 0$ or ± 1 for any $k \geqslant 0$. Tamura [7] generalized this result by proving the following theorem: Let $\{R_k\}_{k\geqslant 0}$ be a linear recurrence of positive integers satisfying

$$R_{k+n} = R_{k+n-1} + \cdots + R_k \quad (k \geqslant 0)$$

with $n \geqslant 2$ and let

$$P(z) = \prod_{k \geqslant 0} (1 - z^{R_k}) = \sum_{k \geqslant 0} \varepsilon(k) z^k.$$

Then, if n is even, $\{\varepsilon(k) \mid k \geqslant 0\}$ is a finite set; if in addition $R_k = 2^k$ $(0 \leqslant k \leqslant n-1)$, $\varepsilon(k) = 0$ or ± 1 for any $k \geqslant 0$. He also showed that $P(g^{-1})$ is irrational for any integer g with $|g| \geqslant 2$. In the same paper, he studied a Lambert-type series

$$\Theta(z) = \sum_{k \geqslant 0} \frac{z^{R_0 + R_1 + \cdots + R_k}}{(1 - z^{R_0})(1 - z^{R_1}) \cdots (1 - z^{R_k})}$$

and proved, using its continued fraction expansion

$$\Theta(z) = \cfrac{z^{R_0}}{1 - z^{R_0} + \cfrac{-z^{R_1}(1 - z^{R_0})}{1 + \cfrac{-z^{R_2}(1 - z^{R_1})}{1 + \ddots + \cfrac{-z^{R_n}(1 - z^{R_{n-1}})}{1 + \ddots}}}},$$

that $\Theta(g^{-1})$ is irrational for any integer $g \geqslant 2$. It is conjectured in [7] that $P(\alpha)$ and $\Theta(\alpha)$ are transcendental for any algebraic number α with $0 < |\alpha| < 1$. We note that the transcendency of $P(\alpha)$, and even the algebraic independence of the values of $P(z)$ at distinct algebraic numbers, can be deduced from Theorem 5 in [9]: Let $\alpha_1, \ldots, \alpha_r$ be algebraic numbers with $0 < |\alpha_i| < 1$ $(1 \leqslant i \leqslant r)$ such that none of α_i/α_j $(1 \leqslant i < j \leqslant r)$ is a root of unity. Then $P(\alpha_i)$ $(1 \leqslant i \leqslant r)$ are algebraically independent. In this paper we prove the algebraic independency of the values at algebraic numbers of $\Theta(z)$ defined by a linear recurrence which is more general than $\{R_k\}_{k \geqslant 0}$. Such values can be reduced to those of Mahler functions of several variables, which satisfies a more general type of functional equation than that discussed in [9], so that we need new techniques in this paper to treat these functions.

Let $\{a_k\}_{k \geqslant 0}$ be a linear recurrence of positive integers satisfying

$$a_{k+n} = c_1 a_{k+n-1} + \cdots + c_n a_k \quad (k \geqslant 0), \tag{1}$$

where $c_1, \ldots, c_n$ are nonnegative integers with $c_n \neq 0$. For any $k \geqslant 0$, let N_k be the greatest common divisor of n consecutive terms $a_k, a_{k+1}, \ldots, a_{k+n-1}$. We define a polynomial associated with (1) by

$$\Phi(X) = X^n - c_1 X^{n-1} - \cdots - c_n. \tag{2}$$

Theorem. *Let* $\{a_k\}_{k\geqslant 0}$ *be a linear recurrence satisfying* (1). *Suppose that* $\{a_k\}_{k\geqslant 0}$ *is not a geometric progression. Assume that* $\Phi(\pm 1)\neq 0$ *and the ratio of any pair of distinct roots of* $\Phi(X)$ *is not a root of unity. Define*

$$\begin{aligned} f(z) &= \sum_{k\geqslant 0} \frac{z^{a_0+a_1+\cdots+a_k}}{(1-z^{a_0})(1-z^{a_1})\cdots(1-z^{a_k})} \\ &= \cfrac{z^{a_0}}{1-z^{a_0}+\cfrac{-z^{a_1}(1-z^{a_0})}{1+\cfrac{-z^{a_2}(1-z^{a_1})}{1+\ddots+\cfrac{-z^{a_n}(1-z^{a_{n-1}})}{1+\ddots}}}}. \end{aligned} \tag{3}$$

Let $\alpha_1,\dots,\alpha_r$ *be algebraic numbers with* $0<|\alpha_i|<1$ $(1\leqslant i\leqslant r)$. *Then* $f(\alpha_1),\dots,f(\alpha_r)$ *are algebraically dependent if and only if there exist some* $k\geqslant 0$ *and distinct* i,j $(1\leqslant i,j\leqslant r)$ *such that* $\alpha_i^{N_k}=\alpha_j^{N_k}$.

