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Monomial maps on $\mathbb{P}^2$ and their arithmetic dynamics

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ABSTRACT

We say that a rational map on $\mathbb{P}^n$ is a monomial map if it can be expressed in some coordinate system as $[F_0 : \cdots : F_n]$ where each F_i is a monomial. We consider arithmetic dynamics of monomial maps on $\mathbb{P}^2$. In particular, as Silverman (1993) explored for rational maps on $\mathbb{P}^1$, we determine when orbits contain only finitely many integral points. Our first result is that if some iterate of a monomial map on $\mathbb{P}^2$ is a polynomial, then the first such iterate is 1, 2, 3, 4, 6, 8, or 12. We then completely determine all monomial maps whose orbits always contain just finitely many integral points. Our condition is based on the exponents in the monomials. In cases when there are finitely many integral points in all orbits, we also show that the sizes of the numerators and the denominators are comparable. The main ingredients of the proofs are linear algebra, such as Perron–Frobenius theorem.

Dynamics is a study of iterative behaviors of a rational map

$$\phi = \frac{a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0}{b_d z^d + b_{d-1} z^{d-1} + \cdots + b_0}.$$

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Namely, it is a study of the limiting behavior of the n-th iterate $\phi^{(n)} = \underbrace{\phi \circ \cdots \circ \phi}_{n \text{ times}}$ as n goes to ∞. Traditionally, this has been explored over the field of complex numbers, that is, when a_i and b_i are in $\mathbb{C}$. On the other hand, rational maps are really algebraic morphisms on $\mathbb{P}^1$, and when they are defined over a field of arithmetic interests, one can also explore arithmetic questions related to the dynamics (see [5] for an introduction to the subject). Silverman [4] was one of the first to explore these directions. In particular, he investigated when the orbit $\mathcal{O}_\phi(P) = \{P, \phi(P), \phi^{(2)}(P), \phi^{(3)}(P), \ldots\}$ of a point $P \in \mathbb{Q}$ under $\phi \in \mathbb{Q}(z)$ contains infinitely many distinct integers. If $\phi^{(n)}$ is ever a polynomial, then just plugging in an integer produces infinitely many integers. Silverman's result in [4] is that aside from this obvious exception, we always get only finitely many integers in each orbit.

There have been some extensions and improvements of [4]. For example, [2] shows that we get new primes dividing numerators and the denominators of $\phi^{(n)}(P)$ as $n \to \infty$. This is more precise than [4], involving non-archimedean absolute values. As for extending to spaces other than $\mathbb{P}^1$, [1] deals with the analog for maps on $(\mathbb{P}^1)^g$.

In this article, we consider the analog of [4] for monomial maps on $\mathbb{P}^2$. These are the maps of the form $[F_0 : F_1 : F_2]$ where each F_i is a monomial. Most of these are merely rational maps, not morphisms. We will work mostly with the dehomogenized form $\phi = (x^i y^j, x^k y^l)$ with $i, j, k, l \in \mathbb{Z}$, obtained by dividing by F_2. These are precisely the algebraic-group morphisms on $\mathbb{G}_m \times \mathbb{G}_m$, but we want to analyze integral points, so we will view the monomial maps as special examples of rational maps on $\mathbb{P}^2$. A natural notion of "integral points" in this setting is $(\mathbb{P}^2 \setminus (Z = 0))(\mathbb{Z}) = \{[X : Y : 1]\colon\ X, Y \in \mathbb{Z}\}$, and in the dehomogenized form, we will often refer to this set as the set of integers. With this setup, we analyze the analog of [4] for monomial maps on $\mathbb{P}^2$.

Theorem 1 determines which iterations monomial maps can become polynomials. For $\mathbb{P}^1$, Riemann–Hurwitz guarantees that if $\phi^{(2)}$ is not a polynomial then no $\phi^{(n)}$ will be. For monomial maps on $\mathbb{P}^2$, the analog turns out to be $\phi^{(24)}$. In fact, we classify exactly when the first polynomial can occur. Theorem 2 and Theorem 4 are generalizations of Silverman's result to monomial maps on $\mathbb{P}^2$. Theorem 2 characterizes monomial maps which *always* have finitely many integers in the orbits no matter the initial point. Unlike the situation of $\mathbb{P}^1$, where we are guaranteed finiteness of integers in all orbits if we avoid polynomials, it is not nearly as common for monomial maps. Our characterization is in terms of the exponent matrix $\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$ of $\phi = (x^i y^j, x^k y^l)$ and its eigenvalues. Theorem 4 is the analog of the deeper result of Silverman's; that is, when we are guaranteed finiteness of integers in orbits, how "far away" from integers the orbit points actually are. Silverman's result says that the numerator and the denominator of $\phi^{(n)}(P)$ become comparable logarithmically as $n \to \infty$. For monomial maps on $\mathbb{P}^2$, we show in Theorem 4 that when we write both the x-coordinate and the y-coordinate in reduced fractions, the product of the numerators and the product of the denominators become comparable logarithmically. The proofs of Theorems 1, 2, and 4 all utilize the asymptotic analyses of the n-th power of the exponent matrix via the eigenvalues.

This paper is organized as follows. The first section contains some background material on monomial maps, and then we give the precise statements of all the theorems. We also remark there that the question of finiteness of integral points in orbits was also explored by the second author [7] *assuming* a very deep conjecture in Diophantine geometry by Vojta [6]. The monomial maps do *not* satisfy the hypotheses of the main theorem in [7] so this article is not a direct example. On the other hand, our original motivation for exploring the dynamics (in particular integers in orbits) of monomial maps was trying to find explicit maps where questions of the form in [7] can be answered unconditionally, so we will discuss the connections. The next three sections contain the proofs of the three theorems, respectively. After the proofs, we include some concrete numerical examples.

