

# Congruences involving Bernoulli and Euler numbers

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## Abstract

Let  $[x]$  be the integral part of  $x$ . Let  $p > 5$  be a prime. In the paper we mainly determine  $\sum_{x=1}^{[p/4]} \frac{1}{x^k} \pmod{p^2}$ ,  $\binom{p-1}{[p/4]} \pmod{p^3}$ ,  $\sum_{k=1}^{p-1} \frac{2^k}{k} \pmod{p^3}$  and  $\sum_{k=1}^{p-1} \frac{2^k}{k^2} \pmod{p^2}$  in terms of Euler and Bernoulli numbers. For example, we have

$$\sum_{x=1}^{[p/4]} \frac{1}{x^2} \equiv (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}) + \frac{14}{3} p B_{p-3} \pmod{p^2},$$

where  $E_n$  is the  $n$ th Euler number and  $B_n$  is the  $n$ th Bernoulli number.

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## 1. Introduction

The Bernoulli numbers  $\{B_n\}$  and Bernoulli polynomials  $\{B_n(x)\}$  are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n \geq 2) \quad \text{and} \quad B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad (n \geq 0).$$

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The Euler numbers  $\{E_n\}$  and Euler polynomials  $\{E_n(x)\}$  are defined by

$$\frac{2e^t}{e^{2t} + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \left( |t| < \frac{\pi}{2} \right) \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi),$$

which are equivalent to (see [MOS])

$$E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^n \binom{2n}{2r} E_{2r} = 0 \quad (n \geq 1)$$

and

$$E_n(x) = \frac{1}{2^n} \sum_{r=0}^n \binom{n}{r} (2x-1)^{n-r} E_r.$$

Let  $[x]$  be the integral part of  $x$ . For a given prime  $p$  let  $\mathbb{Z}_p$  denote the set of rational  $p$ -integers (those rational numbers whose denominator is not divisible by  $p$ ). For  $a \in \mathbb{Z}_p$  with  $a \not\equiv 0 \pmod{p}$ , as usual we define the Fermat quotient  $q_p(a) = (a^{p-1} - 1)/p$ . In the paper we establish some congruences involving Bernoulli and Euler numbers. In particular, in  $\mathbb{Z}_p$  we have

$$\sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{1}{k} \equiv q_p(2) - p \left( \frac{1}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} (E_{2p-4} - 2E_{p-3}) \right) + \frac{1}{3} p^2 q_p(2)^3 \pmod{p^3},$$

$$(-1)^{[\frac{p}{4}]} \binom{p-1}{[\frac{p}{4}]} \equiv 1 + 3p q_p(2) + p^2 (3q_p(2)^2 - (-1)^{\frac{p-1}{2}} E_{p-3}) \pmod{p^3},$$

$$\sum_{\substack{1 \leq k < p \\ 4 \nmid k+p}} \frac{1}{k} \equiv \frac{1}{4} q_p(2) - \frac{1}{8} p q_p(2)^2 + \frac{1}{12} p^2 q_p(2)^3 - \frac{7}{192} p^2 B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) - \frac{7}{12} p^2 B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -q_p(2)^2 + p \left( \frac{2}{3} q_p(2)^3 + \frac{7}{6} B_{p-3} \right) \pmod{p^2},$$

where  $p$  is a prime greater than 5.

In addition to the above notation, we also use throughout this paper the following notation:  $\mathbb{Z}$ —the set of integers,  $\mathbb{N}$ —the set of positive integers,  $\{x\}$ —the fractional part of  $x$ ,  $\varphi(n)$ —Euler's totient function.

## 2. Basic lemmas

We begin with a useful identity involving Bernoulli polynomials.

**Lemma 2.1.** Let  $p, m \in \mathbb{N}$  and  $k, r \in \mathbb{Z}$  with  $k \geq 0$ . Then

$$\sum_{\substack{x=0 \\ x \equiv r \pmod{m}}}^{p-1} x^k = \frac{m^k}{k+1} \left( B_{k+1} \left( \frac{p}{m} + \left\{ \frac{r-p}{m} \right\} \right) - B_{k+1} \left( \left\{ \frac{r}{m} \right\} \right) \right).$$

In the case  $m = 1$  Lemma 2.1 is well known. See [MOS,IR]. Lemma 2.1 was established by the author in 1991. A proof is given in [S4], and a generalization was published by author's brother Z.W. Sun [Su]. From [S2, Lemma 2.3] and [IR, Proposition 15.2.4, p. 238] we have

**Lemma 2.2.** Suppose that  $k, p \in \mathbb{N}$  with  $p > 1$ . If  $x, y \in \mathbb{Z}_p$ , then  $pB_k(x) \in \mathbb{Z}_p$  and  $(B_k(x) - B_k(y))/k \in \mathbb{Z}_p$ . If  $p$  is an odd prime such that  $p-1 \nmid k$ , then  $B_k(x)/k \in \mathbb{Z}_p$ .

**Lemma 2.3.** (See [MOS].) Let  $x$  and  $y$  be variables and  $n \in \mathbb{N}$ . Then

- (i)  $B_{2n+1} = 0$ .
- (ii)  $B_n(1-x) = (-1)^n B_n(x)$ .
- (iii)  $B_n(x+y) = \sum_{r=0}^n \binom{n}{r} B_{n-r}(y) x^r$ .
- (iv)  $E_{n-1}(x) = \frac{2^n}{n} (B_n(\frac{x+1}{2}) - B_n(\frac{x}{2}))$ .

**Lemma 2.4.** (See [MOS,GS].) Let  $n \in \mathbb{N}$ . Then

$$B_{2n}\left(\frac{1}{4}\right) = B_{2n}\left(\frac{3}{4}\right) = \frac{2-2^{2n}}{4^{2n}} B_{2n}, \quad B_{2n}\left(\frac{1}{3}\right) = B_{2n}\left(\frac{2}{3}\right) = \frac{3-3^{2n}}{2 \cdot 3^{2n}} B_{2n}$$

and

$$B_{2n}\left(\frac{1}{6}\right) = B_{2n}\left(\frac{5}{6}\right) = \frac{(2-2^{2n})(3-3^{2n})}{2 \cdot 6^{2n}} B_{2n}.$$

**Lemma 2.5.** For any nonnegative integer  $n$  we have

$$E_{2n} = -4^{2n+1} \frac{B_{2n+1}(\frac{1}{4})}{2n+1}.$$

**Proof.** It is well known that  $E_{2n} = 2^{2n} E_{2n}(\frac{1}{2})$ . Thus applying Lemma 2.3 we see that

$$\begin{aligned} E_{2n} &= 2^{2n} E_{2n+1-1} \left( \frac{1}{2} \right) = 2^{2n} \cdot \frac{2^{2n+1}}{2n+1} \left( B_{2n+1} \left( \frac{3}{4} \right) - B_{2n+1} \left( \frac{1}{4} \right) \right) \\ &= \frac{2^{4n+1}}{2n+1} \left( -B_{2n+1} \left( \frac{1}{4} \right) - B_{2n+1} \left( \frac{1}{4} \right) \right) = -\frac{2^{4n+2}}{2n+1} B_{2n+1} \left( \frac{1}{4} \right). \end{aligned}$$

This proves the lemma.  $\square$

From [S3, Corollary 3.1 and Theorem 4.2] we have:

**Lemma 2.6.** Let  $p$  be an odd prime,  $x \in \mathbb{Z}_p$  and  $k, b \in \mathbb{N}$  with  $p-1 \nmid b$ . Then

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b} \equiv \frac{B_b(x)}{b} \pmod{p} \quad \text{for } b \geq 2$$

and

$$\frac{B_{k(p-1)+b}(x)}{k(p-1)+b} \equiv k \frac{B_{p-1+b}(x)}{p-1+b} - (k-1) \frac{B_b(x)}{b} \pmod{p^2} \quad \text{for } b > 2.$$

**Lemma 2.7.** Let  $p > 3$  be a prime,  $r \in \mathbb{Z}$  and  $k, m \in \mathbb{N}$  with  $k < p-3$  and  $p \nmid m$ . Then

$$\begin{aligned} \sum_{\substack{x=1 \\ x \equiv r \pmod{m}}}^{p-1} \frac{1}{x^k} &\equiv \frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{m^k(\varphi(p^3) - k + 1)} \\ &\quad + \frac{kp}{m^{k+1}} \left( \frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p-2-k} - 2 \frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p-1-k} \right) \\ &\quad - \frac{k(k+1)p^2}{2(k+2)m^{k+2}} B_{p-2-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \pmod{p^3}. \end{aligned}$$

**Proof.** From Lemmas 2.1, 2.3(iii) and Euler's theorem we see that

$$\begin{aligned} \sum_{\substack{x=1 \\ x \equiv r \pmod{m}}}^{p-1} \frac{1}{x^k} &\equiv \sum_{\substack{x=0 \\ x \equiv r \pmod{m}}}^{p-1} x^{\varphi(p^3)-k} \\ &= \frac{m^{\varphi(p^3)-k}}{\varphi(p^3) - k + 1} \left( B_{\varphi(p^3)-k+1} \left( \frac{p}{m} + \left\{ \frac{r-p}{m} \right\} \right) - B_{\varphi(p^3)-k+1} \left( \left\{ \frac{r}{m} \right\} \right) \right) \\ &= \frac{m^{\varphi(p^3)-k}}{\varphi(p^3) - k + 1} \left\{ \sum_{j=0}^{\varphi(p^3)-k+1} \binom{\varphi(p^3)-k+1}{j} \frac{p^j}{m^j} B_{\varphi(p^3)-k+1-j} \left( \left\{ \frac{r-p}{m} \right\} \right) \right. \\ &\quad \left. - B_{\varphi(p^3)-k+1} \left( \left\{ \frac{r}{m} \right\} \right) \right\} \\ &= m^{\varphi(p^3)-k} \left\{ \frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3) - k + 1} + \frac{p}{m} B_{\varphi(p^3)-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \right. \\ &\quad \left. + \frac{\varphi(p^3) - k}{2} \cdot \frac{p^2}{m^2} B_{\varphi(p^3)-k-1} \left( \left\{ \frac{r-p}{m} \right\} \right) \right. \\ &\quad \left. + \sum_{j=3}^{\varphi(p^3)-k+1} \frac{p^{j-3}}{j} \binom{\varphi(p^3)-k}{j-1} \frac{p^3}{m^j} B_{\varphi(p^3)-k+1-j} \left( \left\{ \frac{r-p}{m} \right\} \right) \right\} \pmod{p^3}. \end{aligned}$$

As  $k < p - 3$  we have  $p - 1 \nmid \varphi(p^3) - k - 2$  and so  $B_{\varphi(p^3)-k-2}(\{\frac{r-p}{m}\}) \in \mathbb{Z}_p$  by Lemma 2.2. For  $j \geq 4$  we have  $p^{j-3}/j \equiv 0 \pmod{p}$ . Thus  $\frac{p^{j-3}}{j} B_{\varphi(p^3)-k+1-j}(\{\frac{r-p}{m}\}) \in \mathbb{Z}_p$  for  $j \geq 3$ . Hence, by the above we obtain

$$m^k \sum_{\substack{x=1 \\ x \equiv r \pmod{m}}}^{p-1} \frac{1}{x^k} \equiv \frac{B_{\varphi(p^3)-k+1}(\{\frac{r-p}{m}\}) - B_{\varphi(p^3)-k+1}(\{\frac{r}{m}\})}{\varphi(p^3) - k + 1} + \frac{p}{m} B_{\varphi(p^3)-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \\ + \frac{\varphi(p^3) - k}{2} \cdot \frac{p^2}{m^2} B_{\varphi(p^3)-k-1} \left( \left\{ \frac{r-p}{m} \right\} \right) \pmod{p^3}.$$