Remark 1. Theorem with $r=1$ implies that $f(\alpha)$ and so in particular $\Theta(\alpha)$ is transcendental, since the characteristic polynomial $X^n-(X^{n-1}+\cdots+1)$ of $\{R_k\}_{k\geqslant 0}$ is irreducible over $\boldsymbol{Q}$ and its roots $\rho_1,\dots,\rho_n$ satisfy $\rho_1>1>\max\{|\rho_2|,\dots,|\rho_n|\}$ (cf. Lemma 10 in [6]) and so, by Remark 1 in [8], none of ρ_i/ρ_j $(i\neq j)$ is a root of unity, which means that Theorem can be applied to $\Theta(z)$. Thus, both of the Tamura's problems mentioned above has been completely settled.

As a corollary of Theorem, we find a new class of functions each of which takes algebraically independent values at any given distinct algebraic numbers different from zero.

Corollary. *Let* $\{a_k\}_{k\geqslant 0}$ *be as in Theorem. Suppose in addition that* $N_k=\text{g.c.d.}(a_k,a_{k+1},\dots,a_{k+n-1})=1$ *for any* $k\geqslant 0$. *Let* $f(z)$ *be the function of the variable* z *defined by* (3) *and let* $\alpha_1,\dots,\alpha_r$ *be algebraic numbers with* $0<|\alpha_i|<1$ $(1\leqslant i\leqslant r)$. *Then* $f(\alpha_1),\dots,f(\alpha_r)$ *are algebraically independent if* $\alpha_1,\dots,\alpha_r$ *are distinct.*

Remark 2. The condition that $N_k=1$ for any $k\geqslant 0$ is satisfied if $c_n=1$ in (1) and $\text{g.c.d.}(a_0,\dots,a_{n-1})=1$. For instance, the linear recurrence $\{R_k\}_{k\geqslant 0}$ defined above satisfies this condition if $\text{g.c.d.}(R_0,\dots,R_{n-1})=1$.

Example. If g.c.d.$(R_0, \ldots, R_{n-1}) = 1$, then $\Theta(\alpha_1), \ldots, \Theta(\alpha_r)$ are algebraically independent for any distinct algebraic numbers $\alpha_1, \ldots, \alpha_r$ with $0 < |\alpha_i| < 1$ $(1 \leqslant i \leqslant r)$ by the corollary with Remarks 1 and 2.

2. Lemmas

Let $F(z_1, \ldots, z_n)$ and $F[[z_1, \ldots, z_n]]$ denote the field of rational functions and the ring of formal power series in variables $z_1, \ldots, z_n$ with coefficients in a field F, respectively, and $F^\times$ the multiplicative group of nonzero elements of F. Let $\Omega = (\omega_{ij})$ be an $n \times n$ matrix with nonnegative integer entries. Then the maximum ρ of the absolute values of the eigenvalues of Ω is itself an eigenvalue (cf. [2, Theorem 3, p. 66]). If $\boldsymbol{z} = (z_1, \ldots, z_n)$ is a point of $\boldsymbol{C}^n$ with $\boldsymbol{C}$ the set of complex numbers, we define a transformation $\Omega : \boldsymbol{C}^n \to \boldsymbol{C}^n$ by

$$\Omega \boldsymbol{z} = \left(\prod_{j=1}^{n} z_j^{\omega_{1j}}, \ldots, \prod_{j=1}^{n} z_j^{\omega_{nj}} \right). \tag{4}$$

We suppose that Ω and an algebraic point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$, where α_i are nonzero algebraic numbers, have the following four properties:

(I) Ω is nonsingular and none of its eigenvalues is a root of unity, so that in particular $\rho > 1$.

(II) Every entry of the matrix Ω^k is $O(\rho^k)$ as k tends to infinity.

(III) If we put $\Omega^k \boldsymbol{\alpha} = (\alpha_1^{(k)}, \ldots, \alpha_n^{(k)})$, then

$$\log |\alpha_i^{(k)}| \leqslant -c\rho^k \quad (1 \leqslant i \leqslant n)$$

for all sufficiently large k, where c is a positive constant.

