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Journal of Number Theory 108 (2004) 268–286
JOURNAL OF Number Theory
http://www.elsevier.com/locate/jnt

On the field of definition for modularity of CM elliptic curves

Naoki Murabayashi

Department of Mathematical Sciences, Faculty of Science, Yamagata University, Yamagata, 990-8560, Japan

Received 6 January 2004; revised 12 February 2004

Available online 1 September 2004

Communicated by K.A. Ribet

Abstract

The purpose of this paper is to decide the conditions under which a CM elliptic curve is modular over its field of definition.
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Keywords: Field of definition; Modularity; Complex multiplication; Elliptic curve

1. Introduction

Let E be a CM elliptic curve defined over an algebraic number field $F \subseteq \mathbb{C}$ whose $\mathbb{Q}$-algebra of endomorphisms defined over $\overline{\mathbb{Q}}$, denoted by $\mathrm{End}_{\overline{\mathbb{Q}}}^{0}(E)$, is isomorphic to an imaginary quadratic field $K \subseteq \mathbb{C}$. We take an integral ideal $\mathfrak{m}$ in K and denote by $I_K(\mathfrak{m})$ the group of fractional ideals in K prime to $\mathfrak{m}$. We consider a homomorphism $\lambda : I_K(\mathfrak{m}) \to \mathbb{C}^\times$ such that (i) $\lambda((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \bmod^\times \mathfrak{m}$; (ii) λ is primitive, i.e. there is no proper divisor $\mathfrak{n}$ of $\mathfrak{m}$ such that λ has a extension $\widetilde{\lambda}$ to $I_K(\mathfrak{n})$ with the property: $\widetilde{\lambda}((\alpha)) = \alpha$ for any $\alpha \in K^\times$ s.t. $\alpha \equiv 1 \bmod^\times \mathfrak{n}$. Then we put

$$f_\lambda(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{m}) \\ \mathfrak{a}:\text{integral}}} \lambda(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z} \quad (z \in \mathfrak{H}, \text{ the complex upper plane}),$$

E-mail address: murabaya@sci.kj.yamagata-u.ac.jp.

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doi:10.1016/j.jnt.2004.05.004

where $N(\mathfrak{a})$ denotes the absolute norm of an ideal $\mathfrak{a}$. Let $-D$ be the discriminant of K and put $N := DN(\mathfrak{m})$. We define a Dirichlet character $\varepsilon : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ by

$$\bar{a} \mapsto \left(\frac{-D}{a}\right)\frac{\lambda((a))}{a} \quad (a \in \mathbb{Z}, \quad (a, N) = 1),$$

where if $a = p_1^{e_1} \cdots p_r^{e_r}$ is the factorization of a into prime factors,

$$\left(\frac{-D}{a}\right) = \prod_{i=1}^{r} \left(\frac{-D}{p_i}\right)^{e_i}, \quad \left(\frac{-D}{p_i}\right) = \begin{cases} 1 & \text{if } p_i \text{ splits in } K/\mathbb{Q}, \\ -1 & \text{if } p_i \text{ is inert in } K/\mathbb{Q}. \end{cases}$$

By Hecke–Shimura, we have the following:

Fact 1. *f_λ is a normalized newform of weight two on $\Gamma_1(N)$ and ε is the Nebentypus of f_λ.*

By the Eichler–Shimura theory, for any normalized newform f of weight two on $\Gamma_1(M)$, we can associate the abelian variety J_f defined over $\mathbb{Q}$ which is a $\mathbb{Q}$-simple factor of $J_1(M)$, the jacobian variety of the modular curve $X_1(M)$. Shimura proved the following (see Proposition 1.6 and Remark 1.7 in [5]):

Fact 2. $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \neq \{0\}$ *if and only if there exists an above λ such that $f = f_\lambda$, where* $\mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$ *denotes the additive group of homomorphisms from E to J_f defined over* $\overline{\mathbb{Q}}$.

For any imaginary quadratic field K, if we take an integral ideal $\mathfrak{m}_0$ in K such that

$$\zeta \in K, \ \zeta \text{ is a root of unity}, \ \zeta \equiv 1 \bmod^\times \mathfrak{m}_0 \Rightarrow \zeta = 1$$

holds (we can always do so), there exists a homomorphism $\lambda : I_K(\mathfrak{m}_0) \to \mathbb{C}^\times$ satisfying the condition (i). Replacing $\mathfrak{m}_0$ by the minimal divisor $\mathfrak{m}$ of $\mathfrak{m}_0$ such that λ has an extension $\widetilde{\lambda}$ to $I_K(\mathfrak{m})$ and $\widetilde{\lambda}$ has also the property (i), we may assume that λ is primitive. Therefore, we have

Fact 3. *For any CM elliptic curve E defined over an algebraic number field F, there exists a newform f such that a non-zero homomorphism $\varphi : E \to J_f$ defined over $\overline{\mathbb{Q}}$ exists, that is, E is modular over $\overline{\mathbb{Q}}$.*

In this paper, we will consider the following questions.

Question 1. *Let E/F be as above. Under what condition does there exist a newform f such that a non-zero homomorphism $\varphi : E \to J_f$ defined over F exists, that is, when is E modular over F?*

Question 2. *Assume that E/F is modular over F. Therefore, there exists a newform f with* $\mathrm{Hom}_F(E, J_f) \neq \{0\}$. *Then, how large is* $\mathrm{Hom}_F(E, J_f)$? *In other words, decide the multiplicity of E as F-simple factor of J_f.*

2. Preliminaries

Let E/F, K, $\lambda : I_K(\mathfrak{m}) \to \mathbb{C}^\times$, and $f = f_\lambda$ be as in the introduction. Let $f = \sum_{m \geqslant 1} a_m q^m$ $(q = e^{2\pi i z})$ be the Fourier expansion at $i\infty$ and put $H := \mathbb{Q}(a_m | m \geqslant 1)$ $(\subseteq \mathbb{C})$. Let n be the dimension of J_f, then H is an algebraic number field with $[H : \mathbb{Q}] = n$. A $\mathbb{Q}$-algebra isomorphism $\theta : H \tilde{\to} \mathrm{End}^0_{\mathbb{Q}}(J_f) = \mathrm{End}_{\mathbb{Q}}(J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ is defined by

$$a_m \mapsto \text{the endomorphism of } J_f \text{ induced by the } m\text{th Hecke operator w.r.t. } \Gamma_1(N)$$

$(m = 1, 2, \ldots)$. In [3] Shimura proved that J_f is isogenous to $E^n = E \times \cdots \times E$ (n terms) over $\overline{\mathbb{Q}}$, expressed by $J_f \sim_{\overline{\mathbb{Q}}} E^n$. So we have $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f) \cong M_n(K)$, the algebra of $n \times n$-matrices with entries in K. Let Z be the center of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$. Then we have $Z \cong K$. We denote by T the sub $\mathbb{Q}$-algebra of $\mathrm{End}^0_{\overline{\mathbb{Q}}}(J_f)$ generated by Z and $\theta(H)$. Shimura used the following facts in the proof of Proposition 1.6 in [5] and we state them as a lemma without proof.

Lemma 2.1. (1) $Z \cap \theta(H) = \mathbb{Q}$. *Especially this implies that* $\dim_H T = 2$.
(2) $\mathrm{End}^0_K(J_f) = T$.

Therefore, as for the structure of T, we have the possibility of the following two cases:

Case 1: T is isomorphic to an algebraic number field with degree $2n$ (over $\mathbb{Q}$) $\Leftrightarrow K \not\subseteq H$;

Case 2: $T \cong H \oplus H \Leftrightarrow K \subseteq H$.

Let $F' = \langle F, K \rangle$ be the subfield of $\mathbb{C}$ generated by F and K. It is well known that $\mathrm{End}^0_{\overline{\mathbb{Q}}}(E) = \mathrm{End}^0_{F'}(E)$ $(\cong K)$. We put $\mathcal{M} := \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ over F acts on $\mathcal{M}$ by the action on coefficients of homomorphisms. If we know the structure of $\mathcal{M}$ as Galois module, we will be able to answer Questions 1 and 2. Therefore, our purpose in this paper is to determine the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-module. On the other hand we have the following.

Lemma 2.2. $\mathrm{Hom}_{F'}(E, J_f) \neq \{0\} \Leftrightarrow \mathrm{Hom}_F(E, J_f) \neq \{0\}$.

