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Explicit functorial correspondences for level zero representations of p-adic linear groups ☆

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ABSTRACT

Let F be a non-Archimedean local field and D a central F-division algebra of dimension n^2, $n \geqslant 1$. We consider first the irreducible smooth representations of $D^\times$ trivial on 1-units, and second the indecomposable, n-dimensional, semisimple, Weil–Deligne representations of F which are trivial on wild inertia. The sets of equivalence classes of these two sorts of representations are in canonical (functorial) bijection via the composition of the Jacquet–Langlands correspondence and the Langlands correspondence. They are also in canonical bijection via explicit parametrizations in terms of tame admissible pairs. This paper gives the relation between these two bijections. It is based on analysis of the discrete series of the general linear group $\mathrm{GL}_n(F)$ in terms of a classification by extended simple types.

1. Let F be a non-Archimedean local field. Let $n \geqslant 1$ be an integer and D a central F-division algebra of dimension n^2. We denote by $\mathfrak{o}_D$ the discrete valuation ring in D and by $\mathfrak{p}_D$ the maximal ideal of $\mathfrak{o}_D$. Let U_D^1 denote the group $1+\mathfrak{p}_D$ of principal units in D: this is a compact, open, normal subgroup of the locally profinite group $D^\times = \mathrm{GL}_1(D)$. We denote by $\mathcal{A}_1(D)$ the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_1(D)$ and by $\mathcal{A}_1(D)_0$ the set of $\pi \in \mathcal{A}_1(D)$ such that $U_D^1 \subset \operatorname{Ker}\pi$.

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On the other hand, let $\mathcal{G}_n(F)$ denote the set of equivalence classes of semisimple, smooth Weil–Deligne representations of F of dimension n. Let $\mathcal{G}_n^{\square}(F)$ denote the subset of $\mathcal{G}_n(F)$ consisting of classes of *indecomposable* representations. A Weil–Deligne representation $\rho = (\sigma, \mathfrak{n})$ is *of level zero* if the linear component σ is trivial on the wild inertia subgroup of the Weil group of F. We denote by $\mathcal{G}_n(F)_0$ the subset of $\mathcal{G}_n(F)$ consisting of classes of representations of level zero, and put $\mathcal{G}_n^{\square}(F)_0 = \mathcal{G}_n(F)_0 \cap \mathcal{G}_n^{\square}(F)$.

2. The *Jacquet–Langlands correspondence* [1,25] is a canonical bijection

$$\boldsymbol{j} : \mathcal{A}_1(D) \longrightarrow \mathcal{A}_n^{\square}(F),$$

where $\mathcal{A}_n^{\square}(F)$ denotes the set of equivalence classes of irreducible smooth representations of the group $\mathrm{GL}_n(F)$ which are *essentially square-integrable* modulo the centre $F^{\times}$ of $\mathrm{GL}_n(F)$. Following custom and practice, we refer to the elements of $\mathcal{A}_n^{\square}(F)$ as the *discrete series* of $\mathrm{GL}_n(F)$.

Let $\mathcal{A}_n(F)$ be the set of equivalence classes of irreducible smooth representations of $\mathrm{GL}_n(F)$. The Langlands correspondence [21,15,16] is a canonical bijection

$$\boldsymbol{l} : \mathcal{A}_n(F) \longrightarrow \mathcal{G}_n(F)$$

which, one knows, satisfies $\boldsymbol{l}(\mathcal{A}_n^{\square}(F)) = \mathcal{G}_n^{\square}(F)$. Composing, we get a bijection $\boldsymbol{lj} : \mathcal{A}_1(D) \to \mathcal{G}_n^{\square}(F)$ which, we shall see, induces a bijection

$$(\star) \qquad \boldsymbol{lj} : \mathcal{A}_1(D)_0 \longrightarrow \mathcal{G}_n^{\square}(F)_0.$$

This paper revolves around the fact that each of the sets $\mathcal{G}_n^{\square}(F)_0$, $\mathcal{A}_1(D)_0$ admits a canonical, explicit parametrization in terms of the same set of objects. It is therefore practical, and interesting, to seek an explicit description of the bijection $(\star)$.

3. These parametrizations are in terms of *admissible tame pairs*. An admissible tame pair over F consists of a finite unramified field extension E/F and a tamely ramified character θ of $E^{\times}$ such that the conjugates θ^{γ}, $\gamma \in \mathrm{Gal}(E/F)$, are distinct. We denote by $\mathcal{T}(F; n)$ the set of isomorphism classes of admissible tame pairs $(E/F, \theta)$ for which $[E:F]$ divides n.

A well-known construction, going back to [14] and reviewed in 1.5 below, yields a canonical bijection

$$\begin{aligned} \mathcal{T}(F; n) &\longrightarrow \mathcal{A}_1(D)_0, \\ (E/F, \theta) &\longmapsto \pi_D(\theta). \end{aligned}$$

On the other hand, a variation on a standard procedure, reviewed in 1.4 below, yields a canonical bijection

$$\begin{aligned} \mathcal{T}(F; n) &\longrightarrow \mathcal{G}_n^{\square}(F)_0, \\ (E/F, \theta) &\longmapsto \rho_n(\theta). \end{aligned}$$

The objective of the paper is to prove the following.

Main Theorem. *Let e, f be positive integers such that $ef = n$. Let $(E/F, \theta)$ be an admissible tame pair such that $[E:F] = f$. If η_E is the unramified quadratic character of $E^{\times}$, then*

$$\boldsymbol{lj}\pi_D(\theta) = \rho_n\big(\eta_E^{e(f-1)}\theta\big).$$

This result is required in work [22] of D. Prasad and D. Ramakrishnan on the self-dual representations of groups $D^\times$. We have written this paper in response to their questions on the matter.

4. A key component of our method is provided by work [30] of A. Silberger and E.-W. Zink. As part of a more general programme, they produce a canonical bijection

$$(\dagger)\qquad \begin{aligned} \mathcal{T}(F;n) &\longrightarrow \mathcal{A}_n^\square(F)_0, \\ (E/F,\theta) &\longmapsto \pi_n(E/F,\theta), \end{aligned}$$

where $\mathcal{A}_n^\square(F)_0$ is the set of $\pi \in \mathcal{A}_n^\square(F)$ which are "of level zero" in a sense explained in Section 3 below. They show that the Jacquet–Langlands correspondence gives a bijection $\mathcal{A}_1(D)_0 \to \mathcal{A}_n^\square(F)_0$ and work out the relation between the maps π_n, $\boldsymbol{j} \circ \pi_D$: it is quoted as Theorem 1 in 6.1 below. That result, it must be remarked, is effectively unprecedented. Practically all other work on explicit Jacquet–Langlands correspondences, notably [6,10], deals only with representations corresponding to *cuspidal* representations of $\mathrm{GL}_n(F)$.

For this paper, it remains only to compute the relationship between ρ_n and $\boldsymbol{l} \circ \pi_n$. This is given as Theorem 2 in 6.3.

5. The resolution of this rather simple question seems to require a substantial technical apparatus, concerned with the structure theory for the discrete series $\mathcal{A}_n^\square(F)$ of $\mathrm{GL}_n(F)$. This theory has two main aspects, to which we have to add a third.

First, one may describe the representations $\pi \in \mathcal{A}_n^\square(F)$ in terms of parabolic induction. We refer to Jacquet and Shalika [18] for the main results and an overview of the literature. The earlier survey article [24] of Rodier may also be found helpful. The second aspect relies on the theory of simple types in $\mathrm{GL}_n(F)$ developed in [11]. The isomorphisms of Hecke algebras attached to such types reduce the study of $\mathcal{A}_n^\square(F)$ to the case of representations with Iwahori-fixed vector. That may be treated "by hand" or viewed as an instance of more general ideas in [2,19]. The basis of this approach is Section 7.7 of [11], supplemented by the discussion in [12, 8.3]. We use results and insights from both aspects.

We also introduce a third, and more novel, approach. In [30], Silberger and Zink show, at least in a relevant special case, that discrete series representations may be parametrized by certain *extended* simple types. This generalizes the description [11] of cuspidal representations as induced from extended *maximal* simple types. The construction of the bijection (†) proceeds via a parametrization of the extended simple types in terms of admissible tame pairs. Our task is to relate this approach to the first aspect above, using the second, along with the general theory of types and covers set out in [13].

6. The initial formulation of the problem in Section 1 is quite straightforward. In Section 2, we review the theory of simple types in $\mathrm{GL}_n(F)$ and their Hecke algebras. We can confine ourselves to the level zero case, so avoiding much of the general apparatus of [11]. We then specify some particular elements of Hecke algebras and work out relations between them. This level of details goes beyond what is directly available from [11].

In Section 3, we review as briefly as possible the theory of the discrete series and its relation with the simple types. We introduce the extended simple types of [30] in Section 4, and use the special functions of Section 2 to connect them with the standard classification of the discrete series. In Section 5, we give the connection between admissible tame pairs and extended simple types. We then apply the general machinery of the earlier sections to prove the Main Theorem in Section 6. By that stage, it has been reduced to a simple check of certain numerical parameters.

A reader familiar with the general theories mentioned above will notice that practically everything we do is susceptible of substantial generalization. Certainly there is no difficulty treating general simple types in $\mathrm{GL}_n(F)$. Using the more recent work of Sécherre [26–28] and his paper [29] with Stevens, the potential for handling all simple types in any inner form of $\mathrm{GL}_n(F)$ is equally clear. As we have no direct application in mind, we have avoided straying into these areas.

Comment. The relation between ρ_n and $\boldsymbol{l} \circ \pi_n$ (6.3, Theorem 2 below) is also the subject of a speculative aside in [30, Remark, p. 182]. The estimate given there is incorrect, but it has no functional relation with the main material of [30], and so has no effect on the validity of its results.

1. Tame admissible pairs and representations

Throughout, F denotes a non-Archimedean local field with discrete valuation ring $\mathfrak{o}_F$. We let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$ and $\Bbbk_F = \mathfrak{o}_F/\mathfrak{p}_F$ the residue field. We set $|\Bbbk_F| = q$. We write U_F for the group $\mathfrak{o}_F^\times$ of units of $\mathfrak{o}_F$ and U_F^1 for the group $1+\mathfrak{p}_F$ of principal units.

We give a more detailed account of the concept of *admissible tame pair* used in the introduction. We show how it can be used to parametrize two of the three classes of representations in which we shall be interested.