From Lemma 2.6 we see that

$$\frac{B_{\varphi(p^3)-k-1}(\{\frac{r-p}{m}\})}{\varphi(p^3) - k - 1} = \frac{B_{(p^2-1)(p-1)+p-2-k}(\{\frac{r-p}{m}\})}{(p^2-1)(p-1) + p - 2 - k} \equiv \frac{B_{p-2-k}(\{\frac{r-p}{m}\})}{p - 2 - k} \pmod{p}$$

and

$$\frac{B_{\varphi(p^3)-k}(\{\frac{r-p}{m}\})}{\varphi(p^3) - k} = \frac{B_{(p^2-1)(p-1)+p-1-k}(\{\frac{r-p}{m}\})}{(p^2-1)(p-1) + p - 1 - k} \\ \equiv (p^2 - 1) \frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p - 2 - k} - (p^2 - 2) \frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p - 1 - k} \pmod{p^2}.$$

Thus

$$B_{\varphi(p^3)-k-1} \left( \left\{ \frac{r-p}{m} \right\} \right) \equiv \frac{k+1}{k+2} B_{p-2-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \pmod{p}$$

and

$$B_{\varphi(p^3)-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \equiv k \frac{B_{2p-2-k}(\{\frac{r-p}{m}\})}{2p - 2 - k} - 2k \frac{B_{p-1-k}(\{\frac{r-p}{m}\})}{p - 1 - k} \pmod{p^2}.$$

Now putting all the above together we obtain the result.  $\square$

**Lemma 2.8.** Let  $p$  be an odd prime,  $a \in \mathbb{Z}_p$ ,  $a \not\equiv 0 \pmod{p}$ ,  $n \in \mathbb{N}$  and  $p > n + 1$ . Then

$$\frac{a^{\varphi(p^n)} - 1}{p^n} \equiv \sum_{s=1}^n \frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s \pmod{p^n}.$$

**Proof.** As  $p > n + 1 \geq 2$  we see that  $p^{s-1}/s! \in \mathbb{Z}_p$  and  $p^{s-1}/s! \equiv 0 \pmod{p}$  for  $s \geq 2$ . Thus,

$$\frac{a^{\varphi(p^n)} - 1}{p^n} = \frac{(1 + pq_p(a))^{p^{n-1}} - 1}{p^n} = \frac{1}{p^n} \sum_{s=1}^{p^{n-1}} \binom{p^{n-1}}{s} p^s q_p(a)^s$$

$$\begin{aligned}
&= q_p(a) + \sum_{s=2}^{p^{n-1}} (p^{n-1} - 1) \cdots (p^{n-1} - s + 1) \cdot \frac{p^{s-1}}{s!} q_p(a)^s \\
&\equiv q_p(a) + \sum_{s=2}^{p^{n-1}} (-1)(-2) \cdots (-s + 1) \cdot \frac{p^{s-1}}{s!} q_p(a)^s \\
&= \sum_{s=1}^{p^{n-1}} \frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s \pmod{p^n}.
\end{aligned}$$

As  $p > n + 1$  we see that  $p^{s-n-1}/s \in \mathbb{Z}_p$  for  $s \geq n + 1$ . Thus for  $s \geq n + 1$  we have

$$\frac{(-1)^{s-1}}{s} p^{s-1} q_p(a)^s = (-1)^{s-1} \cdot \frac{p^{s-n-1}}{s} \cdot q_p(a)^s \cdot p^n \equiv 0 \pmod{p^n}.$$

Now putting the above together we obtain the result.  $\square$

**Lemma 2.9.** Let  $p$  be an odd prime and  $k \in \{0, 1, \dots, p-1\}$ . Then

$$\begin{aligned}
(-1)^k \binom{p-1}{k} &\equiv 1 - p \sum_{i=1}^k \frac{1}{i} + \frac{p^2}{2} \left\{ \left( \sum_{i=1}^k \frac{1}{i} \right)^2 - \sum_{i=1}^k \frac{1}{i^2} \right\} \\
&\quad - \frac{p^3}{6} \left\{ \left( \sum_{i=1}^k \frac{1}{i} \right)^3 - 3 \left( \sum_{i=1}^k \frac{1}{i} \right) \left( \sum_{i=1}^k \frac{1}{i^2} \right) + 2 \sum_{i=1}^k \frac{1}{i^3} \right\} \pmod{p^4}.
\end{aligned}$$

**Proof.** For  $k = 0, 1, 2$  it is easy to verify the result. Now assume  $k \geq 3$ . Clearly

$$\begin{aligned}
\binom{p-1}{k} &= \frac{(p-1)(p-2) \cdots (p-k)}{k!} \\
&= \frac{1}{k!} \left\{ p^k + \cdots + p^3 \sum_{1 \leq i < j < l \leq k} \frac{(-1)(-2) \cdots (-k)}{(-i)(-j)(-l)} \right. \\
&\quad \left. + p^2 \sum_{1 \leq i < j \leq k} \frac{(-1)(-2) \cdots (-k)}{(-i)(-j)} + p \sum_{i=1}^k \frac{(-1)(-2) \cdots (-k)}{-i} + (-1)^k k! \right\} \\
&\equiv (-1)^k \left( -p^3 \sum_{1 \leq i < j < l \leq k} \frac{1}{ijl} + p^2 \sum_{1 \leq i < j \leq k} \frac{1}{ij} - p \sum_{i=1}^k \frac{1}{i} + 1 \right) \pmod{p^4}.
\end{aligned}$$

Observe that

$$\left( \sum_{i=1}^k \frac{1}{i} \right)^2 = \sum_{i=1}^k \frac{1}{i^2} + 2 \sum_{1 \leq i < j \leq k} \frac{1}{ij}$$

and

$$\begin{aligned}
\left(\sum_{i=1}^k \frac{1}{i}\right)^3 &= \sum_{1 \leq i, j, l \leq k} \frac{1}{ijl} = 6 \sum_{1 \leq i < j < l \leq k} \frac{1}{ijl} + 3 \sum_{i=1}^k \sum_{\substack{j=1 \\ j \neq i}}^k \frac{1}{i^2 j} + \sum_{i=1}^k \frac{1}{i^3} \\
&= 6 \sum_{1 \leq i < j < l \leq k} \frac{1}{ijl} + 3 \left(\sum_{j=1}^k \frac{1}{j}\right) \left(\sum_{i=1}^k \frac{1}{i^2}\right) - 3 \sum_{i=1}^k \frac{1}{i^3} + \sum_{i=1}^k \frac{1}{i^3}.
\end{aligned}$$

We then have

$$\sum_{1 \leq i < j \leq k} \frac{1}{ij} = \frac{1}{2} \left\{ \left(\sum_{i=1}^k \frac{1}{i}\right)^2 - \sum_{i=1}^k \frac{1}{i^2} \right\}$$

and

$$\sum_{1 \leq i < j < l \leq k} \frac{1}{ijl} = \frac{1}{6} \left\{ \left(\sum_{i=1}^k \frac{1}{i}\right)^3 - 3 \left(\sum_{i=1}^k \frac{1}{i}\right) \left(\sum_{i=1}^k \frac{1}{i^2}\right) + 2 \sum_{i=1}^k \frac{1}{i^3} \right\}.$$

Now putting all the above together we obtain the result.  $\square$

We remark that the congruence for  $\binom{p-1}{k} \pmod{p^3}$  was given by Lehmer in [L, p. 360].

### 3. Congruences for $\sum_{1 \leq x < p, m|x-p} \frac{1}{x^k}$ ( $m = 3, 4, 6$ ) and $\sum_{x=1}^{[p/4]} \frac{1}{x^k}$

**Theorem 3.1.** *Let  $p > 3$  be a prime. Then*

- (i) 
$$\sum_{\substack{k=1 \\ k \equiv p \pmod{3}}}^{p-1} \frac{1}{k} \equiv \frac{1}{2} q_p(3) - \frac{1}{4} p q_p(3)^2 + \frac{1}{6} p^2 q_p(3)^3 - \frac{p^2}{81} B_{p-3} \pmod{p^3}.$$
- (ii) 
$$\sum_{\substack{k=1 \\ k \equiv p \pmod{4}}}^{p-1} \frac{1}{k} \equiv \frac{3}{4} q_p(2) - \frac{3}{8} p q_p(2)^2 + \frac{1}{4} p^2 q_p(2)^3 - \frac{p^2}{192} B_{p-3} \pmod{p^3}.$$
- (iii) 
$$\begin{aligned} \sum_{\substack{k=1 \\ k \equiv p \pmod{6}}}^{p-1} \frac{1}{k} &\equiv \frac{1}{3} q_p(2) + \frac{1}{4} q_p(3) - p \left( \frac{1}{6} q_p(2)^2 + \frac{1}{8} q_p(3)^2 \right) \\ &\quad + p^2 \left( \frac{1}{9} q_p(2)^3 + \frac{1}{12} q_p(3)^3 - \frac{1}{648} B_{p-3} \right) \pmod{p^3}. \end{aligned}$$

**Proof.** Note that  $B_n(0) = B_n$  and  $B_{2n+1} = 0$  for  $n \in \mathbb{N}$ . Taking  $k = 1$  and  $r = p$  in Lemma 2.7 we see that if  $m \in \mathbb{N}$  and  $p \nmid m$ , then

$$\sum_{\substack{k=1 \\ k \equiv p \pmod{m}}}^{p-1} \frac{1}{k} \equiv \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\{\frac{p}{m}\})}{mp^2(p-1)} - \frac{p^2}{3m^3} B_{p-3} \pmod{p^3}.$$

As  $B_{2n}(x) = B_{2n}(1-x)$ , for  $m = 3, 4, 6$  we have  $B_{2n}(\{\frac{p}{m}\}) = B_{2n}(\frac{1}{m})$ . Since  $pB_{p^2(p-1)} \equiv p-1 \pmod{p^3}$  by [S2, Corollary 4.1], using Lemmas 2.4, 2.8 and Euler's theorem we see that

$$\begin{aligned} \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{3})}{p^2(p-1)} &= \left(1 - \frac{3-3^{p^2(p-1)}}{2 \cdot 3^{p^2(p-1)}}\right) \frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{3}{2 \cdot 3^{p^2(p-1)}} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \equiv \frac{3}{2} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \\ &\equiv \frac{3}{2} \left( q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3 \right) \pmod{p^3}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} &= \left(1 - \frac{2-2^{p^2(p-1)}}{4^{p^2(p-1)}}\right) \frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{2^{p^2(p-1)} + 2}{4^{p^2(p-1)}} \cdot \frac{2^{p^2(p-1)} - 1}{p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \equiv 3 \cdot \frac{2^{p^2(p-1)} - 1}{p^3} \\ &\equiv 3 \left( q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3 \right) \pmod{p^3} \end{aligned}$$

and

$$\begin{aligned} \frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{6})}{p^2(p-1)} &= \left(1 - \frac{(2-2^{p^2(p-1)})(3-3^{p^2(p-1)})}{2 \cdot 6^{p^2(p-1)}}\right) \frac{B_{p^2(p-1)}}{p^2(p-1)} \\ &= \frac{(2^{p^2(p-1)} - 1)(3^{p^2(p-1)} - 1) + 4(2^{p^2(p-1)} - 1) + 3(3^{p^2(p-1)} - 1)}{2 \cdot 6^{p^2(p-1)} \cdot p^3} \cdot \frac{pB_{p^2(p-1)}}{p-1} \\ &\equiv 2 \cdot \frac{2^{p^2(p-1)} - 1}{p^3} + \frac{3}{2} \cdot \frac{3^{p^2(p-1)} - 1}{p^3} \\ &\equiv 2 \left( q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3 \right) + \frac{3}{2} \left( q_p(3) - \frac{1}{2}pq_p(3)^2 + \frac{1}{3}p^2q_p(3)^3 \right) \pmod{p^3}. \end{aligned}$$

Now combining all the above we obtain the result.  $\square$

**Corollary 3.1.** Let  $p > 3$  be a prime. Then

$$\sum_{\substack{k=1 \\ k \equiv -p \pmod{4}}}^{p-1} \frac{1}{k} \equiv \frac{1}{4}q_p(2) - \frac{1}{8}pq_p(2)^2 + \frac{1}{12}p^2q_p(2)^3 - \frac{7}{192}p^2B_{p-3} \pmod{p^3}.$$