Proof. When $F = F'$, it is trivial. So we will consider the case: $F \neq F'$. We set $\mathrm{Gal}(F'/F) = \langle \tau_0 \rangle$ and fix $\iota : K \tilde{\to} \mathrm{End}^0_{F'}(E)$. Since $(\Leftarrow)$ is trivial, we will prove $(\Rightarrow)$.

By assumption there exists $\eta\,(\neq 0)\in \mathrm{Hom}_{F'}(E, J_f)$. Then $\eta' := \eta + {}^{\tau_0}\eta$ is defined over F. Therefore, if $\eta'\neq 0$, we have $\mathrm{Hom}_F(E, J_f)\neq\{0\}$. Now we assume that $\eta' = 0$. We take a positive integer m such that $\iota(m\sqrt{-D})\in \mathrm{End}_{F'}(E)$. Then it is well known that ${}^{\tau_0}(\iota(m\sqrt{-D})) = -\iota(m\sqrt{-D})$. Hence $\eta'' := \eta\circ\iota(m\sqrt{-D})$ is defined over F. Since $\iota(m\sqrt{-D})$ is surjective, $\eta''\neq 0$. This completes the proof. □

By this lemma, for answer to Question 1, it is enough to study the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. But, for answer to Question 2, this does not seem to be enough. Nevertheless, as we will see later, under assumption $\mathrm{Hom}_{F'}(E, J_f)\neq\{0\}$ the structure of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-module can be easily recovered from that of $\mathcal{M}$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. Therefore, in the following we will study the $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module structure.

By composition of homomorphisms, $\mathcal{M}$ has the structure of left T- and right K-module:

$$T = \mathrm{End}^0_K(J_f) \curvearrowright \mathcal{M} \curvearrowleft \mathrm{End}^0_{F'}(E) \cong K.$$

As $J_f \sim_{\overline{\mathbb{Q}}} E^n$, we have

$$\mathcal{M} \cong \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, E^n)\otimes_{\mathbb{Z}}\mathbb{Q} \cong K^{\oplus n}$$

as $\mathbb{Q}$-vector space. In particular we have $\dim_{\mathbb{Q}}\mathcal{M} = n\times\dim_{\mathbb{Q}} K = 2n$. On the other hand $H \xrightarrow{\sim} \theta(H)\subseteq T$, we can view $\mathcal{M}$ as H-vector space. Since $[H:\mathbb{Q}]\times\dim_H\mathcal{M} = \dim_{\mathbb{Q}}\mathcal{M} = 2n$, we have $\dim_H\mathcal{M} = 2$.

Proposition 2.3. *$\mathcal{M}$ is a free left T-module of rank* 1.

Proof. Case 1 is trivial, so we will prove Case 2, i.e. $T\cong H\oplus H$. Since $\mathcal{M}$ is a left T- and right K-module, K acts H-linearly on $\mathcal{M}$. Therefore, $\mathcal{M}\curvearrowleft K$ corresponds to a $\mathbb{Q}$-algebra homomorphism

$$\varrho : K\to \mathrm{End}_H\mathcal{M}\cong M_2(H).$$

Since $K\subseteq H$, by taking a suitable H-basis of $\mathcal{M}$, we have

$$\varrho(a) = \begin{pmatrix} i_1(a) & 0 \\ 0 & i_2(a)\end{pmatrix},\quad i_k : K\hookrightarrow H : \text{ring homomorphism } (k=1,2)$$

for any $a\in K$. First we prove that $i_1\neq i_2$. Assume $i_1 = i_2$. We can take some $a_0\in \mathrm{End}_{\overline{\mathbb{Q}}}(E)\backslash\mathbb{Z}(\hookrightarrow K$ by $\iota^{-1})$ such that $\varrho(a_0)$ has the form

$$\begin{pmatrix}\alpha_0 & 0\\ 0 & \alpha_0\end{pmatrix},\quad \alpha_0\in\theta^{-1}(\,\mathrm{End}_{\mathbb{Q}}(J_f)\backslash\mathbb{Z})(\subseteq H).$$

We have $\theta(\alpha_0) \notin Z$ because $Z \cap \theta(H) = \mathbb{Q}$. On the other hand, for any homomorphism $\varphi : E \to J_f$, one has

$$\varphi \circ a_0 = \theta(\alpha_0) \circ \varphi.$$

For any $\alpha \in \mathrm{End}_{\overline{\mathbb{Q}}}(J_f)$, $\alpha \circ \varphi \in \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$, so we get

$$(\alpha \circ \theta(\alpha_0)) \circ \varphi = (\alpha \circ \varphi) \circ a_0 = \theta(\alpha_0) \circ (\alpha \circ \varphi) = (\theta(\alpha_0) \circ \alpha) \circ \varphi.$$

Hence, we have $(\alpha \circ \theta(\alpha_0) - \theta(\alpha_0) \circ \alpha)(\varphi(x)) = 0$ for any $x \in E$. Since $J_f \sim_{\overline{\mathbb{Q}}} E^n$, the abelian subvariety of J_f generated by $\varphi(E)$ for all $\varphi \in \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$ coincides with J_f. Therefore we have $\alpha \circ \theta(\alpha_0) = \theta(\alpha_0) \circ \alpha$, i.e. $\theta(\alpha_0) \in Z$, so this is a contradiction.

Putting

$$\mathcal{M}_k := \{\eta \in \mathcal{M} \mid \eta \circ a = \theta(i_k(a)) \circ \eta \text{ for } {}^{\forall} a \in K \cong \mathrm{End}_{\overline{\mathbb{Q}}}(E)\} \quad (k = 1, 2),$$

it holds that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\dim_H \mathcal{M}_k = 1$. Take $\eta_k (\neq 0) \in \mathcal{M}_k \cap \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$ and fix it $(k = 1, 2)$. Then, for any $\alpha \in T$, $\alpha \circ \eta_k \in \mathcal{M}_k$, so there exists a unique lement α' of H such that $\alpha \circ \eta_k = \theta(\alpha') \circ \eta_k$. Defining $\varepsilon_k : T \ni \alpha \mapsto \alpha' \in H$, ε_k is a $\mathbb{Q}$-algebra homomorphism. We consider a $\mathbb{Q}$-algebra homomorphism $\varepsilon := \varepsilon_1 \oplus \varepsilon_2 : T \to H \oplus H$ and will prove that it is an isomorphism. Since $\dim_{\mathbb{Q}} T = \dim_{\mathbb{Q}} H \oplus H$, it is sufficient to prove that it is injective. Suppose $\varepsilon(\alpha) = (0, 0)$ for some $\alpha \in T$. Then, for any $\varphi \in \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$, $\alpha \circ \varphi = 0$. By the above argument, we get $\alpha = 0$. So we have proved that $\eta_0 := \eta_1 + \eta_2$ is a free basis of $\mathcal{M}$ over T. □

Let ℓ be a prime number and put

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad V_\ell(J_f) := T_\ell(J_f) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \quad \mathcal{M}_\ell := \mathcal{M} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell,$$

where $T_\ell(E)$ and $T_\ell(J_f)$ are Tate modules. We can consider the following actions:

- $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by diagonal;
- $H \hookrightarrow T \curvearrowright \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$ by the action on $\mathcal{M}$.

