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An arithmetical property of powers of Salem numbers ☆

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Abstract

Let ζ be a nonzero real number and let α be a Salem number. We show that the difference between the largest and smallest limit points of the fractional parts of the numbers $\zeta\alpha^n$, when n runs through the set of positive rational integers, can be bounded below by a positive constant depending only on α if and only if the algebraic integer $\alpha - 1$ is a unit.

MSC: 11J71; 11R04; 11R06

Keywords: Salem numbers; Fractional parts

1. Introduction

The problem of studying the distribution mod 1 of the powers of a fixed real number α greater than 1, has been of interest for some time. In his monograph [4], R. Salem considered the case of certain special real numbers α. For instance, he showed that if α is a Pisot number then α^n mod 1 tends to zero, whereas if α is a Salem number then the sequence α^n mod 1 is dense in the unit interval. Recall that a Pisot (respectively a Salem) number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1 (respectively are of modulus at most 1 and with a conjugate of modulus 1). Throughout, when we speak about a conjugate, the minimal polynomial or the degree of an algebraic number we mean over the field of the rationals $\mathbb{Q}$.

A slightly finer problem concerns studying the distribution of the fractional parts $\{\zeta\alpha^n\}$ of the numbers $\zeta\alpha^n$, where ζ is a fixed real number and where n runs through the set of non-negative

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rational integers $\mathbb{N}$. Recall that $\{\zeta\alpha^n\}$ is the difference $\zeta\alpha^n - [\zeta\alpha^n]$ and $[\zeta\alpha^n]$ is the greatest rational integer less than or equal to $\zeta\alpha^n$. In a recent paper [1], A. Dubickas has shown that if α is algebraic and ζ is not in the field $\mathbb{Q}(\alpha)$ when α is either a Pisot or a Salem number, then the distance $\Delta(\zeta,\alpha)$ between the largest limit point of the sequence $(\{\zeta\alpha^n\})_{n\in\mathbb{N}}$ and the smallest one satisfies

$$\Delta(\zeta,\alpha) \geqslant \frac{1}{L(\alpha)}, \tag{1}$$

where $L(\alpha)$ is the sum of the absolute values of the coefficients of the minimal polynomial of α. The aim of this paper is to show when α is a Salem number, $\zeta \in \mathbb{Q}(\alpha)$ and $\zeta \neq 0$, that inequality (1) remains true only if $\alpha - 1$ is a unit:

Theorem. *Let α be a Salem number and let ζ be a nonzero element of the field $\mathbb{Q}(\alpha)$. Then,*

(i) $\Delta(\zeta,\alpha) > 0$;
(ii) *if ζ is an algebraic integer, then* $\Delta(\zeta,\alpha) = 1$;
(iii) *if $\alpha - 1$ is a unit, then* $\Delta(\zeta,\alpha) \geqslant \frac{1}{L(\alpha)}$;
(iv) *if $\alpha - 1$ is not a unit, then* $\inf_\zeta \Delta(\zeta,\alpha) = 0$.

We prove this theorem in Section 3. In the next one we show some auxiliary results. As usual we denote the ring of rational integers, the field of complex numbers and the ring of polynomials with rational integer coefficients by $\mathbb{Z}$, $\mathbb{C}$ and $\mathbb{Z}[X]$, respectively.

It is worth noting that the case $\zeta = 1$ of the theorem (ii) is a corollary of the much stronger result of R. Salem: the sequence $(\{\zeta\alpha^n\})_{n\in\mathbb{N}}$ is dense in the unit interval. Using the same argument as in [4, p. 33], it is easy to check that the sequence $(\{\zeta\alpha^n\})_{n\in\mathbb{N}}$ is dense in the unit interval when $\zeta \in \mathbb{Z}$ and $\zeta \neq 0$. However, we shall show in Remark 1 that for any Salem number α there is a nonzero integer ζ of the field $\mathbb{Q}(\alpha)$ and a subinterval I_ζ of the unit interval such that the sequence $(\{\zeta\alpha^n\})_{n\in\mathbb{N}}$ has no limit point in I_ζ; moreover, the number ζ can be chosen so that the length of the interval I_ζ is close to 1.

2. Some lemmas

Lemma 0. *If α is a Salem number of degree d, then d is even, $d \geqslant 4$ and $\frac{1}{\alpha}$ is the only conjugate of α with modulus less than* 1 (*all the other conjugates are of modulus* 1).

Proof. The proof follows immediately from the definition of Salem numbers. □

Lemma 1. [3] *Let α_1 be an algebraic number with conjugates $\alpha_1, \alpha_2, \dots, \alpha_d$, where $d \geqslant 2$. Assume that for some $N \in \mathbb{N}$, some subset $\{i_1, i_2, \dots, i_k\}$ of $\{1, 2, \dots, d\}$ with $1 \leqslant k \leqslant d-1$, and some $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_k} \in \mathbb{C}$ we have*

$$\sum_{1\leqslant j\leqslant k} \lambda_{i_j}\alpha_{i_j}^n \in \mathbb{Q}$$

for all rational integers $n \geqslant N$. Then,

$$\lambda_{i_1} = \lambda_{i_2} = \dots = \lambda_{i_k} = 0.$$

Lemma 2. *Let $\varepsilon > 0$ and let $\alpha_1, \alpha_2, \ldots, \alpha_{d-2}$ be the conjugates with modulus 1 of a Salem number α labelled so that $\alpha_{2j} = \overline{\alpha_{2j-1}}$ for all $j \in \{1, 2, \ldots, \frac{d}{2} - 1\}$. Then,*