(IV) For any nonzero $f(\boldsymbol{z}) \in \boldsymbol{C}[[z_1, \ldots, z_n]]$ which converges in some neighborhood of the origin, there are infinitely many positive integers k such that $f(\Omega^k \boldsymbol{\alpha}) \neq 0$.

We note that property (II) is satisfied if every eigenvalue of Ω of absolute value ρ is a simple root of the minimal polynomial of Ω.

Lemma 1 (Tanaka [8, Lemma 4, Proof of Theorem 2]). *Suppose that $\Phi(\pm 1) \neq 0$ and the ratio of any pair of distinct roots of $\Phi(X)$ is not a root of unity, where $\Phi(X)$ is the polynomial defined by* (2). *Let*

$$\Omega = \begin{pmatrix} c_1 & 1 & 0 & \ldots & 0 \\ c_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 1 \\ c_n & 0 & \ldots & \ldots & 0 \end{pmatrix} \tag{5}$$

and let $\beta_1, \dots, \beta_s$ *be multiplicatively independent algebraic numbers with* $0 < |\beta_j| < 1$ $(1 \leqslant j \leqslant s)$. *Let* p *be a positive integer and put*

$$\Omega' = \mathrm{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{s}).$$

Then the matrix Ω' *and the point*

$$\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s)$$

have properties (I)–(IV).

Lemma 2 (Kubota [3], see also Nishioka [5]). *Let* K *be an algebraic number field. Suppose that* $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}) \in K[[z_1, \dots, z_n]]$ *converge in an* n*-polydisc* U *around the origin and satisfy the functional equations*

$$f_i(\Omega \boldsymbol{z}) = a_i(\boldsymbol{z}) f_i(\boldsymbol{z}) + b_i(\boldsymbol{z}) \quad (1 \leqslant i \leqslant m),$$

where $a_i(\boldsymbol{z}), b_i(\boldsymbol{z}) \in K(z_1, \dots, z_n)$ *and* $a_i(\boldsymbol{0})$ *is defined and nonzero. Assume that the* $n \times n$ *matrix* Ω *and a point* $\boldsymbol{\alpha} \in U$ *whose components are nonzero algebraic numbers have properties* (I)–(IV) *and that* $a_i(\boldsymbol{z})$ *are defined and nonzero at* $\Omega^k \boldsymbol{\alpha}$ *for all* $k \geqslant 0$. *If* $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z})$ *are algebraically independent over* $K(z_1, \dots, z_n)$, *then* $f_1(\boldsymbol{\alpha}), \dots, f_m(\boldsymbol{\alpha})$ *are algebraically independent*.

Lemma 2 is essentially due to Kubota [3] and improved by Nishioka [5].

In what follows, C denotes a field of characteristic 0. Let $L = C(z_1, \dots, z_n)$ and let M be the quotient field of $C[[z_1, \dots, z_n]]$. Let Ω be an $n \times n$ matrix with nonnegative integer entries having property (I). We define an endomorphism $\tau : M \to M$ by

$$f^\tau(\boldsymbol{z}) = f(\Omega \boldsymbol{z}) \quad (f(\boldsymbol{z}) \in M)$$

and a subgroup H of $L^\times$ by

$$H = \{ g^\tau g^{-1} \mid g \in L^\times \}.$$

Lemma 3 (Kubota [3], see also Nishioka [5]). *Let* $f_i \in M$ $(i = 1, \dots, m)$ *satisfy*

$$f_i^\tau = a_i f_i + b_i,$$

where $a_i \in L^\times$, $b_i \in L$ $(1 \leqslant i \leqslant m)$. *Suppose that* a_i, b_i $(1 \leqslant i \leqslant m)$ *have the following properties*:

(i) *For any* i $(1 \leqslant i \leqslant m)$, *there is no element* g *of* L *satisfying*

$$g^\tau = a_i g + c b_i, \quad c \in C^\times.$$

(ii) *For any distinct* i, j $(1 \leqslant i, j \leqslant m)$, $a_i a_j^{-1} \notin H$.

Then the functions f_i $(1 \leqslant i \leqslant m)$ *are algebraically independent over* L.

We adopt the usual vector notation, that is, if $I = (i_1, \ldots, i_n) \in \boldsymbol{N}_0^n$ with $\boldsymbol{N}_0$ the set of nonnegative integers, we write $\boldsymbol{z}^I = z_1^{i_1} \cdots z_n^{i_n}$. We denote by $C[z_1, \ldots, z_n]$ the ring of polynomials in variables $z_1, \ldots, z_n$ with coefficients in C.