1. Background and precise statements of the theorems

Here we discuss the background material on monomial maps and then present precise statements of the results. We say a map $\phi = [F_0 : F_1 : F_2]$ on $\mathbb{P}^2$ is a *monomial map* if the F_i's are monomials of same degree. Unless $\{F_0, F_1, F_2\} = \{X^d, Y^d, Z^d\}$ as sets, ϕ is not a morphism. On the other hand, the points where ϕ might not be defined are restricted to just those on the triangle $XYZ = 0$. In particular, as long as $\alpha\beta\gamma \neq 0$, the orbit of $[\alpha : \beta : \gamma]$ is well-defined.

Since F_0/F_2 and F_1/F_2 are degree-zero, they can be expressed as $(X/Z)^i(Y/Z)^j$ for some $i, j \in \mathbb{Z}$. For most of this paper, we will use this dehomogenized notation $\phi = (x^i y^j, x^k y^l)$. We define the *exponent matrix* of ϕ to be $\left(\begin{smallmatrix} i & j \\ k & l \end{smallmatrix}\right)$, and denote it by A_ϕ. This is the Jacobian of ϕ at $(1, 1)$. Since

$$\phi\left(x^p y^q, x^r y^s\right) = \left(x^{ip+jr} y^{iq+js}, x^{kp+lr} y^{kq+ls}\right),$$

we can check by induction that $A_{\phi^{(n)}} = (A_\phi)^n$. Therefore, the exponent matrix A_ϕ and its eigenvalues play crucial roles in the dynamics of the monomial map ϕ.

Our first result is determining the first iterate of ϕ that is a polynomial (if ever). Here, a polynomial means that in the dehomogenized representation of ϕ, both coordinates are polynomials, that is, we do not have any negative entries in A_ϕ. Our interest is analyzing integer-points $(\mathbb{P}^2\backslash(Z = 0))(\mathbb{Z}) = \{[X : Y : 1]\colon\ X, Y \in \mathbb{Z}\}$ in orbits, and a polynomial trivially creates infinitely many integers in the orbits of $(\alpha, \beta) \in \mathbb{Z}^2$. So just as in $\mathbb{P}^1$, we would like to know up to what iterations we have to check to see if we ever get a polynomial in $\phi^{(n)}$. By the previous paragraph, this reduces to determining powers of A_ϕ which can have all entries nonnegative. There is a useful criterion for entrywise nonnegative matrices in linear algebra, namely Perron–Frobenius theorem (Theorem 5 in Section 2). Together with some cyclotomic field theory, we prove the following result.

Theorem 1. *Let ϕ be a monomial map on $\mathbb{P}^2$. If $\phi^{(n)}$ is a polynomial for some n, then the first such n is* 1, 2, 3, 4, 6, 8 *or* 12.

Hence, if $\phi^{(24)}$ is not a polynomial, then $\phi^{(n)}$ will never be a polynomial. All of the numbers listed in the theorem do occur:

ϕ (dehomogeneous)	ϕ (homogeneous)	First polynomial iterate
(xy, y)	$[XY : YZ : Z^2]$	$\phi^{(1)} = (xy, y)$
$(x^{-1}y^{-1}, x^{-1})$	$[Z^2 : YZ : XY]$	$\phi^{(2)} = (x^2y, xy)$
$(x^{-1}y^{-1}, x)$	$[Z^2 : X^2 : XY]$	$\phi^{(3)} = (x, y)$
(y^{-1}, x)	$[Z^2 : XY : YZ]$	$\phi^{(4)} = (x, y)$
(y^{-1}, xy)	$[Z^3 : XY^2 : YZ^2]$	$\phi^{(6)} = (x, y)$
$(x^{-1}y, x^{-1}y^{-1})$	$[Y^2Z : Z^3 : XYZ]$	$\phi^{(8)} = (x^{16}, y^{16})$
(xy^{-1}, xy^2)	$[XZ^3 : XY^2Z : YZ^3]$	$\phi^{(12)} = (x^{729}, y^{729})$

These simple examples already indicate a pattern: if a monomial map becomes a polynomial for the first time after three or more iterations, then the first polynomial seems to be of the form (x^m, y^m), i.e. corresponding to a scalar matrix. This observation, proved in Lemma 7, is crucial to our proof of Theorem 1.

Our second result gives a characterization of monomial maps whose orbits always contain just finitely many integers. Unlike the $\mathbb{P}^1$ case, there exist many monomial maps which do not ever become polynomials but which have an orbit that contains infinitely many distinct integers. Whether or not an orbit with infinitely many integers exists is controlled by the eigenvalues of the matrix A_ϕ.

Theorem 2. *Let $\phi = (x^i y^j, x^k y^l)$ be a monomial map on $\mathbb{P}^2$, and let $A = A_\phi$ be its exponent matrix. Suppose that at least some orbit of ϕ contains infinitely many distinct rational points. In the following circumstances, all orbits contain just finitely many distinct integers, i.e. $\mathcal{O}_\phi(P) \cap (\mathbb{P}^2\backslash(Z = 0))(\mathbb{Z})$ is a finite set for any $P \in \mathbb{P}^2(\mathbb{Q})\backslash(XYZ = 0)$:*

(1) *A has two real eigenvalues $\lambda_1, \lambda_2 \notin \mathbb{Q}$ with $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| > 1$, satisfying $(i - \lambda_1)j > 0$.*
(2) *A is diagonalizable with two rational eigenvalues λ_1, λ_2 with $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| > 1$, satisfying $(i - \lambda_1)j > 0$ and either $|\lambda_2| \leqslant 1$ or $(i - \lambda_2)j > 0$.*

(3) *A is not diagonalizable with the unique eigenvalue* λ *satisfying* $|\lambda| > 1$, *and* $(i - \lambda)j > 0$.
(4) $\phi = (x/y^m, y)$, $(y^m/x, 1/y)$, $(x, y/x^m)$, *or* $(1/x, x^m/y)$ *for some* $m \geqslant 1$.