We define a homomorphism $v : \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E) \to V_\ell(J_f)$ by

$$(\varphi \otimes a) \otimes x \mapsto a\varphi(x).$$

Proposition 2.4. *v is an isomorphism of (left) $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F)]$-modules and is also an isomorphism of (left) $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F')]$-modules, where $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F)]$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F')]$) denotes the group algebra of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ (resp. $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$) over $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ (resp. $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$).*

Proof. It is clear that v is a homomorphism of $H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F)]$-modules and $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/F')]$-modules. So we will prove that v is bijective. We take

an isogeny $\psi : J_f \to E \times \cdots \times E$ (n terms) defined over $\overline{\mathbb{Q}}$ and fix it. Let $\psi_i : J_f \to E$ be the compositon of ψ and the projection to the ith component ($i = 1, \ldots, n$). Then

$$\mu_1 : \mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E) \to V_\ell(E)^{\oplus n}, \quad (\varphi \otimes a) \otimes x \mapsto (a(\psi_1 \circ \varphi)(x), \ldots, a(\psi_n \circ \varphi)(x))$$

turns to be an isomorphism of $\mathbb{Q}_\ell$-modules. On the other hand

$$\mu_2 : V_\ell(J_f) \to V_\ell(E)^{\oplus n}, \quad x \mapsto (\psi_1(x), \ldots, \psi_n(x))$$

also turns to be an isomorphism of $\mathbb{Q}_\ell$-modules and it holds that $\mu_2 \circ v = \mu_1$. Hence v is bijective. □

3. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell \otimes_{K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell} V_\ell(E)$

We review the known results about the structure of $V_\ell(E)$ as $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$-module. By changing $\iota : K \xrightarrow{\sim} \mathrm{End}^0_{F'}(E)$ if necessary, we may assume that the CM-type of (E, ι) is $(K; \{id\})$. Then there exists a lattice $\mathfrak{a}$ of K such that the following commutative diagram holds:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a} & \longrightarrow & K_{\mathbb{R}} & \longrightarrow & K_{\mathbb{R}}/\mathfrak{a} & \longrightarrow & 0 \quad \text{(exact)} \\
 & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} & & \\
0 & \longrightarrow & q(\mathfrak{a}) & \longrightarrow & \mathbb{C} & \xrightarrow[\xi]{} & E(\mathbb{C}) & \longrightarrow & 0 \quad \text{(exact)},
\end{array}$$

where $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$ and $q(a \otimes x) = ax$. By the theory of complex multiplication, the following is well known (see Theorem 19.8, p. 134 in [6]).

Theorem 3.1. (1) *Every point of $E(\mathbb{C})$ with finite order is F'_{ab}-rational, where F'_{ab} denotes the maximal abelian extension of F'.*

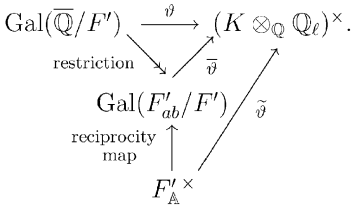
(2) *There exists a unique homomorphism $\alpha_{E/F'} : F'^{\times}_{\mathbb{A}} \to K^{\times}$ (where $F'^{\times}_{\mathbb{A}}$ denotes the idele group of F') such that*

- $\mathrm{Ker}(\alpha_{E/F'})$ *is open in* $F'^{\times}_{\mathbb{A}}$;
- *For any* $x \in F'^{\times}_{\mathbb{A}}$, $\alpha_{E/F'}(x) N_{F'/K}(x)^{-1} \mathfrak{a} = \mathfrak{a}$, *where* $N_{F'/K}$ *is the norm map from* $F'^{\times}_{\mathbb{A}}$ *to* $K^{\times}_{\mathbb{A}}$;
- *For any* $x \in F'^{\times}_{\mathbb{A}}$, $\alpha_{E/F'}(x) \rho(\alpha_{E/F'}(x)) = N(il(x))$, *where* $\rho(v)$ *is the complex conjugate of a complex number* v *and* $il(x)$ *is the fractional ideal of* F' *associated to an idele element* x;
- *For any* $x \in F'^{\times}_{\mathbb{A}}$ *and* $w \in K/\mathfrak{a}$, ${}^{[x, F']} r(w) = r(\alpha_{E/F'}(x) N_{F'/K}(x)^{-1} w)$, *where* $[x, F']$ *is the element of* $\mathrm{Gal}(F'_{ab}/F')$ *corresponding to* x *by the reciprocity law of class field theory*.

Since $V_\ell(E)$ is viewed as free $K\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$-module of rank 1 by ι, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(E)$ determines the homomorphism

$$\vartheta : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \to (K\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times.$$

Then ϑ factors through the restriction map to F'_{ab}. So we denote by $\bar{\vartheta}$ the induced map from $\mathrm{Gal}(F'_{ab}/F')$ to $(K\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times$ and by $\widetilde{\vartheta}$ the composition of the reciprocity map for F' and $\bar{\vartheta}$. Thus we have the following commutative diagram:



Then Theorem 3.1 implies the following:

Corollary 3.2. *For any* $x\in F'^{\times}_{\mathbb{A}}$, $\widetilde{\vartheta}(x) = (\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_\ell$, *where* $(\ )_\ell$ *denotes the* ℓ*-component*.

By Proposition 2.3, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}$ determines the homomorphism

$$\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \to T^\times.$$

Let χ_ℓ be the composition of χ and the canonical map $T^\times \to (T\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)^\times$, then χ_ℓ corresponds to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $\mathcal{M}_\ell$. In other words, taking a basis η of $\mathcal{M}$ over T, we have ${}^\sigma\eta = \chi(\sigma)\circ\eta$ for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

Firstly, we consider Case 1. Since K acts T-linearly on $\mathcal{M}$, we can take a $\mathbb{Q}$-algebra isomorphism $\kappa : K \xrightarrow{\sim} Z\subseteq T$ such that $\eta\circ\iota(a) = \kappa(a)\circ\eta$ for any $a\in K$, denoted by $\eta a = a\eta$ for short. We take a basis v of $V_\ell(E)$ over $K\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. Then $\omega := \eta\otimes v$ becomes a free basis of $\mathcal{M}_\ell\otimes_{K\otimes_{\mathbb{Q}}\mathbb{Q}_\ell} V_\ell(E)$ over $T\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ and it holds that

$$\begin{aligned}{}^\sigma\omega &= {}^\sigma\eta\otimes{}^\sigma v = (\chi_\ell(\sigma)\circ\eta)\otimes(\vartheta(\sigma)v) = (\chi_\ell(\sigma)\circ\eta\circ(\iota\otimes 1)(\vartheta(\sigma)))\otimes v\\ &= (\chi_\ell(\sigma)\vartheta(\sigma)\,\eta)\otimes v = \chi_\ell(\sigma)\vartheta(\sigma)\,\omega\end{aligned}$$

for any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

Next, we consider Case 2. $\sqrt{-D}$ $(\in K)$ acts T-linearly on $\mathcal{M}$, so there exists some $t\in T$ such that $\eta'\circ\iota(\sqrt{-D}) = t\circ\eta'$ for any $\eta'\in\mathcal{M}$. We will show that $t\in Z$ (one should note that in Case 2, T has two $\mathbb{Q}$-subalgebras isomorphic to K, so it is not

trivial that $t \in Z$). For any $\varphi \in \mathrm{End}^0_{\mathbb{Q}}(J_f)$ and $\eta' \in \mathcal{M}$, we have

$$(\varphi \circ t) \circ \eta' = \varphi \circ (t \circ \eta') = \varphi \circ (\eta' \circ \iota(\sqrt{-D})) = (\varphi \circ \eta') \circ \iota(\sqrt{-D})$$
$$= t \circ (\varphi \circ \eta') = (t \circ \varphi) \circ \eta',$$

therefore $t \circ \varphi = \varphi \circ t$ in $\mathrm{End}^0_{\mathbb{Q}}(J_f)$, hence $t \in Z$. This concludes that similarly with Case 1, there exists a $\mathbb{Q}$-algebra isomorphism $\kappa : K \xrightarrow{\sim} Z \subseteq T$ with the same property. Let $\gamma_1 : K \hookrightarrow H$ be the map induced by the inclusion $K \subseteq H$ and $\gamma_2 : K \hookrightarrow H$ be the other homomorphism. We define an isomorphism of $\mathbb{Q}$-algebras $\varepsilon : T \xrightarrow{\sim} H \oplus H$ by

$$z(\in Z) \mapsto (\gamma_1(\kappa^{-1}(z)), \quad \gamma_2(\kappa^{-1}(z))), \quad \theta(a)(\in \theta(H)) \mapsto (a, a).$$

For $k = 1, 2$, we set

$$\chi_\ell^{(k)} : \mathrm{Gal}(\overline{\mathbb{Q}}/F') \xrightarrow[\chi_\ell]{} (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[\varepsilon \otimes 1]{\sim} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times \xrightarrow[k\text{th component}]{\text{projection to}} (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times.$$

These arguments imply the following:

Proposition 3.3. *Let the notations be as above. We regard $K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \subseteq T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by injection $\kappa \otimes 1$.*

(1) *In Case* 1, *it holds that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$,*

$$^\sigma\omega = \chi_\ell(\sigma)\vartheta(\sigma)\omega.$$

(2) *In Case* 2, *identifying $T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ with $(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\oplus 2}$ by $\varepsilon \otimes 1$, it holds that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$,*

$$^\sigma\omega = (\chi_\ell^{(1)}(\sigma)\gamma_1(\vartheta(\sigma)), \chi_\ell^{(2)}(\sigma)\gamma_2(\vartheta(\sigma)))\omega,$$

where we denote $\gamma_k \otimes 1 : K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \hookrightarrow H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ by γ_k $(k = 1, 2)$ for simplicity.