Lemma 4 (Nishioka [5]). *If* $A, B \in C[z_1, \ldots, z_n]$ *are coprime, then* $(A^\tau, B^\tau) = \boldsymbol{z}^I$, *where* $I \in \boldsymbol{N}_0^n$.

Lemma 5 (Tanaka [9]). *Let* Ω *be an* $n \times n$ *matrix with nonnegative integer entries which has property* (I). *Let* $R(\boldsymbol{z})$ *be a nonzero polynomial in* $C[z_1, \ldots, z_n]$. *If* $R(\Omega \boldsymbol{z})$ *divides* $R(\boldsymbol{z})\boldsymbol{z}^I$, *where* $I \in \boldsymbol{N}_0^n$, *then* $R(\boldsymbol{z})$ *is a monomial in* $z_1, \ldots, z_n$.

Lemma 6. *Let* $P(\boldsymbol{z})$ *be a nonconstant polynomial in* $\boldsymbol{z} = (z_1, \ldots, z_n)$ *with* $n \geqslant 2$. *Let* Ω *be an* $n \times n$ *matrix with positive integer entries which has property* (I). *Then*

$$\deg_{\boldsymbol{z}} P(\Omega \boldsymbol{z}) > \deg_{\boldsymbol{z}} P(\boldsymbol{z}).$$

Proof. Let $c\boldsymbol{z}^J$ be a term of $P(\boldsymbol{z})$ for which $\deg_{\boldsymbol{z}} P(\boldsymbol{z}) = J^t\mathbf{1}$ holds, where $\mathbf{1} = (1, \ldots, 1) \in \boldsymbol{N}_0^n$. Then $c\boldsymbol{z}^{J\Omega}$ is a term of $P(\Omega \boldsymbol{z})$ and so

$$\deg_{\boldsymbol{z}} P(\Omega \boldsymbol{z}) \geqslant J\Omega^t\mathbf{1} \geqslant nJ^t\mathbf{1} > J^t\mathbf{1}.$$

This completes the proof of the lemma. □

Let $\{a_k\}_{k \geqslant 0}$ be a linear recurrence satisfying (1) and define a monomial

$$P(\boldsymbol{z}) = z_1^{a_{n-1}} \cdots z_n^{a_0}, \tag{6}$$

which is denoted similarly to (4) by

$$P(\boldsymbol{z}) = (a_{n-1}, \ldots, a_0)\boldsymbol{z}. \tag{7}$$

Let Ω be the matrix defined by (5). It follows from (1), (4), and (7) that

$$P(\Omega^k \boldsymbol{z}) = z_1^{a_{k+n-1}} \cdots z_n^{a_k} \quad (k \geqslant 0).$$

Lemma 7 (Tanaka [9]). *Suppose that* $\{a_k\}_{k \geqslant 0}$ *is not a geometric progression. Assume that* $\Phi(\pm 1) \neq 0$ *and the ratio of any pair of distinct roots of* $\Phi(X)$ *is not a root of unity. Let* $\bar{C}$ *be an algebraically closed field of characteristic* 0. *Suppose that* $G(\boldsymbol{z})$ *is an element of the quotient field of* $\bar{C}[[z_1, \ldots, z_n]]$ *satisfying the functional equation of the* *form*

$$G(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} Q_k(P(\Omega^k \boldsymbol{z}))\right) G(\Omega^p \boldsymbol{z}),$$

where Ω *is defined by* (5), $p>0$, $q\geqslant 0$ *are integers, and* $Q_k(X)\in \bar{C}(X)$ $(q\leqslant k\leqslant p+q-1)$ *are defined and nonzero at* $X=0$. *If* $G(\boldsymbol{z})\in \bar{C}(z_1,\dots,z_n)$, *then* $G(\boldsymbol{z})\in \bar{C}$ *and* $Q_k(X)\in \bar{C}^{\times}$ $(q\leqslant k\leqslant p+q-1)$.