**Proof.** Using [S3, Remark 5.3] we know that

$$\begin{aligned} & \sum_{\substack{k=1 \\ k \equiv -p \pmod{4}}}^{p-1} \frac{1}{k} + \sum_{\substack{k=1 \\ k \equiv p \pmod{4}}}^{p-1} \frac{1}{k} \\ &= \sum_{\substack{k=1 \\ 2 \nmid k}}^{p-1} \frac{1}{k} \equiv q_p(2) - \frac{1}{2} p q_p(2)^2 + \frac{1}{3} p^2 q_p(2)^3 - \frac{1}{24} p^2 B_{p-3} \pmod{p^3}. \end{aligned}$$

Thus applying Theorem 3.1(ii) we deduce the result.  $\square$

**Remark 3.1.** Let  $m \in \{3, 4, 6\}$ . In 1938 E. Lehmer [L] obtained the congruences for

$$\sum_{\substack{k=1 \\ k \equiv p \pmod{m}}}^{p-1} \frac{1}{k} \pmod{p^2}.$$

Using the formulas for  $\sum_{k \equiv r \pmod{m}} \binom{p}{k}$ , in [S1] the author gave congruences for

$$\sum_{\substack{k=1 \\ k \equiv r \pmod{m}}}^{p-1} \frac{1}{k} \pmod{p}.$$

**Corollary 3.2.** Let  $p > 3$  be a prime. Then

$$\begin{aligned} \sum_{\substack{k=1 \\ k \equiv -p \pmod{3}}}^{\frac{p-1}{2}} \frac{1}{k} &\equiv -\frac{2}{3} q_p(2) + \frac{1}{2} q_p(3) + p \left( \frac{1}{3} q_p(2)^2 - \frac{1}{4} q_p(3)^2 \right) \\ &+ p^2 \left( -\frac{2}{9} q_p(2)^3 + \frac{1}{6} q_p(3)^3 - \frac{7}{324} B_{p-3} \right) \pmod{p^3}. \end{aligned}$$

**Proof.** Clearly

$$\begin{aligned} \sum_{\substack{k=1 \\ k \equiv -p \pmod{3}}}^{\frac{p-1}{2}} \frac{1}{k} &= 2 \sum_{\substack{k=1 \\ 2k \equiv p+3 \pmod{6}}}^{\frac{p-1}{2}} \frac{1}{2k} = 2 \sum_{\substack{k=1 \\ k \equiv p+3 \pmod{6}}}^{p-1} \frac{1}{k} \\ &= 2 \left( \sum_{\substack{k=1 \\ k \equiv p \pmod{3}}}^{p-1} \frac{1}{k} - \sum_{\substack{k=1 \\ k \equiv p \pmod{6}}}^{p-1} \frac{1}{k} \right). \end{aligned}$$

Thus appealing to Theorem 3.1 we obtain the result.  $\square$

**Theorem 3.2.** Let  $p > 3$  be a prime. Then

$$\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} \equiv -3q_p(2) + p \left( \frac{3}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} (E_{2p-4} - 2E_{p-3}) \right) \\ - p^2 \left( q_p(2)^3 + \frac{7}{12} B_{p-3} \right) \pmod{p^3}$$

and

$$\sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{1}{k} \equiv q_p(2) - p \left( \frac{1}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} (E_{2p-4} - 2E_{p-3}) \right) + \frac{1}{3} p^2 q_p(2)^3 \pmod{p^3}.$$

**Proof.** Taking  $k = 1$ ,  $r = 0$  and  $m = 4$  in Lemma 2.7 we find

$$\sum_{\substack{x=1 \\ 4|x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\{\frac{-p}{4}\}) - B_{p^2(p-1)}}{4p^2(p-1)} + \frac{p}{16} \left( \frac{B_{2p-3}(\{\frac{-p}{4}\})}{2p-3} - 2 \frac{B_{p-2}(\{\frac{-p}{4}\})}{p-2} \right) \\ - \frac{p^2}{192} B_{p-3} \left( \left\{ \frac{-p}{4} \right\} \right) \pmod{p^3}.$$

As  $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$  we then have

$$\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} = 4 \sum_{\substack{x=1 \\ 4|x}}^{p-1} \frac{1}{x} \equiv \frac{B_{p^2(p-1)}(\frac{1}{4}) - B_{p^2(p-1)}}{p^2(p-1)} + (-1)^{\frac{p+1}{2}} \frac{p}{4} \left( \frac{B_{2p-3}(\frac{1}{4})}{2p-3} - 2 \frac{B_{p-2}(\frac{1}{4})}{p-2} \right) \\ - \frac{p^2}{48} B_{p-3} \left( \frac{1}{4} \right) \pmod{p^3}.$$

From the proof of Theorem 3.1 we know that

$$\frac{B_{p^2(p-1)} - B_{p^2(p-1)}(\frac{1}{4})}{p^2(p-1)} \equiv 3q_p(2) - \frac{3}{2} p q_p(2)^2 + p^2 q_p(2)^3 \pmod{p^3}.$$

By Lemmas 2.5 and 2.6 we have

$$E_{2p-4} = -4^{2p-3} \frac{B_{2p-3}(\frac{1}{4})}{2p-3} \equiv -4^{p-2} \frac{B_{p-2}(\frac{1}{4})}{p-2} = E_{p-3} \pmod{p}. \quad (3.1)$$

Observe that  $a^{s(p-1)} = (1 + p q_p(a))^s \equiv 1 + s p q_p(a) \pmod{p^2}$  for  $a, s \in \mathbb{Z}$  with  $p \nmid a$ . We then have

$$\frac{1}{4^{2p-2}} \equiv 1 - 2 p q_p(4) \pmod{p^2} \quad \text{and} \quad \frac{1}{4^{p-1}} \equiv 1 - p q_p(4) \pmod{p^2}.$$

Hence

$$\begin{aligned}
 & -\frac{1}{4} \left( \frac{B_{2p-3}(\frac{1}{4})}{2p-3} - 2 \frac{B_{p-2}(\frac{1}{4})}{p-2} \right) \\
 &= \frac{E_{2p-4}}{4^{2p-2}} - 2 \frac{E_{p-3}}{4^{p-1}} \equiv (1 - 2pq_p(4))E_{2p-4} - 2(1 - pq_p(4))E_{p-3} \\
 &= E_{2p-4} - 2E_{p-3} - 2pq_p(4)(E_{2p-4} - E_{p-3}) \\
 &\equiv E_{2p-4} - 2E_{p-3} \pmod{p^2}.
 \end{aligned}$$

On the other hand, by Lemma 2.4 we have

$$B_{p-3} \left( \frac{1}{4} \right) = \frac{2 - 2^{p-3}}{4^{p-3}} B_{p-3} = \frac{32 - 4 \cdot 2^{p-1}}{4^{p-1}} B_{p-3} \equiv 28B_{p-3} \pmod{p}.$$

Thus combining the above we obtain

$$\begin{aligned}
 \sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} &\equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 - p^2q_p(2)^3 + (-1)^{\frac{p-1}{2}}p(E_{2p-4} - 2E_{p-3}) \\
 &\quad - \frac{p^2}{48} \cdot 28B_{p-3} \pmod{p^3}.
 \end{aligned}$$

From [S3, Theorem 5.2] we know that

$$\sum_{1 \leq k < \frac{p}{2}} \frac{1}{k} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Observe that

$$\sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{1}{k} = \sum_{1 \leq k < \frac{p}{2}} \frac{1}{k} - \sum_{1 \leq k < \frac{p}{4}} \frac{1}{k}.$$

We then obtain the remaining result.  $\square$

**Remark 3.2.** For any prime  $p > 3$ , the congruence  $\sum_{k=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{k} \equiv -3q_p(2) \pmod{p}$  was first established by Lerch (see [D]), and a simple proof concerning the formula for  $\sum_{4|k} \binom{p}{k}$  was given by the author in [S1].

**Corollary 3.3.** Let  $p > 3$  be a prime. Then

$$\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} \equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 - (-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{\frac{p}{4} < k < \frac{p}{2}} \frac{1}{k} \equiv q_p(2) - \frac{1}{2} p q_p(2)^2 + (-1)^{\frac{p-1}{2}} p E_{p-3} \pmod{p^2}.$$

**Lemma 3.1.** Let  $p > 5$  be a prime,  $r \in \mathbb{Z}$ ,  $k, m \in \mathbb{N}$ ,  $p \nmid m$  and  $1 < k < p - 3$ . Then

$$\begin{aligned} m^k \sum_{\substack{x=1 \\ x \equiv r \pmod{m}}}^{p-1} \frac{1}{x^k} \\ \equiv \frac{B_{2p-1-k}(\{\frac{r}{m}\}) - B_{2p-1-k}(\{\frac{r-p}{m}\})}{2p-1-k} - 2 \frac{B_{p-k}(\{\frac{r}{m}\}) - B_{p-k}(\{\frac{r-p}{m}\})}{p-k} \\ + \frac{kp}{m(k+1)} B_{p-1-k} \left( \left\{ \frac{r-p}{m} \right\} \right) \pmod{p^2}. \end{aligned}$$

**Proof.** From Lemma 2.6 we see that if  $x \in \mathbb{Z}_p$ , then

$$\frac{B_{2p-2-k}(x)}{2p-2-k} \equiv \frac{B_{p-1-k}(x)}{p-1-k} \pmod{p}$$

and

$$\begin{aligned} \frac{B_{\varphi(p^3)-k+1}(x)}{\varphi(p^3)-k+1} &= \frac{B_{(p^2-1)(p-1)+p-k}(x)}{(p^2-1)(p-1)+p-k} \\ &\equiv (p^2-1) \frac{B_{2p-1-k}(x)}{2p-1-k} - (p^2-2) \frac{B_{p-k}(x)}{p-k} \\ &\equiv -\frac{B_{2p-1-k}(x)}{2p-1-k} + 2 \frac{B_{p-k}(x)}{p-k} \pmod{p^2}. \end{aligned}$$

This together with Lemma 2.7 gives the result.  $\square$

Putting  $r = 0$ ,  $p$  in Lemma 3.1 and noting that  $B_{2n+1} = 0$  ( $n \geq 1$ ) we deduce the following result.

**Theorem 3.3.** Let  $p > 5$  be a prime,  $k, m \in \mathbb{N}$ ,  $p \nmid m$  and  $1 < k < p - 3$ . Then

$$\begin{aligned} m^k \sum_{\substack{x=1 \\ x \equiv p \pmod{m}}}^{p-1} \frac{1}{x^k} \\ \equiv \begin{cases} \frac{B_{2p-1-k}(\{\frac{p}{m}\}) - B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}(\{\frac{p}{m}\}) - B_{p-k}}{p-k} \pmod{p^2} & \text{if } 2 \nmid k, \\ \frac{B_{2p-1-k}(\{\frac{p}{m}\}) - 2 \frac{B_{p-k}(\{\frac{p}{m}\})}{p-k} + \frac{kp}{m(k+1)} B_{p-1-k} \pmod{p^2} & \text{if } 2 \mid k. \end{cases} \end{aligned}$$

and

$$\sum_{x=1}^{[p/m]} \frac{1}{x^k} \equiv \begin{cases} \frac{B_{2p-1-k} - B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k} - 2 \frac{B_{p-k} - B_{p-k}(\{\frac{-p}{m}\})}{p-k} \\ \quad + \frac{kp}{m(k+1)} B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \text{if } 2 \nmid k, \\ -\frac{B_{2p-1-k}(\{\frac{-p}{m}\})}{2p-1-k} + 2 \frac{B_{p-k}(\{\frac{-p}{m}\})}{p-k} \\ \quad + \frac{kp}{m(k+1)} B_{p-1-k}(\{\frac{-p}{m}\}) \pmod{p^2} & \text{if } 2 \mid k. \end{cases}$$

**Corollary 3.4.** Let  $p > 5$  be a prime and  $k \in \{3, 5, \dots, p-4\}$ . Then

$$\begin{aligned} \text{(i)} \quad & \sum_{x=1}^{[p/3]} \frac{1}{x^k} \equiv -\frac{3^k-3}{2k} B_{p-k} \pmod{p}, \\ \text{(ii)} \quad & \sum_{x=1}^{[p/4]} \frac{1}{x^k} \equiv -\frac{2^{2k-1}-2^{k-1}-1}{k} B_{p-k} \pmod{p}, \\ \text{(iii)} \quad & \sum_{x=1}^{[p/6]} \frac{1}{x^k} \equiv -\frac{(2^k-1)(3^k-1)-2}{2k} B_{p-k} \pmod{p}, \\ \text{(iv)} \quad & \sum_{\frac{p}{6} < x < \frac{p}{4}} \frac{1}{x^k} \equiv \frac{(2^k-1)(3^k-2^k-1)}{2k} B_{p-k} \pmod{p}, \\ \text{(v)} \quad & \sum_{\frac{p}{4} < x < \frac{p}{3}} \frac{1}{x^k} \equiv \frac{2^{2k}-2^k-3^k+1}{2k} B_{p-k} \pmod{p}. \end{aligned}$$