4. On relation between Eichler–Shimura theory and complex multiplication theory about J_f

In this section, we will describe a relation between λ in $f = f_\lambda$ and the homomorphism corresponding to $\alpha_{E/F'}$ in higher-dimensional case. The content of this section is essentially stated in the proof of Proposition 1.6 in [5] without detailed proof. We present the results in a slightly different form to be convenient to our purpose.

Firstly, we consider Case 1. Then $L := \langle K, H \rangle$ $(\subseteq \mathbb{C})$ is a CM-field with $[L:\mathbb{Q}] = 2n$. We define an isomorphism of $\mathbb{Q}$-algebras $\iota' : L \xrightarrow{\sim} T = \mathrm{End}^0_K(J_f)$ by

$$a(\in K) \mapsto \kappa(a)(\in Z), \quad x(\in H) \mapsto \theta(x).$$

Then (J_f, ι') is an abelian variety with complex multiplication defined over K in the sense of Shimura (see Section 19.7 in [6]). Since $\theta(H) \subseteq \mathrm{End}^0_{\mathbb{Q}}(J_f)$, the characteristic polynomial of any element of H acting on $H^0(J_f, \Omega^1_{/\mathbb{C}}) = H^0(J_f, \Omega^1_{/\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$ has $\mathbb{Q}$-rational coefficients. Therefore, by Lemma 1 in [7] (p. 38), the representation of H on $H^0(J_f, \Omega^1_{/\mathbb{C}})$ is equivalent to the regular representation of H over $\mathbb{Q}$. It is also proved that Z acts on $H^0(J_f, \Omega^1_{/\mathbb{C}})$ by scalar multiple. Let $(L, \{\varpi_1, \ldots, \varpi_n\})$ be the CM-type of (J_f, ι'), then we have

$$\{\varpi_{1|H}, \ldots, \varpi_{n|H}\} = \{\varpi \mid \varpi : H \hookrightarrow \mathbb{C}\}, \quad \varpi_{i|K} = id_K \ \ (i = 1, \ldots, n)$$

by changing the identification of K as subfield of $\mathbb{C}$ if necessary. Hence the reflex of $(L, \{\varpi_1, \ldots, \varpi_n\})$ is $(K, \{id_K\})$. Let $g' : K^\times_{\mathbb{A}} \to L^\times_{\mathbb{A}}$ be the canonical map induced from the inclusion $K \subseteq L$. Similarly with case of E/F', the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_\ell(J_f)$ determines the homomorphism

$$\delta : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to (L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$$

and we define $\widetilde{\delta} : K^\times_{\mathbb{A}} \to (L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^\times$ by the same manner as defining $\widetilde{\vartheta}$. The theory of complex multiplication also implies the following:

Corollary 4.1. *For any $x \in K^\times_{\mathbb{A}}$, $\widetilde{\delta}(x) = (\alpha_{J_f/K}(x) g'(x)^{-1})_\ell$, where $\alpha_{J_f/K} : K^\times_{\mathbb{A}} \to L^\times$ is the homomorphism corresponding to $\alpha_{E/F'}$ in higher dimensional case.*

Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ be the set of all bad primes of J_f/K. For every $\mathfrak{p}_k$ $(1 \leqslant k \leqslant s)$, we take the least positive integer t_k such that

$$x \in K^\times_{\mathfrak{p}_k} \subseteq K^\times_{\mathbb{A}}, \quad x - 1 \in \mathfrak{p}_k^{t_k} \Rightarrow \alpha_{J_f/K}(x) = 1.$$

We set $\mathfrak{n} := \mathfrak{p}_1^{t_1} \cdots \mathfrak{p}_s^{t_s}$, $G_K(\mathfrak{n}) := \{x \in K^\times_{\mathbb{A}} \mid x_\infty = 1, x_{\mathfrak{p}_k} = 1 \ (1 \leqslant k \leqslant s)\}$, $U_K := \{x \in K^\times_{\mathbb{A}} \mid x_{\mathfrak{p}} \in \mathcal{O}^\times_{K_{\mathfrak{p}}}$ for any finite prime $\mathfrak{p}\}$, and $U_K(\mathfrak{n}) := G_K(\mathfrak{n}) \cap U_K$. We consider the canonical isomorphism $G_K(\mathfrak{n})/U_K(\mathfrak{n}) \xrightarrow{\sim} I_K(\mathfrak{n})$ by which the class represented by $x \in G_K(\mathfrak{n})$ is sent to $il(x) \in I_K(\mathfrak{n})$. Since $U_K(\mathfrak{n}) \subseteq \mathrm{Ker}(\alpha_{J_f/K})$, we obtain the homomorphism

$$\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \to L^\times$$

induced from $\alpha_{J_f/K}$. By the two properties of $\alpha_{J_f/K}$: (i) $x \in K^\times_\infty = \mathbb{C}^\times \subseteq K^\times_{\mathbb{A}} \Rightarrow \alpha_{J_f/K}(x) = 1$; (ii) $x \in K^\times \subseteq K^\times_{\mathbb{A}} \Rightarrow \alpha_{J_f/K}(x) = g'(x) = x$, it holds that

$$\alpha \in K^\times, \quad \alpha \equiv 1 \bmod^\times \mathfrak{n} \Rightarrow \widetilde{\alpha_{J_f/K}}((\alpha)) = \alpha.$$

It is clear that $\widetilde{\alpha_{J_f/K}} : I_K(\mathfrak{n}) \to L^\times \subseteq \mathbb{C}^\times$ is primitive. By Hecke–Shimura, we have that

$$h(z) := \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{n}) \\ \mathfrak{a}:\text{integral}}} \widetilde{\alpha_{J_f/K}}(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z}$$

is a normalized newform of weight two on $\Gamma_1(N')$ $(N' := DN(\mathfrak{n}))$.

Proposition 4.2. *In Case* 1, *we have* $\lambda = \widetilde{\alpha_{J_f/K}}$ *and* $\mathfrak{m} = \mathfrak{n}$.

Proof. Letting ∞_1 be the archimedean place of L corresponding to the inclusion $L \subseteq \mathbb{C}$, we define the Grössen-character $\beta_{J_f/K} : K_{\mathbb{A}}^\times \to \mathbb{C}^\times$ of J_f/K by

$$\beta_{J_f/K}(x) = (\alpha_{J_f/K}(x) g'(x)^{-1})_{\infty_1}.$$

(By a Grössen-character of $K_{\mathbb{A}}^\times$, we mean a continuous homomorphism of $K_{\mathbb{A}}^\times$ into $\mathbb{C}^\times$ whose kernel contains $K^\times$.) By Theorem 12. p. 529 in [4], we have an equality of zeta-functions:

$$\begin{aligned} \zeta(s; J_f/\mathbb{Q}, H) = L(s, \beta_{J_f/K}) &= \prod_{\substack{\mathfrak{p}:\text{prime ideal in } K \\ (\mathfrak{p},\mathfrak{n})=1}} \left(1 - \frac{\widetilde{\alpha_{J_f/K}}(\mathfrak{p})}{N(\mathfrak{p})^s}\right)^{-1} \\ &= \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{n}) \\ \mathfrak{a}:\text{integral}}} \frac{\widetilde{\alpha_{J_f/K}}(\mathfrak{a})}{N(\mathfrak{a})^s} = L(s, h). \end{aligned}$$

On the other hand, by Eichler–Shimura theory for modular forms, we have $\zeta(s; J_f/\mathbb{Q}, H) = L(s, f_\lambda)$, hence $L(s, f_\lambda) = L(s, h)$. So we have that $f_\lambda = h$ and $N = N'$. By the statement in the third line on p. 203 of [3], we obtain that $\lambda = \widetilde{\alpha_{J_f/K}}$ and $\mathfrak{m} = \mathfrak{n}$. □

Next we investigate Case 2. Since J_f is defined over $\mathbb{Q}$, $\rho_{|K}$ $(\in \mathrm{Gal}(K/\mathbb{Q}))$ acts on $T = \mathrm{End}_K^0(J_f)$. Identifying T with $H \oplus H$ by ε, this action corresponds to the automorphism of $H \oplus H$ defined by $(x, y) \mapsto (y, x)$. Let ξ_1, ξ_2 be the elements of T which correspond to $(1, 0)$, $(0, 1)$ respectively. We take a positive integer r such that $r\xi_k \in \mathrm{End}_K(J_f)$ $(k = 1, 2)$ and set $\xi_k' := r\xi_k$. Then $C := \mathrm{Im}(\xi_1')$ is an abelian subvariety of J_f defined over K. Since ${}^\rho\xi_1' = \xi_2'$, we have ${}^\rho C = \mathrm{Im}(\xi_2')$. So we can define an isogeny $\varphi : J_f \to C \times {}^\rho C$ defined over K by $x \mapsto (\xi_1'(x), \xi_2'(x))$ and this implies $J_f \sim_K C \times {}^\rho C$.