In all other situations (including the case when A has complex eigenvalues), there exists $P \in \mathbb{P}^2(\mathbb{Q}) \backslash (XYZ = 0)$ *such that* $\mathcal{O}_\phi(P)$ *contains infinitely many distinct integral points.*

Example 3. Let $\phi = (y/x, x)$, so the exponent matrix $A = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. It is not clear by an immediate inspection if all orbits have finitely many integers. The eigenvalues are $\lambda_1 = \frac{-1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{-1+\sqrt{5}}{2}$, so $(i - \lambda_1)j = \frac{\sqrt{5}-1}{2} > 0$. Thus, this ϕ is in category (1) of the theorem, and so all orbits have just finitely many integers. For this particular example, we can explicitly see what prevents orbits from having infinitely many integers. We can prove by induction that

$$A^n = (-1)^n \begin{pmatrix} F_{n+1} & -F_n \\ -F_n & F_{n-1} \end{pmatrix},$$

where $\{F_0, F_1, F_2, \ldots\} = \{0, 1, 1, 2, 3, 5, 8, 13, \ldots\}$ is the Fibonacci sequence. Therefore, we have

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right),$$

and $\frac{F_{n-1}}{F_n} > \frac{F_n}{F_{n+1}}$ if n is even and the inequality is reversed if n is odd. Suppose on the contrary that the orbit of (α, β) under ϕ contains infinitely many integer points. Since α and β cannot both be ± 1, there is a prime p such that $\operatorname{ord}_p(\alpha)$ or $\operatorname{ord}_p(\beta)$ is nonzero. Let us consider the case when $\operatorname{ord}_p(\beta) \neq 0$ and there are infinitely many *even* n's such that $\phi^{(n)}(\alpha, \beta)$ is an integer; the cases of infinitely many odd n's and/or $\operatorname{ord}_p(\alpha) \neq 0$ will be similar. As the exponent matrix for $\phi^{(n)}$ is A^n and being an integer is equivalent to having nonnegative valuations at every prime, we must have

$$F_{n+1} \operatorname{ord}_p(\alpha) - F_n \operatorname{ord}_p(\beta) \geqslant 0$$
$$-F_n \operatorname{ord}_p(\alpha) + F_{n-1} \operatorname{ord}_p(\beta) \geqslant 0$$

for infinitely many even n. Thus, if $\operatorname{ord}_p(\beta) > 0$, then $\frac{F_n}{F_{n+1}} \leqslant \frac{\operatorname{ord}_p(\alpha)}{\operatorname{ord}_p(\beta)} \leqslant \frac{F_{n-1}}{F_n}$, and as the two ends go to the same irrational number (the reciprocal of the golden ratio), while the middle is a fixed rational number, this is a contradiction. If $\operatorname{ord}_p(\beta) < 0$, then a similar argument forces $\frac{F_n}{F_{n+1}} > \frac{F_{n-1}}{F_n}$, an immediate contradiction. We will discuss this example further in Example 10.

As in the above example, the proof of Theorem 2 involves an analysis of the asymptotics of $A_{\phi^{(n)}} = (A_\phi)^n$, and we will use the Jordan normal form of A. Although it is stated over $\mathbb{Q}$ here, one can generalize Theorem 2 to number fields. One care has to be taken: other than $\mathbb{Q}$ and quadratic imaginary fields, we have infinitely many units, and monomial maps send units to units, so the corresponding statement for general number fields would characterize when one gets an orbit with infinitely many distinct non-unit integers.

Our final result is the analog of the deepest result in [4]. When there are only finitely many integers in every orbit, one can ask how the sizes of the denominators and the numerators compare as we iterate the map. Silverman uses Roth's theorem to show that the ratio of $\log|a_n|$ and $\log|b_n|$ goes toward 1 as n goes to infinity, where a_n/b_n is $\phi^{(n)}(P)$ in a reduced fraction. We get a similar result for monomial maps, but now the limit might not be 1:

Theorem 4. *Assume that a monomial map* ϕ *belongs to one of the cases* (1)–(3) *of Theorem* 2. *Let* $P \in \mathbb{P}^2(\mathbb{Q}) \backslash (XYZ = 0)$ *such that the orbit* $\mathcal{O}_\phi(P)$ *is an infinite set. Writing both the x- and the y-coordinates* *of $\phi^{(n)}(P)$ in reduced fractions, let N_n be the product of the numerators and let D_n be the product of the denominators. Then*

$$\lim_{n\to\infty} \frac{\max(\log|N_n|, \log|D_n|)}{\min(\log|N_n|, \log|D_n|)}$$

exists and is a strictly positive number.

This limit is computable from the exponent matrix and its eigenvalues, together with primes at which the point P has nonzero valuations. For example, if A has nonrational real eigenvalues and if $P = (\alpha, \beta)$ is such that $\mathrm{ord}_p(\alpha)$ and $\mathrm{ord}_p(\beta)$ are all zero except for one prime, then the limit is $\frac{j}{i-\lambda}$ (or its reciprocal if this is less than 1), where λ is the eigenvalue with the bigger absolute value. More generally, the limit is a ratio of positive linear combinations of logarithms of primes.