(i) *for any subset $\{\eta_1, \eta_2, \ldots, \eta_{d-2}\}$ of $\mathbb{C}$, where $\eta_{2j} = \overline{\eta_{2j-1}}$ and $|\eta_{2j}| = 1$ for all $j \in \{1, 2, \ldots, \frac{d}{2} - 1\}$, there is $n \in \mathbb{N}$ arbitrarily large such that*

$$\left|\alpha_k^n - \eta_k\right| < \varepsilon$$

for all $k \in \{1, 2, \ldots, d-2\}$;

(ii) *there exists a nonzero element $R \in \mathbb{Z}[X]$ of degree at most $d-1$ satisfying*

$$\left|R(\alpha_k)\right| < \varepsilon$$

for all $k \in \{1, 2, \ldots, d-2\}$, and

$$R(1) \equiv 1 \mod \big(P(1)\big),$$

where P is the minimal polynomial of α.

Proof. (i) By Lemma 0 the conjugates of α can be labelled as in Lemma 2. Let $\alpha_1 = e^{i\pi\theta_1}$, $\alpha_3 = e^{i\pi\theta_3}, \ldots, \alpha_{d-3} = e^{i\pi\theta_{d-3}}$ (where $i^2 = -1$) be the conjugates of α in the upper half plane. By a result of Pisot [4, p. 32], the numbers $1, \theta_1, \theta_3, \ldots, \theta_{d-3}$ are linearly independent over $\mathbb{Q}$. It follows by Kronecker's theorem [4, Appendix 8] that there are $n \in \mathbb{N}$ arbitrarily large and $p_1, p_3, \ldots, p_{d-3} \in \mathbb{Z}$ such that

$$\left|n\theta_j - \frac{\phi_j}{2} + p_j\right| < \frac{\varepsilon}{2\pi},$$

where $\eta_j = e^{i\pi\phi_j}$ and $j \in \{1, 3, \ldots, d-3\}$; thus

$$\left|\alpha_j^{2n} - \eta_j\right| = \left|e^{i2n\pi\theta_j} - e^{i\pi\phi_j}\right| < \varepsilon$$

for all $j \in \{1, 3, \ldots, d-3\}$ and by complex conjugation we obtain the result.

(ii) By the same argument as above and with the same notation, there are $p_1, p_3, \ldots, p_{d-3} \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that for all $j \in \{1, 3, \ldots, d-3\}$ we have

$$\left|n\theta_j - \left(-\frac{\theta_j}{2|P(1)|}\right) - p_j\right| < \varepsilon \frac{r}{2|P(1)|\pi},$$

where $r = \min_{j\in\{1,2,\ldots,d-2\}} |\alpha_j - 1|$. It follows that

$$\left|\alpha_j^{1+2n|P(1)|} - 1\right| = \left|e^{i(1+2n|P(1)|)\pi\theta_j} - e^{i2\pi|P(1)|p_j}\right| < \varepsilon r$$

and so by complex conjugation we obtain for all $j \in \{1, 2, \ldots, d-2\}$ that $|C(\alpha_j)| < \varepsilon r$, where

$$C(x) = x^{2n|P(1)|+1} - 1 \in \mathbb{Z}[X].$$

Let R be the remainder of the Euclidean division of the polynomial

$$\frac{C(x)}{x-1} = 1 + x + \cdots + x^{2n|P(1)|}$$

by the monic polynomial P. Then, $R \in \mathbb{Z}[X]$, R is of degree at most $d-1$, $R(1) \equiv 1 \bmod (P(1))$ and

$$0 < \left|R(\alpha_j)\right| = \left|\frac{C(\alpha_j)}{\alpha_j - 1}\right| < \varepsilon$$

for all $j \in \{1, 2, \ldots, d-2\}$, since $\alpha > 1$ and $|R(\alpha)| = |\frac{C(\alpha)}{\alpha-1}| \neq 0$. $\square$

The following result is a corollary of [1, Lemma 1].

Lemma 3. [1] *If $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 = a_d \prod_{1 \leqslant i \leqslant d}(x - \alpha_i)$ is the minimal polynomial of an algebraic number, then the linear system*

$$\sum_{1 \leqslant i \leqslant d} X_i \alpha_i^n = q_n \quad (n = 0, 1, \ldots, d-1),$$

where $q_0, q_1, \ldots, q_{d-1}$ are fixed in $\mathbb{Q}$, has a unique solution $(X_1, X_2, \ldots, X_d)$, where

$$X_1 = \frac{\sum_{0 \leqslant k \leqslant d-1} q_k \beta_k}{P'(\alpha_1)}, \qquad \beta_k = \sum_{k+1 \leqslant l \leqslant d} a_l \alpha_1^{l-k-1},$$

$X_i = \sigma_i(X_1)$, σ_i is the embedding of $\mathbb{Q}(\alpha_1)$ into $\mathbb{C}$ sending α_1 to α_i and $i \in \{1, 2, \ldots, d\}$.