Lemma 4.3. *We have* $J_f \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C) \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}({}^\rho C)$, *where* $R_{K/\mathbb{Q}}(C)$ *denotes the Weil restriction from* K *to* $\mathbb{Q}$ *of* C.

Proof. By Theorem 3, p. 185 in [1], $R_{K/\mathbb{Q}}(J_{f/K}) \sim_{\mathbb{Q}} J_f^2$. On the other hand, since $J_f \sim_K C \times {}^{\rho}C$, we have $R_{K/\mathbb{Q}}(J_{f/K}) \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C \times {}^{\rho}C)$ and $R_{K/\mathbb{Q}}(C \times {}^{\rho}C)$ is isomorphic to $R_{K/\mathbb{Q}}(C) \times R_{K/\mathbb{Q}}({}^{\rho}C)$ over $\mathbb{Q}$. We get the claim, because $J_f^2 \sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C) \times R_{K/\mathbb{Q}}({}^{\rho}C)$ and J_f is simple over $\mathbb{Q}$. □

To understand the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_\ell(J_f)$, it is sufficient to do so for that of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ on $V_\ell(C)$ by this lemma. Putting $R := \theta^{-1}(\mathrm{End}_{\mathbb{Q}}(J_f))$, we define a ring homomorphism $\iota'' : R \to \mathrm{End}_K(C)$ by

$$a \mapsto (C \ni x \mapsto (\theta(a))(x) \in C)$$

and denote $\iota'' \otimes 1 : H = R \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{End}_K^0(C)$ by the same notation ι''. In Case 2, $K \subseteq H$, so H is a CM-field. Then (C, ι'') is an abelian variety with complex multiplication defined over K. Let H_0 be the maximal real subfield of H and $(H, \{\tau_1, \ldots, \tau_{n_1}\})$ $\left(n_1 := \frac{n}{2}\right)$ be the CM-type of (C, ι''). Since $H_0 \subseteq \mathrm{End}_K^0(C)$, the characteristic polynomial of any element of H_0 acting on $H^0(C, \Omega^1_{/\mathbb{C}})$ has K-rational coefficients. Since H_0 is totally real, its coefficients also lie in $\mathbb{R}$. So it has $\mathbb{Q}$-rational coefficients. It is also proved that $K \subseteq H$ acts on $H^0(C, \Omega^1_{/\mathbb{C}})$ by scalar mutiple because $\iota''(K)$ coincides with the center of $\mathrm{End}_{\mathbb{Q}}^0(C) \cong M_{n_1}(K)$. Therefore we have

$$\{\tau_{1|H_0}, \ldots, \tau_{n_1|H_0}\} = \{\tau \mid \tau : H_0 \hookrightarrow \mathbb{R}\}, \quad \tau_{i|K} = id_K \ \ (i = 1, \ldots, n_1)$$

by changing the identification of K as subfield of $\mathbb{C}$ if necessary. Hence the reflex of $(H, \{\tau_1, \ldots, \tau_{n_1}\})$ is $(K, \{id_K\})$. Let $g'' : K_{\mathbb{A}}^{\times} \to H_{\mathbb{A}}^{\times}$ be the canonical map induced from $\gamma_1 : K \hookrightarrow H$. Similary with Case 1, we have

$$\delta' : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}, \quad \widetilde{\delta'} : K_{\mathbb{A}}^{\times} \to (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times},$$

and the following:

Corollary 4.4. *For any* $x \in K_{\mathbb{A}}^{\times}$, $\widetilde{\delta'}(x) = (\alpha_{C/K}(x) g''(x)^{-1})_\ell$.

Let $\mathfrak{n}'$ be the one corresponding to $\mathfrak{n}$ in case of C/K. Then , as Case 1, we can define

$$\widetilde{\alpha_{C/K}} : I_K(\mathfrak{n}') \to H^{\times}, \quad h'(z) = \sum_{\substack{\mathfrak{a} \in I_K(\mathfrak{n}') \\ \mathfrak{a} \text{ :integral}}} \widetilde{\alpha_{C/K}}(\mathfrak{a}) e^{2\pi i N(\mathfrak{a}) z}$$

and $h'(z)$ is a normalized newform of weight two on $\Gamma_1(N'')(N'' := DN(\mathfrak{n}'))$.

Proposition 4.5. *In Case* 2, *we have* $\lambda = \widetilde{\alpha_{C/K}}$ *and* $\mathfrak{m} = \mathfrak{n}'$.

Proof. We take a free basis v_1 of $V_\ell(C)$ over $H\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. Since $J_f\sim_{\mathbb{Q}} R_{K/\mathbb{Q}}(C)$,

$$V_\ell(J_f)\cong\mathbb{Q}_\ell\,[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]\otimes_{\mathbb{Q}_\ell\,[\mathrm{Gal}(\overline{\mathbb{Q}}/K)]}V_\ell(C)$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module. By this identification, $\{1\otimes v_1,\rho\otimes v_1\}$ becomes a free basis of $V_\ell(J_f)$ over $H\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$. Using this basis, we can express the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $V_\ell(J_f)$ by matrices and we denote this by $\Phi_\ell:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to GL_2(H\otimes_{\mathbb{Q}}\mathbb{Q}_\ell)$. For any $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, if $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, we have

$$\begin{aligned}\sigma(1\otimes v_1)&=\sigma\otimes v_1=1\otimes{}^{\sigma}v_1=1\otimes\delta'(\sigma)v_1=\delta'(\sigma)(1\otimes v_1);\\ \sigma(\rho\otimes v_1)&=\sigma\rho\otimes v_1=(\rho\cdot\rho\sigma\rho)\otimes v_1=\rho\otimes{}^{\rho\sigma\rho}v_1=\rho\otimes(\delta'(\rho\sigma\rho)v_1)\\ &=\delta'(\rho\sigma\rho)(\rho\otimes v_1)\end{aligned}$$

We set $h'=\sum_{m\geqslant 1}b_mq^m$. We take any prime p with $(p,\ell NN'')=1$ and denote by $\sigma_p(\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))$ a Frobenius element at p. If p remains prime in K, then $\sigma_p\notin\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, so we have $Tr(\Phi_\ell(\sigma_p))=0=b_p$ by the above calculations. Next, we consider the case: p splits in K, i.e. $(p)=\mathfrak{p}\mathfrak{p}'$. By changing $\mathfrak{p}$ and $\mathfrak{p}'$ if necessary, we may assume that σ_p is also a Frobenius element at $\mathfrak{p}$, i.e. $\sigma_p=\sigma_{\mathfrak{p}}(\in\mathrm{Gal}(\overline{\mathbb{Q}}/K))$. Then we have

$$Tr(\Phi(\sigma_p))=\delta'(\sigma_p)+\delta'(\rho\sigma_p\rho)=\delta'(\sigma_{\mathfrak{p}})+\delta'(\sigma_{\mathfrak{p}'})=\widetilde{\alpha_{C/K}}(\mathfrak{p})+\widetilde{\alpha_{C/K}}(\mathfrak{p}')=b_p.$$

On the other hand, by the Eichler–Shimura theory, we have $Tr(\Phi(\sigma_p))=a_p$. So it holds that $a_p=b_p$ for almost all primes p. This implies $f_\lambda=h'$. □

5. Main results

Let $\beta_{E/F'}:F'^{\times}_{\mathbb{A}}\to\mathbb{C}^{\times}$ be the Grössen-character of E/F' (By definition, $\beta_{E/F'}(x)=(\alpha_{E/F'}(x)N_{F'/K}(x)^{-1})_\infty$.)