1.1. A *tame pair* over F consists of a finite unramified field extension E/F and a character θ of $E^\times$ which is *tamely ramified*, in that it is trivial on the group $U_E^1 = 1+\mathfrak{p}_E$.

A tame pair $(E/F, \theta)$ is said to be *admissible* if the conjugates θ^γ, $\gamma \in \mathrm{Gal}(E/F)$, are distinct.

Two tame pairs $(E_i/F, \theta_i)$, $i = 1, 2$, are deemed to be *F-isomorphic* if there is an F-isomorphism $\alpha : E_1 \to E_2$ such that $\theta_1 = \theta_2 \circ \alpha$.

The *degree* of a tame pair $(E/F, \theta)$ is the degree $[E:F]$ of the extension E/F. We denote by $\mathcal{T}_f(F)$ the set of F-isomorphism classes of admissible tame pairs over F of degree f. If n is a positive integer, we also write

$$\mathcal{T}(F;n) = \bigcup_{f|n} \mathcal{T}_f(F).$$

Remark. A tame pair $(E/F, \theta)$ is admissible if and only if the restricted characters $\theta^\gamma|_{U_E}$, $\gamma \in \mathrm{Gal}(E/F)$, are distinct.

1.2. We choose a separable algebraic closure $\bar F/F$ of F, and let $\mathcal{W}_F$ denote the Weil group of $\bar F/F$. We write $x \mapsto \|x\|$ for the homomorphism $\mathcal{W}_F \to q^{\mathbb{Z}} \subset \mathbb{R}^\times$ which is trivial on the inertia subgroup of $\mathcal{W}_F$ and maps a geometric Frobenius element to q^{-1}.

Let $\rho = (\sigma, \mathfrak{n})$ be a semisimple Weil–Deligne representation of F. (See [8, Chapter 7] for an introduction to this concept.) Thus σ is a linear representation of $\mathcal{W}_F$, say $\sigma : \mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(V)$, which is finite-dimensional, smooth and semisimple, while $\mathfrak{n}$ is a nilpotent endomorphism of V satisfying

$$\sigma(x)\mathfrak{n}\sigma(x)^{-1} = \|x\|\mathfrak{n}, \quad x \in \mathcal{W}_F. \tag{1.2.1}$$

Defining equivalence in the obvious way and setting $\dim\rho = \dim\sigma$, we denote by $\mathcal{G}_n(F)$ the set of equivalence classes of semisimple Weil–Deligne representations of F of dimension n.

We also write $\mathcal{G}_n^\square(F)$ (resp. $\mathcal{G}_n^0(F)$) for the set of *indecomposable* (resp. *irreducible*) elements of $\mathcal{G}_n(F)$.

For an integer $e \geqslant 1$, let $\mathrm{Sp}_e(F)$ denote the *special* Weil–Deligne representation of F of dimension e. We recall the definition. The underlying vector space V has basis $v_0, v_1, \dots, v_{e-1}$. We define a nilpotent endomorphism $\mathfrak{n}_e$ of V by

$$\mathfrak{n}_e v_i = v_{i+1}, \quad 0 \leqslant i < e-1,$$
$$\mathfrak{n}_e v_{e-1} = 0.$$

We define a semisimple smooth representation ς_e of $\mathcal{W}_F$ on V by

$$\varsigma_e(x) v_i = \|x\|^{i+(1-n)/2} v_i, \quad 0 \leqslant i \leqslant e-1.$$

The pair $(\varsigma_e, \mathfrak{n}_e)$ is then a semisimple Weil–Deligne representation of F, and is what we denote by $\mathrm{Sp}_e(F)$.

If σ is an irreducible smooth representation of $\mathcal{W}_F$, say $\sigma : \mathcal{W}_F \to \mathrm{Aut}_{\mathbb{C}}(X)$, we may form the semisimple Weil–Deligne representation

$$\mathrm{Sp}_e(\sigma) = \sigma \otimes \mathrm{Sp}_e(F) = (\sigma \otimes \varsigma_e, 1_X \otimes \mathfrak{n}_e). \tag{1.2.2}$$

We summarize the basic properties of this construction [8, 31.2].

(1.2.3).

(1) *The representation* $\mathrm{Sp}_e(\sigma)$ *of* (1.2.2) *is indecomposable.*
(2) *Let* $\rho \in \mathcal{G}_n^{\square}(F)$. *There is a positive divisor* f *of* n *and an irreducible,* f*-dimensional, smooth representation* σ *of* $\mathcal{W}_F$ *such that* $\rho \cong \mathrm{Sp}_e(\sigma)$, $ef = n$. *The equivalence class of* σ *is uniquely determined by the equivalence class of* ρ.

1.3. Let $\rho = (\sigma, \mathfrak{n}) \in \mathcal{G}_n(F)$. We say that ρ is *of level zero* if σ is trivial on the wild inertia subgroup of $\mathcal{W}_F$. We let $\mathcal{G}_n(F)_0$ denote the set of $\rho \in \mathcal{G}_n(F)$ which are of level zero, and set

$$\mathcal{G}_n^{\square}(F)_0 = \mathcal{G}_n^{\square}(F) \cap \mathcal{G}_n(F)_0, \qquad \mathcal{G}_n^{0}(F)_0 = \mathcal{G}_n(F) \cap \mathcal{G}_n(F)_0.$$

In the context of (1.2.3), surely $\mathrm{Sp}_e(\sigma)$ has level zero if and only if $\sigma \in \mathcal{G}_f^0(F)$ has level zero.

1.4. Let $(E/F, \theta)$ be an admissible tame pair of degree f. We choose an F-embedding of E in $\bar{F}$ and use it to identify E with a subfield of $\bar{F}$. The subgroup of $\mathcal{W}_F$ which fixes E is then open and may be identified with the Weil group $\mathcal{W}_E$ of $\bar{F}/E$. Composing with the Artin Reciprocity map $\mathcal{W}_E \to E^{\times}$ (normalized to take geometric Frobenius elements to prime elements), θ yields a smooth character of $\mathcal{W}_E$ which we again denote by θ. We form the smoothly induced representation

$$\sigma(\theta) = \mathrm{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F} \theta \tag{1.4.1}$$

of $\mathcal{W}_F$. Straightforward arguments yield the following result: see, for example, [7, 2.2 Proposition].

(1.4.2).

(1) *The representation* $\sigma(\theta)$ *of* (1.4.1) *is irreducible and of level zero. Its equivalence class depends only on the isomorphism class of* $(E/F, \theta)$.
(2) *The map*

$$\begin{aligned} \mathcal{T}_f(F) &\longrightarrow \mathcal{G}_f^0(F)_0, \\ (E/F, \theta) &\longmapsto \sigma(\theta), \end{aligned}$$

is a bijection.

Let e, f be positive integers, $n = ef$. If $(E/F, \theta) \in \mathcal{T}_f(F)$, we set

$$\rho_n(\theta) = \mathrm{Sp}_e\bigl(\sigma(\theta)\bigr). \tag{1.4.3}$$

Proposition 1. *Let $n \geqslant 1$. The map*

$$\begin{aligned} \mathcal{T}(F;n) &\longrightarrow \mathcal{G}_n^{\square}(F)_0, \\ (E/F,\theta) &\longmapsto \rho_n(\theta), \end{aligned}$$

is a canonical bijection.

Proof. The result follows on combining (1.2.3) with (1.4.2). □

1.5. We consider a second family of representations. We fix the integer $n \geqslant 1$ and a central F-division algebra D of dimension n^2. We write $\mathfrak{o}_D$ for the discrete valuation ring in D and $\mathfrak{p}_D$ for the maximal ideal of $\mathfrak{o}_D$. We set $U_D^1 = 1 + \mathfrak{p}_D$.

An irreducible smooth representation π of $D^\times$ is *of level zero* if $U_D^1 \subset \operatorname{Ker} \pi$. We write $\mathcal{A}_1(D)$ for the set of equivalence classes of irreducible smooth representations of $D^\times$ and $\mathcal{A}_1(D)_0$ for the subset of $\mathcal{A}_1(D)$ consisting of classes of level zero.

Let $(E/F,\theta)$ be an admissible tame pair of degree f dividing n, and set $n = ef$. There exists an F-embedding $E \to D$, unique up to conjugation by an element of $D^\times$. We choose such an embedding and use it to identify E with an F-subalgebra of D. Let B denote the D-centralizer of E. Thus B is a central E-division algebra of dimension e^2.

Let $\mathrm{Nrd}_B : B^\times \to E^\times$ be the reduced norm map. It satisfies $\mathrm{Nrd}_B(U_B^1) = U_E^1$, while $U_B^1 = B \cap U_D^1$. These two properties allow us to define a smooth character Λ of the group $\boldsymbol{J} = B^\times U_D^1$ by

$$\Lambda(bu) = \theta(\mathrm{Nrd}_B\, b), \quad b \in B^\times,\ u \in U_D^1.$$

In particular, Λ is trivial on U_D^1.

Proposition 2.

(1) *The representation*

$$\pi_D(\theta) = \operatorname{Ind}_{\boldsymbol{J}}^{D^\times} \Lambda \tag{1.5.1}$$

is irreducible, smooth and of level zero. Its equivalence class depends only on the isomorphism class of $(E/F,\theta)$.

(2) *The map*

$$\begin{aligned} \mathcal{T}(F;n) &\longrightarrow \mathcal{A}_1(D)_0, \\ (E/F,\theta) &\longmapsto \pi_D(\theta), \end{aligned} \tag{1.5.2}$$

is a bijection.

The result is classical and may be found in any of [14,20,4,23] or [3]. We therefore give no details.

2. Simple types and special functions in level zero

In this section, we recall the definition of a *simple type of level zero* in the group $\mathrm{GL}_n(F)$, in the sense of [11]. We specify some particular functions from the Hecke algebra of such a type, and work out relations between them.

2.1. Let $G = \mathrm{GL}_n(F)$ and $A = \mathrm{M}_n(F)$. We recall Definition 5.5.10(b) of [11].

Definition 1. A simple type of level zero in G is a pair (J, λ) as follows.