Lemma 4. *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the roots of a polynomial P with coefficients in $\mathbb{C}$, where $\alpha_i \neq \alpha_j$ for all $1 \leqslant i < j \leqslant d$. If $P(1) \neq 0$, then*

$$\sum_{1 \leqslant i \leqslant d} \frac{\alpha_i^j}{(1-\alpha_i)P'(\alpha_i)} = \frac{1}{P(1)}$$

for all $j \in \{0, 1, \ldots, d-1\}$.

Proof. From the known equalities [5, p. 56]

$$\sum_{1 \leqslant i \leqslant d} \frac{1}{(1-\alpha_i)P'(\alpha_i)} = \frac{1}{P(1)}, \quad \text{and} \quad \sum_{1 \leqslant i \leqslant d} \frac{\alpha_i^k}{P'(\alpha_i)} = 0,$$

where $k \in \{0, 1, \ldots, d-2\}$, we have when $j \in \{1, 2, \ldots, d-1\}$

$$\sum_{1 \leqslant i \leqslant d} \frac{1-\alpha_i^j}{(1-\alpha_i)P'(\alpha_i)} = \sum_{1 \leqslant i \leqslant d} \sum_{0 \leqslant k \leqslant j-1} \frac{\alpha_i^k}{P'(\alpha_i)} = \sum_{0 \leqslant k \leqslant j-1} \sum_{1 \leqslant i \leqslant d} \frac{\alpha_i^k}{P'(\alpha_i)} = 0$$

and

$$\sum_{1\leqslant i\leqslant d}\frac{\alpha_i^j}{(1-\alpha_i)P'(\alpha_i)}=\sum_{1\leqslant i\leqslant d}\frac{\alpha_i^j-1}{(1-\alpha_i)P'(\alpha_i)}+\sum_{1\leqslant i\leqslant d}\frac{1}{(1-\alpha_i)P'(\alpha_i)}=\frac{1}{P(1)}. \qquad \square$$

Lemma 5. *Let $P(x)=x^d+a_{d-1}x^{d-1}+\cdots+a_0=\prod_{1\leqslant i\leqslant d}(x-\alpha_i)$ be the minimal polynomial of an algebraic integer and let $(X_1,X_2,\ldots,X_d)$ be the solution of the linear system*

$$\sum_{1\leqslant i\leqslant d}X_i\alpha_i^n=t_n+\frac{s}{P(1)}\quad (n=0,1,\ldots,d-1),$$

where $s,t_0,t_1,\ldots,t_{d-1}\in\mathbb{Z}$. Then, the sequence $(t_n)_{n\in\mathbb{N}}$ defined by

$$t_{n+d}=-(a_{d-1}t_{n+d-1}+a_{d-2}t_{n+d-2}+\cdots+a_0t_n+s)$$

satisfies

$$\sum_{1\leqslant i\leqslant d}X_i\alpha_i^n=t_n+\frac{s}{P(1)}$$

and $t_n\in\mathbb{Z}$ for all $n\in\mathbb{N}$.

Proof. We use induction on n. By hypothesis Lemma 5 is true for $n\in\{0,1,\ldots,d-1\}$. Assume that for some $n\in\mathbb{N}$ we have $\sum_{1\leqslant i\leqslant d}X_i\alpha_i^{n+j}=t_{n+j}+\frac{s}{P(1)}$ for all $j\in\{0,1,\ldots,d-1\}$. Then,

$$\sum_{0\leqslant j\leqslant d-1}a_j\biggl(\sum_{1\leqslant i\leqslant d}X_i\alpha_i^{n+j}\biggr)=\sum_{0\leqslant j\leqslant d-1}a_jt_{n+j}+s\frac{\sum_{0\leqslant j\leqslant d-1}a_j}{P(1)}$$

and so

$$-\sum_{1\leqslant i\leqslant d}X_i\alpha_i^{n+d}=\sum_{0\leqslant j\leqslant d-1}a_jt_{n+j}+s-\frac{s}{P(1)}=-t_{n+d}-\frac{s}{P(1)},$$

since

$$\sum_{0\leqslant j\leqslant d-1}a_j\biggl(\sum_{1\leqslant i\leqslant d}X_i\alpha_i^{n+j}\biggr)=\sum_{1\leqslant i\leqslant d}X_i\alpha_i^n\biggl(\sum_{0\leqslant j\leqslant d-1}a_j\alpha_i^j\biggr)=\sum_{1\leqslant i\leqslant d}X_i\alpha_i^n\bigl(-\alpha_i^d\bigr). \qquad \square$$

Lemma 6. *Let $(a_0,a_1,\ldots,a_d)$ be a finite sequence of rational integers, where $a_0=a_d=1$, and let $R\in\mathbb{Z}[X]$ of degree at most $d-1$. Then, R can be written*

$$R(x)=1-t_0-t_0\biggl(\sum_{1\leqslant l\leqslant d}a_lx^{l-1}\biggr)+(x-1)\sum_{1\leqslant k\leqslant d-1}t_k\sum_{k+1\leqslant l\leqslant d}a_lx^{l-(k+1)},$$

for some $t_0,t_1,\ldots,t_{d-1}\in\mathbb{Z}$, if and only if $R(1)\equiv 1 \bmod (\sum_{0\leqslant l\leqslant d}a_l)$.