3. Proof of Theorem

Proof of Theorem. First, we prove that if $\alpha_{i_1}^{N_{k_0}} = \alpha_{i_2}^{N_{k_0}}$ for some $k_0\geqslant 0$ and distinct i_1, i_2 $(1\leqslant i_1, i_2\leqslant r)$, then $f(\alpha_{i_1})$ and $f(\alpha_{i_2})$ are algebraically dependent. We see by (1) that N_{k_0} divides a_k for any $k\geqslant k_0$. Hence, if $\alpha_{i_1}^{N_{k_0}} = \alpha_{i_2}^{N_{k_0}}$, then $\alpha_{i_1}^{a_k} = \alpha_{i_2}^{a_k}$ for any $k\geqslant k_0$, so that

$$\begin{aligned}
&\prod_{k=0}^{k_0-1} \frac{1-\alpha_{i_1}^{a_k}}{\alpha_{i_1}^{a_k}} \left(f(\alpha_{i_1}) - \sum_{k=0}^{k_0-1} \prod_{l=0}^{k} \frac{\alpha_{i_1}^{a_l}}{1-\alpha_{i_1}^{a_l}} \right) \\
&\quad = \sum_{k\geqslant k_0} \prod_{l=k_0}^{k} \frac{\alpha_{i_1}^{a_l}}{1-\alpha_{i_1}^{a_l}} \\
&\quad = \sum_{k\geqslant k_0} \prod_{l=k_0}^{k} \frac{\alpha_{i_2}^{a_l}}{1-\alpha_{i_2}^{a_l}} \\
&\quad = \prod_{k=0}^{k_0-1} \frac{1-\alpha_{i_2}^{a_k}}{\alpha_{i_2}^{a_k}} \left(f(\alpha_{i_2}) - \sum_{k=0}^{k_0-1} \prod_{l=0}^{k} \frac{\alpha_{i_2}^{a_l}}{1-\alpha_{i_2}^{a_l}} \right),
\end{aligned}$$

which means that $f(\alpha_{i_1})$ and $f(\alpha_{i_2})$ are algebraically dependent.

Next we prove that if $f(\alpha_1),\dots,f(\alpha_r)$ are algebraically dependent, then there exist some $k\geqslant 0$ and distinct i_1, i_2 $(1\leqslant i_1, i_2\leqslant r)$ such that $\alpha_{i_1}^{N_k} = \alpha_{i_2}^{N_k}$. Suppose that $f(\alpha_1),\dots,f(\alpha_r)$ are algebraically dependent. There exist multiplicatively independent algebraic numbers $\beta_1,\dots,\beta_s$ with $0<|\beta_j|<1 (1\leqslant j\leqslant s)$ such that

$$\alpha_i = \zeta_i \prod_{j=1}^{s} \beta_j^{e_{ij}} \quad (1\leqslant i\leqslant r), \tag{8}$$

where $\zeta_1,\dots,\zeta_r$ are roots of unity and e_{ij} $(1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s)$ are nonnegative integers (cf. [4,5]). Take a positive integer N such that $\zeta_i^N = 1$ for any i $(1\leqslant i\leqslant r)$. We can choose a positive integer p and a sufficiently large integer q such that $a_{k+p}\equiv a_k \pmod N$ for any $k\geqslant q$. Let $y_{j\lambda}$ $(1\leqslant j\leqslant s,\ 1\leqslant \lambda\leqslant n)$ be variables and let $\boldsymbol{y}_j =$

$(y_{j1}, \dots, y_{jn})$ $(1 \leqslant j \leqslant s)$, $\boldsymbol{y} = (\boldsymbol{y}_1, \dots, \boldsymbol{y}_s)$. Define

$$g_i(\boldsymbol{y}) = \sum_{k \geqslant q} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \boldsymbol{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \boldsymbol{y}_j)^{e_{ij}}} \quad (1 \leqslant i \leqslant r),$$

where $P(\boldsymbol{z})$ and Ω are defined by (6) and (5), respectively. Letting

$$\boldsymbol{\beta} = (\underbrace{1, \dots, 1}_{n-1}, \beta_1, \dots\dots, \underbrace{1, \dots, 1}_{n-1}, \beta_s),$$

we see that

$$g_i(\boldsymbol{\beta}) = \sum_{k \geqslant q} \prod_{l=q}^{k} \frac{\alpha_i^{a_l}}{1 - \alpha_i^{a_l}}$$

and so

$$f(\alpha_i) = \left(\prod_{k=0}^{q-1} \frac{\alpha_i^{a_k}}{1 - \alpha_i^{a_k}} \right) g_i(\boldsymbol{\beta}) + \sum_{k=0}^{q-1} \prod_{l=0}^{k} \frac{\alpha_i^{a_l}}{1 - \alpha_i^{a_l}}.$$