(1) The group J is the unit group $U_{\mathfrak{a}}$ of a hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A satisfying

$$\mathfrak{a}/\mathfrak{p}_{\mathfrak{a}} \cong \mathrm{M}_f(\Bbbk_F) \times \mathrm{M}_f(\Bbbk_F) \times \cdots \times \mathrm{M}_f(\Bbbk_F), \tag{2.1.1}$$

where there are e factors, $ef = n$, and $\mathfrak{p}_{\mathfrak{a}}$ denotes the Jacobson radical of $\mathfrak{a}$.
(2) There is an irreducible cuspidal representation $\tilde{\lambda}_0$ of $\mathrm{GL}_f(\Bbbk_F)$ so that λ is the inflation, via the map $J = U_{\mathfrak{a}} \to U_{\mathfrak{a}}/1 + \mathfrak{p}_{\mathfrak{a}} \cong \mathrm{GL}_f(\Bbbk_F)^e$, of the representation $\tilde{\lambda}_0 \otimes \tilde{\lambda}_0 \otimes \cdots \otimes \tilde{\lambda}_0$.

In the context of the definition, we will usually write $1 + \mathfrak{p}_{\mathfrak{a}} = U^1_{\mathfrak{a}}$.

We note some elementary properties of the hereditary order $\mathfrak{a}$. For such background, see the early pages of [11] and [5].

(2.1.2). *Let $\mathfrak{a}$ be a hereditary $\mathfrak{o}_F$-order in A satisfying* (2.1.1).

(1) *Any hereditary $\mathfrak{o}_F$-order in A, satisfying* (2.1.1), *is G-conjugate to $\mathfrak{a}$.*
(2) *The radical $\mathfrak{p}_{\mathfrak{a}}$ is a principal ideal of $\mathfrak{a}$, in that there exists $\varpi_{\mathfrak{a}} \in G$ such that $\mathfrak{p}_{\mathfrak{a}} = \varpi_{\mathfrak{a}}\mathfrak{a} = \mathfrak{a}\varpi_{\mathfrak{a}}$.*
(3) *The group*

$$\mathcal{K}_{\mathfrak{a}} = \{g \in G : g\mathfrak{a}g^{-1} = \mathfrak{a}\}$$

is the G-normalizer of $U_{\mathfrak{a}}$. Moreover, $\mathcal{K}_{\mathfrak{a}} = \varpi_{\mathfrak{a}}^{\mathbb{Z}} \ltimes U_{\mathfrak{a}}$, for any element $\varpi_{\mathfrak{a}}$ such that $\mathfrak{p}_{\mathfrak{a}} = \varpi_{\mathfrak{a}}\mathfrak{a}$.

Any element $\varpi_{\mathfrak{a}}$ as in part (2) of (2.1.2) will be called a *prime element* of $\mathfrak{a}$. We also observe that $\mathfrak{p}_{\mathfrak{a}}^e = \mathfrak{p}_F\mathfrak{a}$. The integer e is thus the *F-period* of $\mathfrak{a}$.

2.2. It is only the G-conjugacy class of the pair (J, λ) which is of concern. This allows us to impose a *standard form* on the order $\mathfrak{a}$.

(2.2.1). *Let $\mathfrak{a} = \mathfrak{a}_F(e, f)$ be the set of matrices $(x_{ij})_{1 \leqslant i, j \leqslant e}$ such that $x_{ij} \in \mathrm{M}_f(\mathfrak{o}_F)$ and $x_{ij} \in \mathfrak{p}_F\mathrm{M}_f(\mathfrak{o}_F)$ when $i > j$. The set $\mathfrak{a}$ is then a hereditary $\mathfrak{o}_F$-order in A satisfying* (2.1.1). *The radical $\mathfrak{p}_{\mathfrak{a}}$ of $\mathfrak{a}$ consists of those block matrices $(x_{ij}) \in \mathfrak{a}$ for which $x_{ii} \in \mathfrak{p}_F\mathrm{M}_f(\mathfrak{o}_F)$, $1 \leqslant i \leqslant e$.*

Any hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A satisfying (2.1.1) is G-conjugate to $\mathfrak{a}_F(e, f)$.

For notational convenience, we introduce another element of structure via which we can rewrite the definition of $\mathfrak{a}_F(e, f)$. Let E/F be an unramified field extension of degree f and set $B = \mathrm{End}_E(E^e) = \mathrm{M}_e(E)$. Let $\mathfrak{b} = \mathfrak{a}_E(e, 1)$: this is the standard *minimal* hereditary $\mathfrak{o}_E$-order in B. Write $A = \mathrm{End}_F(E^e)$. We think of E as an F-subalgebra of A via the canonical inclusions $E \to B \to A$. There is then a unique hereditary $\mathfrak{o}_F$-order $\mathfrak{a}$ in A such that $E^\times \subset \mathcal{K}_{\mathfrak{a}}$ and $\mathfrak{a} \cap B = \mathfrak{b}$.

If we choose an F-basis of E, the algebra A becomes identified with $\mathrm{M}_n(F)$. If we choose this basis to be an $\mathfrak{o}_F$-basis of $\mathfrak{o}_E$, the order $\mathfrak{a}$ becomes identified with $\mathfrak{a}_F(e, f)$. We use this scheme to identify specific elements of G and functions in the Hecke algebras of types.

2.3. We take a simple type (J, λ) as in 2.1. We assume that J is the group of units of $\mathfrak{a} = \mathfrak{a}_F(e, f)$, constructed from a field extension E/F as in 2.2. We set $H = B^\times = \mathrm{GL}_e(E)$. Thus H is the G-centralizer of $E^\times$. Moreover, if $\mathfrak{b} = \mathfrak{a} \cap B$, the group $I = U_{\mathfrak{b}}$ is the standard Iwahori subgroup of H.

In this and the next subsections, we summarize the account in [11, 5.6] of the intertwining properties of the representation λ.

Working first in the group H, let W_0 be the group of permutation matrices. In particular, W_0 is generated by the involutions s_i, $1 \leqslant i \leqslant e-1$, where s_i is the permutation matrix realizing the transposition $(i, i+1)$.

We choose a *standard* prime element Π of $\mathfrak{a}$ as follows. We choose, once for all, a prime element ϖ of F (*not* E) and let $\Pi = (x_{ij}) \in H$ be the matrix given by $x_{i,i+1} = 1$ for $1 \leqslant i \leqslant e-1$, $x_{e1} = \varpi$, all other entries being 0.

The element Π satisfies $\mathfrak{p}_{\mathfrak{a}} = \Pi\mathfrak{a} = \mathfrak{a}\Pi$ as required, and Π^e is the scalar matrix $\varpi 1_A$. The G-normalizer of J is the semi-direct product $\Pi^{\mathbb{Z}} \ltimes J$, as in (2.1.2). Conjugation by Π also fixes the representation λ. Likewise, the H-normalizer of I is $\Pi^{\mathbb{Z}} \ltimes I$.

Let W denote the group generated by Π and W_0. The map

$$\begin{aligned} W &\longrightarrow I\backslash H/I, \\ w &\longmapsto IwI, \end{aligned} \tag{2.3.1}$$

is then a bijection. The discussion in [11, 5.6] yields the following facts.

(2.3.2).

(1) *The set $I_G(\lambda)$ of elements of G which intertwine λ is given by*

$$I_G(\lambda) = JHJ = JWJ.$$

(2) *The map*

$$\begin{aligned} W &\longrightarrow J\backslash I_G(\lambda)/J, \\ w &\longmapsto JwJ, \end{aligned}$$

is a bijection and, moreover, $JwJ \cap H = IwI$, for every $w \in W$.

2.4. We refine (2.3.2) into an isomorphism of *Hecke algebras.*

Let λ act on the vector space X, and let $(\check\lambda, \check X)$ denote the contragredient representation. If f is an endomorphism of X (resp. $\check X$), we write $\check f$ for the transpose endomorphism of $\check X$ (resp. X). Thus $\check{\check f} = f$ and $(\lambda(x))^\vee = \check\lambda(x^{-1})$, $x \in J$.

Let $\mathcal{H}_\lambda = \mathcal{H}(G, \lambda)$ be the space of compactly supported functions $\phi : G \to \mathrm{End}_{\mathbb{C}}(\check X)$ satisfying

$$\phi(j_1 g j_2) = \check\lambda(j_1)\phi(g)\check\lambda(j_2),$$

for $j, k \in J$ and $g \in G$. Let μ_G be the Haar measure on G for which $\mu_G(J) = 1$. We use the measure μ_G to define convolution of elements of $\mathcal{H}_\lambda$ and henceforward treat $\mathcal{H}_\lambda$ as a unital, associative $\mathbb{C}$-algebra relative to this operation.

Similarly, let $\mathcal{H}_E = \mathcal{H}(H, 1_I)$ be the space of compactly supported functions $\psi : H \to \mathbb{C}$ such that $\psi(xhy) = \psi(h)$, for $h \in H$ and $x, y \in I$. We regard $\mathcal{H}_E$ as $\mathbb{C}$-algebra under convolution defined relative to the Haar measure μ_H on H for which $\mu_H(I) = 1$.

For $w \in W$, let $[w]$ denote the characteristic function of IwI. In particular, $[w] \in \mathcal{H}_E$.

We summarize from [11, 5.6] the points we need concerning the algebra $\mathcal{H}_E$ and its relationship with $\mathcal{H}_\lambda$.

(2.4.1) Facts.

(1) *The functions $[w]$, $w \in W$, form a basis of the vector space $\mathcal{H}_E$.*

(2) *For any $w \in W$, we have*

$$[w] * [\Pi] = [w\Pi] \quad \text{and} \quad [\Pi] * [w] = [\Pi w].$$

In particular, the function $[\Pi]$ is invertible in $\mathcal{H}_E$, with inverse $[\Pi^{-1}]$.

(3) *The functions $[s_i]$, $1 \leqslant i \leqslant e-1$, satisfy the relation*

$$\big([s_i]+1\big) * \big([s_i]-q^f\big) = 0, \tag{2.4.2}$$

where $q = q_F = |\Bbbk_F|$.

(4) *The elements $[s_i]$, $1 \leqslant i \leqslant e-1$ and $[\Pi]$, $[\Pi^{-1}]$ together generate the $\mathbb{C}$-algebra $\mathcal{H}_E$.*

(5) *There exists an algebra isomorphism $\Upsilon : \mathcal{H}_\lambda \to \mathcal{H}_E$ such that, if $\phi \in \mathcal{H}_\lambda$ has support JwJ, $w \in W$, then $\Upsilon\phi$ has support IwI. That is,*

$$\operatorname{supp} \Upsilon\psi = \operatorname{supp} \psi \cap H, \tag{2.4.3}$$

for all $\psi \in \mathcal{H}_\lambda$.