Proof. It is clear that if R has the form as in Lemma 6, then $R(1) = 1 - t_0(\sum_{0\leqslant l\leqslant d} a_l)$, since $a_0 = 1$, and so $R(1) \equiv 1 \bmod (\sum_{0\leqslant l\leqslant d} a_l)$. Conversely, a simple induction shows that if $R \in \mathbb{Z}[X]$ of degree at most $d-1$, then there are $b_0, b_1, \dots, b_{d-1} \in \mathbb{Z}$ such that

$$R(x) = \sum_{0\leqslant k\leqslant d-2} b_k \sum_{k+2\leqslant l\leqslant d} a_l x^{l-(k+1)} + b_{d-1}.$$

Now, for $k \in \{0, 1, \dots, d-2\}$ set

$$t_{k+1} = t_0 + b_0 + b_1 + \dots + b_k,$$

where t_0 is the rational integer defined by the equality $R(1) = 1 - t_0 \sum_{0\leqslant i\leqslant d} a_i$. Then, $t_{k+1} \in \mathbb{Z}$, $t_{k+1} - t_k = b_k$ for all $k \in \{0, 1, \dots, d-2\}$ and the polynomial R can be written in this case

$$R(x) = \sum_{0\leqslant k\leqslant d-2} (t_{k+1} - t_k) \sum_{k+2\leqslant l\leqslant d} a_l x^{l-(k+1)} + b_{d-1},$$

where

$$b_{d-1} = 1 - t_0\Bigg(\sum_{0\leqslant l\leqslant d} a_l\Bigg) - \sum_{0\leqslant k\leqslant d-2} (t_{k+1} - t_k) \sum_{k+2\leqslant l\leqslant d} a_l = 1 - t_0 - \sum_{0\leqslant k\leqslant d-1} t_k a_{k+1}.$$

Hence,

$$R(x) = 1 - \Bigg(t_0 + \sum_{0\leqslant k\leqslant d-1} t_k a_{k+1}\Bigg) + \sum_{0\leqslant k\leqslant d-2} (t_{k+1} - t_k) \sum_{k+2\leqslant l\leqslant d} a_l x^{l-(k+1)}$$

and the result follows by a simple computation. □

3. Proof of the theorem

Let

$$P(x) = a_0 + a_1 x + \dots + a_d x^d = \prod_{1\leqslant i\leqslant d} (x - \alpha_i)$$

be the minimal polynomial of a Salem number $\alpha = \alpha_1$. Then, by Lemma 0, $\frac{1}{\alpha}$ is a conjugate, say α_2, of α and $a_0 = a_d = 1$. Let ζ be a nonzero element of the field $\mathbb{Q}(\alpha)$, $n \in \mathbb{N}$, $y_n = \{\zeta\alpha^n\}$ and $x_n = [\zeta\alpha^n]$. Then, from the equality

$$a_0\zeta\alpha^n + a_1\zeta\alpha^{n+1} + \dots + a_d\zeta\alpha^{n+d} = 0,$$

we have

$$a_0(x_n + y_n) + a_1(x_{n+1} + y_{n+1}) + \dots + a_d(x_{n+d} + y_{n+d}) = 0$$

and the real number

$$s_n = a_0 y_n + a_1 y_{n+1} + \cdots + a_d y_{n+d} \tag{2}$$

satisfies

$$s_n = -a_0 x_n - a_1 x_{n+1} - \cdots - a_d x_{n+d};$$

thus $s_n \in \mathbb{Z}$. Let $\mu = \limsup y_n$ and $\lambda = \liminf y_n$. Then, for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\lambda - \varepsilon \leqslant y_n \leqslant \mu + \varepsilon,$$

and so by (2) we obtain

$$(\lambda - \varepsilon)\Big(\sum_{a_i>0} a_i\Big) + (\mu + \varepsilon)\Big(\sum_{a_i<0} a_i\Big) \leqslant s_n \leqslant (\mu + \varepsilon)\Big(\sum_{a_i>0} a_i\Big) + (\lambda - \varepsilon)\Big(\sum_{a_i<0} a_i\Big).$$

It follows when sequence $(s_n)_{n\in\mathbb{N}}$ takes infinitely many times at least two distinct values that

$$(\mu - \lambda + 2\varepsilon)L(\alpha) = (\mu - \lambda + 2\varepsilon)\Big(\sum_{a_i>0} a_i\Big) + (\lambda - \mu - 2\varepsilon)\Big(\sum_{a_i<0} a_i\Big) \geqslant 1$$

and so

$$\Delta(\zeta, \alpha) = \mu - \lambda \geqslant \frac{1}{L(\alpha)} > 0.$$

Assume now that there are $N \in \mathbb{N}$ and $s \in \mathbb{Z}$ such that $s_n = s$ for all $n \geqslant N$. Then, by (2) we have

$$a_0(y_{n+1} - y_n) + a_1(y_{n+2} - y_{n+1}) + \cdots + a_d(y_{n+1+d} - y_{n+d}) = 0$$

and the characteristic equation of the linearly recurrent sequence $(y_{n+1} - y_n)_{n\geqslant N}$ is $P(x) = 0$. Hence, there are d complex numbers $\gamma_1, \gamma_2, \ldots, \gamma_d$ such that

$$y_{n+1} - y_n = \gamma_1\alpha_1^n + \gamma_2\alpha_2^n + \cdots + \gamma_d\alpha_d^n$$

for all $n \geqslant N$. Moreover, we have $\gamma_1 = 0$, since $|y_{n+1} - y_n| \leqslant 1$, and so

$$y_{n+1} - y_n = \gamma_2\alpha_2^n + \gamma_3\alpha_3^n + \cdots + \gamma_d\alpha_d^n. \tag{3}$$