Theorem 5.1. *Let E be an elliptic curve with complex multiplication defined over an algebraic number field F $(\subseteq\mathbb{C})$ with $\mathrm{End}^0_{\overline{\mathbb{Q}}}(E)\cong K$ $(\subseteq\mathbb{C})$. Put $F':=\langle F,K\rangle(\subseteq\mathbb{C})$. Then the following three conditions are equivalent*:

(1) *E is modular over F.*
(2) *There exists a Grössen-character $\gamma:K^{\times}_{\mathbb{A}}\to\mathbb{C}^{\times}$ such that $\gamma\circ N_{F'/K}=\beta_{E/F'}$.*
(3) *All the points of E of finite order are rational over $\langle F',K_{ab}\rangle=\langle F,K_{ab}\rangle$.*

Proof. The equivalence of (2) and (3) is a special case of Theorem 4. p. 511 in [4].

We will prove that (1) implies (2). By assumption, there exists a normalized newform f of weight two (obtained by some $\lambda:I_K(\mathfrak{m})\to\mathbb{C}^{\times}$ as $f=f_\lambda$) such that

$\mathrm{Hom}_{F'}(E, J_f) \neq \{0\}$. From f we define H as above. Firstly, we consider Case 1. We define $\widetilde{\chi_\ell} : F'^{\times}_{\mathbb{A}} \to (T \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}$ from χ_ℓ by the same manner as defining $\widetilde{\vartheta}$ from ϑ in Section 3. By the commutative diagram

$$\begin{array}{ccc} F'^{\times}_{\mathbb{A}} & \xrightarrow{\text{norm}} & K^{\times}_{\mathbb{A}} \\ {\scriptstyle \text{reciprocity law}}\downarrow & & \downarrow{\scriptstyle \text{reciprocity law}} \\ \mathrm{Gal}(F'_{ab}/F') & \xrightarrow[\text{restriction}]{} & \mathrm{Gal}(K_{ab}/K), \end{array}$$

Propositions 2.4, 3.3, Corollaries 3.2 and 4.1, we have that

$$\widetilde{\chi_\ell}(x) = \alpha_{E/F'}(x)^{-1} \alpha_{J_f/K}(N_{F'/K}(x)) \quad \text{for any } x \in F'^{\times}_{\mathbb{A}}.$$

(We identify L with T by ι'.) In Case 1, T is a field, so we have

$$\mathrm{Hom}_{F'}(E, J_f) \neq \{0\} \Leftrightarrow \chi = 1 \Leftrightarrow \widetilde{\chi_\ell} = 1 \Leftrightarrow \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'}.$$

We note that Proposition 4.2 is rephrased to that the map

$$G_K(\mathfrak{m}) \xrightarrow{il} I_K(\mathfrak{m}) \xrightarrow{\lambda} \mathbb{C}^{\times}$$

can be continuously extended to $K^{\times}_{\mathbb{A}}$ by the manner: any $x \in K^{\times} (\subseteq K^{\times}_{\mathbb{A}})$ is mapped to 1 and this extended map, denoted by $\bar{\lambda}$, coincides with $\beta_{J_f/K}$. Then it holds that

$$\alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} \Leftrightarrow \bar{\lambda} \circ N_{F'/K} = \beta_{E/F'},$$

so we can take $\bar{\lambda}$ as γ in (2).

Next we consider Case 2. By the argument in the proof of Proposition 4.5, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ on $V_\ell(J_f)$ corresponds to the homomorphism

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \xrightarrow{\Phi_\ell} & GL_2(H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell) \\ \cup & & \cup \\ \mathrm{Gal}(\overline{\mathbb{Q}}/F') & \longrightarrow & (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \oplus (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \\ \cup & & \cup \\ \sigma & \longmapsto & (\delta'(\sigma),\ \delta'(\rho\sigma\rho)). \end{array}$$

By Propositions 2.4 and 3.3, we have that one of the following two statements holds:

(a) $\chi_\ell^{(1)}(\sigma) = \gamma_1(\vartheta(\sigma))^{-1}\delta'(\sigma)$, $\chi_\ell^{(2)}(\sigma) = \gamma_2(\vartheta(\sigma))^{-1}\delta'(\rho\sigma\rho)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$;
(b) $\chi_\ell^{(1)}(\sigma) = \gamma_1(\vartheta(\sigma))^{-1}\delta'(\rho\sigma\rho)$, $\chi_\ell^{(2)}(\sigma) = \gamma_2(\vartheta(\sigma))^{-1}\delta'(\sigma)$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/F')$.

We will prove that (b) is impossible. For this we assume that (b) holds. For $k = 1, 2$, we define $\widetilde{\chi_\ell^{(k)}}$ similary with $\widetilde{\chi_\ell}$. If $\sigma_{|F'_{ab}} = [x, F']$ ($[x, F']$ denotes the image of $x \in F'^{\times}_{\mathbb{A}}$ by the reciprocity law of F'), then we have $\rho\sigma\rho_{|K_{ab}} = [\rho(N_{F'/K}(x)), K]$ by the

class field theory. Therefore, for any $x \in F'^{\times}_{\mathbb{A}}$, we have

$$\widetilde{\chi_\ell^{(1)}}(x) = \gamma_1(\widetilde{\vartheta}(x)^{-1})\widetilde{\delta'}(\rho(N_{F'/K}(x)))$$
$$= \gamma_1(\alpha_{E/F'}(x))^{-1}\gamma_1((N_{F'/K}(x))_\ell)\alpha_{C/K}(\rho(N_{F'/K}(x)))\gamma_1((\rho(N_{F'/K}(x)))_\ell)^{-1}.$$

Since $\gamma_1 \circ \rho = \gamma_2$, this is rephrased to that

$$\frac{\gamma_1((N_{F'/K}(x))_\ell)}{\gamma_2((N_{F'/K}(x))_\ell)} = \frac{\widetilde{\chi_\ell^{(1)}}(x)\alpha_{E/F'}(x)}{\alpha_{C/K}(\rho(N_{F'/K}(x)))}.$$

We can take a transcendental element π of $\mathbb{Q}_\ell$ over $\mathbb{Q}$ and put $x_0 := 1 \otimes 1 + \sqrt{-D} \otimes \pi \in (K \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \subseteq (F' \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times} \subseteq F'^{\times}_{\mathbb{A}}$. Now we suppose that ℓ splits completely in H. Since $K \subseteq H$, we can view $K \subseteq \mathbb{Q}_\ell$. By the isomorphism

$$\left(\prod_{\substack{j:H\hookrightarrow\mathbb{Q}_\ell \\ j(\sqrt{-D})=\sqrt{-D}}} j\otimes 1\right) \oplus \left(\prod_{\substack{r:H\hookrightarrow\mathbb{Q}_\ell \\ r(\sqrt{-D})=-\sqrt{-D}}} r\otimes 1\right) : H\otimes_{\mathbb{Q}}\mathbb{Q}_\ell \xrightarrow{\sim} (\underbrace{\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell}_{\frac{n}{2}}) \oplus (\underbrace{\mathbb{Q}_\ell \oplus \cdots \oplus \mathbb{Q}_\ell}_{\frac{n}{2}}),$$

the element

$$\frac{\gamma_1((N_{F'/K}(x_0))_\ell)}{\gamma_2((N_{F'/K}(x_0))_\ell)} = \frac{\gamma_1(x_0^d)}{\gamma_2(x_0^d)}$$

is mapped to

$$\left(\left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d, \ldots, \left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d, \left(\frac{1-\pi\sqrt{-D}}{1+\pi\sqrt{-D}}\right)^d, \ldots, \left(\frac{1-\pi\sqrt{-D}}{1+\pi\sqrt{-D}}\right)^d\right),$$

where $d = [F' : K]$. Putting

$$\xi := \frac{\widetilde{\chi_\ell^{(1)}}(x_0)\alpha_{E/F'}(x_0)}{\alpha_{C/K}(\rho(N_{F'/K}(x_0)))} \in H^{\times} \subseteq (H \otimes_{\mathbb{Q}} \mathbb{Q}_\ell)^{\times}$$

and taking $j : H \hookrightarrow \mathbb{Q}_\ell$ with $j(\sqrt{-D}) = \sqrt{-D}$, we have that

$$\left(\frac{1+\pi\sqrt{-D}}{1-\pi\sqrt{-D}}\right)^d = j(\xi) \text{ in } \mathbb{Q}_\ell.$$

We note that $j(\xi)$ is algebraic over $\mathbb{Q}$. So we have that

$$\pi = \frac{\sqrt[d]{j(\xi)} - 1}{\sqrt{-D}(1 + \sqrt[d]{j(\xi)})} \in \overline{\mathbb{Q}}.$$

This is a contradiction. Hence we have proved that (a) holds.