**Proof.** From Lemma 2.6 and Theorem 3.3 we see that for  $m = 3, 4, 6$ ,

$$\sum_{x=1}^{[p/m]} \frac{1}{x^k} \equiv -\frac{B_{p-k} - B_{p-k}(\{\frac{-p}{m}\})}{p-k} = \frac{B_{p-k}(\frac{1}{m}) - B_{p-k}}{p-k} \pmod{p}.$$

Now applying Lemma 2.4 we deduce (i)–(iii). (iv) follows from (ii) and (iii), and (v) follows from (i) and (ii).  $\square$

**Theorem 3.4.** Let  $p > 5$  be a prime and  $k \in \{3, 5, \dots, p-4\}$ . Then

$$\begin{aligned} \text{(i)} \quad & \sum_{\substack{x=1 \\ x \equiv p \pmod{3}}}^{p-1} \frac{1}{x^k} \equiv \frac{3^{k-1}-1}{2 \cdot 3^{k-1}} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}, \\ \text{(ii)} \quad & \sum_{\substack{x=1 \\ x \equiv p \pmod{4}}}^{p-1} \frac{1}{x^k} \equiv \frac{2^{2k-1}-2^{k-1}-1}{2^{2k}} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}, \\ \text{(iii)} \quad & \sum_{\substack{x=1 \\ x \equiv p \pmod{6}}}^{p-1} \frac{1}{x^k} \equiv \frac{(2^k-1)(3^k-1)-2}{2 \cdot 6^k} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}. \end{aligned}$$

**Proof.** Let  $m \in \{3, 4, 6\}$ . As  $B_{2n}(1-x) = B_{2n}(x)$ , we see that  $B_{2n}(\{\frac{p}{m}\}) = B_{2n}(\frac{1}{m})$ . Hence, applying Theorem 3.3 we have

$$m^k \sum_{\substack{x=1 \\ x \equiv p \pmod{m}}}^{p-1} \frac{1}{x^k} \equiv \frac{B_{2p-1-k}(\frac{1}{m}) - B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}(\frac{1}{m}) - B_{p-k}}{p-k} \pmod{p^2}. \quad (3.2)$$

By Lemma 2.4 we have  $B_{2n}(\frac{1}{3}) - B_{2n} = \frac{1}{2}(3^{1-2n} - 3)B_{2n}$ . Note that  $3^{2-2p} \equiv 2 \cdot 3^{1-p} - 1 \pmod{p^2}$ . By the above we obtain

$$\begin{aligned} 3^k \sum_{\substack{x=1 \\ x \equiv p \pmod{3}}}^{p-1} \frac{1}{x^k} &\equiv \frac{3^{1-(2p-1-k)} - 3}{2} \cdot \frac{B_{2p-1-k}}{2p-1-k} - 2 \cdot \frac{3^{1-(p-k)} - 3}{2} \cdot \frac{B_{p-k}}{p-k} \\ &\equiv \frac{3^k(2 \cdot 3^{1-p} - 1) - 3}{2} \cdot \frac{B_{2p-1-k}}{2p-1-k} - (3^{k+1-p} - 3) \frac{B_{p-k}}{p-k} \\ &= 3^{k+1-p} \left( \frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k} \right) - \frac{3^k + 3}{2} \cdot \frac{B_{2p-1-k}}{2p-1-k} + 3 \frac{B_{p-k}}{p-k} \\ &\equiv 3^k \left( \frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k} \right) - \frac{3^k + 3}{2} \cdot \frac{B_{2p-1-k}}{2p-1-k} + 3 \frac{B_{p-k}}{p-k} \\ &= \frac{3^k - 3}{2} \cdot \frac{B_{2p-1-k}}{2p-1-k} - (3^k - 3) \frac{B_{p-k}}{p-k} \pmod{p^2}. \end{aligned}$$

This proves (i). Now we consider (ii). From Lemma 2.4 we know that  $B_{2n}(\frac{1}{4}) - B_{2n} = (2^{1-4n} - 2^{-2n} - 1)B_{2n}$ . Observe that  $2^{s(p-1)} = (1 + pq_p(2))^s \equiv 1 + spq_p(2) \pmod{p^2}$ . Using (3.2) we see that

$$\begin{aligned} 4^k \sum_{\substack{x=1 \\ x \equiv p \pmod{4}}}^{p-1} \frac{1}{x^k} &\equiv (2^{2k-1-4(p-1)} - 2^{k-1-2(p-1)} - 1) \frac{B_{2p-1-k}}{2p-1-k} \\ &\quad - 2(2^{2k-1-2(p-1)} - 2^{k-1-(p-1)} - 1) \frac{B_{p-k}}{p-k} \\ &\equiv (2^{2k-1}(1 - 4pq_p(2)) - 2^{k-1}(1 - 2pq_p(2)) - 1) \frac{B_{2p-1-k}}{2p-1-k} \\ &\quad - 2(2^{2k-1}(1 - 2pq_p(2)) - 2^{k-1}(1 - pq_p(2)) - 1) \frac{B_{p-k}}{p-k} \\ &= (2^k - 2^{2k+1})q_p(2)p \left( \frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k} \right) \\ &\quad + (2^{2k-1} - 2^{k-1} - 1) \frac{B_{2p-1-k}}{2p-1-k} - 2(2^{2k-1} - 2^{k-1} - 1) \frac{B_{p-k}}{p-k} \\ &\equiv (2^{2k-1} - 2^{k-1} - 1) \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}. \end{aligned}$$

This proves (ii). Finally we consider (iii). As  $B_{2n}(\frac{1}{6}) = \frac{1}{2}(2^{1-2n} - 1)(3^{1-2n} - 1)B_{2n}$ , by (3.2) we have

$$\begin{aligned}
 & 6^k \sum_{\substack{x=1 \\ x \equiv p \pmod{6}}}^{p-1} \frac{1}{x^k} \\
 & \equiv \frac{1}{2} \left( (2^{k-2(p-1)} - 1)(3^{k-2(p-1)} - 1) - 2 \right) \frac{B_{2p-1-k}}{2p-1-k} \\
 & \quad - \left( (2^{k-(p-1)} - 1)(3^{k-(p-1)} - 1) - 2 \right) \frac{B_{p-k}}{p-k} \\
 & \equiv \frac{1}{2} \left\{ (2^k(1 - 2pq_p(2)) - 1)(3^k(1 - 2pq_p(3)) - 1) - 2 \right\} \frac{B_{2p-1-k}}{2p-1-k} \\
 & \quad - \left\{ (2^k(1 - pq_p(2)) - 1)(3^k(1 - pq_p(3)) - 1) - 2 \right\} \frac{B_{p-k}}{p-k} \\
 & \equiv \left( \frac{(2^k - 1)(3^k - 1) - 2}{2} - 2^k(3^k - 1)pq_p(2) - 3^k(2^k - 1)pq_p(3) \right) \frac{B_{2p-1-k}}{2p-1-k} \\
 & \quad - \left( (2^k - 1)(3^k - 1) - 2 - 2^k(3^k - 1)pq_p(2) - 3^k(2^k - 1)pq_p(3) \right) \frac{B_{p-k}}{p-k} \\
 & = \frac{(2^k - 1)(3^k - 1) - 2}{2} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \\
 & \quad - (2^k(3^k - 1)q_p(2) + 3^k(2^k - 1)q_p(3))p \left( \frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k} \right) \\
 & \equiv \frac{(2^k - 1)(3^k - 1) - 2}{2} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}.
 \end{aligned}$$

This proves (iii) and hence the theorem is proved.  $\square$

**Corollary 3.5.** *Let  $p > 5$  be a prime and  $k \in \{3, 5, \dots, p-4\}$ . Then*

$$\begin{aligned}
 & \sum_{\substack{x=1 \\ x \equiv p \pmod{3}}}^{p-1} \frac{1}{x^k} \equiv \frac{3^{k-1} - 1}{2k \cdot 3^{k-1}} B_{p-k} \pmod{p}, \\
 & \sum_{\substack{x=1 \\ x \equiv p \pmod{4}}}^{p-1} \frac{1}{x^k} \equiv \frac{2^{2k-1} - 2^{k-1} - 1}{k \cdot 2^{2k}} B_{p-k} \pmod{p}, \\
 & \sum_{\substack{x=1 \\ x \equiv p \pmod{6}}}^{p-1} \frac{1}{x^k} \equiv \frac{(2^k - 1)(3^k - 1) - 2}{2k \cdot 6^k} B_{p-k} \pmod{p}.
 \end{aligned}$$

**Corollary 3.6.** Let  $p > 5$  be a prime and  $k \in \{3, 5, \dots, p-4\}$ . Then

$$\sum_{\substack{x=1 \\ x \equiv -p \pmod{4}}}^{p-1} \frac{1}{x^k} \equiv \frac{2^{2k-1} - 3 \cdot 2^{k-1} + 1}{2^{2k}} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}.$$

**Proof.** From [S3, Theorems 5.1 and 5.2] we see that

$$\sum_{\substack{x=1 \\ 2 \nmid x}}^{p-1} \frac{1}{x^k} = \sum_{x=1}^{p-1} \frac{1}{x^k} - \frac{1}{2^k} \sum_{x=1}^{(p-1)/2} \frac{1}{x^k} \equiv -\frac{2^k-2}{2^k} \left( 2 \frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k} \right) \pmod{p^2}.$$

Thus

$$\sum_{\substack{x=1 \\ x \equiv p \pmod{4}}}^{p-1} \frac{1}{x^k} + \sum_{\substack{x=1 \\ x \equiv -p \pmod{4}}}^{p-1} \frac{1}{x^k} \equiv \frac{2^k-2}{2^k} \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) \pmod{p^2}.$$

Now applying Theorem 3.4 we deduce the result.  $\square$

**Theorem 3.5.** Let  $p > 5$  be a prime and  $k \in \{2, 4, \dots, p-5\}$ . Then

$$\begin{aligned} \sum_{1 \leq x < \frac{p}{4}} \frac{1}{x^k} &\equiv (-1)^{\frac{p-1}{2}} 4^{k-1} (2E_{p-1-k} - E_{2p-2-k}) \\ &\quad + \frac{2^{k-2}(2^{k+1}-1)k}{k+1} p B_{p-1-k} \pmod{p^2} \end{aligned}$$

and

$$\begin{aligned} \sum_{\frac{p}{4} < x < \frac{p}{2}} \frac{1}{x^k} &\equiv (-1)^{\frac{p+1}{2}} 4^{k-1} (2E_{p-1-k} - E_{2p-2-k}) \\ &\quad - \frac{k(2^{k-1}-1)(2^{k+1}-1)}{2(k+1)} p B_{p-1-k} \pmod{p^2}. \end{aligned}$$

**Proof.** As  $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$ , putting  $m = 4$  in Theorem 3.3 we see that

$$\sum_{x=1}^{[p/4]} \frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}} \left( \frac{B_{2p-1-k}(\frac{1}{4})}{2p-1-k} - 2 \frac{B_{p-k}(\frac{1}{4})}{p-k} \right) + \frac{kp}{4(k+1)} B_{p-1-k} \left( \frac{1}{4} \right) \pmod{p^2}.$$