Hence the values $g_i(\boldsymbol{\beta})$ $(1 \leqslant i \leqslant r)$ are algebraically dependent. Let

$$\Omega' = \operatorname{diag}(\underbrace{\Omega^p, \dots, \Omega^p}_{s}),$$

where p is replaced by its multiple such that all the entries of Ω^p are positive. (We can choose such a p. For the proof see [8].) Then each $g_i(\boldsymbol{y})$ satisfies the functional equation

$$\begin{aligned} g_i(\boldsymbol{y}) = {} & \left(\prod_{k=q}^{p+q-1} \frac{\zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \boldsymbol{y}_j)^{e_{ij}}} \right) g_i(\Omega' \boldsymbol{y}) \\ & + \sum_{k=q}^{p+q-1} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \boldsymbol{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \boldsymbol{y}_j)^{e_{ij}}}, \end{aligned}$$

where $\Omega' \boldsymbol{y} = (\Omega^p \boldsymbol{y}_1, \dots, \Omega^p \boldsymbol{y}_s)$. Let $D = |\det(\Omega - I)|$, where I is the identity matrix. Then D is a positive integer, since $\Phi(1) \neq 0$, where $\Phi(X)$ is the polynomial defined by (2). Let $y'_{j\lambda} = y_{j\lambda}^{1/D}$ $(1 \leqslant j \leqslant s,\ 1 \leqslant \lambda \leqslant n)$, $\boldsymbol{y}'_j = (y'_{j1}, \dots, y'_{jn})$ $(1 \leqslant j \leqslant s)$, and $\boldsymbol{y}' = (\boldsymbol{y}'_1, \dots, \boldsymbol{y}'_s)$. Noting that $\prod_{j=1}^{s} P((\Omega - I)^{-1} \Omega^q \boldsymbol{y}_j)^{e_{ij}} = \prod_{j=1}^{s} P(D(\Omega - I)^{-1} \Omega^q \boldsymbol{y}'_j)^{e_{ij}} \in \bar{\boldsymbol{Q}}(\boldsymbol{y}')$, we define

$$\begin{aligned} h_i(\boldsymbol{y}') &= \left(\prod_{j=1}^{s} P((\Omega - I)^{-1} \Omega^q \boldsymbol{y}_j)^{e_{ij}} \right) g_i(\boldsymbol{y}) - R_i(\boldsymbol{y}') \\ &= \left(\prod_{j=1}^{s} P(D(\Omega - I)^{-1} \Omega^q \boldsymbol{y}'_j)^{e_{ij}} \right) g_i(\boldsymbol{y}') - R_i(\boldsymbol{y}') \quad (1 \leqslant i \leqslant r), \end{aligned}$$

where

$$g_i(\mathbf{y}') = \sum_{k \geqslant q} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}} \in \bar{\mathbf{Q}}[[\mathbf{y}']],$$

$$R_i(\mathbf{y}') = \left(\prod_{j=1}^{s} P(D(\Omega - I)^{-1} \Omega^q \mathbf{y}'_j)^{e_{ij}} \right) \sum_{k=q}^{k_1} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}} \in \bar{\mathbf{Q}}(\mathbf{y}'),$$

and k_1 is such a large integer that $h_i(\mathbf{y}') \in \bar{\mathbf{Q}}[[\mathbf{y}']]$ $(1 \leqslant i \leqslant r)$. Then each $h_i(\mathbf{y}')$ satisfies the functional equation

$$\begin{aligned} h_i(\mathbf{y}') = & \left(\prod_{k=q}^{p+q-1} \frac{\zeta_i^{a_k}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}'_j)^{De_{ij}}} \right) h_i(\Omega' \mathbf{y}') \\ & + \left(\prod_{j=1}^{s} P(D(\Omega - I)^{-1} \Omega^q \mathbf{y}'_j)^{e_{ij}} \right) \sum_{k=q}^{p+q-1} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}'_j)^{De_{ij}}} \\ & + \left(\prod_{k=q}^{p+q-1} \frac{\zeta_i^{a_k}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}'_j)^{De_{ij}}} \right) R_i(\Omega' \mathbf{y}') - R_i(\mathbf{y}'), \end{aligned}$$

where $\Omega' \mathbf{y}' = (\Omega^p \mathbf{y}'_1, \ldots, \Omega^p \mathbf{y}'_s)$. Since $g_i(\boldsymbol{\beta})$ $(1 \leqslant i \leqslant r)$ are algebraically dependent, so are $h_i(\boldsymbol{\beta}')$ $(1 \leqslant i \leqslant r)$, where

$$\boldsymbol{\beta}' = (\underbrace{1, \ldots, 1}_{n-1}, \beta_1^{1/D}, \ldots \ldots, \underbrace{1, \ldots, 1}_{n-1}, \beta_s^{1/D}).$$