(6) *Let $\Upsilon' : \mathcal{H}_\lambda \to \mathcal{H}_E$ be an algebra isomorphism with the property* (2.4.3). *Let $\psi \in \mathcal{H}_\lambda$ have support ΠJ.*
   (a) *There exists a constant $a \in \mathbb{C}^\times$ such that $\Upsilon'\psi = a\Upsilon\psi$.*
   (b) *The algebra isomorphisms Υ, Υ' coincide if and only if $a = 1$.*

We delineate as "support-preserving" the family of algebra isomorphisms Υ satisfying (2.4.3).

2.5. We exhibit some particular elements of the Hecke algebra $\mathcal{H}_\lambda$. Observe that, as a consequence of (2.4.1)(5), each coset JwJ, $w \in W$, supports only a one-dimensional space of elements of $\mathcal{H}_\lambda$. This space may be described in terms of intertwining operators, as in [11, 4.1.2]. We exhibit a sequence of special cases.

Looking back at Definition 1 in 2.1, the representation λ is determined by the irreducible cuspidal representation $\tilde\lambda_0$, say $\tilde\lambda_0 : \mathrm{GL}_f(\Bbbk_F) \to \mathrm{Aut}_{\mathbb{C}}(X_0)$. Thus λ acts on the space $X = X_0 \otimes X_0 \otimes \cdots \otimes X_0$ and $\check\lambda$ on $\check X = \check X_0 \otimes \check X_0 \otimes \cdots \otimes \check X_0$.

For $1 \leqslant i \leqslant e-1$, define $t_i \in \mathrm{Aut}_{\mathbb{C}}(\check X)$ to be the automorphism which interchanges the i-th and $(i+1)$-th tensor factors $\check X_0$.

Lemma 1. *For $1 \leqslant i \leqslant e-1$, there exists a unique function $\phi_i \in \mathcal{H}_\lambda$ with support Js_iJ and such that $\phi_i(s_i) = t_i$.*

Next, let Γ denote the automorphism

$$v_1 \otimes v_2 \otimes \cdots \otimes v_e \longmapsto v_2 \otimes \cdots \otimes v_e \otimes v_1$$

of $\check X = \check X_0 \otimes \cdots \otimes \check X_0$.

Lemma 2. *There is a unique function $\phi_\Pi \in \mathcal{H}_\lambda$ with support ΠJ and such that $\phi_\Pi(\Pi) = \Gamma$.*

Finally, let $\Delta \in H$ denote the diagonal matrix

$$\Delta = \operatorname{diag}(1, 1, \ldots, 1, \varpi).$$

Lemma 3. *There is a unique function $\phi_\Delta \in \mathcal{H}_\lambda$ with support $J\Delta J$ and such that $\phi_\Delta(\Delta) = 1$, the identity automorphism of $\check X$.*

All of these lemmas are elementary applications of [11, (4.1.2)]. Before passing on, we note a useful property of the function ϕ_Π. For any $f \in \mathcal{H}_\lambda$, $g \in G$, we have

$$\operatorname{supp}(\phi_\Pi * f) = \Pi \operatorname{supp}(f) \quad \text{and} \quad \phi_\Pi * f(\Pi g) = \phi_\Pi(\Pi) f(g). \tag{2.5.1}$$

This follows from (2.4.1)(2) and a simple computation. Similarly for $f * \phi_\Pi$.

2.6. In the notation of Definition 1 (2.1), let λ_0 denote the representation of $\mathrm{GL}_f(\mathfrak{o}_F)$ obtained by inflating $\tilde{\lambda}_0$. The following calculation is central to the paper.

Proposition 3. *Let θ_0 denote the unique character of U_F occurring in $\lambda_0|_{U_F}$. If $\Upsilon : \mathcal{H}_\lambda \to \mathcal{H}_E$ is a support-preserving algebra isomorphism, then*

$$\Upsilon \phi_i = \theta_0(-1) q^{f(f-1)/2} [s_i], \quad 1 \leqslant i \leqslant e-1. \tag{2.6.1}$$

Moreover,

$$\phi_\Pi * \phi_1 * \phi_2 * \cdots * \phi_{e-1} = \phi_\Delta. \tag{2.6.2}$$

Proof. The relation $\Pi^{-1} s_i \Pi = s_{i+1}$ and (2.5.1) together imply that any two of the functions ϕ_i are conjugate by a power of ϕ_Π,

$$\phi_i * \phi_\Pi = \phi_\Pi * \phi_{i+1}, \quad 1 \leqslant i < e-1,$$

while $\Upsilon \phi_\Pi = a[\Pi]$, for some $a \in \mathbb{C}^\times$. To prove (2.6.1), therefore, it is enough to treat the case $i = 1$. Define $\sigma \in \mathcal{H}_\lambda$ by the relation $\Upsilon \sigma = [s_1]$. Thus (2.4.2)

$$\sigma * \sigma = q^f + \big(q^f - 1\big)\sigma.$$

Abbreviating $\phi = \phi_1$, we have $\phi = a\sigma$ for some $a \in \mathbb{C}^\times$ such that

$$\phi * \phi = a^2 q^f + a\big(q^f - 1\big)\phi.$$

Equivalently,

$$\phi * \phi(1) = a^2 q^f, \qquad \phi * \phi(s_1) = a\big(q^f - 1\big) t_1.$$

Computing the first expression, we find

$$\phi * \phi(1) = \int_G \phi(x)\phi\big(x^{-1}\big)\, d\mu_G(x) = \sum_g \phi(g s_1)\phi\big(s_1 g^{-1}\big),$$

where, in the sum, g ranges over $J/J \cap s_1 J s_1$. To get such a set of coset representatives, we let x range over a set of representatives for $\mathrm{M}_f(\mathfrak{o}_F)/\mathfrak{p}_F \mathrm{M}_f(\mathfrak{o}_F)$ and take g_x of the form $1 + x'$, where x' is the $e \times e$ block matrix having x in the 12-place, zeros elsewhere. In particular, $g_x \in \operatorname{Ker} \lambda$, whence

$$\phi * \phi(1) = q^{f^2} t_1^2 = q^{f^2}.$$

That is, $a = \pm q^{f(f-1)/2}$.

To determine the sign, we have to compute

$$\begin{aligned}\phi * \phi(s_1) &= \int_G \phi(x)\phi\big(x^{-1}s_1\big)\, d\mu_G(x)\\ &= \sum_g \phi(gs_1)\phi\big(s_1 g^{-1} s_1\big),\end{aligned}$$

with g ranging as before. For our particular choice of coset representatives g, we have $g \in \operatorname{Ker}\lambda$ so $\phi(gs_1) = \phi(s_1) = t_1$. Also, the elements g^{-1} constitute a set of representatives for $J/J \cap s_1 J s_1$. Thus

$$\phi * \phi(s_1) = t_1 \sum_g \phi(s_1 g s_1),$$

with g ranging as before. We write g in the form

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \qquad s_1 g s_1 = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},$$

showing only the first $2f$ rows and columns. This matrix contributes to the integral if and only if it lies in Js_1J. This condition is equivalent to the matrix $x \in \mathrm{M}_f(\mathfrak{o}_F)$ lying in $\mathrm{GL}_f(\mathfrak{o}_F)$, following the identity

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix} s_1 \begin{pmatrix} x & 1 \\ 0 & -x^{-1} \end{pmatrix}.$$

Therefore, writing μ for the contragredient of $\tilde{\lambda}_0$,

$$\begin{aligned}\phi * \phi(s_1) &= \theta_0(-1)t_1 \sum_y t_1 \circ \big(\mu(y) \otimes \mu\big(y^{-1}\big)\big)\\ &= \theta_0(-1) \sum_y \mu(y) \otimes \mu\big(y^{-1}\big),\end{aligned}$$

the sum being taken over $y \in \mathrm{GL}_f(\Bbbk_F)$. To simplify further, we take $v_1, v_2 \in \check{X}_0$, $x_1, x_2 \in X_0$ and evaluate the inner product

$$\begin{aligned}\big\langle \phi * \phi(s_1)(v_1 \otimes v_2), x_1 \otimes x_2 \big\rangle &= \theta_0(-1) \sum_y \big\langle \mu(y)v_1 \otimes \mu\big(y^{-1}\big)v_2, x_1 \otimes x_2 \big\rangle\\ &= \theta_0(-1) \sum_y \langle \mu(y)v_1, x_1 \rangle \big\langle \mu\big(y^{-1}\big)v_2, x_2 \big\rangle,\end{aligned}$$

which, applying the first Schur orthogonality relation for μ, reduces to

$$\big\langle \phi * \phi(s_1)(v_1 \otimes v_2), x_1 \otimes x_2 \big\rangle = \theta_0(-1)k\langle v_1, x_2\rangle\langle v_2, x_1\rangle,$$

where k is a positive constant. In other words, $\phi * \phi(s_1) = \theta_0(-1)kt_1$, for a constant $k > 0$. The relation (2.6.1) has been proved.

We turn to the identity (2.6.2). For $1 \leqslant i \leqslant e-1$, we define

$$c_i = s_i s_{i+1} \cdots s_{e-1} \in W_0.$$

Thus $c_i = s_i c_{i+1}$, $i \geqslant 1$, and $\Pi c_1 = \Delta$. If γ_i denotes the automorphism of $\check{X}$ corresponding to the index permutation c_i, then $\gamma_i = t_i t_{i+1} \cdots t_{e-1}$ and $\gamma_1 = \Gamma^{-1}$.

Lemma 4. *Let $1 \leqslant i \leqslant e-1$. The function $\Phi_i = \phi_i * \phi_{i+1} * \cdots * \phi_{e-1}$ has support Jc_iJ and $\Phi_i(c_i) = \gamma_i$.*

Proof. We proceed by induction on i. We first note that $\Phi_{e-1} = \phi_{e-1}$ and the assertion is the definition of ϕ_{e-1}. We therefore take $1 \leqslant i < e-1$ and assume the result is valid for Φ_{i+1}.