From Lemmas 2.4 and 2.5 we have



$$\begin{aligned}\frac{B_{2p-1-k}(\frac{1}{4})}{2p-1-k} &= -\frac{4^{k-1}E_{2p-2-k}}{4^{2(p-1)}} \equiv -4^{k-1}(1-2pq_p(4))E_{2p-2-k} \pmod{p^2}, \\ \frac{B_{p-k}(\frac{1}{4})}{p-k} &= -\frac{4^{k-1}E_{p-1-k}}{4^{p-1}} \equiv -4^{k-1}(1-pq_p(4))E_{p-1-k} \pmod{p^2}, \\ B_{p-1-k}\left(\frac{1}{4}\right) &= \frac{2-2^{p-1-k}}{4^{p-1-k}}B_{p-1-k} \equiv 2^k(2^{k+1}-1)B_{p-1-k} \pmod{p}.\end{aligned}$$

As  $B_{2p-1-k}(\frac{1}{4})/(2p-1-k) \equiv B_{p-k}(\frac{1}{4})/(p-k) \pmod{p}$ , we see that  $E_{2p-2-k} \equiv E_{p-1-k} \pmod{p}$  and so

$$\begin{aligned}\sum_{x=1}^{[p/4]} \frac{1}{x^k} &\equiv (-1)^{\frac{p-1}{2}}(-4^{k-1})\{(1-2pq_p(4))E_{2p-2-k} - 2(1-pq_p(4))E_{p-1-k}\} \\ &\quad + \frac{kp}{4(k+1)} \cdot 2^k(2^{k+1}-1)B_{p-1-k} \\ &\equiv (-1)^{\frac{p-1}{2}}4^{k-1}(2E_{p-1-k} - E_{2p-2-k}) + \frac{2^{k-2}(2^{k+1}-1)k}{k+1}pB_{p-1-k} \pmod{p^2}.\end{aligned}$$

By [S3, Corollary 5.2(a)], we have

$$\sum_{1 \leq x < \frac{p}{2}} \frac{1}{x^k} \equiv \frac{k(2^{k+1}-1)}{2(k+1)}pB_{p-1-k} \pmod{p^2}.$$

Note that

$$\sum_{\frac{p}{4} < x < \frac{p}{2}} \frac{1}{x^k} = \sum_{1 \leq x < \frac{p}{2}} \frac{1}{x^k} - \sum_{1 \leq x < \frac{p}{4}} \frac{1}{x^k}.$$

By the above we obtain the remaining result.  $\square$

**Corollary 3.7.** (Lehmer [L, (20)].) Let  $p > 5$  be a prime and  $k \in \{2, 4, \dots, p-5\}$ . Then

$$\sum_{x=1}^{[p/4]} \frac{1}{x^k} \equiv (-1)^{\frac{p-1}{2}}4^{k-1}E_{p-1-k} \pmod{p}.$$

**Corollary 3.8.** Let  $p > 5$  be a prime. Then

$$\sum_{x=1}^{[p/4]} \frac{1}{x^2} \equiv (-1)^{\frac{p-1}{2}}(8E_{p-3} - 4E_{2p-4}) + \frac{14}{3}pB_{p-3} \pmod{p^2}.$$

**Theorem 3.6.** Let  $p > 5$  be a prime and  $k \in \{3, 5, \dots, p-4\}$ . Then

$$\sum_{1 \leq x < \frac{p}{4}} \frac{1}{x^k} \equiv (2^{2k-1} - 2^{k-1} - 1) \left( 2 \frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k} \right) \\ - (-1)^{\frac{p-1}{2}} 4^{k-1} k p E_{p-2-k} \pmod{p^2}$$

and

$$\sum_{\frac{p}{4} < x < \frac{p}{2}} \frac{1}{x^k} \equiv -(2^{2k-1} - 3 \cdot 2^{k-1} + 1) \left( 2 \frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k} \right) \\ + (-1)^{\frac{p-1}{2}} 4^{k-1} k p E_{p-2-k} \pmod{p^2}.$$

**Proof.** As  $B_n(\frac{3}{4}) = (-1)^n B_n(\frac{1}{4})$ , putting  $m = 4$  in Theorem 3.3 we see that

$$\sum_{1 \leq x < \frac{p}{4}} \frac{1}{x^k} \equiv \frac{B_{2p-1-k} - B_{2p-1-k}(\frac{1}{4})}{2p-1-k} - 2 \frac{B_{p-k} - B_{p-k}(\frac{1}{4})}{p-k} \\ + \frac{k p}{4(k+1)} \cdot (-1)^{\frac{p+1}{2}} B_{p-1-k} \left( \frac{1}{4} \right) \pmod{p^2}.$$

According to Lemmas 2.4 and 2.5 we have

$$B_{2n} - B_{2n} \left( \frac{1}{4} \right) = (1 + 2^{-2n} - 2^{1-4n}) B_{2n} \quad \text{and} \quad B_{2n+1} \left( \frac{1}{4} \right) = -\frac{2n+1}{4^{2n+1}} E_{2n}.$$

Thus,

$$\sum_{1 \leq x < \frac{p}{4}} \frac{1}{x^k} \\ \equiv (1 + 2^{k-1-2(p-1)} - 2^{2k-1-4(p-1)}) \frac{B_{2p-1-k}}{2p-1-k} \\ - 2(1 + 2^{k-1-(p-1)} - 2^{2k-1-2(p-1)}) \frac{B_{p-k}}{p-k} \\ + \frac{k p}{4(k+1)} \cdot (-1)^{\frac{p+1}{2}} (-4^{k-(p-1)} (p-1-k)) E_{p-2-k} \\ \equiv (1 + 2^{k-1}(1 - 2pq_p(2)) - 2^{2k-1}(1 - 4pq_p(2))) \frac{B_{2p-1-k}}{2p-1-k} \\ - 2(1 + 2^{k-1}(1 - pq_p(2)) - 2^{2k-1}(1 - 2pq_p(2))) \frac{B_{p-k}}{p-k} + (-1)^{\frac{p+1}{2}} 4^{k-1} k p E_{p-2-k} \\ = (1 + 2^{k-1} - 2^{2k-1}) \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right)$$

$$\begin{aligned}
& + (2^{2k+1} - 2^k)q_p(2)p \left( \frac{B_{2p-1-k}}{2p-1-k} - \frac{B_{p-k}}{p-k} \right) + (-1)^{\frac{p+1}{2}} 4^{k-1}kpE_{p-2-k} \\
& \equiv (1 + 2^{k-1} - 2^{2k-1}) \left( \frac{B_{2p-1-k}}{2p-1-k} - 2 \frac{B_{p-k}}{p-k} \right) + (-1)^{\frac{p+1}{2}} 4^{k-1}kpE_{p-2-k} \pmod{p^2}.
\end{aligned}$$

From [S3, Theorem 5.2(b)] we have

$$\sum_{1 \leq x < \frac{p}{2}} \frac{1}{x^k} \equiv (2^k - 2) \left( 2 \frac{B_{p-k}}{p-k} - \frac{B_{2p-1-k}}{2p-1-k} \right) \pmod{p^2}.$$

Now combining the above we deduce the result.  $\square$

**Theorem 3.7.** *Let  $p > 5$  be a prime. Then*

$$\begin{aligned}
\sum_{\frac{p}{2} < k < \frac{3p}{4}} \frac{1}{k} & \equiv -q_p(2) + p \left( \frac{1}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(6E_{p-3} - 3E_{2p-4}) \right) \\
& - \frac{1}{3}p^2(q_p(2)^3 + 14B_{p-3}) \pmod{p^3}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\frac{3p}{4} < k < p} \frac{1}{k} & \equiv 3q_p(2) - p \left( \frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(6E_{p-3} - 3E_{2p-4}) \right) \\
& + p^2 \left( q_p(2)^3 + \frac{59}{12}B_{p-3} \right) \pmod{p^3}.
\end{aligned}$$

**Proof.** It is clear that

$$\begin{aligned}
\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} + \sum_{\frac{3p}{4} < k < p} \frac{1}{k} & = \sum_{1 \leq k < \frac{p}{4}} \left( \frac{1}{k} + \frac{1}{p-k} \right) = \sum_{1 \leq k < \frac{p}{4}} \frac{p}{k(p-k)} \\
& = p \sum_{1 \leq k < \frac{p}{4}} \frac{p+k}{kp^2 - k^3} \equiv p \sum_{1 \leq k < \frac{p}{4}} \frac{p+k}{-k^3} \\
& = -p^2 \sum_{1 \leq k < \frac{p}{4}} \frac{1}{k^3} - p \sum_{1 \leq k < \frac{p}{4}} \frac{1}{k^2} \pmod{p^3}.
\end{aligned}$$

By Corollaries 3.4 and 3.8 we have

$$\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k^3} \equiv -9B_{p-3} \pmod{p}$$

and

$$\sum_{1 \leq k < \frac{p}{4}} \frac{1}{k^2} \equiv 4(-1)^{\frac{p-1}{2}} (2E_{p-3} - E_{2p-4}) + \frac{14}{3} p B_{p-3} \pmod{p^2}.$$

Thus

$$\begin{aligned} \sum_{1 \leq k < \frac{p}{4}} \frac{1}{k} + \sum_{\frac{3p}{4} < k < p} \frac{1}{k} \\ &\equiv -p^2(-9B_{p-3}) - p \left( 4(-1)^{\frac{p-1}{2}} (2E_{p-3} - E_{2p-4}) + \frac{14}{3} p B_{p-3} \right) \\ &= -4(-1)^{\frac{p-1}{2}} (2E_{p-3} - E_{2p-4}) p + \frac{13}{3} p^2 B_{p-3} \pmod{p^3}. \end{aligned}$$

Hence applying Theorem 3.2 we obtain

$$\begin{aligned} \sum_{\frac{3p}{4} < k < p} \frac{1}{k} &\equiv -4(-1)^{\frac{p-1}{2}} (2E_{p-3} - E_{2p-4}) p + \frac{13}{3} p^2 B_{p-3} + 3q_p(2) \\ &\quad - p \left( \frac{3}{2} q_p(2)^2 - (-1)^{\frac{p-1}{2}} (2E_{p-3} - E_{2p-4}) \right) + p^2 \left( q_p(2)^3 + \frac{7}{12} B_{p-3} \right) \\ &= 3q_p(2) - p \left( \frac{3}{2} q_p(2)^2 + (-1)^{\frac{p-1}{2}} (6E_{p-3} - 3E_{2p-4}) \right) \\ &\quad + p^2 \left( q_p(2)^3 + \frac{59}{12} B_{p-3} \right) \pmod{p^3}. \end{aligned}$$

From [L, p. 353] or [S3, Theorem 5.1(a)] we have

$$\sum_{1 \leq k < p} \frac{1}{k} \equiv -\frac{1}{3} p^2 B_{p-3} \pmod{p^3}.$$

By [S3, Theorem 5.2(c)],

$$\sum_{1 \leq k < \frac{p}{2}} \frac{1}{k} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3} p^2 q_p(2)^3 - \frac{7}{12} p^2 B_{p-3} \pmod{p^3}.$$

Thus

$$\begin{aligned} \sum_{\frac{p}{2} < k < p} \frac{1}{k} &= \sum_{1 \leq k < p} \frac{1}{k} - \sum_{1 \leq k < \frac{p}{2}} \frac{1}{k} \\ &\equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3} p^2 q_p(2)^3 + \frac{1}{4} p^2 B_{p-3} \pmod{p^3}. \end{aligned}$$

Observing that

$$\sum_{\frac{p}{2} < k < \frac{3p}{4}} \frac{1}{k} = \sum_{\frac{p}{2} < k < p} \frac{1}{k} - \sum_{\frac{3p}{4} < k < p} \frac{1}{k}$$

and applying the above we obtain the remaining result.  $\square$

**Corollary 3.9.** *Let  $p > 5$  be a prime. Then*

$$\sum_{\frac{p}{2} < k < \frac{3p}{4}} \frac{1}{k} \equiv -q_p(2) + \frac{1}{2}pq_p(2)^2 + 3(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{\frac{3p}{4} < k < p} \frac{1}{k} \equiv 3q_p(2) - \frac{3}{2}pq_p(2)^2 - 3(-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}.$$