Let $\zeta_1 = \zeta, \zeta_2, \ldots, \zeta_d$ be the conjugates of ζ, where $\zeta_i = \sigma_i(\zeta)$ and σ_i is the embedding of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ sending α to α_i, and let

$$z_n = \zeta_1\alpha_1^n + \zeta_2\alpha_2^n + \cdots + \zeta_d\alpha_d^n.$$

Then, $z_n \in \mathbb{Q}$,

$$y_n = z_n - x_n - \big(\zeta_2\alpha_2^n + \zeta_3\alpha_3^n + \cdots + \zeta_d\alpha_d^n\big)$$

and so for $n \geqslant N$ we have

$$y_{n+1} - y_n = (z_{n+1} - x_{n+1}) - (z_n - x_n) - \zeta_2(\alpha_2 - 1)\alpha_2^n - \cdots - \zeta_d(\alpha_d - 1)\alpha_d^n.$$

It follows by the equality (3) that

$$\big(\gamma_2 + \zeta_2(\alpha_2 - 1)\big)\alpha_2^n + \cdots + \big(\gamma_d + \zeta_d(\alpha_d - 1)\big)\alpha_d^n = z_{n+1} - x_{n+1} - z_n + x_n \in \mathbb{Q}$$

and so by Lemma 1 we obtain $z_{n+1} - x_{n+1} = z_n - x_n$. Consequently, there is $c \in \mathbb{Q}$ such that

$$y_n = c - \big(\zeta_2\alpha_2^n + \zeta_3\alpha_3^n + \cdots + \zeta_d\alpha_d^n\big), \tag{4}$$

for all $n \geqslant N$. Now, we claim that

$$\mu = c + |\zeta_3| + |\zeta_4| + \cdots + |\zeta_d| \tag{5}$$

and

$$\lambda = c - |\zeta_3| - |\zeta_4| - \cdots - |\zeta_d|. \tag{6}$$

Indeed, for any $\varepsilon > 0$ there is $M \in \mathbb{N}$, such that $|\zeta_2\alpha_2^n| < \varepsilon$ for all $n \geqslant M$. Since, relation (4) implies $|y_n - c| \leqslant |\zeta_2\alpha_2^n| + |\zeta_3| + |\zeta_4| + \cdots + |\zeta_d|$ when $n \geqslant N$, we see that for all $n \geqslant \max(N, M)$ we have

$$|y_n - c| < \varepsilon + |\zeta_3| + |\zeta_4| + \cdots + |\zeta_d|.$$

So $\mu \leqslant c + |\zeta_3| + |\zeta_4| + \cdots + |\zeta_d|$ and $\lambda \geqslant c - |\zeta_3| - |\zeta_4| - \cdots - |\zeta_d|$. Conversely, Lemma 2(i) asserts that there are infinitely many $n \in \mathbb{N}$ such that

$$\big|\zeta_i\alpha_i^n + |\zeta_i|\big| < \varepsilon,$$

respectively such that

$$\big|\zeta_i\alpha_i^n - |\zeta_i|\big| < \varepsilon$$

for all $i \in \{3, 4, \ldots, d\}$. Let $\varepsilon_i = \zeta_i\alpha_i^n + |\zeta_i|$ (respectively $\varepsilon_i = \zeta_i\alpha_i^n - |\zeta_i|$), where $i \in \{3, 4, \ldots, d\}$. Then, by (4) we have when $n \geqslant \max(N, M)$

$$y_n = c + |\zeta_3| + \cdots + |\zeta_d| - \big(\zeta_2\alpha_2^n + \varepsilon_3 + \cdots + \varepsilon_d\big)$$

and

$$y_n > c + |\zeta_3| + \cdots + |\zeta_d| - (d-1)\varepsilon,$$

respectively

$$y_n = c - |\zeta_3| - \cdots - |\zeta_d| - \big(\zeta_2\alpha_2^n + \varepsilon_3 + \cdots + \varepsilon_d\big)$$

and

$$y_n < c - |\zeta_3| - \cdots - |\zeta_d| + (d-1)\varepsilon,$$

as the real number $\zeta_2\alpha_2^n + \varepsilon_3 + \cdots + \varepsilon_d = c + |\zeta_3| + \cdots + |\zeta_d| - y_n$ (respectively $\zeta_2\alpha_2^n + \varepsilon_3 + \cdots + \varepsilon_d = c - |\zeta_3| - \cdots - |\zeta_d| - y_n$) satisfies $|\zeta_2\alpha_2^n + \varepsilon_3 + \cdots + \varepsilon_d| < (d-1)\varepsilon$. Hence, $\mu \geqslant c + |\zeta_3| + |\zeta_4| + \cdots + |\zeta_d|$ (respectively $\lambda \leqslant c - |\zeta_3| - \cdots - |\zeta_d|$) and that proves the claim.