In the algebra $\mathcal{H}_E$, an elementary calculation yields $[s_i][c_{i+1}] = [c_i]$. The support of $\Phi_i = \phi_i * \Phi_{i+1}$ is therefore $Js_iJc_{i+1}J = Jc_iJ$. It remains to compute

$$\begin{aligned} \phi_i * \Phi_{i+1}(c_i) &= \int_G \phi_i(x)\Phi_{i+1}\big(x^{-1}s_ic_{i+1}\big)\,d\mu_G(x) \\ &= \sum_g \phi_i(gs_i)\Phi_{i+1}\big(s_ig^{-1}s_ic_{i+1}\big), \end{aligned}$$

where g ranges over $J/J \cap s_iJs_i$. Only the trivial coset representative g satisfies $s_igs_ic_{i+1} \in Jc_{i+1}J$, giving

$$\Phi_i(c_i) = \phi_i * \Phi_{i+1}(c_i) = t_i\Phi_{i+1}(c_{i+1}) = c_i,$$

by inductive hypothesis. $\square$

As in (2.5.1) therefore, $\phi_\Pi * \Phi_1$ has support $J\Pi c_1J = J\Delta J$ and $\phi_\Pi * \Phi_1(\Delta) = \phi_\Pi(\Pi)\Phi_1(c_1) = 1$, as required for (2.6.2). $\square$

3. Discrete series and types

Let $G = \mathrm{GL}_n(F)$. We write $\mathcal{A}_n(F)$ for the set of equivalence classes of irreducible smooth representations of G. We let $\mathcal{A}_n^\square(F)$ be the set of classes of representations $\pi \in \mathcal{A}_n(F)$ which are *essentially square-integrable* modulo the centre $F^\times$ of G, and $\mathcal{A}_n^0(F)$ the set of *cuspidal* classes $\pi \in \mathcal{A}_n(F)$. In particular, $\mathcal{A}_n^0(F) \subset \mathcal{A}_n^\square(F)$. We tend to refer to the elements of $\mathcal{A}_n^\square(F)$ as the "discrete series" of G.

In this section, we recall and develop aspects of the relation between the discrete series and simple types in G. Most of the time, we specialize to simple types of level zero, but this restriction serves only to allow us the use of the notation of Section 2 (and so avoid recalling the full machinery of [11]). A reader familiar with [11] and [13] will have little difficulty generalizing the discussion.

3.1. We first establish some notation. We fix a positive divisor e of n, say $n = ef$, and define some subgroups of $G = \mathrm{GL}_n(F)$.

Let P_e denote the standard (upper triangular) parabolic subgroup of G such that, if N_e is the unipotent radical of P_e, then $P_e/N_e \cong \mathrm{GL}_f(F)^e$. We let P'_e denote the transpose of P_e. Thus P'_e is a parabolic subgroup of G whose unipotent radical N'_e is the transpose of N_e. The group $M_e = P_e \cap P'_e$ is a Levi component of both P_e and P'_e. We have $P_e = M_eN_e$ and $P'_e = M_eN'_e$, while M_e is the group of matrices in G which are diagonal in $f \times f$ blocks. In particular, $M_e \cong \mathrm{GL}_f(F)^e$.

When e is fixed for a period, we tend to write simply $P = P_e$ and so on. With this abbreviated notation, we write $\boldsymbol{\iota}_P^G$ for the *normalized* smooth induction functor from P to G. Likewise r_N is the *normalized* Jacquet functor relative to N, adjoint to $\boldsymbol{\iota}_P^G$. Similarly for N' but, since we need it several times, we prefer the abbreviated form

$$r'_e = r_{N'_e}. \tag{3.1.1}$$

3.2. We recall, with no proofs, the standard description of the discrete series in terms of parabolic induction. For the basic results, and an overview of the literature, we refer to Jacquet and Shalika [18] and Rodier [24].

Again we set $G = \mathrm{GL}_n(F)$. We take positive integers e, f such that $ef = n$. We let $P = P_e = MN$ and $P' = MN'$ be the standard parabolic subgroups of G defined in 3.1.

Let $\tau \in \mathcal{A}^0_f(F)$. If a is a real number, we write τ^a for the representation $x \mapsto \|\det x\|^a \tau(x)$ of $\mathrm{GL}_f(F)$. We form the representation

$$\tau_M = \tau^{(1-e)/2} \otimes \tau^{(3-e)/2} \otimes \cdots \otimes \tau^{(e-1)/2}$$

of M. We inflate τ_M to a representation of P, trivial on N, and induce to obtain a smooth representation $\boldsymbol{\iota}_P^G \tau_M$ of G. The major result we need is the following.

(3.2.1).

(1) *The representation $\boldsymbol{\iota}_P^G \tau_M$ admits a unique irreducible G-quotient $\mathrm{St}_e(\tau)$. The representation $\mathrm{St}_e(\tau)$ lies in the discrete series $\mathcal{A}^\square_n(F)$ of G.*

(2) *Let $\pi \in \mathcal{A}^\square_n(F)$. There exist a positive divisor e of n and a representation $\tau \in \mathcal{A}^0_{n/e}(F)$ such that $\pi \cong \mathrm{St}_e(\tau)$. The pair (e, τ) is determined uniquely by the equivalence class of π.*

For the genesis of this result, see [24, Proposition 11] and the discussion in [18, p. 209]. A different approach (reversing the tactic adopted in these pages) is summarized in [12, 8.3].

We need a supplementary detail.

Lemma 5. *If $\pi = \mathrm{St}_e(\tau)$, as in (3.2.1), then $r'_e\pi \cong \tau_M$.*

Proof. See [24, Proposition 9]. □

3.3. We recall some general facts. For the moment, G could be any locally profinite group and J a compact open subgroup of G. Let λ be an irreducible smooth representation of J, $\lambda : J \to \mathrm{Aut}_{\mathbb{C}}(X)$ say. We form the Hecke algebra $\mathcal{H}(G, \lambda) = \mathcal{H}_\lambda$ in the standard way, as in Section 2. We form the contragredient $\check\lambda$ of λ, and use the same notation for duals as in Section 2. In particular, if $\phi \in \mathcal{H}_\lambda$, we define $\check\phi \in \mathcal{H}(G, \check\lambda)$ by

$$\check\phi(g) = \phi\big(g^{-1}\big)^\vee, \quad g \in G.$$

Let (π, V) be a smooth representation of G. We define

$$V_\lambda = \mathrm{Hom}_J(\lambda, \pi).$$

This complex vector space carries the structure of left $\mathcal{H}_\lambda$-module by (*cf.* [13, 2.7] *et seq.*)

$$\phi \cdot f(x) = \int_G \pi(g) f\big(\check\phi\big(g^{-1}\big)x\big)\, d\mu_G(g), \tag{3.3.1}$$

for $\phi \in \mathcal{H}_\lambda$, $f \in V_\lambda$, $x \in X$.

We revert to the case where $G = \mathrm{GL}_n(F)$ and (J,λ) is a simple type in G in the sense of [11, 5.5.10]. (There is no advantage in assuming λ has level zero.) We exploit the fact that (J,λ) is a *G-type* in the sense of [13, 4.1]. (See also the discussion in [13, 9.3] for more details.)

If (π, V) is a smooth representation of G, we let V^λ denote the sum of the spaces $f(X)$, $f \in V_\lambda$, and $V(\lambda)$ the G-subspace of V generated by V^λ.

Lemma 6. *Let (π, V), (π', V') be smooth representations of G. The $\mathcal{H}_\lambda$-modules V_λ, V'_λ are isomorphic if and only if the G-spaces $V(\lambda)$, $V'(\lambda)$ are isomorphic. In particular, if (π, V) is irreducible and $V_\lambda \neq 0$, the module V_λ determines π up to isomorphism.*

Proof. As (J,λ) is a G-type, the lemma follows from [13, 4.3]. $\square$

3.4. Let (J,λ) be a simple type of level zero in G. Thus $J = U_{\mathfrak{a}}$ where, we may assume, the hereditary order $\mathfrak{a}$ is $\mathfrak{a}_F(e,f)$, $ef = n$ (2.2.1). The representation λ is derived from an irreducible cuspidal representation $\tilde{\lambda}_0$ of $\mathrm{GL}_f(\Bbbk_F)$, as in (2.1.1). We use the associated notation introduced in Section 2.

We recall some machinery from [13], adapted to the case in hand.

We set $J_0 = \mathrm{GL}_f(\mathfrak{o}_F)$ and let λ_0 be the inflation of $\tilde{\lambda}_0$ to a representation of J_0. The pair (J_0,λ_0) is then a *maximal* simple type in $\mathrm{GL}_f(F)$, in the sense of [11, 6.2], as well as being of level zero. We recall [11] some basic features of this situation.

(3.4.1). *Let $\kappa \in \mathcal{A}_f(F)$ contain λ_0.*

(1) *The representation κ is cuspidal.*
(2) *Set $\boldsymbol{J}_0 = F^\times J_0$ and $G_0 = \mathrm{GL}_f(F)$. There exists a unique representation Λ_0 of $\boldsymbol{J}_0$ occurring in κ and such that $\Lambda_0|_{J_0} \cong \lambda_0$. Moreover,*

$$\kappa \cong c\text{-}\mathrm{Ind}_{\boldsymbol{J}_0}^{G_0} \Lambda_0.$$

(3) *If $\kappa' \in \mathcal{A}_f(F)$ contains λ_0, then $\kappa' \cong \kappa$ if and only if the central characters $\omega_{\kappa'}$, ω_κ coincide.*

Here, of course, (3) follows directly from (2).

We note the following general fact, obvious in the present case.

(3.4.2). *Let $P = P_e = MN$, $P' = MN'$, as in* 3.1.

(1) *Set $J_M = J \cap M$, $J_N = J \cap N$, $J_{N'} = J \cap N'$. The group J then satisfies*

$$J = J_{N'} \cdot J_M \cdot J_N,$$

$$J_M = J_0 \times J_0 \times \cdots \times J_0.$$

(2) *The kernel of λ contains both J_N and $J_{N'}$, while*

$$\lambda|_{J_M} \cong \lambda_0 \otimes \lambda_0 \otimes \cdots \otimes \lambda_0.$$

As remarked in the proof of Lemma 6 (3.3), the pair (J,λ) is a G-type. Likewise, (J_0,λ_0) is a G_0-type. It follows readily that, on setting $\lambda_M = \lambda|_{J_M}$, the pair (J_M,λ_M) is an M-type. Moreover, (J,λ) is a *G-cover of* (J_M,λ_M) in the sense of [13, 8.1]. (Again see [13, 9.3] for a more detailed discussion.) We summarize the relevant consequences of this observation.