**Theorem 3.8.** *Let  $p > 5$  be a prime. Then*

$$(-1)^{\lfloor \frac{p}{4} \rfloor} \binom{p-1}{\lfloor \frac{p}{4} \rfloor} \equiv 1 + 3pq_p(2) + p^2(3q_p(2)^2 - (-1)^{\frac{p-1}{2}}E_{p-3}) \pmod{p^3}$$

and

$$\begin{aligned} (-1)^{\lfloor \frac{p}{4} \rfloor} \binom{p-1}{\lfloor \frac{p}{4} \rfloor} &\equiv 1 + 3pq_p(2) + p^2(3q_p(2)^2 - (-1)^{\frac{p-1}{2}}(2E_{p-3} - E_{2p-4})) \\ &\quad + p^3\left(q_p(2)^3 - 3(-1)^{\frac{p-1}{2}}q_p(2)E_{p-3} + \frac{5}{4}B_{p-3}\right) \pmod{p^4}. \end{aligned}$$

**Proof.** From Theorem 3.2 we have

$$\begin{aligned} \sum_{i=1}^{\lfloor p/4 \rfloor} \frac{1}{i} &\equiv -3q_p(2) + p\left(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p-1}{2}}(E_{2p-4} - 2E_{p-3})\right) \\ &\quad - p^2\left(q_p(2)^3 + \frac{7}{12}B_{p-3}\right) \pmod{p^3}. \end{aligned}$$

By Corollaries 3.3 and 3.8 we have

$$\sum_{i=1}^{\lfloor p/4 \rfloor} \frac{1}{i} \equiv -3q_p(2) + \frac{3}{2}pq_p(2)^2 - (-1)^{\frac{p-1}{2}}pE_{p-3} \pmod{p^2}$$

and

$$\sum_{i=1}^{\lfloor p/4 \rfloor} \frac{1}{i^2} \equiv (-1)^{\frac{p-1}{2}}(8E_{p-3} - 4E_{2p-4}) + \frac{14}{3}pB_{p-3} \pmod{p^2}.$$

From this we deduce

$$\begin{aligned} \left( \sum_{i=1}^{[p/4]} \frac{1}{i} \right)^2 - \sum_{i=1}^{[p/4]} \frac{1}{i^2} &\equiv 9q_p(2)^2 - (-1)^{\frac{p-1}{2}} (8E_{p-3} - 4E_{2p-4}) \\ &\quad + p \left( -9q_p(2)^3 + 6(-1)^{\frac{p-1}{2}} q_p(2)E_{p-3} - \frac{14}{3}B_{p-3} \right) \pmod{p^2}. \end{aligned}$$

By Corollary 3.4 we have  $\sum_{i=1}^{[p/4]} \frac{1}{i^3} \equiv -9B_{p-3} \pmod{p}$ . Thus,

$$\begin{aligned} \left( \sum_{i=1}^{[p/4]} \frac{1}{i} \right)^3 - 3 \left( \sum_{i=1}^{[p/4]} \frac{1}{i} \right) \left( \sum_{i=1}^{[p/4]} \frac{1}{i^2} \right) + 2 \sum_{i=1}^{[p/4]} \frac{1}{i^3} \\ \equiv (-3q_p(2))^3 - 3(-3q_p(2)) \cdot 4(-1)^{\frac{p-1}{2}} E_{p-3} + 2(-9B_{p-3}) \\ = 6 \left( -\frac{9}{2}q_p(2)^3 + 6(-1)^{\frac{p-1}{2}} q_p(2)E_{p-3} - 3B_{p-3} \right) \pmod{p}. \end{aligned}$$

Now putting all the above together with Lemma 2.9 and the fact  $E_{2p-4} \equiv E_{p-3} \pmod{p}$  yields the result.  $\square$

**Remark 3.3.** The congruence  $(-1)^{[\frac{p}{4}]} \binom{p-1}{[\frac{p}{4}]} \equiv 1 + 3pq_p(2) \pmod{p^2}$  was known to Lehmer. See [L, (51)].

For any prime  $p > 3$  we recall the Legendre symbol

$$\left( \frac{p}{3} \right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3}, \\ -1 & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

**Theorem 3.9.** Let  $p$  be a prime greater than 5. Then

- (i)  $\sum_{k=1}^{[p/6]} \frac{1}{k^2} \equiv 5 \sum_{k=1}^{[p/3]} \frac{1}{k^2} \equiv \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p}$ .
- (ii)  $\sum_{k=1}^{[p/6]} \frac{1}{k} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p(q_p(2)^2 + \frac{3}{4}q_p(3)^2) - \frac{p}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^2}$ .
- (iii)  $\sum_{k=1}^{[p/3]} \frac{1}{k} \equiv -\frac{3}{2}q_p(3) + \frac{3}{4}pq_p(3)^2 - \frac{p}{30} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^2}$ .
- (iv)  $\sum_{k=1}^{[2p/3]} \frac{(-1)^{k-1}}{k} \equiv 9 \sum_{k=1, 3|k+p}^{p-1} \frac{1}{k} \equiv \frac{p}{10} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^2}$ .
- (v) We have

$$\begin{aligned} (-1)^{[\frac{p}{6}]} \binom{p-1}{[\frac{p}{6}]} &\equiv 1 + p \left( 2q_p(2) + \frac{3}{2}q_p(3) \right) + p^2 \left( q_p(2)^2 + 3q_p(2)q_p(3) \right. \\ &\quad \left. + \frac{3}{8}q_p(3)^2 - \frac{1}{6} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) \pmod{p^3} \end{aligned}$$

and

$$(-1)^{\lfloor \frac{p}{3} \rfloor} \binom{p-1}{\lfloor \frac{p}{3} \rfloor} \equiv 1 + \frac{3}{2} p q_p(3) + \frac{3}{8} p^2 q_p(3)^2 - \frac{p^2}{60} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^3}.$$

**Proof.** Taking  $k = 2$  and  $m = 3, 6$  in Theorem 3.3 and using Lemma 2.6 we see that

$$\sum_{k=1}^{\lfloor p/6 \rfloor} \frac{1}{k^2} \equiv \frac{B_{p-2}(\{\frac{-p}{6}\})}{p-2} = -\left(\frac{p}{3}\right) \frac{B_{p-2}(\frac{1}{6})}{p-2} \equiv \frac{1}{2} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{6}\right) \pmod{p}$$

and

$$\sum_{k=1}^{\lfloor p/3 \rfloor} \frac{1}{k^2} \equiv \frac{B_{p-2}(\{\frac{-p}{3}\})}{p-2} = -\left(\frac{p}{3}\right) \frac{B_{p-2}(\frac{1}{3})}{p-2} \equiv \frac{1}{2} \left(\frac{p}{3}\right) B_{p-2} \left(\frac{1}{3}\right) \pmod{p}.$$

By Raabe's theorem (cf. [S2, Lemma 2.2]) we have  $B_{p-2}(\frac{1}{6}) + B_{p-2}(\frac{1}{6} + \frac{1}{2}) = 2^{1-(p-2)} B_{p-2}(\frac{1}{3})$ . Thus

$$B_{p-2} \left( \frac{1}{6} \right) = 2^{3-p} B_{p-2} \left( \frac{1}{3} \right) - B_{p-2} \left( \frac{2}{3} \right) = (2^{3-p} + 1) B_{p-2} \left( \frac{1}{3} \right) \equiv 5 B_{p-2} \left( \frac{1}{3} \right) \pmod{p}.$$

Hence (i) holds.

Suppose  $m \in \{3, 6\}$ . Taking  $k = 1$ ,  $r = 0$  and  $m = 3, 6$  in Lemma 2.7 and using Lemma 2.6 we see that

$$\begin{aligned} \sum_{k=1}^{\lfloor \frac{p}{m} \rfloor} \frac{1}{k} &= m \sum_{\substack{x=1 \\ x \equiv 0 \pmod{m}}}^{p-1} \frac{1}{x} \equiv \frac{B_{\varphi(p^3)}(\{\frac{-p}{m}\}) - B_{\varphi(p^3)}}{\varphi(p^3)} - \frac{p}{m} \cdot \frac{B_{p-2}(\{\frac{-p}{m}\})}{p-2} \\ &= \frac{B_{\varphi(p^3)}(\frac{1}{m}) - B_{\varphi(p^3)}}{\varphi(p^3)} + \frac{p}{m} \left( \frac{p}{3} \right) \frac{B_{p-2}(\frac{1}{m})}{p-2} \pmod{p^2}. \end{aligned}$$

From the proof of Theorem 3.1 we have

$$\frac{B_{\varphi(p^3)} - B_{\varphi(p^3)}(\frac{1}{m})}{\varphi(p^3)} \equiv \begin{cases} \frac{3}{2} q_p(3) - \frac{3}{4} p q_p(3)^2 & \pmod{p^2} & \text{if } m = 3, \\ 2 q_p(2) - p q_p(2)^2 + \frac{3}{2} q_p(3) - \frac{3}{4} p q_p(3)^2 & \pmod{p^2} & \text{if } m = 6. \end{cases}$$

Hence (ii) and (iii) follow from the above and the fact  $B_{p-2}(\frac{1}{3}) \equiv \frac{1}{5} B_{p-2}(\frac{1}{6}) \pmod{p}$ .

Now we consider (iv). Since

$$\sum_{1 \leq k < \frac{2p}{3}} \frac{1 - (-1)^{k-1}}{k} = \sum_{\substack{1 \leq k < \frac{2p}{3} \\ 2|k}} \frac{2}{k} = \sum_{1 \leq k < \frac{p}{3}} \frac{1}{k} \quad \text{and} \quad \sum_{1 \leq k < \frac{p}{2}} \frac{1}{k^2} \equiv 0 \pmod{p}$$

by [S3, Corollary 5.2], using (i) we see that

$$\begin{aligned}
\sum_{1 \leq k < \frac{2p}{3}} \frac{(-1)^{k-1}}{k} &= \sum_{1 \leq k < \frac{2p}{3}} \frac{1}{k} - \sum_{1 \leq k < \frac{p}{3}} \frac{1}{k} = \sum_{\frac{p}{3} < k < \frac{2p}{3}} \frac{1}{k} \\
&= \sum_{\frac{p}{3} < k < \frac{p}{2}} \left( \frac{1}{k} + \frac{1}{p-k} \right) = \sum_{\frac{p}{3} < k < \frac{p}{2}} \frac{p}{k(p-k)} \\
&\equiv -p \sum_{\frac{p}{3} < k < \frac{p}{2}} \frac{1}{k^2} \equiv p \left( \sum_{1 \leq k < \frac{p}{2}} \frac{1}{k^2} - \sum_{\frac{p}{3} < k < \frac{p}{2}} \frac{1}{k^2} \right) \\
&= p \sum_{1 \leq k < \frac{p}{3}} \frac{1}{k^2} \equiv \frac{p}{10} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^2}.
\end{aligned}$$

On the other hand, noting that  $\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$  (cf. [L,S3]) and using (iii) and Theorem 3.1(i) we see that

$$\begin{aligned}
\sum_{\substack{k=1 \\ 3|k+p}}^{p-1} \frac{1}{k} &= \sum_{k=1}^{p-1} \frac{1}{k} - \sum_{\substack{k=1 \\ k \equiv 0 \pmod{3}}}^{p-1} \frac{1}{k} - \sum_{\substack{k=1 \\ k \equiv p \pmod{3}}}^{p-1} \frac{1}{k} \\
&\equiv -\frac{1}{3} \sum_{k=1}^{\lfloor \frac{p}{3} \rfloor} \frac{1}{k} - \sum_{\substack{k=1 \\ k \equiv p \pmod{3}}}^{p-1} \frac{1}{k} \\
&\equiv -\frac{1}{3} \left( -\frac{3}{2} q_p(3) + \frac{3}{4} p q_p(3)^2 - \frac{p}{30} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) - \left( \frac{1}{2} q_p(3) - \frac{1}{4} p q_p(3)^2 \right) \\
&= \frac{p}{90} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \pmod{p^2}.
\end{aligned}$$

Thus (iv) is true. Finally we consider (v). By Lemma 2.9, for  $m = 3, 6$  we have

$$(-1)^{\lfloor \frac{p}{m} \rfloor} \binom{p-1}{\lfloor \frac{p}{m} \rfloor} \equiv 1 - p \sum_{k=1}^{\lfloor \frac{p}{m} \rfloor} \frac{1}{k} + \frac{p^2}{2} \left( \left( \sum_{k=1}^{\lfloor \frac{p}{m} \rfloor} \frac{1}{k} \right)^2 - \sum_{k=1}^{\lfloor \frac{p}{m} \rfloor} \frac{1}{k^2} \right) \pmod{p^3}.$$

