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Artin's L-functions and one-dimensional characters

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Abstract

Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group G, let $\chi_1, \dots, \chi_r$ be the irreducible non-trivial characters of G, and let $\mathcal{A}$ be the $\mathbb{C}$-algebra generated by the Artin L-functions $L(s, \chi_1), \dots, L(s, \chi_r)$. Let $\mathcal{B}$ be the subalgebra of $\mathcal{A}$ generated by the L-functions corresponding to induced characters of non-trivial one-dimensional characters of subgroups of G. We prove: (1) $\mathcal{B}$ is of Krull dimension r and has the same quotient field as $\mathcal{A}$; (2) $\mathcal{B} = \mathcal{A}$ iff G is M-group; (3) the integral closure of $\mathcal{B}$ in $\mathcal{A}$ equals $\mathcal{A}$ iff G is quasi-M-group.

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Let $K/\mathbb{Q}$ be a finite Galois extension with the Galois group G, and let χ be a non-trivial irreducible character of G. Artin's conjecture predicts that the L-function $L(s, \chi, K/\mathbb{Q})$ is holomorphic in the whole complex plane [1, p. 105].

Let $\chi_1, \dots, \chi_r$ be the irreducible non-trivial characters of G. The corresponding L-functions $L(s, \chi_1), \dots, L(s, \chi_r)$ are algebraically independent over $\mathbb{C}$ [2, Corollary 4, p. 183)]. Let $\mathcal{A} := \mathbb{C}[L(s, \chi_1), \dots, L(s, \chi_r)]$ be the $\mathbb{C}$-algebra generated by the meromorphic functions $L(s, \chi_1), \dots, L(s, \chi_r)$. It is isomorphic to the algebra of polynomials in r variables over $\mathbb{C}$. Let $\mathcal{O}(\mathbb{C})$ be the $\mathbb{C}$-algebra of holomorphic functions in $\mathbb{C}$. Artin's conjecture is:

$$\mathcal{A} \subseteq \mathcal{O}(\mathbb{C}).$$

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Let $\mathcal{S}$ be the set of all subgroups of G. For a subgroup $H \in \mathcal{S}$ let $\hat{H}_0^1$ be the set of all non-trivial one-dimensional complex characters of H, that is, the set of all non-constant group homomorphisms of H in the multiplicative group $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$. For a subgroup $H \in \mathcal{S}$ and a character $\varphi \in \hat{H}_0^1$ let φ^G be the induced character of G. The Artin L-function $L(s, \varphi^G, K/\mathbb{Q})$ is holomorphic, being equal to the Hecke L-function $L(s, \varphi, K_2/K_1)$ of the abelian extension K_2/K_1, K_1 the fixed field of H, K_2 the fixed field of $\operatorname{Ker}\varphi \subseteq H$, so:

$$\mathcal{B} \subseteq \mathcal{H} \subseteq \mathcal{A},$$

where $\mathcal{B}$ is the $\mathbb{C}$-subalgebra of $\mathcal{A}$ generated by the functions $L(s, \varphi^G, K/\mathbb{Q})$, $H \in \mathcal{S}$, $\varphi \in \hat{H}_0^1$, and $\mathcal{H} := \mathcal{A} \cap \mathcal{O}(\mathbb{C})$. How large is the algebra $\mathcal{B}$?

Theorem 1. *The finitely generated $\mathbb{C}$-algebra $\mathcal{B}$ is of Krull dimension r. The quotient field of $\mathcal{B}$ equals the quotient field of $\mathcal{A}$.*

Proof. Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. By [3, p. 209], there exist subgroups $H_1, \ldots, H_l$ of G, non-trivial one-dimensional characters φ_i of H_i, $i = 1, \ldots, l$, and integers $m_1, \ldots, m_l$ such that

$$\chi = m_1 \varphi_1^G + \cdots + m_l \varphi_l^G.$$

It follows that

$$L(s, \chi) = L\big(s, \varphi_1^G\big)^{m_1} \cdot \ldots \cdot L\big(s, \varphi_l^G\big)^{m_l}$$

belongs to the quotient field of $\mathcal{B}$, hence $\mathcal{A}$ is contained in the quotient field of $\mathcal{B}$. Since $\mathcal{B}$ is contained in $\mathcal{A}$ it follows that the quotient field of $\mathcal{B}$ equals the quotient field of $\mathcal{A}$. The Krull dimension of the finitely generated $\mathbb{C}$-algebra $\mathcal{B}$ equals the transcendence degree of its quotient field, that is, the transcendence degree of the quotient field of $\mathcal{A}$, which is r. $\square$

Definition. A finite group G is M-group if for every irreducible character χ of G there exist a subgroup $H \subseteq G$ and a one-dimensional character $\varphi : H \to \mathbb{C}^\times$ such that

$$\chi = \varphi^G.$$

Theorem 2. *The following assertions are equivalent*:

(a) $\mathcal{B} = \mathcal{A}$.
(b) *The Galois group G is M-group.*

Proof. (a) $\Rightarrow$ (b): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since $L(s, \chi) \in \mathcal{B}$ there exist subgroups $H_1, \ldots, H_l$ of G, one-dimensional non-trivial irreducible characters φ_j of H_j, $j = 1, \ldots, l$, and a polynomial

$$P(X_1, \ldots, X_l) = \sum_{i_1 \geqslant 0, \ldots, i_l \geqslant 0} a_{i_1 \ldots i_l} X_1^{i_1} \ldots X_l^{i_l} \in \mathbb{C}[X_1, \ldots, X_l]$$

such that

$$L(s, \chi) = P\big(L\big(s, \varphi_1^G\big), \ldots, L\big(s, \varphi_l^G\big)\big),$$

that is

$$L(s,\chi)=\sum_{i_1\geqslant 0,\ldots,i_l\geqslant 0} a_{i_1\ldots i_l} L\big(s, i_1\varphi_1^G+\cdots+i_l\varphi_l^G\big).$$

By the linear independence of L-functions corresponding to different characters [2, Theorem 1, p. 179] it follows that there exist $i_1,\ldots,i_l$ such that

$$\chi=i_1\varphi_1^G+\cdots+i_l\varphi_l^G.$$

Since χ is irreducible there exist $1\leqslant j\leqslant l$ such that

$$\chi=\varphi_j^G,$$

hence G is M-group.