By Lemma 1, the matrix Ω' and $\boldsymbol{\beta}'$ have properties (I)–(IV). Then the functions $h_i(\mathbf{y}')$ $(1 \leqslant i \leqslant r)$ are algebraically dependent over $\bar{\mathbf{Q}}(\mathbf{y}')$ by Lemma 2. Hence $g_i(\mathbf{y})$ $(1 \leqslant i \leqslant r)$ are algebraically dependent over $\bar{\mathbf{Q}}(\mathbf{y}')$ and so they are algebraically dependent over $\bar{\mathbf{Q}}(\mathbf{y})$. Therefore by Lemma 3, at least one of the following two cases arises:

(i) For some i $(1 \leqslant i \leqslant r)$, there exist an algebraic number $c \neq 0$ and $F(\mathbf{y}) \in \bar{\mathbf{Q}}(\mathbf{y})$ such that

$$\begin{aligned} F(\mathbf{y}) = & \left(\prod_{k=q}^{p+q-1} \frac{\zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{ij}}} \right) F(\Omega' \mathbf{y}) \\ & + c \sum_{k=q}^{p+q-1} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}_j)^{e_{ij}}}{1 - \zeta_i^{a_l} \prod_{j=1}^{s} P(\Omega^l \mathbf{y}_j)^{e_{ij}}}. \end{aligned} \tag{9}$$

(ii) For some distinct i_1, i_2 $(1 \leqslant i_1, i_2 \leqslant r)$, there exists $G(\mathbf{y}) \in \bar{\mathbf{Q}}(\mathbf{y}) \setminus \{0\}$ such that

$$G(\mathbf{y}) = \left(\prod_{k=q}^{p+q-1} \frac{\zeta_{i_1}^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{i_1 j}} (1 - \zeta_{i_2}^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{i_2 j}})}{\zeta_{i_2}^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{i_2 j}} (1 - \zeta_{i_1}^{a_k} \prod_{j=1}^{s} P(\Omega^k \mathbf{y}_j)^{e_{i_1 j}})} \right) G(\Omega' \mathbf{y}). \tag{10}$$

Let M be a positive integer and let

$$\boldsymbol{y}_j = (y_{j1}, \dots, y_{jn}) = (z_1^{M^j}, \dots, z_n^{M^j}) \quad (1 \leqslant j \leqslant s),$$

where M is so large that the following two properties are both satisfied:

(a) If $(e_{i1}, \dots, e_{is}) \neq (e_{i'1}, \dots, e_{i's})$, then $\sum_{j=1}^{s} e_{ij} M^j \neq \sum_{j=1}^{s} e_{i'j} M^j$.

(b) $F^*(\boldsymbol{z}) = F(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \bar{\boldsymbol{Q}}(z_1, \dots, z_n)$,

$$G^*(\boldsymbol{z}) = G(z_1^M, \dots, z_n^M, \dots, z_1^{M^s}, \dots, z_n^{M^s}) \in \bar{\boldsymbol{Q}}(z_1, \dots, z_n) \backslash \{0\}.$$

Then by (9) and (10), at least one of the following two functional equations holds:

$$F^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \frac{\zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}}{1 - \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}} \right) F^*(\Omega^p \boldsymbol{z}) + c \sum_{k=q}^{p+q-1} \prod_{l=q}^{k} \frac{\zeta_i^{a_l} P(\Omega^l \boldsymbol{z})^{E_i}}{1 - \zeta_i^{a_l} P(\Omega^l \boldsymbol{z})^{E_i}}, \tag{11}$$

$$G^*(\boldsymbol{z}) = \left(\prod_{k=q}^{p+q-1} \frac{\zeta_{i_1}^{a_k} P(\Omega^k \boldsymbol{z})^{E_{i_1}} (1 - \zeta_{i_2}^{a_k} P(\Omega^k \boldsymbol{z})^{E_{i_2}})}{\zeta_{i_2}^{a_k} P(\Omega^k \boldsymbol{z})^{E_{i_2}} (1 - \zeta_{i_1}^{a_k} P(\Omega^k \boldsymbol{z})^{E_{i_1}})} \right) G^*(\Omega^p \boldsymbol{z}), \tag{12}$$

where $E_i = \sum_{j=1}^{s} e_{ij} M^j$ for any i $(1 \leqslant i \leqslant r)$.