Since the number of digits of the product of the numerators and the number of digits of the product of the denominators have to become roughly the same, this theorem says that the orbit points must be very far from being integers. So this is an analog of Silverman's result. On the other hand, a more natural generalization would be the comparability of the logarithm of the maximum of the homogeneous coordinates of $\phi^{(n)}(P)$ and the logarithm of the Z-th coordinate. In fact, we can see from Theorem 4 that these two logarithms are also comparable: if the point $(r/s, t/u) = [ru : st : su]$ satisfies $\log(rt) \sim \log(su)$, then we have $\log(ru) \leqslant \log(rt) + \log(su)$ and $\log(st) \leqslant \log(rt) + \log(su)$, so the logarithms of both the X-coordinate and the Y-coordinate are comparable to the logarithm of the Z-coordinate. More precisely, in the remark following the proof of Theorem 4, we will show that the limit

$$\frac{\log(\max(|X_n|, |Y_n|, |Z_n|))}{\log|Z_n|}$$

also exists, where $\phi^{(n)}(P) = [X_n : Y_n : Z_n]$ written with integers without common divisors. However, the expression for this limit is much more complicated than the limit for Theorem 4.

Note that Theorem 4 does not hold for ϕ in case (4) of Theorem 2: if $\phi = (x/y, y)$, then $\phi^{(n)} = (x/y^n, y)$, so $\phi^{(n)}(1, 2) = (1/2^n, 2)$.

We end this section with two remarks. First, our original motivation for this paper comes from trying to find unconditional examples of the previous work [7] on integer orbits by the second author, which *assumed* a powerful conjecture in Diophantine geometry formulated by Vojta [6]. The main theorem [7, Theorem 1.1] has two parts: (1) if $\phi : \mathbb{P}^N \to \mathbb{P}^N$ is a *morphism* of degree d such that there exists k with $d^k > N + 1$ making the pullback $(\phi^{(k)})^*(H)$ of a hyperplane divisor H a normal crossings divisor, then for any $P \in \mathbb{P}^N(\mathbb{Q})$, the orbit $\mathcal{O}_\phi(P)$ contains only finitely many integer points $(\mathbb{P}^N \backslash H)(\mathbb{Z})$ (2) if ϕ is a morphism such that $(\phi^{(k)})^*(H)$ is a normal crossings divisor for all k, then the ratio of the logarithm of the H-coordinate of $\phi^{(n)}(P)$ to the logarithmic height of $\phi^{(n)}(P)$ goes to 1 as $n \to \infty$. Vojta's conjecture is a powerful height inequality in Diophantine geometry, and it can be viewed as an extension of Roth's theorem on $\mathbb{P}^1$. Therefore, [7] is a generalization of Silverman's result to higher-dimensional projective spaces, replacing the nonpolynomial condition (i.e. not totally ramified) of Silverman by the normal-crossings condition (which implies not ramified at all).

Viewed in this context, Theorems 2 and 4 offer similar results as [7] for the case of monomial maps, without assuming any conjectures. On the other hand, they are not examples of [7], since monomial maps are *rational* maps and not morphisms. Moreover, H in our case is $Z = 0$, but $(\phi^{(k)})^*(H)$ for a monomial map ϕ is the triangle $XYZ = 0$ with high multiplicity, so it is never normal-crossings. So the key hypotheses of the main theorem of [7] are not satisfied by the monomial maps on $\mathbb{P}^2$, and accordingly the results are different: in Theorem 2, we show that many maps can have infinitely many distinct integers without their iterations ever becoming a polynomial, and in Theorem 4, we show that the log ratio of the sizes of the coordinates can go to a number not equal to 1. Nevertheless, monomial maps offer concrete examples where one can explore similar types of questions, possibly gathering some evidence for generalizations of [7] to rational maps.

Our second remark is about generalizing Theorems 1, 2, and 4 to monomial maps on $\mathbb{P}^3$ or even higher-dimensional projective spaces. At a first glance, this seems to be a lot more difficult. For one thing, it is not even clear if anything similar to Theorem 1 even holds. Just by using exponents between -2 and 2, for any n between 1 and 46, we can find a monomial map whose first polynomial iterate occurs at n. For example, if $\phi = (xy^{-1}, x^2y^2z^{-1}, x^{-1}y^{-1}z^{-2})$, then the first polynomial iterate is $\phi^{(46)}$, which is

$$\begin{aligned}&\bigl(x^{889794221451447}y^{427541822476455}z^{1281621242409692},\\ &\ x^{426537597456782}y^{1743873641384684}z^{4272405549705531},\\ &\ x^{2990784307295839}y^{5554026792115223}z^{21396738325026192}\bigr).\end{aligned} \tag{1}$$

We give a few more examples: the first polynomial iterate of $(x^{-2}z, xy^{-2}, xyz^2)$ is the 43rd one, the first polynomial iterate of $(x^{-1}y^{-1}z^{-2}, xy^2z^{-2}, x^{-2}y^{-1})$ is the 44th one, and the first polynomial iterate of $(y^{-1}z^2, x^2y^{-2}z^{-1}, x^2yz)$ is the 45th one. We did not find any monomial whose first polynomial iterate occurs at 47, but this is probably caused by the limited range of integers used. So the situation is very different from the $\mathbb{P}^2$ case. Moreover, as one can even see in (1), the first polynomial iterate does not seem to correspond to any sort of a special class of matrices, as we obtain the scalars for $\mathbb{P}^2$ (Lemma 7). One more difficulty is that our proofs of Theorems 2 and 4 are based on Jordan normal forms for 2 by 2 matrices, making the generalization of the arguments difficult.

2. Iteration of monomial maps and polynomials

We will now prove Theorem 1. This is really a corollary of a linear algebra result (applied to the matrix A_ϕ), so we will state the results in this setting. We first make the following definition.

Definition. If A is a real matrix, we say A is *entrywise nonnegative* if all entries of A are nonnegative. For brevity, we may refer to A as *nonnegative* or write $A \geqslant 0$, but this should not be confused with the usual definition of a nonnegative matrix (self-adjoint with nonnegative eigenvalues). If A_1 and A_2 are matrices of the same size then we say $A_1 \geqslant A_2$ if $A_1 - A_2$ is entrywise nonnegative. Entrywise nonpositivity, positivity, and negativity are similarly defined.