It follows immediately by (5) and (6) that $0 < c < 1$, since $\mu \leqslant 1$, $0 \leqslant \lambda$ and $\zeta \neq 0$. Furthermore, we have in this case that $\alpha - 1$ is not a unit, because if $|P(1)| = 1$, then by relations (2) and (4), we obtain $c = \frac{s}{P(1)}$ and $c \in \mathbb{Z}$; so the proof of the theorem (iii) is now complete. Note also by (5) and (6) that

$$\Delta(\zeta, \alpha) = 2\big(|\zeta_3| + |\zeta_4| + \cdots + |\zeta_d|\big) \tag{7}$$

and so the theorem (i) is true.

To prove the theorem (iv), we have to show that for any $\varepsilon > 0$ there is $\zeta \in \mathbb{Q}(\alpha)$ and $\zeta \neq 0$ satisfying the two conditions:

(C1) There are $s \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $a_0x_n + a_1x_{n+1} + \cdots + a_dx_{n+d} = -s$, for all $n \geqslant N$.
(C2) If $i \in \{3, 4, \ldots, d\}$, then $|\sigma_i(\zeta)| < \varepsilon$ (and we conclude by (7)).

First, consider the linear system

$$\sum_{1 \leqslant i \leqslant d} X_i\alpha_i^n = t_n - \frac{1}{P(1)} \quad (n = 0, 1, \ldots, d-1), \tag{8}$$

where $t_0, t_1, \ldots, t_{d-1} \in \mathbb{Z}$. Then, by Lemma 3 we know that this system has a unique solution $X(t_0, t_1, \ldots, t_{d-1}) = (X_1, X_2, \ldots, X_d)$, where

$$X(t_0, t_1, \ldots, t_{d-1}) = -\frac{\sum_{0 \leqslant k \leqslant d-1} \beta_k}{P(1)P'(\alpha)} + \frac{\sum_{0 \leqslant k \leqslant d-1} t_k\beta_k}{P'(\alpha)},$$
$$\beta_k = \sum_{k+1 \leqslant l \leqslant d} a_l\alpha^{l-(k+1)}$$

and $X_i = \sigma_i(X_1)$ for all $i \in \{1, 2, \ldots, d\}$. Note also that $X(t_0, t_1, \ldots, t_{d-1}) \neq 0$, since $t_0 \in \mathbb{Z}$ and $P(1) \leqslant -2$ ($P(1) \leqslant -2$ because $P(1) < 0$ and $\alpha - 1$ is not a unit). Moreover, by Lemma 4 we have

$$X(0, 0, \ldots, 0) = \frac{1}{(\alpha - 1)P'(\alpha)}.$$

It follows that

$$X(t_0, t_1, \ldots, t_{d-1}) = \frac{1}{(\alpha - 1)P'(\alpha)} + \frac{\sum_{0 \leqslant k \leqslant d-1} t_k\beta_k}{P'(\alpha)}$$

and so

$$X(t_0, t_1, \dots, t_{d-1}) = \frac{1 + (\alpha - 1)\sum_{0\leqslant k\leqslant d-1} t_k\beta_k}{(\alpha - 1)P'(\alpha)}. \tag{9}$$

Now, we claim that if for some $t_0, t_1, \dots, t_{d-1} \in \mathbb{Z}$ system (8) has a solution $(X_1, X_2, \dots, X_d)$ satisfying $\sum_{3\leqslant i\leqslant d} |X_i| < \frac{1}{|P(1)|}$, then the condition (C1) holds with $\zeta = X_1$ and $s = -1$. Indeed, if $(\zeta = \zeta_1, \zeta_2, \dots, \zeta_d)$ is a solution of (8), then Lemma 5 shows that for all $n \in \mathbb{N}$ we have

$$\sum_{1\leqslant i\leqslant d} \zeta_i\alpha_i^n = t_n - \frac{1}{P(1)}, \tag{10}$$

where $t_n \in \mathbb{Z}$ and $t_{n+d} + a_{d-1}t_{n+d-1} + \dots + a_0t_n = 1$. Moreover, the equality $\zeta\alpha^n = x_n + y_n$ together with relation (10), yield

$$|t_n - x_n| \leqslant \left|y_n + \frac{1}{P(1)}\right| + \sum_{3\leqslant i\leqslant d} |\zeta_i| + \frac{|\zeta_2|}{\alpha^n}$$

and so $|t_n - x_n| < 1$ for n large, since $P(1) \leqslant -2$, $|y_n + \frac{1}{P(1)}| \leqslant 1 + \frac{1}{P(1)}$ and if $\sum_{3\leqslant i\leqslant d} |\zeta_i| < -\frac{1}{P(1)}$ then there is $N \in \mathbb{N}$ such that $\sum_{3\leqslant i\leqslant d} |\zeta_i| + \frac{|\zeta_2|}{\alpha^n} < -\frac{1}{P(1)}$ for all $n \geqslant N$. Hence, there is $N \in \mathbb{N}$, such that if $n \geqslant N$, then $t_n = x_n$, and so