Thus appealing to (i)–(iii) we obtain

$$\begin{aligned}
&(-1)^{\lfloor \frac{p}{6} \rfloor} \binom{p-1}{\lfloor \frac{p}{6} \rfloor} \\
&\equiv 1 - p \left( -2q_p(2) - \frac{3}{2} q_p(3) + p \left( q_p(2)^2 + \frac{3}{4} q_p(3)^2 \right) - \frac{p}{12} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) \\
&\quad + \frac{p^2}{2} \left( \left( -2q_p(2) - \frac{3}{2} q_p(3) \right)^2 - \frac{1}{2} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) \pmod{p^3}
\end{aligned}$$

and



$$\begin{aligned}
(-1)^{[\frac{p}{3}]} \binom{p-1}{[\frac{p}{3}]} &\equiv 1 - p \left( -\frac{3}{2} q_p(3) + \frac{3}{4} p q_p(3)^2 - \frac{p}{30} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) \\
&\quad + \frac{p^2}{2} \left( \left( -\frac{3}{2} q_p(3) \right)^2 - \frac{1}{10} \left( \frac{p}{3} \right) B_{p-2} \left( \frac{1}{6} \right) \right) \pmod{p^3}.
\end{aligned}$$

This yields (v) and hence the theorem is proved.  $\square$

**Remark 3.4.** Let  $p > 5$  be a prime. The congruences

$$\begin{aligned}
\sum_{k=1}^{[p/3]} \frac{1}{k} &\equiv -\frac{3}{2} q_p(3) \pmod{p}, & \sum_{k=1}^{[p/6]} \frac{1}{k} &\equiv -2q_p(2) - \frac{3}{2} q_p(3) \pmod{p}, \\
(-1)^{[p/3]} \binom{p-1}{[p/3]} &\equiv 1 + \frac{3}{2} p q_p(3) \pmod{p^2}, \\
(-1)^{[p/6]} \binom{p-1}{[p/6]} &\equiv 1 + 2p q_p(2) + \frac{3}{2} p q_p(3) \pmod{p^2}
\end{aligned}$$

were known to Lehmer [L], and the congruence

$$\sum_{k=1}^{[p/3]} \frac{1}{k^2} \equiv \frac{1}{5} \sum_{k=1}^{[p/6]} \frac{1}{k^2} \pmod{p}$$

is due to Schwindt (cf. [R,L]). In [S1], using the formulas for  $\sum_{k \equiv r \pmod{3}} \binom{p}{k}$  the author proved that

$$\sum_{k=1}^{[\frac{2p}{3}]} \frac{(-1)^{k-1}}{k} \equiv 0 \pmod{p}.$$

**Corollary 3.10.** Let  $p > 5$  be a prime. Then

- (i)  $\sum_{k=1}^{[p/6]} \frac{1}{k} + \frac{p}{6} \sum_{k=1}^{[p/6]} \frac{1}{k^2} \equiv -2q_p(2) - \frac{3}{2} q_p(3) + p(q_p(2)^2 + \frac{3}{4} q_p(3)^2) \pmod{p^2}.$
- (ii)  $\sum_{k=1}^{[p/3]} \frac{1}{k} + \frac{p}{3} \sum_{k=1}^{[p/3]} \frac{1}{k^2} \equiv -\frac{3}{2} q_p(3) + \frac{3}{4} p q_p(3)^2 \pmod{p^2}.$
- (iii) We have

$$\begin{aligned}
&(-1)^{[\frac{p}{6}]} \binom{p-1}{[\frac{p}{6}]} - 10(-1)^{[\frac{p}{3}]} \binom{p-1}{[\frac{p}{3}]} \\
&\equiv -9 + p \left( 2q_p(2) - \frac{27}{2} q_p(3) \right) + p^2 \left( q_p(2)^2 + 3q_p(2)q_p(3) - \frac{27}{8} q_p(3)^2 \right) \pmod{p^3}.
\end{aligned}$$

**Corollary 3.11.** *Let  $p > 5$  be a prime. Then*

- (i) 
$$\sum_{1 \leq k < \frac{2p}{3}} \frac{(-1)^{k-1}}{k} \equiv 9 \sum_{\substack{1 \leq k < p \\ 3|k+p}} \frac{1}{k} \equiv \frac{p}{5} \sum_{k=1}^{\lfloor \frac{p}{6} \rfloor} \frac{1}{k^2} \pmod{p^2},$$
- (ii) 
$$\sum_{1 \leq k < \frac{2p}{3}} \frac{(-1)^{k-1}}{k} + 3 \sum_{1 \leq k < \frac{p}{3}} \frac{1}{k} \equiv -\frac{9}{2} q_p(3) + \frac{9}{4} p q_p(3)^2 \pmod{p^2},$$
- (iii) 
$$\sum_{1 \leq k < \frac{2p}{3}} \frac{(-1)^{k-1}}{k} - 2 \sum_{\frac{p}{6} < k < \frac{p}{3}} \frac{1}{k} \equiv -4 q_p(2) + 2 p q_p(2)^2 \pmod{p^2},$$
- (iv) 
$$3 \sum_{1 \leq k < \frac{p}{6}} \frac{1}{k} + 5 \sum_{\frac{p}{6} < k < \frac{p}{3}} \frac{1}{k} \equiv 4 q_p(2) - \frac{9}{2} q_p(3) + p \left( -2 q_p(2)^2 + \frac{9}{4} q_p(3)^2 \right) \pmod{p^2}.$$

#### 4. Congruences for $\sum_{k=1}^{p-1} \frac{2^k}{k}$ and $\sum_{k=1}^{p-1} \frac{2^k}{k^2}$

Let  $p > 3$  be a prime. For  $n \in \mathbb{N}$  let

$$G_n(x) = \sum_{k=1}^{p-1} \frac{x^k}{k^n}.$$

Then  $G_n(x) \in \mathbb{Z}_p[x]$ . In [Gl] Glaisher showed that

$$G_1(2) = \sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2 q_p(2) \pmod{p}. \quad (4.1)$$

In 2004 Granville [Gr] proved the following Skula's conjecture:

$$G_2(2) = \sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -q_p(2)^2 \pmod{p}. \quad (4.2)$$

In the section we determine  $G_1(2) \pmod{p^3}$  and  $G_2(2) \pmod{p^2}$ .

**Lemma 4.1.** *Let  $p$  be an odd prime. In  $\mathbb{Z}_p[x]$  we have*

$$G_2(x) \equiv \frac{1}{p} \left( \frac{1 + (x-1)^p - x^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k} \right) + p \sum_{r=2}^{p-1} \frac{x^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2}.$$

**Proof.** From Lemma 2.9 we know that

$$(-1)^k \binom{p-1}{k} \equiv 1 - p \sum_{s=1}^k \frac{1}{s} \pmod{p^2}$$

for  $k \leq p-1$ . Thus

$$\sum_{r=1}^{p-1} \frac{x^r}{r^2} \left( 1 - p \sum_{s=1}^{r-1} \frac{1}{s} \right) \equiv \sum_{r=1}^{p-1} \frac{x^r}{r^2} (-1)^{r-1} \binom{p-1}{r-1} = \frac{1}{p} \sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r} \binom{p}{r} x^r \pmod{p^2}.$$

Since  $\int_0^1 t^{r-1} dt = \frac{1}{r}$ , setting  $y = 1 - xt$  we see that

$$\begin{aligned} & \sum_{r=1}^{p-1} \frac{(-1)^{r-1}}{r} \binom{p}{r} x^r \\ &= \int_0^1 \sum_{r=1}^{p-1} (-1)^{r-1} \binom{p}{r} x^r t^{r-1} dt = \int_0^1 \frac{(1-xt)^p - 1 - (-xt)^p}{-t} dt \\ &= - \int_1^{1-x} \frac{y^p - 1 - (y-1)^p}{y-1} dy = \int_1^{1-x} \left( (y-1)^{p-1} - \sum_{k=1}^p y^{k-1} \right) dy \\ &= \left( \frac{(y-1)^p}{p} - \sum_{k=1}^p \frac{y^k}{k} \right) \Big|_1^{1-x} = -\frac{x^p}{p} - \sum_{k=1}^p \frac{(1-x)^k}{k} + \sum_{k=1}^p \frac{1}{k} \\ &= \frac{1-x^p + (x-1)^p}{p} - \sum_{k=1}^{p-1} \frac{(1-x)^k - 1}{k}. \end{aligned}$$

Now combining the above we obtain the result.  $\square$

**Lemma 4.2.** Let  $p > 3$  be a prime. Then

$$G_2(2) \equiv -q_p(2)^2 + p \left( \sum_{r=2}^{p-1} \frac{2^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} + \frac{2}{3} q_p(2)^3 - \frac{1}{12} B_{p-3} \right) \pmod{p^2}.$$

**Proof.** From [S3, Remark 5.3] we know that

$$\sum_{k=1}^{p-1} \frac{1 - (-1)^k}{k} = 2 \sum_{\substack{k=1 \\ 2 \nmid k}}^{p-1} \frac{1}{k} \equiv 2q_p(2) - pq_p(2)^2 + \frac{2}{3} p^2 q_p(2)^3 - \frac{1}{12} p^2 B_{p-3} \pmod{p^3}.$$

Thus putting  $x = 2$  in Lemma 4.1 and applying the above gives the result.  $\square$

**Lemma 4.3.** Let  $p > 3$  be a prime and  $n \in \mathbb{N}$ . Then

$$npG_{n+1}(x) \equiv (-1)^n x^p G_n(1/x) - G_n(x) \pmod{p^2}.$$

**Proof.** Clearly we have

$$\begin{aligned} x^p G_n\left(\frac{1}{x}\right) &= \sum_{i=1}^{p-1} \frac{x^p}{x^i \cdot i^n} = \sum_{k=1}^{p-1} \frac{x^k}{(p-k)^n} \equiv \sum_{k=1}^{p-1} \frac{x^k}{(-k)^n + n(-k)^{n-1}p} \\ &= \sum_{k=1}^{p-1} \frac{(k+np)x^k}{(-k)^{n-1}(n^2 p^2 - k^2)} \equiv (-1)^n \sum_{k=1}^{p-1} \frac{(k+np)x^k}{k^{n+1}} \\ &= (-1)^n (npG_{n+1}(x) + G_n(x)) \pmod{p^2}. \end{aligned}$$

This yields the result.  $\square$

**Theorem 4.1.** Let  $p > 3$  be a prime. Then

- (i) 
$$\sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) - \frac{7}{12}p^2 B_{p-3} \pmod{p^3}.$$
- (ii) 
$$\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -q_p(2)^2 + p\left(\frac{2}{3}q_p(2)^3 + \frac{7}{6}B_{p-3}\right) \pmod{p^2}.$$
- (iii) 
$$\sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \equiv q_p(2) - \frac{p}{2}q_p(2)^2 \pmod{p^2}.$$
- (iv) 
$$\sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} \equiv -\frac{1}{2}q_p(2)^2 \pmod{p}.$$

**Proof.** Suppose  $x \in \mathbb{Z}_p$  and  $x \not\equiv 0 \pmod{p}$ . From Lemma 4.1 we see that

$$\begin{aligned} x^p G_2(1/x) &\equiv \frac{1}{p} \left( \frac{x^p + (1-x)^p - 1}{p} + \sum_{k=1}^{p-1} \frac{x^k - (x-1)^k}{k} \cdot x^{p-k} \right) \\ &\quad + p \sum_{r=1}^{p-1} \frac{x^{p-r}}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2}. \end{aligned}$$

As

$$\sum_{s=1}^{p-1-k} \frac{1}{s} = \sum_{s=1}^{p-1} \frac{1}{s} - \sum_{r=1}^k \frac{1}{p-r} \equiv 0 - \sum_{r=1}^k \frac{1}{p-r} \equiv \sum_{r=1}^k \frac{1}{r} \pmod{p}, \quad (4.3)$$

we see that

$$\sum_{r=1}^{p-1} \frac{x^{p-r}}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} = \sum_{k=1}^{p-1} \frac{x^k}{(p-k)^2} \sum_{s=1}^{p-k-1} \frac{1}{s} \equiv \sum_{k=1}^{p-1} \frac{x^k}{k^2} \sum_{s=1}^k \frac{1}{s} \pmod{p}.$$