Clearly if A is entrywise nonnegative and $v \geqslant 0$, then $Av \geqslant 0$. Conversely, A is entrywise nonnegative if it takes all nonnegative vectors to nonnegative vectors.

We will use the following Perron–Frobenius characterization of entrywise nonnegative matrices [3, Theorem 16.4]:

Theorem 5 *(Perron–Frobenius). Every nonnegative $l \times l$ matrix A, $A \neq 0$, has an eigenvalue λ such that*

(1) *λ is real and nonnegative,*
(2) *there exists a nonnegative eigenvector of λ,*
(3) *the absolute value of every other eigenvalue is $\leqslant \lambda$,*
(4) *if κ is another eigenvalue of A with $|\kappa| = \lambda$, then $\kappa = e^{2\pi ik/m}\lambda$, where k, m are positive integers with $m \leqslant l$.*

In the case of 2×2 matrices, the final condition implies that $\kappa = \pm\lambda$, since $m = 1$ or 2. We also use the following standard terminology.

Definition. Suppose A is a nondiagonalizable 2×2 matrix, with an eigenvalue λ and an eigenvector v_1. Then v_2 is a *generalized eigenvector* of A if v_2 is not a scalar multiple of v_1 and $(A-\lambda I)v_2 = v_1$.

If we write A using the basis $\{v_1, v_2\}$, then the matrix is $\left(\begin{smallmatrix}\lambda & 1\\ 0 & \lambda\end{smallmatrix}\right)$, and we can easily check that $A^k v_2 = \lambda^k v_2 + k\lambda^{k-1} v_1$.

Lemma 6. *If $A \in M_2(\mathbb{R})$ has real eigenvalues, and A^n is entrywise nonnegative for some n, then A^2 is entrywise nonnegative.*

Proof. Note that since $A^n \geqslant 0$, $A^{2rn} \geqslant 0$ for all r, so $A^k \geqslant 0$ for infinitely many even k. Let λ_1, λ_2 be the eigenvalues of A, and first let us consider the case when $|\lambda_1| > |\lambda_2|$ with v_i corresponding eigenvectors. Since v_1 is an eigenvector of A^n corresponding to the eigenvalue with bigger absolute value, by Theorem 5, we can choose v_1 to be nonnegative. Let v be any nonnegative vector, and we can uniquely write $v = c_1 v_1 + c_2 v_2$. We will prove that $A^2 v \geqslant 0$. If $c_1 < 0$, then $A^k v = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 < 0$ for any sufficiently large even k. Therefore, we may assume $c_1 \geqslant 0$. Then

$$A^2 v = c_1 \lambda_1^2 v_1 + c_2 \lambda_2^2 v_2 \geqslant c_1 \lambda_2^2 v_1 + c_2 \lambda_2^2 v_2 = \lambda_2^2 v \geqslant 0,$$

so A^2 must be nonnegative.

Now suppose $|\lambda_1| = |\lambda_2|$, i.e., $\lambda_1 = \pm\lambda_2$. If A is diagonalizable, then A^2 is diagonalizable with both eigenvalues equal to λ_1^2, so $A^2 \geqslant 0$ as desired. If A is not diagonalizable, then let $\lambda = \lambda_1 = \lambda_2$ and let v_1 be an eigenvector of A and v_2 be its generalized eigenvector. For any nonnegative $v = c_1 v_1 + c_2 v_2$,

$$\begin{aligned} A^k v &= c_1 A^k v_1 + c_2 A^k v_2 = c_1 \lambda^k v_1 + c_2 \lambda^k v_2 + k c_2 \lambda^{k-1} v_1 \\ &= \lambda^k v + k c_2 \lambda^{k-1} v_1 = \lambda^{k-2}\big(\lambda^2 v + k c_2 \lambda v_1\big). \end{aligned}$$

In order for above to be nonnegative for infinitely many even k, we must have $c_2 \lambda v_1 \geqslant 0$. But then $A^2 v = \lambda^2 v + 2c_2 \lambda v_1 \geqslant 0$. $\square$

Lemma 7. *Let $A \in M_2(\mathbb{Q})$. Suppose that for some $n > 0$, $A^n \geqslant 0$, but for $m < n$, $A^m \ngeqslant 0$. Then n is* 1, 2, 3, 4, 6, 8, *or* 12. *Moreover, if such an n is greater than* 2, *then A^n is a scalar matrix.*

Proof. If $n = 1$ or 2, then we are done. Otherwise, by Lemma 6 A has non-real eigenvalues λ, $\bar{\lambda}$. By Theorem 5, λ^n is real and $\lambda^n > 0$. Since A is diagonalizable, so is A^n, and $(\bar{\lambda})^n = \overline{\lambda^n} = \lambda^n$ implies that A^n is a scalar matrix. Note that if $m < n$, we cannot have λ^m real and positive, since then $\bar{\lambda}^m = \lambda^m$ would be real and positive, and A^m would be a nonnegative matrix. Therefore, $(\lambda/|\lambda|)^n = 1$, but $(\lambda/|\lambda|)^m \neq 1$ for any $m < n$. So ω, defined to be $\lambda/|\lambda|$, is a primitive nth root of unity.