$$x_{n+d} + a_{d-1}x_{n+d-1} + \dots + a_0x_n = 1 \quad (s = -1)$$

for all $n \geqslant N$. Consequently, it suffices to show that for any $0 < \varepsilon < \frac{1}{(d-2)|P(1)|}$, there are $t_0, t_1, \dots, t_{d-1} \in \mathbb{Z}$ such that the corresponding solution of (8) satisfies (C2). In fact, we will show that for any $\varepsilon > 0$, in particular, when

$$\varepsilon < \frac{\min_{j\in\{3,4,\dots,d\}} |(\alpha_j - 1)P'(\alpha_j)|}{(d-2)|P(1)|}$$

there are $t_0, t_1, \dots, t_{d-1} \in \mathbb{Z}$ such that

$$\left|1 + (\alpha_j - 1)\sum_{0\leqslant k\leqslant d-1} t_k\sigma_j(\beta_k)\right| < \varepsilon$$

for all $j \in \{3, 4, \dots, d\}$, and we will conclude by (9). Indeed, we know by Lemma 2(ii) that for any $\varepsilon > 0$ there is $R \in \mathbb{Z}[X]$ of degree at most $d - 1$ satisfying

$$\left|R(\alpha_j)\right| < \varepsilon \tag{11}$$

for all $j \in \{3, 4, \dots, d\}$. Moreover, we have $R(1) \equiv 1 \bmod (P(1))$ and so by Lemma 6, there are $t_0, t_1, \dots, t_{d-1} \in \mathbb{Z}$ such that

$$R(x) = 1 - t_0 - t_0\left(\sum_{1\leqslant l\leqslant d} a_l x^{l-1}\right) + (x - 1)\sum_{1\leqslant k\leqslant d-1} t_k \sum_{k+1\leqslant l\leqslant d} a_l x^{l-(k+1)}.$$

It follows by the equality $\beta_0 = -\frac{1}{\alpha}$ that

$$R(\alpha) = 1 - t_0 - t_0\beta_0 + (\alpha - 1) \sum_{1 \leqslant k \leqslant d-1} t_k \beta_k = 1 + (\alpha - 1) \sum_{0 \leqslant k \leqslant d-1} t_k \beta_k$$

and so

$$R(\alpha_j) = 1 + (\alpha_j - 1) \sum_{0 \leqslant k \leqslant d-1} t_k \sigma_j(\beta_k).$$

The last relation together with inequality (11) yield the result and this ends the proof of the theorem (iv).

To prove the theorem (ii), we have to show that 0 and 1 are limit points of the sequence $(\{\zeta\alpha^n\})$, when ζ is an integer of the field $\mathbb{Q}(\alpha)$. With the same notation (although the proof is independent of the one above), we have $z_n = \zeta_1\alpha_1^n + \zeta_2\alpha_2^n + \cdots + \zeta_d\alpha_d^n \in \mathbb{Z}$, and

$$y_n = z_n - x_n - \left(\zeta_2\alpha_2^n + \zeta_3\alpha_3^n + \cdots + \zeta_d\alpha_d^n\right).$$

Let $\alpha_3 = e^{i\theta_3}, \alpha_5 = e^{i\theta_5}, \ldots, \alpha_{d-1} = e^{i\theta_{d-1}}$ (where $i^2 = -1$) be the conjugates of α in the upper half plane and let $\zeta_3 = \rho_3 e^{i\phi_3}, \zeta_5 = \rho_5 e^{i\phi_5}, \ldots, \zeta_{d-1} = \rho_{d-1} e^{i\phi_{d-1}}$ be the corresponding conjugates of ζ. Then,

$$y_n = z_n - x_n - \zeta_2\alpha_2^n - 2 \sum_{2 \leqslant j \leqslant d/2} \rho_{2j-1} \cos(n\theta_{2j-1} + \phi_{2j-1}). \tag{12}$$

Now, let δ be a real number satisfying $0 < \delta < \min\{\frac{1}{2}, 2\rho_3\}$. Then, by the same argument as in the proof of Lemma 2(i) we have that for any $\varepsilon > 0$ there is $n \in \mathbb{N}$ arbitrarily large such that

$$\left|2\rho_3 \cos(n\theta_3 + \phi_3) + \delta\right| < \varepsilon,$$

respectively such that

$$\left|2\rho_3 \cos(n\theta_3 + \phi_3) - \delta\right| < \varepsilon,$$

and

$$\left|2\rho_j \cos(n\theta_j + \phi_j)\right| < \varepsilon$$

for all $j \in \{5, 7, \ldots, d-1\}$. It follows by (12) that there are infinitely many n such that

$$\left|y_n - (z_n - x_n) - \delta\right| < \frac{d}{2}\varepsilon,$$

respectively such that

$$\left|y_n - (z_n - x_n) + \delta\right| < \frac{d}{2}\varepsilon,$$

as $|\zeta_2\alpha_2^n| < \varepsilon$ when n is sufficiently large. Thus, we have for these n's, when $\varepsilon < \frac{2}{d}\delta$,