Thus

$$\begin{aligned} x^p G_2(1/x) &\equiv \frac{1}{p} \left( \frac{x^p - (x-1)^p - 1}{p} + \sum_{k=1}^{p-1} \frac{x^p - x^{p-k}(x-1)^k}{k} \right) \\ &\quad + p \sum_{k=1}^{p-1} \frac{x^k}{k^2} \sum_{s=1}^k \frac{1}{s} \pmod{p^2}. \end{aligned}$$

Hence applying Lemmas 4.1 and 4.3 we see that

$$\begin{aligned} 2pG_3(x) &\equiv x^p G_2(1/x) - G_2(x) \\ &\equiv \frac{1}{p} \left\{ \frac{2(x^p - (x-1)^p - 1)}{p} + \sum_{k=1}^{p-1} \left( \frac{x^p - x^{p-k}(x-1)^k}{k} + \frac{(1-x)^k - 1}{k} \right) \right\} \\ &\quad + p \sum_{k=1}^{p-1} \frac{x^k}{k^2} \left( \sum_{s=1}^k \frac{1}{s} - \sum_{s=1}^{k-1} \frac{1}{s} \right) \pmod{p^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} G_3(x) &\equiv \frac{1}{p^2} \left\{ \frac{2(x^p - (x-1)^p - 1)}{p} \right. \\ &\quad \left. + \sum_{k=1}^{p-1} \frac{1}{k} (x^p - 1 + (1-x)^k - x^{p-k}(x-1)^k) \right\} \pmod{p}. \end{aligned} \quad (4.4)$$

Taking  $x = 2$  we find

$$G_3(2) \equiv \frac{1}{p^2} \left\{ 4q_p(2) + (2^p - 2) \sum_{k=1}^{p-1} \frac{1}{k} + \sum_{k=1}^{p-1} \frac{1 + (-1)^k}{k} - 2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \right\} \pmod{p}.$$

It is well known that  $\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$  (cf. [L, p. 353]). From [S3, Theorem 5.2(c)] we also know that

$$\sum_{k=1}^{p-1} \frac{1 + (-1)^k}{k} = \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k} \equiv -2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \pmod{p^3}.$$

Thus

$$G_3(2) \equiv \frac{1}{p^2} \left\{ 2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} - 2^p \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} \right\} \pmod{p}. \quad (4.5)$$

As

$$\begin{aligned} -\sum_{k=1}^{p-1} \frac{2^{p-k}}{k} &= -\sum_{k=1}^{p-1} \frac{2^k}{p-k} = -\sum_{k=1}^{p-1} \frac{2^k(p^2 + kp + k^2)}{p^3 - k^3} \equiv \sum_{k=1}^{p-1} \frac{2^k(p^2 + kp + k^2)}{k^3} \\ &= p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3}, \end{aligned} \quad (4.6)$$

by (4.5) we have

$$\begin{aligned} p^2G_3(2) &\equiv 2q_p(2) + pq_p(2)^2 - \frac{2}{3}p^2q_p(2)^3 - \frac{7}{12}p^2B_{p-3} \\ &\quad + p^2G_3(2) + pG_2(2) + G_1(2) \pmod{p^3}. \end{aligned}$$

Namely,

$$pG_2(2) \equiv -G_1(2) - 2q_p(2) - pq_p(2)^2 + \frac{2}{3}p^2q_p(2)^3 + \frac{7}{12}p^2B_{p-3} \pmod{p^3}. \quad (4.7)$$

According to Lemma 4.1 we have

$$G_2(-1) \equiv \frac{1}{p} \left( -2q_p(2) - \sum_{k=1}^{p-1} \frac{2^k}{k} + \sum_{k=1}^{p-1} \frac{1}{k} \right) + p \sum_{r=1}^{p-1} \frac{(-1)^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} \pmod{p^2}.$$

As  $p > 3$ , by [S3, Corollary 5.2 and Theorem 5.1] or [L] we have

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^3} \equiv -2B_{p-3} \pmod{p} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{1}{k} \equiv -\frac{1}{3}p^2B_{p-3} \pmod{p^3}.$$

Thus applying (4.3) we see that

$$\begin{aligned} \sum_{r=1}^{p-1} \frac{(-1)^r}{r^2} \sum_{s=1}^{r-1} \frac{1}{s} &= \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(2k)^2} \sum_{s=1}^{2k-1} \frac{1}{s} - \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(p-2k)^2} \sum_{s=1}^{p-2k-1} \frac{1}{s} \\ &\equiv \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{(2k)^2} \left( \sum_{s=1}^{2k-1} \frac{1}{s} - \sum_{s=1}^{2k} \frac{1}{s} \right) = -\frac{1}{8} \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k^3} \equiv \frac{1}{4}B_{p-3} \pmod{p} \end{aligned}$$

and hence

$$G_2(-1) \equiv \frac{1}{p} \left( -2q_p(2) - \sum_{k=1}^{p-1} \frac{2^k}{k} - \frac{1}{3}p^2B_{p-3} \right) + \frac{p}{4}B_{p-3} \pmod{p^2}.$$

On the other hand, using [S3, Corollaries 5.1 and 5.2] we have

$$\begin{aligned} G_2(-1) &= \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = \sum_{\substack{k=1 \\ 2 \nmid k}}^{p-1} \frac{2}{k^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} = \frac{1}{2} \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^2} - \sum_{k=1}^{p-1} \frac{1}{k^2} \\ &\equiv \frac{1}{2} \cdot \frac{2(2^3-1)}{2(2+1)} p B_{p-3} - \frac{2}{3} p B_{p-3} = \frac{1}{2} p B_{p-3} \pmod{p^2}. \end{aligned}$$

Hence

$$-2q_p(2) - \sum_{k=1}^{p-1} \frac{2^k}{k} - \frac{1}{3} p^2 B_{p-3} + \frac{p^2}{4} B_{p-3} \equiv p G_2(-1) \equiv \frac{p^2}{2} B_{p-3} \pmod{p^3}.$$

This yields

$$G_1(2) = \sum_{k=1}^{p-1} \frac{2^k}{k} \equiv -2q_p(2) - \frac{7}{12} p^2 B_{p-3} \pmod{p^3}.$$

So (i) holds. Substituting this into (4.7) gives

$$p G_2(2) \equiv -p q_p(2)^2 + \frac{7}{6} p^2 B_{p-3} + \frac{2}{3} p^2 q_p(2)^3 \pmod{p^3}.$$

That is,

$$G_2(2) \equiv -q_p(2)^2 + p \left( \frac{2}{3} q_p(2)^3 + \frac{7}{6} B_{p-3} \right) \pmod{p^2}.$$

Thus (ii) is true. By (4.6) we have

$$\sum_{k=1}^{p-1} \frac{2^{p-k}}{k} \equiv -p G_2(2) - G_1(2) \pmod{p^2}.$$

From the above we know that

$$G_1(2) \equiv -2q_p(2) \pmod{p^2} \quad \text{and} \quad G_2(2) \equiv -q_p(2)^2 \pmod{p}.$$

Thus

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} &\equiv \frac{-G_1(2) - p G_2(2)}{2^p} \equiv \frac{2q_p(2) + p q_p(2)^2}{2(1 + p q_p(2))} \\ &\equiv \left( q_p(2) + \frac{p}{2} q_p(2)^2 \right) (1 - p q_p(2)) \equiv q_p(2) - \frac{p}{2} q_p(2)^2 \pmod{p^2}. \end{aligned}$$

This proves (iii). From Lemma 4.3 we have

$$2pG_3(2) \equiv 2^p G_2(1/2) - G_2(2) \pmod{p^2}.$$

Thus,

$$G_2\left(\frac{1}{2}\right) \equiv \frac{G_2(2)}{2^p} \equiv \frac{-q_p(2)^2}{2} \pmod{p}.$$

This proves (iv) and hence the theorem is proved.  $\square$

**Corollary 4.1.** *Let  $p > 3$  be a prime. Then*

$$q_p(2) \equiv -\sum_{k=1}^{p-1} \frac{2^{k-1}}{k} \pmod{p^2}.$$

**Remark 4.1.** Let  $p > 3$  be a prime. By [DS, (5)] and [S3, Corollary 5.2(b)] we have

$$G_3(2) = \sum_{k=1}^{p-1} \frac{2^k}{k^3} \equiv -\frac{1}{3}q_p(2)^3 + \frac{7}{48} \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^3} \equiv -\frac{1}{3}q_p(2)^3 - \frac{7}{24}B_{p-3} \pmod{p}.$$

This together with (4.5) and Corollary 3.1 yields

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k \cdot 2^k} &\equiv \frac{q_p(2) + \frac{1}{2}pq_p(2)^2 - \frac{1}{6}p^2q_p(2)^3 - \frac{7}{48}p^2B_{p-3}}{1 + pq_p(2)} \\ &\equiv q_p(2) - \frac{1}{2}pq_p(2)^2 + \frac{1}{3}p^2q_p(2)^3 - \frac{7}{48}p^2B_{p-3} \\ &\equiv 4 \sum_{\substack{k=1 \\ 4|k+p}}^{p-1} \frac{1}{k} \pmod{p^3}. \end{aligned}$$

By the proof of Theorem 4.1, we also have

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{1}{k^2 \cdot 2^k} &= G_2\left(\frac{1}{2}\right) \equiv \frac{pG_3(2) + G_2(2)/2}{2^{p-1}} \\ &\equiv (1 - pq_p(2)) \left( -\frac{1}{3}pq_p(2)^3 - \frac{7}{24}pB_{p-3} + \frac{1}{2} \left( -q_p(2)^2 + \frac{2}{3}pq_p(2)^3 + \frac{7}{6}pB_{p-3} \right) \right) \\ &\equiv -\frac{1}{2}q_p(2)^2 + \frac{1}{2}pq_p(2)^3 + \frac{7}{24}pB_{p-3} \pmod{p^2}. \end{aligned}$$



## References

- [D] L.E. Dickson, History of the Theory of Numbers, vol. I, Chelsea Publ., 1999, pp. 105–112.
- [DS] K. Dilcher, L. Skula, The cube of the Fermat quotient, *Integers: Electron. J. Combin. Number Theory* 6 (2006), #A24.
- [Gl] J.W.L. Glaisher, On the residues of the sums of products of the first  $p - 1$  numbers and their powers to modulus  $p^2$  or  $p^3$ , *Q. J. Math. Oxford* 31 (1900) 321–353.
- [Gr] A. Granville, The square of the Fermat quotient, *Integers: Electron. J. Combin. Number Theory* 4 (2004), #A22.
- [GS] A. Granville, Z.W. Sun, Values of Bernoulli polynomials, *Pacific J. Math.* 172 (1996) 117–137.
- [IR] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., Springer-Verlag, New York, 1990, pp. 228–241.
- [L] E. Lehmer, On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson, *Ann. of Math.* 39 (1938) 350–360.
- [MOS] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, 3rd ed., Springer-Verlag, New York, 1966, pp. 25–32.
- [R] P. Ribenboim, *Thirteen Lectures on Fermat's Last Theorem*, Springer-Verlag, New York, 1979, pp. 139–159.
- [S1] Z.H. Sun, Combinatorial sum  $\sum_{k=0}^n, k \equiv r \pmod{m} \binom{n}{k}$  and its applications in number theory, I, *J. Nanjing Univ. Math. Biquarterly* 9 (1992) 227–240.
- [S2] Z.H. Sun, Congruences for Bernoulli numbers and Bernoulli polynomials, *Discrete Math.* 163 (1997) 153–163.
- [S3] Z.H. Sun, Congruences concerning Bernoulli numbers and Bernoulli polynomials, *Discrete Appl. Math.* 105 (2000) 193–223.
- [S4] Z.H. Sun, Congruences involving Bernoulli polynomials, *Discrete Math.* (2007), doi:10.1016/j.disc.2007.03.038.
- [Su] Z.W. Sun, General congruences for Bernoulli polynomials, *Discrete Math.* 262 (2003) 253–276.