Let $\mathcal{H}_{\lambda_M}$ denote the Hecke algebra $\mathcal{H}(M,\lambda_M)$, formed relative to a Haar measure μ_M on M such that $\mu_M(J_M) = 1$.

If (π, V) is a smooth representation of G, we may apply the Jacquet functor r'_e of (3.1.1) to obtain a smooth representation $(r'_e\pi, r'_eV)$ of $M = M_e$.

(3.4.3). *There is a canonical algebra homomorphism*

$$\mathcal{J}_e = \mathcal{J}^{\lambda}_{N'_e} : \mathcal{H}_{\lambda_M} \longrightarrow \mathcal{H}_{\lambda}$$

with the following property. If (π, V) is a smooth representation of G, with normalized Jacquet module $(r'_e\pi, r'_eV)$ relative to N', the $\mathcal{H}_{\lambda_M}$-module $(r'_eV)_{\lambda_M}$ is isomorphic to the module $\mathcal{J}^*_eV_\lambda$ obtained from the $\mathcal{H}_\lambda$-module V_λ by restriction along the homomorphism $\mathcal{J}_e$. In particular,

$$\dim V_\lambda = \dim\bigl(r'_eV\bigr)_{\lambda_M}, \tag{3.4.4}$$

for any smooth representation (π, V) of G.

Proof. From [13, 7.11], we obtain a canonical map $\mathcal{J}^0_e$ with the required properties, relative to the *unnormalized* Jacquet functor. The map $\mathcal{J}_e$ is obtained by composing $\mathcal{J}^0_e$ with a twist by a positive-valued smooth character of M. □

The result (3.4.3) is a more precise version of a criterion from [11], to the effect that a representation $\pi \in \mathcal{A}_n(F)$ contains λ if and only if its cuspidal support consists of representations $\kappa \in \mathcal{A}_f(F)$ which contain λ_0. Combining this version with (3.2.1), we get the first assertion of:

Lemma 7.

(1) *A representation $\pi \in \mathcal{A}^{\square}_n(F)$ contains λ if and only if $\pi \cong \mathrm{St}_e(\tau)$, for some $\tau \in \mathcal{A}_f(F)$ containing λ_0.*
(2) *Let $\tau \in \mathcal{A}_f(F)$ contain λ_0. The representation $\mathrm{St}_e(\tau)$ contains λ with multiplicity one.*

Proof. Part (2) follows from Lemma 5 of 3.2 and (3.4.4). □

3.5. Let $(\pi, V) \in \mathcal{A}^{\square}_n(F)$. By [11, 8.5.11] (or the remark above), the representation π contains a simple type (J, λ) in G, and this type is uniquely determined by π, up to conjugation in G [11, 8.4.3]. We say that π *has level zero* if (J, λ) has level zero. We write $\mathcal{A}^{\square}_n(F)_0$ for the set of $\pi \in \mathcal{A}^{\square}_n(F)$ of level zero, and similarly define $\mathcal{A}^0_n(F)_0$.

Remark. Let $\pi \in \mathcal{A}_n(F)$. More commonly, one would say that π has level zero if it has a non-trivial fixed vector for the subgroup $1 + \mathrm{M}_n(\mathfrak{p}_F)$ of $\mathrm{GL}_n(\mathfrak{o}_F)$. If π contains some simple type, this is equivalent to the definition above.

3.6. We describe the discrete series in terms of simple types. Let (J, λ) be a simple type of level zero in $G = \mathrm{GL}_n(F)$. Thus $J = U_{\mathfrak{a}}$ where, we may assume, $\mathfrak{a} = \mathfrak{a}_F(e, f)$ in the notation of 2.2. We choose a support-preserving isomorphism $\Upsilon : \mathcal{H}_\lambda \to \mathcal{H}_E$ (2.4.3) and define $\sigma_i \in \mathcal{H}_\lambda$ by

$$\Upsilon\sigma_i = [s_i], \quad 1 \leqslant i \leqslant e-1.$$

We note that the definition of σ_i is independent of the choice of Υ (2.4.1)(6).

The following statement summarizes the structure of the discrete series from the point of view of Hecke algebras and simple types.

(3.6.1). *Let (π, V) be an irreducible smooth representation of G such that $V_\lambda \neq 0$. The following conditions are equivalent.*

(1) *π is a discrete series representation of G;*
(2) $\dim V_\lambda = 1$ *and* $\sigma_i f = -f$, *for* $1 \leqslant i \leqslant e-1$ *and* $f \in V_\lambda$.

Outline of proof. We indicate briefly how this fundamental result may be obtained.

Suppose first that (J,λ) is the "trivial" simple type. This means that J is the standard Iwahori subgroup of G and λ is its trivial character. In the notation of 2.2, $\mathfrak{a} = \mathfrak{a}_F(n,1)$.

Let $(\pi, V) \in \mathcal{A}_n^{\square}(F)$ contain λ. By (3.4.4) and Lemma 7 of 3.4, $\pi \cong \mathrm{St}_n(\xi)$, where ξ is an unramified character of $F^\times = \mathrm{GL}_1(F)$ and, moreover, π contains λ with multiplicity one. By (2.4.2) every σ_i acts on V_λ as -1 or q. The σ_i are all conjugate in $\mathcal{H}_\lambda$ (2.6, proof of Proposition 3). So, there exists $\alpha = -1$ or q such that $\sigma_i v = \alpha v$, for $v \in V_\lambda$ and $1 \leqslant i \leqslant e-1$.

Let $s_e = \Pi^{-1} s_{e-1} \Pi$, and let W_1 be the subgroup of W generated by the involutions s_i, $1 \leqslant i \leqslant e$. A calculation, with the Poincaré series of the Coxeter system $(W_1, \{[s_i]\})$, shows that the case $\alpha = -1$ gives $\pi \in \mathcal{A}_n^{\square}(F)$, the case $\alpha = q$ gives $\pi \notin \mathcal{A}_n^{\square}(F)$.

This deals with the case where (J,λ) is the trivial type. The general case of (3.6.1) then follows from [11, (7.7.1)]. $\square$

Remark 1. The restriction to level zero in (3.6.1) has the sole function of allowing us to use the notation of Section 2. The same proof applies in full generality, using the Hecke algebra isomorphisms of [11, 5.6].

We give an application.

Proposition 4. *Let $\mathcal{A}_n^{\square}(F)_\lambda$ denote the set of $(\pi, V) \in \mathcal{A}_n^{\square}(F)$ which contain λ. For $(\pi, V) \in \mathcal{A}_n^{\square}(F)_\lambda$, define a character $\chi_\pi : \mathcal{H}_\lambda \to \mathbb{C}$ of $\mathcal{H}_\lambda$ by*

$$\phi \cdot f = \chi_\pi(\phi) f, \quad \phi \in \mathcal{H}_\lambda,\ f \in V_\lambda.$$

This character χ_π satisfies

$$\chi_\pi(\sigma_i) = -1, \quad 1 \leqslant i \leqslant e-1. \tag{3.6.2}$$

The map $\pi \mapsto \chi_\pi$ establishes a bijection between $\mathcal{A}_n^{\square}(F)_\lambda$ and the set of characters of $\mathcal{H}_\lambda$ satisfying (3.6.2).

Proof. The proposition is a direct consequence of (3.6.1) and Lemma 6 (3.3). $\square$

Remark 2. It follows from (2.4.1)(4) that the algebra $\mathcal{H}_\lambda$ is generated by the elements σ_i and $\varphi^{\pm 1}$, where φ is a function in $\mathcal{H}_\lambda$ with support ΠJ. A character χ of $\mathcal{H}_\lambda$, satisfying (3.6.2), is therefore completely determined by its value $\chi(\varphi)$.

Proposition 5. *Let $(\pi, V) \in \mathcal{A}_n^{\square}(F)$ contain λ.*

(1) *The representation π is of the form* $\mathrm{St}_e(\tau)$, *where $\tau \in \mathcal{A}_f^0(F)$ contains λ_0.*
(2) *The space $(r'_e V)_{\lambda_M}$ has dimension one and $\mathcal{H}_{\lambda_M}$ acts on it via a character $\chi_{\tau_M} : \mathcal{H}_{\lambda_M} \to \mathbb{C}$.*
(3) *The characters χ_π, χ_{τ_M} are related by*

$$\chi_{\tau_M} = \chi_\pi \circ \mathcal{J}_e. \tag{3.6.3}$$

Proof. The first assertion is Lemma 7 of 3.4. The second follows from this same lemma and (3.4.4). The final one is given by (3.4.3). $\square$

3.7. We combine Proposition 5 of 3.6 with the identities of 2.6.

As in 3.6, we take $\pi \in \mathcal{A}_n^{\square}(F)_0$ containing the simple type (J,λ). Thus $\pi = \mathrm{St}_e(\tau)$, where $\tau \in \mathcal{A}_f^0(F)$ contains λ_0. We recall from (3.4.1)(3) that τ is effectively determined by its central character ω_τ, which satisfies the following condition.

Lemma 8. *If θ_0 is the character of U_F occurring in $\lambda_0|_{U_F}$, then*

$$\omega_\tau|_{U_F} = \theta_0. \tag{3.7.1}$$

We remark that θ_0 is the same as the character, with that name, introduced in Proposition 3 (2.6).

Defining the element Δ as in 2.5, let $\phi_\Delta^M \in \mathcal{H}_{\lambda_M}$ have support ΔJ_M and satisfy $\phi_\Delta^M(\Delta) = 1$ (the identity operator on the representation space $\check{X}$ of $\check{\lambda}$). The definition [13, 7.12, 6.12] of the map $\mathcal{J}_e$ of (3.4.3) gives

$$\mathcal{J}_e \phi_\Delta^M = k_0 \phi_\Delta, \tag{3.7.2}$$

for some constant $k_0 > 0$.

Proposition 6.

(1) *The character χ_{τ_M} satisfies*

$$\chi_{\tau_M}\big(\phi_\Delta^M\big) = k_1 \omega_\tau(\varpi),$$

for some positive constant k_1.