We set

$$\iota''' : H \to \mathrm{End}_K^0({}^{\rho}C), \quad a \mapsto \theta(\rho(a))_{|{}^{\rho}C}.$$

Then $({}^{\rho}C, \iota''')$ is an abelian variety with complex multiplication defined over K which has the same CM-type with (C, ι''). As case of (C, ι''), we have $\alpha_{{}^{\rho}C/K} : K_{\mathbb{A}}^{\times} \to H^{\times}$. Since $\alpha_{{}^{\rho}C/K} = \rho \circ \alpha_{C/K} \circ \rho$, it holds that

$$\text{(a)} \Leftrightarrow \widetilde{\chi_{\ell}^{(1)}}(x) = \alpha_{E/F'}(x)^{-1}\alpha_{C/K}(N_{F'/K}(x)), \quad \widetilde{\chi_{\ell}^{(2)}}(x) = \rho(\alpha_{E/F'}(x)^{-1}\alpha_{{}^{\rho}C/K}(N_{F'/K}(x)))$$

for any $x \in F_{\mathbb{A}}'^{\times}$.

Therefore, we have

$$\mathrm{Hom}_{F'}(E, J_f) \neq \{0\} \Leftrightarrow \chi^{(1)} = 1 \text{ or } \chi^{(2)} = 1 \Leftrightarrow \chi_{\ell}^{(1)} = 1 \text{ or } \chi_{\ell}^{(2)} = 1$$

$$\Leftrightarrow \alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^{\rho}C/K} \circ N_{F'/K} = \alpha_{E/F'}.$$

Set $\lambda' := \rho \circ \lambda \circ \rho : I_K(\rho(\mathfrak{m})) \to \mathbb{C}^{\times}$. As Case 1, we can construct a Grössen-character $\bar{\lambda}$ (resp. $\bar{\lambda}'$) of $K_{\mathbb{A}}^{\times}$ from λ (resp. λ'). Then we have

$$\alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \text{ or } \alpha_{{}^{\rho}C/K} \circ N_{F'/K} = \alpha_{E/F'}$$

$$\Leftrightarrow \bar{\lambda} \circ N_{F'/K} = \beta_{E/F'} \text{ or } \bar{\lambda}' \circ N_{F'/K} = \beta_{E/F'}.$$

Hence, we can take $\bar{\lambda}$ or $\bar{\lambda}'$ as γ in (2).

Finally, we will prove that (2) implies (1). By Lemma 2.2, it is sufficient to show that there exists a normalized newform $f = f_{\lambda}$ of weight two constructed from some $\lambda : I_K(\mathfrak{m}) \to \mathbb{C}^{\times}$ such that $\mathrm{Hom}_{F'}(E, J_f) \neq \{0\}$.

Claim. *Let γ be as in* (2) *and $\mathfrak{n}_0$ be the conductor of γ. As defining $\widetilde{\alpha_{J_f/K}}$ from $\alpha_{J_f/K}$ in Section* 4, *we can also define $\widetilde{\gamma} : I_K(\mathfrak{n}_0) \to \mathbb{C}^{\times}$ from γ. Then it holds that for any $x \in K^{\times}$* s.t. $x \equiv 1 \bmod^{\times} \mathfrak{n}_0$,

$$\widetilde{\gamma}((x)) = x.$$

Proof of Claim. By the equality (2), p. 2 in [8], there exist an integer δ and real number σ, φ such that for any $x \in K^{\times}$ s.t. $x \equiv 1 \bmod^{\times} \mathfrak{n}_0$,

$$\widetilde{\gamma}((x)) = \left(\frac{x}{|x|}\right)^{\delta} |x|^{-2(\sigma + i\varphi)}.$$

Let $\mathfrak{m}_0$ be the conductor of $\beta_{E/F'}$ and set $\widetilde{\mathfrak{n}_0} := \mathrm{lcm}(N_{F'/K}(\mathfrak{m}_0), \mathfrak{n}_0)$. In some case we will view $\widetilde{\mathfrak{n}_0}$ as the integral ideal in F' generated by it, but at this time we will use also the same notation. We note that

$$G_{F'}(\mathfrak{m}_0) \supseteq G_{F'}(\widetilde{\mathfrak{n}_0}) \xrightarrow{N_{F'/K}} G_K(\widetilde{\mathfrak{n}_0}) \subseteq G_K(\mathfrak{n}_0).$$

Associated to any $x \in K^\times$ s.t. $x \equiv 1 \bmod^\times \widetilde{\mathfrak{n}_0}$, we define $y = (y_\mathfrak{p}) \in K_\mathbb{A}^\times$, $z = (z_\mathfrak{P}) \in F_\mathbb{A}'^\times$ by

$$y_\mathfrak{p} = \begin{cases} x & \text{if } \mathfrak{p} \nmid \widetilde{\mathfrak{n}_0} \text{ and } \mathfrak{p} \neq \infty; \\ 1 & \text{otherwise,} \end{cases} \qquad z_\mathfrak{P} = \begin{cases} x & \text{if } \mathfrak{P} \nmid \widetilde{\mathfrak{n}_0} \text{ and } \mathfrak{P} : \text{non-archimedean;} \\ 1 & \text{otherwise.} \end{cases}$$

Then it holds that $y \in G_K(\widetilde{\mathfrak{n}_0})$, $z \in G_{F'}(\widetilde{\mathfrak{n}_0})$, $il(y) = (x)$ in $I_K(\widetilde{\mathfrak{n}_0})$, and $il(z) = (x)$ in $I_{F'}(\widetilde{\mathfrak{n}_0})$. Let $\mathfrak{Q}_1, \ldots, \mathfrak{Q}_t$ be the all prime ideals in F' dividing $\widetilde{\mathfrak{n}_0}$. We denote by ιx (resp. $\iota_\infty x$, $\iota_{\mathfrak{Q}_i} x$ $(1 \leqslant i \leqslant t)$) the elements of $F_\mathbb{A}'^\times$ whose all components are x (resp. the archimedean part of ιx, the element of $F_\mathbb{A}'^\times$ whose $\mathfrak{Q}_i$-component is x and all other components are 1). By assumption, we have $\gamma(N_{F'/K}(z)) = \beta_{E/F'}(z)$. On the other hand,

$$\gamma(N_{F'/K}(z)) = \gamma(y^d) = \widetilde{\gamma}((x))^d = \left(\frac{x}{|x|}\right)^{\delta d} |x|^{-2(\sigma+i\varphi)d} \quad (d = [F' : K]);$$

$$\begin{aligned} \beta_{E/F'}(z) &= \beta_{E/F'}(\iota x)\beta_{E/F'}(\iota_{\mathfrak{Q}_1} x)^{-1} \cdots \beta_{E/F'}(\iota_{\mathfrak{Q}_t} x)^{-1} \beta_{E/F'}(\iota_\infty x)^{-1} \\ &= x^d \quad (\because \text{ Since } \mathfrak{m}_0 \mid \widetilde{\mathfrak{n}_0}, \beta_{E/F'}(\iota_{\mathfrak{Q}_i} x) = 1; \beta_{E/F'}(\iota_\infty x) = N_{F'/K}(x)^{-1} = x^{-d}.) \end{aligned}$$