Now, λ is a root of A's characteristic polynomial, so it is in a quadratic extension of $\mathbb{Q}$. As $\omega^2 = \lambda^2/|\lambda|^2 = \lambda/\bar{\lambda}$, so is ω^2. Hence, ω^2 is a root of unity belonging to a quadratic extension, so its order must be 1, 2, 3, 4, or 6. Therefore, n must be 3, 4, 6, 8 or 12. $\square$

Proof of Theorem 1. Since both components of $\phi^{(n)}$ in a dehomogenized form are polynomials precisely when $A = A_\phi$ has the property that A^n is entrywise nonnegative. Therefore, the result is an immediate consequence of Lemma 7. $\square$

3. Finiteness of integers in orbits

We prove Theorem 2 in this section. Let $A = A_\phi$ be the exponent matrix of $\phi = (x^i y^j, x^k y^l)$. Let λ_1, λ_2 be the two (possibly equal) eigenvalues of A, and we assume without loss of generality that $|\lambda_1| \geqslant |\lambda_2|$. By choosing an eigenvector $\left(\begin{smallmatrix}a\\ c\end{smallmatrix}\right)$ for λ_1 and an eigenvector $\left(\begin{smallmatrix}b\\ d\end{smallmatrix}\right)$ for λ_2 (when A is not diagonalizable then $\left(\begin{smallmatrix}b\\ d\end{smallmatrix}\right)$ is a generalized eigenvector (see Definition 5) of $\lambda_1 = \lambda_2$), we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} A \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda_1 & \delta \\ 0 & \lambda_2 \end{pmatrix},$$

where $\delta = 0$ if A is diagonalizable and $\delta = 1$ if not. Therefore, the exponent matrix for $\phi^{(n)}$ is

$$\begin{aligned} A_{\phi^{(n)}} = A^n &= \frac{1}{ad-bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \lambda_1^n & \delta n\lambda_1^{n-1} \\ 0 & \lambda_2^n \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad\lambda_1^n - bc\lambda_2^n - \delta acn\lambda_1^{n-1} & -ab(\lambda_1^n - \lambda_2^n) + \delta a^2 n\lambda_1^{n-1} \\ cd(\lambda_1^n - \lambda_2^n) - \delta c^2 n\lambda_1^{n-1} & -bc\lambda_1^n + ad\lambda_2^n + \delta acn\lambda_1^{n-1} \end{pmatrix}. \end{aligned} \tag{2}$$

This is an integer-entry matrix. We now divide into three cases.

3.1. Case I: A is diagonalizable with real eigenvalues

Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. Since $\delta = 0$, if $|\lambda_1|$ (and therefore $|\lambda_2|$) is less than or equal to 1, the set $\{A^n\}$ is a finite set of matrices, so all orbits will be finite. Thus, we may assume that $|\lambda_1| > 1$. In the current case, if $|\lambda_1| = |\lambda_2|$, then A^2 is a scalar matrix with diagonal entries bigger than 1, so we trivially get infinitely many distinct integers in the orbit of say $(2, 1)$. We may now thus assume that $|\lambda_1| > |\lambda_2|$.

Let $(\alpha, \beta) \in \mathbb{Q}^2$. Note that a rational number is an integer if and only if it has a nonnegative valuation at every prime. Since the formula for the exponents of $\phi^{(n)}$ is given in (2), $\phi^{(n)}(\alpha, \beta)$ is an integer if and only if

$$\begin{aligned} &\frac{1}{ad-bc}\bigl[\bigl(ad\lambda_1^n - bc\lambda_2^n\bigr)\operatorname{ord}_p(\alpha) - ab\bigl(\lambda_1^n - \lambda_2^n\bigr)\operatorname{ord}_p(\beta)\bigr] \geqslant 0 \\ &\frac{1}{ad-bc}\bigl[cd\bigl(\lambda_1^n - \lambda_2^n\bigr)\operatorname{ord}_p(\alpha) + \bigl(-bc\lambda_1^n + ad\lambda_2^n\bigr)\operatorname{ord}_p(\beta)\bigr] \geqslant 0 \end{aligned} \tag{3}$$

for every prime p. By multiplying the eigenvector $\binom{b}{d}$ by -1 if necessary, we may assume that $ad - bc > 0$. Then (3) can be rewritten as

$$\begin{aligned} &a\bigl(d\operatorname{ord}_p(\alpha) - b\operatorname{ord}_p(\beta)\bigr)\lambda_1^n + b\bigl(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha)\bigr)\lambda_2^n \geqslant 0 \\ &c\bigl(d\operatorname{ord}_p(\alpha) - b\operatorname{ord}_p(\beta)\bigr)\lambda_1^n + d\bigl(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha)\bigr)\lambda_2^n \geqslant 0. \end{aligned} \tag{4}$$

Assume for the moment that $a \neq 0$. If $c/a > 0$, then we can choose integers $\operatorname{ord}_2(\alpha)$ and $\operatorname{ord}_2(\beta)$ so that $a(d\operatorname{ord}_2(\alpha) - b\operatorname{ord}_2(\beta)) > 0$ and $c(d\operatorname{ord}_2(\alpha) - b\operatorname{ord}_2(\beta)) > 0$. For sufficiently large even n, $\lambda_1^n \gg \lambda_2^n > 0$, so both of (4) are satisfied for $p = 2$. If we make the valuations of α and β to be 0 at all odd primes, we can then satisfy (4) for all primes, so we get infinitely many n's where $\phi^{(n)}(\alpha, \beta)$ is an integer point. Since $|\lambda_1| > 1$, it is also clear that the points in the orbit are distinct, so we are guaranteed to have infinitely many distinct integers in this orbit.