(2) *The character χ_π satisfies*

$$\chi_\pi(\phi_\Delta) = k_2 \omega_\tau(\varpi),$$

for some positive constant k_2 and

$$\chi_\pi(\phi_\Pi) = (-1)^{e-1} \omega_\tau\big((-1)^{e-1}\varpi\big). \tag{3.7.3}$$

Proof. Part (1) follows from a simple calculation using (3.3.1). The first relation in (2) then follows from (3.7.2) and (3.6.3). Applying Proposition 3 (2.6), we get the second relation up to a positive constant factor k_3. In particular, $\chi_\pi(\phi_\Pi)^e = \omega_\tau(\varpi)^e k_3^e$. However, ϕ_Π^e has support ϖJ and value 1 at ϖ. It follows that $\chi_\pi(\phi_\Pi)^e = \omega_\pi(\varpi)$, where ω_π is the central character of π. Since $\omega_\pi = \omega_\tau^e|_{F^\times}$, we get $k_3^e = 1$ whence $k_3 = 1$, as required. □

Remark. It is not hard to determine the constants k_i explicitly. Since we have no use for this information, we omit the details.

4. Extended simple types

We recall an elegant idea of Silberger and Zink [30]. This gives another view on the discrete series. Again, the result holds in great generality with much the same proofs.

4.1. Let (J,λ) be a simple type in G of level zero, as in (2.1.1).

The group G acts by conjugation on the set of isomorphism classes of simple types in G. We let $\boldsymbol{J}$ denote the G-stabilizer of (J,λ).

Lemma 9. *Let $\varpi_{\mathfrak{a}}$ be a prime element of $\mathfrak{a}$. The group $\boldsymbol{J}$ is generated by J and the element $\varpi_{\mathfrak{a}}$. Indeed, $\boldsymbol{J} = \varpi_{\mathfrak{a}}^{\mathbb{Z}} \ltimes J$ and, in particular, $\boldsymbol{J}$ contains $JF^\times$ with index e.*

Proof. This has already been observed in 2.3. $\square$

Lemma 10.

(1) *The representation λ admits extension to a representation Λ of $\boldsymbol{J}$.*
(2) *Let Λ, Λ' be representations of $\boldsymbol{J}$ such that $\Lambda|_J \cong \Lambda'|_J \cong \lambda$. There exists an unramified character χ of $F^\times$ such that $\Lambda' \cong \Lambda \otimes \chi_{\boldsymbol{J}}$, where $\chi_{\boldsymbol{J}} = \chi \circ \det|_{\boldsymbol{J}}$.*
(3) *Using the notation of* (2), *the following conditions are equivalent.*
   (a) $\Lambda \otimes \chi_{\boldsymbol{J}} \cong \Lambda$;
   (b) *Λ is G-conjugate to $\Lambda \otimes \chi_{\boldsymbol{J}}$;*
   (c) *$\chi_{\boldsymbol{J}}$ is trivial.*

Proof. Since $\boldsymbol{J}/J$ is cyclic, assertion (1) is immediate. Any two extensions of λ to $\boldsymbol{J}$ are related by tensoring with a character of $\boldsymbol{J}$ trivial on J. Any such character is of the form $\chi_{\boldsymbol{J}}$, whence (2) follows.

In (3), the equivalence of (a) and (c) is given by elementary Clifford theory. In (b), any element g of G which conjugates Λ to $\Lambda \otimes \chi_{\boldsymbol{J}}$ must normalize $J = \boldsymbol{J} \cap \operatorname{Ker}\|\det\|$ and fix λ. That is, $g \in \boldsymbol{J}$. $\square$

We refer to pairs $(\boldsymbol{J},\Lambda)$ obtained this way as *extended simple types of level zero.* We may recover (J,λ) from $(\boldsymbol{J},\Lambda)$: the group J is the unique maximal compact subgroup of $\boldsymbol{J}$ and $\lambda = \Lambda|_J$. We say that $(\boldsymbol{J},\Lambda)$ *lies over* (J,λ).

4.2. The following result derives from [30], but we have changed the proof to better serve our purposes.

Proposition 7. *Let $\pi \in \mathcal{A}_n^\square(F)_0$.*

(1) *There is an extended simple type $(\boldsymbol{J}_\pi, \Lambda_\pi)$ of level zero such that*

$$\operatorname{Hom}_{\boldsymbol{J}_\pi}(\Lambda_\pi, \pi) \neq 0.$$

(2) *The pair $(\boldsymbol{J}_\pi, \Lambda_\pi)$ is uniquely determined by π, up to conjugation in G.*
(3) *The map $\pi \mapsto (\boldsymbol{J}_\pi, \Lambda_\pi)$ establishes a bijection between $\mathcal{A}_n^\square(F)_0$ and the set of G-conjugacy classes of extended simple types of level zero in G.*

Proof. Let $(\pi, V) \in \mathcal{A}_n^\square(F)_0$. The representation π contains a simple type (J,λ) of level zero, and it does so with multiplicity one (3.6.1). Let $\boldsymbol{J}$ be the G-stabilizer of (J,λ). The group $\pi(\boldsymbol{J})$ must stabilize the isotypic space V^λ. As representation of J, the space V^λ affords the original irreducible representation λ. The natural action of $\pi(\boldsymbol{J})$ on V^λ thus provides an extension Λ of λ such that $\operatorname{Hom}_{\boldsymbol{J}}(\Lambda,\pi) \neq 0$. Since (J,λ) is determined by π up to G-conjugacy, the same applies to $(\boldsymbol{J},\Lambda)$. We have proved (1), (2) and found a well-defined map from $\mathcal{A}_n^\square(F)_0$ to the set of conjugacy classes of extended simple types of level zero.

To prove this map is surjective, we take an extended simple type $(\boldsymbol{J},\Lambda)$ of level zero, over a simple type (J,λ). The representation λ occurs in some $\pi \in \mathcal{A}_n^\square(F)_0$ and, by the first part of the proof, this representation π contains some representation Λ' of $\boldsymbol{J}$ such that $\Lambda'|_J \cong \lambda$. We have $\Lambda \cong \Lambda' \otimes \chi_{\boldsymbol{J}}$, for an unramified character χ of $F^\times$ (Lemma 10). The extension Λ thus occurs in the representation $\chi\pi : g \mapsto \chi(\det g)\pi(g)$ of G, and $\chi\pi \in \mathcal{A}_n^\square(F)_0$.

It remains to show that an extended simple type $(\boldsymbol{J}, \Lambda)$ occurs in only one element of $\mathcal{A}_n^{\square}(F)_0$. Let $(\boldsymbol{J}, \Lambda)$ lie over the simple type (J, λ). It is enough to treat the case where $J = U_{\mathfrak{a}}$ and $\mathfrak{a} = \mathfrak{a}_F(e, f)$ (2.2.1). Let λ act on the vector space X, and use the other notation introduced in 2.3.

Let (π, V) contain Λ and consider the natural action of $\mathcal{H}_\lambda$ on the one-dimensional space V_λ. We recall (3.3.1)

$$\phi \cdot f(x) = \int_G \pi(g) f\bigl(\check{\phi}\bigl(g^{-1}\bigr)x\bigr)\, d\mu_G(g), \tag{4.2.1}$$

for $\phi \in \mathcal{H}_\lambda$, $f \in V_\lambda = \mathrm{Hom}_J(X, V)$, $x \in X$. Taking ϕ_Π as in Lemma 2 of 2.5, we find

$$\phi_\Pi \cdot f(x) = \pi(\Pi) f(\check{\Gamma} x).$$

However, the natural action of $\boldsymbol{J}$ on V^λ provides the extension Λ of λ occurring in π. That is,

$$\phi_\Pi \cdot f(x) = \Lambda(\Pi) f(\check{\Gamma} x) = \chi_\pi(\phi_\Pi) f(x).$$

We may take $f \neq 0$. In this case, f is injective and so $\Lambda(\Pi)$ is the operator given by

$$\Lambda(\Pi) = \chi_\pi(\phi_\Pi)\check{\Gamma}^{-1} \in \mathrm{Aut}_{\mathbb{C}}(X). \tag{4.2.2}$$

The character χ_π is determined by its value on ϕ_Π (3.6, Remark 2). Thus Λ determines the character χ_π and also the representation π (Lemma 6 of 3.3). $\square$

5. Parametrization of the discrete series

In Proposition 7 (4.2), we gave a description of the elements of $\mathcal{A}_n^{\square}(F)_0$ in terms of extended simple types of level zero. We convert this into a parametrization in terms of admissible tame pairs, analogous to the discussions of Section 1.

5.1. We review the well-known *cuspidal* case. Section 2.2 of [7] provides further details. Let $(E/F, \theta)$ be an admissible tame pair of degree f.

We set $A_0 = \mathrm{M}_f(F)$ and $G_0 = \mathrm{GL}_f(F)$. We identify $\Bbbk_E$ with a subfield of $\mathrm{M}_f(\Bbbk_F)$. There is a unique character $\tilde{\theta}_0$ of $\Bbbk_E^\times$ which inflates to $\theta|_{U_E}$ via the canonical map $U_E \to \Bbbk_E^\times$. If $\Sigma = \mathrm{Gal}(\Bbbk_E/\Bbbk_F) \cong \mathrm{Gal}(E/F)$, the conjugates $\tilde{\theta}_0^\sigma$, $\sigma \in \Sigma$, are distinct. We use the Green parametrization to construct from $\tilde{\theta}_0$ an irreducible cuspidal representation $\tilde{\lambda}_0$ of $\mathrm{GL}_f(\Bbbk_F)$. This is determined by the character formula

$$\mathrm{tr}\,\tilde{\lambda}_0(x) = (-1)^{f-1} \sum_{\sigma \in \Sigma} \tilde{\theta}_0\bigl(x^\sigma\bigr), \tag{5.1.1}$$

valid for every $x \in \Bbbk_E^\times$ such that the conjugates x^σ, $\sigma \in \Sigma$, are distinct.

We write $J_0 = \mathrm{GL}_f(\mathfrak{o}_F)$ and inflate $\tilde{\lambda}_0$ to a representation λ_0 of J_0. We next set $\boldsymbol{J}_0 = F^\times \mathrm{GL}_f(\mathfrak{o}_F)$ and extend λ_0 to a representation Λ_0 of $\boldsymbol{J}_0$ by deeming that $\Lambda_0|_{F^\times}$ be a multiple of the character $\theta|_{F^\times}$. The pair $(\boldsymbol{J}_0, \Lambda_0)$ is an extended maximal simple type in G_0. The G_0-conjugacy class of $(\boldsymbol{J}_0, \Lambda_0)$ depends only on the isomorphism class of $(E/F, \theta)$.

We set

$$\pi_f(\theta) = c\text{-}\mathrm{Ind}_{\boldsymbol{J}_0}^{G_0} \Lambda_0. \tag{5.1.2}$$

Proposition 8.