Suppose that (11) holds. Letting $F^*(\boldsymbol{z}) = A(\boldsymbol{z})/B(\boldsymbol{z})$, where $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime polynomials in $\bar{\boldsymbol{Q}}[z_1, \dots, z_n]$, we have

$$\begin{aligned} & A(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) \prod_{k=q}^{p+q-1} (1 - \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i}) \\ &= A(\Omega^p \boldsymbol{z}) B(\boldsymbol{z}) \prod_{k=q}^{p+q-1} \zeta_i^{a_k} P(\Omega^k \boldsymbol{z})^{E_i} \\ &\quad + c B(\boldsymbol{z}) B(\Omega^p \boldsymbol{z}) \sum_{k=q}^{p+q-1} \prod_{l=q}^{k} \zeta_i^{a_l} P(\Omega^l \boldsymbol{z})^{E_i} \prod_{m=k+1}^{p+q-1} (1 - \zeta_i^{a_m} P(\Omega^m \boldsymbol{z})^{E_i}) \end{aligned} \tag{13}$$

by (11). We can put $(A(\Omega^p \boldsymbol{z}),\ B(\Omega^p \boldsymbol{z})) = \boldsymbol{z}^I$, where $I \in \boldsymbol{N}_0^n$, by Lemma 4. Then $B(\Omega^p \boldsymbol{z})$ divides $B(\boldsymbol{z}) \boldsymbol{z}^I \prod_{k=q}^{p+q-1} P(\Omega^k \boldsymbol{z})^{E_i}$. Therefore $B(\boldsymbol{z})$ is a monomial in $z_1, \dots, z_n$ by Lemmas 1 and 5. If q is sufficiently large, the right-hand side of (13) is divided by $z_1 \cdots z_n B(\Omega^p \boldsymbol{z})$ and thus $A(\boldsymbol{z})$ is divided by $z_1 \cdots z_n$. Since $A(\boldsymbol{z})$ and $B(\boldsymbol{z})$ are coprime, $B(\boldsymbol{z}) \in \bar{\boldsymbol{Q}}^\times$. If $A(\boldsymbol{z}) \notin \bar{\boldsymbol{Q}}$, then $\deg_{\boldsymbol{z}} A(\Omega^p \boldsymbol{z}) > \deg_{\boldsymbol{z}} A(\boldsymbol{z})$ by Lemma 6, which is a contradiction by comparing the total degrees of both sides of (13). Hence $A(\boldsymbol{z}) \in \bar{\boldsymbol{Q}}$. Letting $z_1 = \cdots = z_n = 0$ in (13), we get $A(\boldsymbol{z}) = 0$. Dividing both sides of (13) by $P(\Omega^q \boldsymbol{z})^{E_i}$ and then letting $z_1 = \cdots = z_n = 0$, we see that $c = 0$, a contradiction. Therefore (12) must hold.

Then by Lemma 7 we see that

$$\frac{\zeta_{i_1}^{a_k} X^{E_{i_1}} (1 - \zeta_{i_2}^{a_k} X^{E_{i_2}})}{\zeta_{i_2}^{a_k} X^{E_{i_2}} (1 - \zeta_{i_1}^{a_k} X^{E_{i_1}})} = \gamma_k \in \bar{\boldsymbol{Q}}^{\times}$$

for any k $(q \leqslant k \leqslant p+q-1)$, where X is a variable. Hence $E_{i_1} = E_{i_2},\ \gamma_k = 1$ and $\zeta_{i_1}^{a_k} = \zeta_{i_2}^{a_k}$ $(q \leqslant k \leqslant p+q-1)$. Therefore $(e_{i_1 1}, \dots, e_{i_1 s}) = (e_{i_2 1}, \dots, e_{i_2 s})$ by the property (a), and $\zeta_{i_1}^{a_k} = \zeta_{i_2}^{a_k}$ $(k \geqslant q)$ since $a_{k+p} \equiv a_k \pmod N$ for any $k \geqslant q$. Hence $\alpha_{i_1}^{a_k} = \alpha_{i_2}^{a_k}$ $(q \leqslant k \leqslant q+n-1)$ by (8) and so $\alpha_{i_1}^{N_q} = \alpha_{i_2}^{N_q}$. This completes the proof of the theorem.

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