(b) $\Rightarrow$ (a): Let $\chi\in\{\chi_1,\ldots,\chi_r\}$. Since G is M-group, there exist a subgroup $H\subseteq G$ and a one-dimensional character $\varphi: H\to\mathbb{C}^\times$ such that

$$\chi=\varphi^G.$$

Since χ is not trivial, the character φ is not trivial, so

$$L(s,\chi)=L\big(s,\varphi^G\big)\in\mathcal{B}.$$

Hence

$$\mathcal{A}\subseteq\mathcal{B}.\qquad\square$$

Definition. A finite group G is *quasi-M-group* if for every irreducible character χ of G there exist a subgroup $H\subseteq G$, a one-dimensional character $\varphi: H\to\mathbb{C}^\times$ and a number $k\geqslant 1$ such that

$$k\chi=\varphi^G.$$

Let $\mathcal{B}'$ be the integral closure of $\mathcal{B}$ in $\mathcal{A}$. It holds

$$\mathcal{B}\subseteq\mathcal{B}'\subseteq\mathcal{H}\subseteq\mathcal{A}.$$

Theorem 3. *The following assertions are equivalent*:

(a) $\mathcal{B}'=\mathcal{A}$.
(b) *The Galois group G is quasi-M-group.*

Proof. (a) $\Rightarrow$ (b): Let $\chi\in\{\chi_1,\ldots,\chi_r\}$. The element $L(s,\chi)$ of $\mathcal{A}$ satisfies a monic equation with coefficients in $\mathcal{B}$:

$$L(s,\chi)^l+b_{l-1}L(s,\chi)^{l-1}+\cdots+b_1L(s,\chi)+b_0=0, \tag{1}$$

$l \geqslant 1$, $b_0, \ldots, b_{l-1} \in \mathcal{B}$. Let $\varphi_1^G, \ldots, \varphi_m^G$ be all pairwise distinct characters of G which are obtained by inducing from non-trivial linear characters of subgroups of G, and let $f_1 := L(s, \varphi_1^G), \ldots, f_m := L(s, \varphi_m^G)$. It holds:

$$\mathcal{B} = \mathbb{C}[f_1, \ldots, f_m].$$

Each coefficient b_j, $j = 0, \ldots, l-1$, is a polynomial in $f_1, \ldots, f_m$:

$$\begin{aligned} b_j = P_j(f_1, \ldots, f_m) &= \sum_{t_1 \geqslant 0, \ldots, t_m \geqslant 0} a^{(j)}_{t_1 \ldots t_m} f_1^{t_1} \cdot \ldots \cdot f_m^{t_m} \\ &= \sum_{t_1 \geqslant 0, \ldots, t_m \geqslant 0} a^{(j)}_{t_1 \ldots t_m} L\big(s, t_1\varphi_1^G + \cdots + t_m\varphi_m^G\big), \end{aligned}$$

and (1) rewrites as

$$L(s, l\chi) + \sum_{j=0}^{l-1} \sum_{t_1 \geqslant 0, \ldots, t_m \geqslant 0} a^{(j)}_{t_1 \ldots t_m} L\big(s, t_1\varphi_1^G + \cdots + t_m\varphi_m^G + j\chi\big) = 0. \tag{2}$$

By the linear independence of L-functions corresponding to different characters [2, Theorem 1, p. 179], and by (2) it follows that there exist $j \in \{0, \ldots, l-1\}$ and $t_1 \geqslant 0, \ldots, t_m \geqslant 0$ such that

$$l\chi = t_1\varphi_1^G + \cdots + t_m\varphi_m^G + j\chi,$$

that is

$$(l-j)\chi = t_1\varphi_1^G + \cdots + t_m\varphi_m^G.$$

Since χ is an irreducible character there exist $u \in \{1, \ldots, m\}$ and $1 \leqslant k \leqslant l-j$ such that $k\chi = \varphi_u^G$, so G is quasi-M-group.

(b) $\Rightarrow$ (a): Let $\chi \in \{\chi_1, \ldots, \chi_r\}$. Since G is quasi-M-group, there exist a subgroup $H \subseteq G$, a 1-dimensional character $\varphi : H \to \mathbb{C}^\times$ and $k \geqslant 1$ such that

$$k\chi = \varphi^G.$$

Since χ is not trivial, the character φ is not trivial. It holds

$$L(s, \chi)^k = L(s, k\chi) = L\big(s, \varphi^G\big) \in \mathcal{B},$$

so $L(s, \chi) \in \mathcal{B}'$. Hence

$$\mathcal{A} \subseteq \mathcal{B}'. \qquad \square$$

It is not known whether there exist quasi-M-groups which are not M-groups. By a theorem of Taketa every M-group is solvable. It is not known whether every quasi-M-group is solvable.

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