(1) *Let $(E/F,\theta)$ be an admissible tame pair of degree f. The representation $\pi_f(\theta)$, defined by* (5.1.2)*, is an irreducible cuspidal representation of $G_0 = \mathrm{GL}_f(F)$. The equivalence class of $\pi_f(\theta)$ depends only on the F-isomorphism class of $(E/F,\theta)$.*
(2) *The map*

$$\begin{aligned} \mathcal{T}_f(F) &\longrightarrow \boldsymbol{\mathcal{A}}^0_f(F)_0, \\ (E/F,\theta) &\longmapsto \pi_f(\theta), \end{aligned} \tag{5.1.3}$$

is a bijection.

Proof. The proposition summarizes 2.2 Proposition of [7]. □

5.2. We generalize the construction of 5.1 to account for all elements of the set $\boldsymbol{\mathcal{A}}^{\square}_n(F)_0$.

Let $(E/F,\theta) \in \mathcal{T}_f(F)$, let $e \geqslant 1$, and set $n = ef$. We use the character θ to define the cuspidal representation $\tilde{\lambda}_0$ of $\mathrm{GL}_f(\Bbbk_F)$, as in (5.1.1). We let $\mathfrak{a} = \mathfrak{a}_F(e,f)$ and $J = U_{\mathfrak{a}}$. If $\mathfrak{p}_{\mathfrak{a}}$ is the Jacobson radical of $\mathfrak{a}$, we have $J/1+\mathfrak{p}_{\mathfrak{a}} = \mathrm{GL}_f(\Bbbk_F)^e$. Following Definition 1 of 2.1, we form the representation λ of J inflated from $\tilde{\lambda}_0 \otimes \tilde{\lambda}_0 \otimes \cdots \otimes \tilde{\lambda}_0$. We so obtain a simple type (J,λ) in $G = \mathrm{GL}_n(F)$. The G-conjugacy class of (J,λ) then depends only on the isomorphism class of $(E/F,\theta)$.

We next choose a prime element ϖ of F and use it to define $\Pi \in \boldsymbol{J}$, just as in 2.3. Therefore $\boldsymbol{J} = \Pi^{\mathbb{Z}} \ltimes J$ is the G-normalizer of (J,λ). Let X_0 be the representation space of $\tilde{\lambda}_0$. Thus $\lambda : J \to \mathrm{Aut}_{\mathbb{C}}(X)$, where $X = X_0 \otimes X_0 \otimes \cdots \otimes X_0$ (with e factors in the tensor product). We define ${}_d\Gamma \in \mathrm{Aut}_{\mathbb{C}}(X)$ by

$${}_d\Gamma : x_1 \otimes x_2 \otimes \cdots \otimes x_e \longmapsto x_2 \otimes \cdots \otimes x_e \otimes x_1, \quad x_k \in X_0.$$

In the notation of 2.5, we have ${}_d\Gamma = \check{\Gamma}^{-1}$.

Lemma 11.

(1) *There is a unique representation Λ_ϖ of $\boldsymbol{J}$ such that $\Lambda_\varpi|_J \cong \lambda$ and $\Lambda_\varpi(\Pi) = {}_d\Gamma$.*
(2) *Let $a \in J$ have image $(a_1, a_2, \ldots, a_e)$ in $\mathrm{GL}_f(\Bbbk_F)^e$. The representation Λ_ϖ then satisfies*

$$\operatorname{tr} \Lambda_\varpi(\Pi a) = \operatorname{tr} \tilde{\lambda}_0(a_1 a_2 \cdots a_e). \tag{5.2.1}$$

Proof. The statement merely recalls the definition and character formula for "tensor induction". □

The defining identity (5.2.1) shows clearly that the representation Λ_ϖ depends on the choice of the prime element $\varpi \in F$. However, the same identity implies that the representation Λ extending λ and given by

$$\Lambda(\Pi) = \theta\big((-1)^{e-1}\varpi\big)\Lambda_\varpi(\Pi) \tag{5.2.2}$$

is *independent* of the choice of ϖ. The pair $(\boldsymbol{J},\Lambda)$ is an extended simple type in G of level zero. We temporarily denote it by $(\boldsymbol{J}_\theta, \Lambda_\theta)$.

Lemma 12.

(1) *The G-conjugacy class of the representation $(\boldsymbol{J}_\theta, \Lambda_\theta)$ depends only on the F-isomorphism class of $(E/F,\theta)$.*

(2) *The map* $(E/F, \theta) \mapsto (\boldsymbol{J}_\theta, \Lambda_\theta)$ *induces a bijection of* $\mathcal{T}(F; n)$ *with the set of conjugacy classes of extended simple types of level zero in* G.

Proof. Let $(\boldsymbol{J}_\theta, \Lambda_\theta)$ lie over the simple type $(J_\theta, \lambda_\theta)$. The conjugacy class of $(J_\theta, \lambda_\theta)$ is visibly independent of any choice made in its construction. The definition (5.2.2) and the character formula (5.2.1) now give (1). Part (2) follows from Lemma 10 of 4.1 and the definition (5.2.2). □

We remark that we make no direct use of part (2) of the lemma. We have included the proof since it is so easy.

Proposition 7 of 4.2 now allows us to make the following definition.

Definition 2. Let $(E/F, \theta) \in \mathcal{T}(F; n)$. Define $\pi_n(\theta)$ to be the unique element of $\mathcal{A}_n^\square(F)_0$ containing the extended simple type $(\boldsymbol{J}_\theta, \Lambda_\theta)$.

We remark that, in the case $n = f$, this definition coincides with (5.1.2).

Proposition 9. *The map*

$$\begin{aligned} \mathcal{T}(F; n) &\longrightarrow \mathcal{A}_n^\square(F)_0, \\ (E/F, \theta) &\longmapsto \pi_n(\theta), \end{aligned} \tag{5.2.3}$$

is a canonical bijection.

Proof. This is a direct consequence of Lemma 12 and Proposition 7 of 4.2. □

Again, we make no direct use of this result.

6. Comparison theorems

We compare the various parametrizations π_D, π_n, ρ_n by admissible tame pairs, elaborated in Section 1 and Section 5.

6.1. Let $n \geqslant 1$ be an integer, and let D be a central F-division algebra of dimension n^2. The *Jacquet–Langlands correspondence* [25,1] gives a canonical bijection

$$\boldsymbol{j} : \mathcal{A}_1(D) \xrightarrow{\approx} \mathcal{A}_n^\square(F). \tag{6.1.1}$$

Theorem 1. *Let* $n = ef$, *where* e *and* f *are positive integers. Let* $(E/F, \theta) \in \mathcal{T}_f(F)$. *If* η_E *denotes the unramified quadratic character of* $E^\times$, *then*

$$\boldsymbol{j}\pi_D(\theta) = \pi_n\big(\eta_E^{f(e-1)}\theta\big). \tag{6.1.2}$$

Allowing for a difference of notation, this result is a case of Theorem 3 on p. 184 of [30].

6.2. For each integer $n \geqslant 1$, the *Langlands correspondence* [21,15,16] gives a canonical bijection

$$\boldsymbol{l} : \mathcal{A}_n(F) \xrightarrow{\approx} \mathcal{G}_n(F). \tag{6.2.1}$$

This restricts to give a canonical bijection

$$\boldsymbol{l} : \mathcal{A}_n^0(F) \xrightarrow{\approx} \mathcal{G}_n^0(F). \tag{6.2.2}$$

Lemma 13. *If e and f are positive integers, and $\tau \in \boldsymbol{\mathcal{A}}_f^0(F)$, then*

$$\boldsymbol{l}\big(\mathrm{St}_e(\tau)\big) = \mathrm{Sp}_e(\boldsymbol{l}\tau). \tag{6.2.3}$$

This is proved in [17, 2.7].

6.3. We arrive at our destination.

Theorem 2. *Let $n = ef$, where e and f are positive integers. Let $(E/F, \theta) \in \mathfrak{T}_f(F)$. If η_E denotes the unramified quadratic character of $E^\times$, then*

$$\boldsymbol{l}\pi_n(\theta) = \rho_n\big(\eta_E^{e-f}\theta\big). \tag{6.3.1}$$

Before proving the theorem, we note that on combining (6.3.1) with (6.1.2), we obtain the following consequence.

Corollary 1. *Let $n = ef$, where e and f are positive integers. Let $(E/F, \theta) \in \mathfrak{T}_f(F)$. If η_E denotes the unramified quadratic character of $E^\times$, then*

$$\boldsymbol{lj}\pi_D(\theta) = \rho_n\big(\eta_E^{e(f-1)}\theta\big).$$

Corollary 1 is, of course, the Main Theorem of the introduction.

6.4. We prove Theorem 2 of 6.3. Consider first the representation $\boldsymbol{l}\pi_f(\theta)$. According to 2.4, Theorem 2 of [9], this is given by

$$\boldsymbol{l}\pi_f(\theta) = \rho_f\big(\eta_E^{f-1}\theta\big). \tag{6.4.1}$$

We turn to the representation $\pi = \pi_n(\theta)$. By Lemma 7 of 3.4, $\pi = \mathrm{St}_e(\tau)$, for some $\tau \in \boldsymbol{\mathcal{A}}_f^0(F)$ containing the representation λ_0 introduced in 5.1.

Let χ_π be the character of $\mathcal{H}_\lambda$ attached to π, as in Proposition 4 of 3.6. Combining (4.2.2) and (5.2.2), we find that

$$\chi_\pi(\phi_\Pi) = \theta\big((-1)^{e-1}\varpi\big). \tag{6.4.2}$$

Applying (3.7.3), we get

$$(-1)^{e-1}\omega_\tau\big((-1)^{e-1}\varpi\big) = \chi_\pi(\phi_\Pi) = \theta\big((-1)^{e-1}\varpi\big).$$

Recalling (3.7.1), this implies $\omega_\tau(\varpi) = (-1)^{e-1}\theta(\varpi)$. We deduce that $\tau = \pi_f(\eta_E^{e-1}\theta)$. We apply (6.4.1) to obtain $\boldsymbol{l}\tau = \rho_f(\eta_E^{e-f}\theta)$. By (6.2.3) and the definition (1.4.3),

$$\boldsymbol{l}\pi = \mathrm{Sp}_e(\boldsymbol{l}\tau) = \rho_n\big(\eta_E^{e-f}\theta\big),$$

whence the theorem follows.

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