$$0 < \delta - \frac{d}{2}\varepsilon < y_n - (z_n - x_n) < \delta + \frac{d}{2}\varepsilon < 2\delta < 1,$$
$$(z_n - x_n) < y_n < (z_n - x_n) + 1$$

and so $z_n - x_n = 0$, respectively

$$-1 < -2\delta < -\delta - \frac{d}{2}\varepsilon < y_n - (z_n - x_n) < -\delta + \frac{d}{2}\varepsilon < 0,$$
$$(z_n - x_n) - 1 < y_n < (z_n - x_n)$$

and so $z_n - x_n = 1$. Consequently, there are infinitely many n such that $|y_n - \delta| \leqslant \frac{d}{2}\varepsilon$ (respectively such that $|y_n - (1-\delta)| \leqslant \frac{d}{2}\varepsilon$) and the number δ (respectively $1-\delta$) is a limit point of the sequence (y_n). Finally, we obtain the result by letting δ tend to 0.

Remark 1. Let α be a Salem number. Then, for any $0 < \varepsilon < 1$ there is an integer ζ of the field $\mathbb{Q}(\alpha)$ and a subinterval I_ζ of the unit interval with length ε such that the sequence $(\{\zeta\alpha^n\})$ has no limit point in I_ζ. We prove this result only when the degree d of α is 4 (the proof is similar for the case where $d > 4$). With the notation of the proof of the theorem (ii), where $\zeta = p^{2t}$, p is a Pisot number satisfying $\mathbb{Q}(p) = \mathbb{Q}(\alpha + \frac{1}{\alpha})$, and t is a rational integer such that the conjugate, say ζ', of ζ in the unit interval satisfies $\zeta' < \frac{1}{6}$ ($\zeta_2 = \zeta$, $\zeta' = \zeta_3 = \zeta_4 = \rho_3$ and $\phi_3 = 0$), we have by (12)

$$y_n = z_n - x_n - \zeta\alpha_2^n - 2\zeta'\cos(n\theta_3).$$

Since, there is $N' \in \mathbb{N}$ such that $|\zeta\alpha_2^n| < \zeta'$ when $n \geqslant N'$, we see (by the last equality) that

$$\left|y_n - (z_n - x_n)\right| < 3\zeta' < \frac{1}{2},$$
$$(z_n - x_n) - 1 < y_n < (z_n - x_n) + 1$$

and so $z_n - x_n = 0$ or 1 for all $n \geqslant N'$. Consequently, there is no limit point of the sequence (y_n) between the positive numbers $3\zeta'$ and $1 - 3\zeta'$, because if $z_n - x_n = 0$, then $y_n < 3\zeta'$ (respectively if $z_n - x_n = 1$, then $1 - y_n < 3\zeta'$ and $y_n > 1 - 3\zeta'$). Letting t tend to infinity we obtain the result, as in this case ζ' tends to 0.

Remark 2. With the hypothesis and the notation of the theorem (iv) and its proof, we have $\inf_{\zeta>0} \Delta(\zeta,\alpha) = 0$. Indeed, if $\zeta < 0$ and $\zeta\alpha^n = x_n + y_n$, then $\{-\zeta\alpha^n\} = 0$ or $\{-\zeta\alpha^n\} = 1 - y_n$, and so there is $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have $\{-\zeta\alpha^n\} = 1 - y_n$ (if there are n and $k \in \mathbb{N}$ with $\{-\zeta\alpha^n\} = \{-\zeta\alpha^k\} = 0$, then $\alpha^{k-n} = q \in \mathbb{Q}$, $\frac{1}{\alpha^{k-n}} = q$, $q^2 = 1$ and $|\alpha| = 1$). It follows that $\limsup\{-\zeta\alpha^n\} = 1 - \lambda$, $\liminf\{-\zeta\alpha^n\} = 1 - \mu$ and $\Delta(-\zeta,\alpha) = \Delta(\zeta,\alpha)$.

Remark 3. By the same arguments and with the same notation, we obtain when α is a Pisot number and $\zeta \in \mathbb{Q}(\alpha)$ that $\Delta(\zeta,\alpha) = 0$ or $\Delta(\zeta,\alpha) \geqslant \frac{1}{L(\alpha)}$. Note also by [4, Theorem 1, p. 3], that we have the better result: $\Delta(\zeta,\alpha) = 0$ or 1, when ζ is an integer of $\mathbb{Q}(\alpha)$. Recently [2], Dubickas showed that when α is Pisot number there is a positive number ζ satisfying $\Delta(\zeta,\alpha) = 0$ if and only if $\alpha = 2$, or $\alpha - 1$ is not a unit, or α is a strong Pisot number (assume that the conjugates α,

$\alpha_2, \ldots, \alpha_d$ of the Pisot number α of degree $d \geqslant 2$ are labelled so that $|\alpha_2| \geqslant |\alpha_3| \geqslant \cdots \geqslant |\alpha_d|$. Then, α is said to be a strong Pisot number if $\alpha_2 > 0$).

Remark 4. It has been proved in [6] (using Salem's construction [4, p. 30]) that any rational integer greater than 1 is a limit of a sequence of Salem numbers (α_n), where the algebraic integers $\alpha_n - 1$ are units for all n.

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