Now assume that $c/a < 0$. If $(\alpha, \beta) \in \mathbb{Q}^2$ is such that for some prime p, $d\operatorname{ord}_p(\alpha) - b\operatorname{ord}_p(\beta) \neq 0$, then it is clear by the same argument that for all sufficiently large n one of (4) will not be satisfied, so we are guaranteed to have finiteness of integers in such an orbit. Now suppose that $d\operatorname{ord}_p(\alpha) - b\operatorname{ord}_p(\beta) = 0$ for all primes p. If $|\lambda_2| \leqslant 1$, then the orbit contains only finitely many distinct points, so we may assume that $|\lambda_2| > 1$. If $|\alpha| = |\beta| = 1$, then the orbit is always finite, so we may assume that $(\operatorname{ord}_p(\alpha), \operatorname{ord}_p(\beta)) \neq (0, 0)$ for some p. For such a p, because $\binom{a}{c}$ and $\binom{b}{d}$ are linearly independent, $a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha) \neq 0$. Therefore, if b and d are both nonzero and $b/d < 0$, then for all n, one of (4) is negative. Moreover, as $|\lambda_2| > 1$, we will have infinitely many distinct points in the orbit. So we have only finitely many integral points despite having infinitely many points in the orbit. If $b = 0$, then $\operatorname{ord}_p(\alpha) = 0$ for every p, so (4) reduce to just $da\operatorname{ord}_p(\beta)\lambda_2^n \geqslant 0$. So by choosing $\operatorname{ord}_p(\beta)$ to have the same sign as da, we obtain integer points for all even n. The case of $d = 0$ is similar. Finally, if $b/d > 0$, then choosing (α, β) so that the parity of $a\operatorname{ord}_2(\beta) - c\operatorname{ord}_2(\alpha)$ is the same as that of b, we can satisfy both of (4) for all even n. Therefore, we have infinitely many distinct integers in this orbit. In summary, when $c/a < 0$, we are guaranteed finiteness of integers in all orbits if $b/d < 0$, or if $|\lambda_2| \leqslant 1$, or if for every (α, β) there exists a prime p so that $d\,\mathrm{ord}_p(\alpha) - b\,\mathrm{ord}_p(\beta) \neq 0$.

Now let us consider the case of $a = 0$. We can then assume that $c = 1$, and by linear independence of two eigenvectors, $b \neq 0$. If we choose $\mathrm{ord}_2(\alpha)$ so that $-bc\,\mathrm{ord}_2(\alpha) > 0$ and then choose $d\,\mathrm{ord}_2(\alpha) - b\,\mathrm{ord}_2(\beta) > 0$, we can satisfy both of (4) for all sufficiently large even n because $|\lambda_1| > |\lambda_2|$ and $|\lambda_1| > 1$. Thus, we get infinitely many distinct integers in this orbit. The case of $c = 0$ is similar.

Since the eigenvector $\binom{a}{c}$ of λ_1 is $\binom{j}{\lambda_1 - i}$ and the eigenvector $\binom{b}{d}$ of λ_2 is $\binom{j}{\lambda_2 - i}$, we obtain cases (1) and (2) of Theorem 2. Note that when eigenvalues are nonrational, for a prime p such that $(\mathrm{ord}_p(\alpha), \mathrm{ord}_p(\beta)) \neq (0, 0)$, $d\,\mathrm{ord}_p(\alpha) - b\,\mathrm{ord}_p(\beta)$ is guaranteed to be nonzero. Therefore, the sign of $a/c = \frac{j}{\lambda_1 - i}$ completely determines whether we always get just finitely many integers in orbits, as stated in case (1) of Theorem 2.

3.2. *Case II: A is nondiagonalizable*

In this case, we have $\lambda = \lambda_1 = \lambda_2$ and $\delta = 1$ in (2), so for $\phi^{(n)}(\alpha, \beta)$ to be an integral point, we must have

$$\left(\lambda^n - \frac{ac}{ad - bc} n\lambda^{n-1}\right) \mathrm{ord}_p(\alpha) + \frac{a^2}{ad - bc} n\lambda^{n-1}\,\mathrm{ord}_p(\beta) \geqslant 0 \tag{5}$$

$$-\frac{c^2}{ad - bc} n\lambda^{n-1}\,\mathrm{ord}_p(\alpha) + \left(\lambda^n + \frac{ac}{ad - bc} n\lambda^{n-1}\right) \mathrm{ord}_p(\beta) \geqslant 0 \tag{6}$$

for all primes p. Since the trace 2λ and the determinant λ^2 of A are integers, $\lambda \in \mathbb{Z}$. If $\lambda = 0$, then $A^2 = 0$, so all orbits are finite sets. We can now assume that $|\lambda| \geqslant 1$.

If $c = 0$, then we may assume that $a = 1$. If $|\lambda| > 1$, then it is easy to see that $(\alpha, \beta) = (2, 1)$ satisfies (5) and (6) for all even n's and (5) goes to infinity as $n \to \infty$, so we get infinitely many distinct integers in its orbit. If $\lambda = \pm 1$, then (5) and (6) are

$$(\pm 1)^n\,\mathrm{ord}_p(\alpha) + \frac{1}{d} n (\pm 1)^{n-1}\,\mathrm{ord}_p(\beta) \geqslant 0, \qquad (\pm 1)^n\,\mathrm{ord}_p(\beta) \geqslant 0.$$

In order for the orbit to contain infinitely many distinct rational points, $\mathrm{ord}_p(\beta)$ must be nonzero for some p. If $\lambda = 1$, then the second inequality forces $\mathrm{ord}_p(\beta) > 0$ for such a p, so we have infinitely many distinct integers if $d > 0$ and we have finitely many integers if $d < 0$. If $\lambda = -1$, then the signs of $\frac{1}{d}(-1)^{n-1}$ and $(-1)^n$ have to agree to have infinitely many integers, so we have infinitely many distinct integers if $d < 0$ and we have finitely many integers if $d > 0$. The case of $a = 0$ (and $c = 1$) can be treated similarly: we have finiteness of integers in every orbit if $\lambda = 1$ and $b < 0$ or $\lambda = -1$ and $b > 0$. Using these information on the eigenvalues and the (generalized) eigenvectors, we can reconstruct the exponent matrix, obtaining $A = \begin{pmatrix} \pm 1 & 1/d \\ 0 & \pm 1 \end{pmatrix}$, $\begin{pmatrix} \pm 1 & 0 \\ 1/b & \pm 1 \end{pmatrix}$. This is precisely case (4) of the theorem.