Therefore, we have that for any $x \in K^\times$ s.t. $x \equiv 1 \bmod^\times \widetilde{\mathfrak{n}_0}$,

$$\left(\frac{x}{|x|}\right)^{\delta d} |x|^{-2(\sigma+i\varphi)d} = x^d.$$

This implies that $\delta = 1$, $\varphi = 0$, $\sigma = -\frac{1}{2}$. □

Now we go back to the proof of the theorem. By Claim, from $\widetilde{\gamma}$ we can construct a normalized newform $f = f_{\widetilde{\gamma}}$ of weight two. Then the arguments in the proof of the statement: (1) ⇒ (2) imply that

$$\gamma \circ N_{F'/K} = \beta_{E/F'} \Leftrightarrow \begin{cases} \alpha_{J_f/K} \circ N_{F'/K} = \alpha_{E/F'} \ (\text{if } K \not\subseteq H) \\ \alpha_{C/K} \circ N_{F'/K} = \alpha_{E/F'} \ (\text{if } K \subseteq H) \end{cases}$$
$$\Rightarrow \mathrm{Hom}_{F'}(E, J_f) \neq \{0\}.$$

So we have proved that (2) ⇒ (1). □

Theorem 5.2. *Let E/F, K, F', and $\beta_{E/F'}$ be as in Theorem* 5.1. *Assume that the condition* (2) *in Theorem* 5.1 *holds. Let $\mathfrak{m}$ be the conductor of γ and set*

$$f(z)=f_{\tilde{\gamma}}(z):=\sum_{\substack{\mathfrak{a}\in I_K(\mathfrak{m})\\ \mathfrak{a}:\text{integral}}}\tilde{\gamma}(\mathfrak{a})q^{N(\mathfrak{a})}=\sum_{m\geqslant 1}a_m q^m \quad (q=e^{2\pi iz}).$$

Put $H:=\mathbb{Q}(a_m\,|\,m\geqslant 1)$. Then we have the followings:

(1) *For any normalized newform g of weight two*, $\mathrm{Hom}_F(E,J_g)\neq\{0\}$ *if and only if there exists some γ as above such that $g=f_{\tilde{\gamma}}$.*

(2) *Case* 1: $K\not\subseteq H$. *Then we have*

$$J_f\sim_F \underbrace{E\times\cdots\times E}_{n}\quad (n=\dim J_f=[H:\mathbb{Q}]).$$

Case 2: $K\subseteq H$.

(a) *If $\gamma=\rho\circ\gamma\circ\rho$ on $P:=K^\times N_{F'/K}(F'^\times_{\mathbb{A}})$, then we have*

$$J_f\sim_F \underbrace{E\times\cdots\times E}_{n}.$$

(b) *If $\gamma\neq\rho\circ\gamma\circ\rho$ on P, then we have that $F=F'$ and there exists an abelian variety A of dimension $\frac{n}{2}$ defined over K such that*

$$J_f\sim_F \underbrace{E\times\cdots\times E}_{\frac{n}{2}}\times A_{/F},\quad \mathrm{Hom}_F(E,A_{/F})=\{0\}.$$

Proof. The statement (1) already has been proved in the proof of Theorem 5.1.

We will prove the statement (2). Firstly, we consider Case 1. By the assumptions and the arguments in the proof of Theorem 5.1, we have that $\mathrm{Hom}_{F'}(E,J_f)=\mathrm{Hom}_{\overline{\mathbb{Q}}}(E,J_f)$. By Lemma 2.2, $V:=\mathrm{Hom}_F(E,J_f)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a non-zero H-vector subspace of $\mathcal{M}$. Since $\dim_H\mathcal{M}=2$, we have $\dim_H V=1$ or 2. If $\dim_H V=2$, then $\mathrm{Hom}_F(E,J_f)=\mathrm{Hom}_{\overline{\mathbb{Q}}}(E,J_f)$, hence $J_f\sim_F E^n$. Therefore we will consider the case of $\dim_H V=1$. Since $[H:\mathbb{Q}]=n$, there exist $\eta_1,\ldots,\eta_n\in\mathrm{Hom}_F(E,J_f)$ such that $\eta_1,\ldots,\eta_n$ are linearly independent over $\mathbb{Q}$. Then

$$\eta: E\times\cdots\times E\to J_f,\quad (x_1,\ldots,x_n)\mapsto\eta_1(x_1)+\cdots+\eta_n(x_n)$$

is a homomorphism defined over F. Since $\mathrm{Hom}_F(E,J_f)\subsetneq\mathrm{Hom}_{F'}(E,J_f)$, we have that $F\neq F'$ and $\iota(\sqrt{-D})\in\mathrm{End}^0_{F'}(E)\backslash\mathrm{End}^0_F(E)$. Now we note that this implies that $\eta_1,\ldots,\eta_n$ are linearly independent over $K\cong\mathrm{End}^0_{F'}(E)$. Suppose that the dimension of the identity component $\mathrm{Ker}(\eta)^0$ of $\mathrm{Ker}(\eta)$ is greater than or equal to one. Then we have $\mathrm{Ker}(\eta)^0\sim_{\overline{\mathbb{Q}}}E^r$ $(r=\dim\mathrm{Ker}(\eta)^0\geqslant 1)$. Therefore, we can take a non-zero homomorphism $\varphi: E\to\mathrm{Ker}(\eta)^0$. Setting

$$t_i : E \xrightarrow{\varphi} \mathrm{Ker}(\eta)^0 \subseteq E \times \cdots \times E \xrightarrow{\text{projection to } i\text{th component}} E, \quad (i = 1, \ldots, n),$$

it holds that $t_1, \ldots, t_n \in \mathrm{End}_{F'}(E)(\hookrightarrow K)$ and $(\eta_1 \circ t_1)(x) + \cdots + (\eta_n \circ t_n)(x) = 0$ $({}^{\forall}x \in E)$. Since φ is non-zero, at least one of t_i is non-zero. So $\eta_1, \ldots, \eta_n$ are linearly dependent over K and this is a contradiction. Hence $\mathrm{Ker}(\eta)$ is a finite set, so η is an isogeny defined over F. Thus $J_f \sim_F E^n$.

Next, we consider Case 2: (a). By the assumptions, $\chi^{(1)} = \chi^{(2)} = 1$, so $\chi = 1$. Hence we have that $\mathrm{Hom}_{F'}(E, J_f) = \mathrm{Hom}_{\overline{\mathbb{Q}}}(E, J_f)$. The same arguments in Case 1 imply $J_f \sim_F E^n$.

Finally, we consider Case 2: (b). By the assumptions, $\chi^{(1)} = 1$ and $\chi^{(2)} \neq 1$, hence we have

$$J_f \sim_{F'} C \times {}^{\rho}C \sim_{F'} E^{\frac{n}{2}} \times {}^{\rho}C, \quad \mathrm{Hom}_{F'}(E, {}^{\rho}C) = \{0\}.$$

In this case $W := \mathrm{Hom}_{F'}(E, J_f) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a one dimensional H-vector space. Suppose that $F' \neq F$. Then $\langle \tau_0 \rangle = \mathrm{Gal}(F'/F)$ acts H-linearly on W. The action of τ_0 must be $(+1)$- or (-1)-multiple. If it is (-1)-multiple, then we have $\mathrm{Hom}_F(E, J_f) = \{0\}$ and this contradicts Lemma 2.2. So it is $(+1)$-multiple, hence $\mathrm{Hom}_F(E, J_f) = \mathrm{Hom}_{F'}(E, J_f)$. We take an element η $(\neq 0)$ of $\mathrm{Hom}_F(E, J_f)$ and $r \in \mathbb{N}$ such that $\iota(r\sqrt{-D}) \in \mathrm{End}_{F'}(E)$. Since $\eta \circ \iota(r\sqrt{-D}) \in \mathrm{Hom}_{F'}(E, J_f) = \mathrm{Hom}_F(E, J_f)$, it holds that

$$-\eta \circ \iota(r\sqrt{-D}) = {}^{\tau_0}\eta \circ {}^{\tau_0}(\iota(r\sqrt{-D})) = {}^{\tau_0}(\eta \circ \iota(r\sqrt{-D})) = \eta \circ \iota(r\sqrt{-D}).$$

Hence we have $\eta \circ \iota(r\sqrt{-D}) = 0$. On the other hand, since $\iota(r\sqrt{-D})$ is surjective and $\eta \neq 0$, we have $\eta \circ \iota(r\sqrt{-D}) \neq 0$. This is a contradiction. Therefore, it is proved that $F' = F$. Taking ${}^{\rho}C$ as A, we get the claim. □

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Further reading

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