Now let us assume that $ac \neq 0$. Note that $|n\lambda^{n-1}|$ is bigger than $|\lambda^n|$ for n sufficiently large, and the coefficient of $n\lambda^{n-1}$ in (6) is c/a times the coefficient of $n\lambda^{n-1}$ in (5). So if $c/a > 0$, by choosing (α, β) so that $\frac{-c\,\mathrm{ord}_2(\alpha) + a\,\mathrm{ord}_2(\beta)}{ad - bc}$ has the same sign as a, we get integers for all sufficiently large odd n, creating infinitely many distinct integers. On the other hand, suppose $c/a < 0$. If $-c\,\mathrm{ord}_p(\alpha) + a\,\mathrm{ord}_p(\beta) \neq 0$ for some prime p, then (5) and (6) have opposite signs for all sufficiently large n, so we get only finitely many integers in its orbit. If $-c\,\mathrm{ord}_p(\alpha) + a\,\mathrm{ord}_p(\beta) = 0$ for all p, then $\mathrm{ord}_p(\alpha)$ and $\mathrm{ord}_p(\beta)$ must have opposite signs whenever they are nonzero. Since (5) says $\lambda^n\,\mathrm{ord}_p(\alpha) \geqslant 0$ and (6) says $\lambda^n\,\mathrm{ord}_p(\beta) \geqslant 0$, this is a contradiction. So we have finiteness of integers in all orbits if $c/a < 0$. This is case (3) of the theorem.

3.3. Case III: A has complex eigenvalues

In this case, we can let $\lambda = \lambda_1$ and then $\lambda_2 = \bar{\lambda}$. Since $\binom{a}{c}$ is an eigenvector for λ, we can let $a = j$ and $c = \lambda - i$ when $j > 0$, and we let $a = -j$ and $c = i - \lambda$ when $j < 0$. We can then take $b = a$ and $d = \bar{c}$ to make $\binom{b}{d}$ an eigenvector for $\bar{\lambda}$. We know that a and b are strictly positive. Interchanging λ and $\bar{\lambda}$ if necessary, we can assume without loss of generality that $\operatorname{Im}(c) > 0$, so that $0 < \arg(c) < \pi$. Substituting these values into (2), we obtain

$$A_{\phi^{(n)}} = A^n = \frac{1}{a(\bar{c} - c)} \begin{pmatrix} a(\bar{c}\lambda^n - \overline{\bar{c}\lambda^n}) & -a^2(\lambda^n - \overline{\lambda^n}) \\ c\bar{c}(\lambda^n - \overline{\lambda^n}) & -a(c\lambda^n - \overline{c\lambda^n}) \end{pmatrix}$$
$$= \frac{1}{-a \operatorname{Im} c} \begin{pmatrix} a \operatorname{Im}(\bar{c}\lambda^n) & -a^2 \operatorname{Im}\lambda^n \\ |c|^2 \operatorname{Im}\lambda^n & -a \operatorname{Im}(c\lambda^n) \end{pmatrix},$$

where all entries are integers as before. If $|\lambda| \leqslant 1$, then by the first line, it is clear that there are only finitely many possibilities for A^n, so all orbits will be finite. Let us now assume that $|\lambda| > 1$. In this case, we will now show that the orbit of $(1, 2)$ always contains infinitely many distinct integers. Taking the 2-adic valuation and using the fact that $\operatorname{Im} c$ and a are both positive, we need to show that

$$\operatorname{Im}\lambda^n \geqslant 0 \quad \text{and} \quad \operatorname{Im}\bigl(c\lambda^n\bigr) \geqslant 0$$

for infinitely many n. In other words, we need infinitely many n so that both λ^n and $c\lambda^n$ lie in the upper half-plane.

Now, if $\arg\lambda$ is a rational multiple of π, λ^n is positive real for infinitely many n, and in this case we satisfy both inequalities, because $\operatorname{Im} c > 0$. For such n, $\operatorname{Im}(c\lambda^n) = |c\lambda^n| \sin(\arg c)$, so as $|\lambda| > 1$, this goes to infinity as n increases. Therefore, these integer points are distinct.

If $\arg\lambda$ is not a rational multiple of π, then $\arg\lambda^n$ is uniformly distributed in $(-\pi, \pi]$, in particular they are dense. Since $0 < \arg c < \pi$, we may therefore choose infinitely many n satisfying $0 < \frac{\pi - \arg c}{2} < \arg\lambda^n < \pi - \arg c < \pi$. Then we have $0 < \frac{\pi - \arg c}{2} < \arg\lambda^n < \arg(c\lambda^n) < \pi$, so both λ^n and $c\lambda^n$ lie in the upper half-plane. Moreover, if we let $t = \min(\sin(\frac{\pi - \arg c}{2}), \sin(\pi - \arg c))$, then $\operatorname{Im}\lambda^n \geqslant |\lambda|^n t$. Therefore, this is not bounded, and hence we have infinitely many *distinct* integer points in the orbit. This finishes the proof in this case, finishing the proof of Theorem 2. $\square$

Using the proof above, we can efficiently find monomial maps which do not satisfy (1)–(4) of Theorem 2 but which has *some* (α, β) such that $\mathcal{O}_\phi(\alpha, \beta)$ is an infinite set containing only finitely many integer points. From the examples below, we see that it is weak to ask for just some orbits to contain finitely many integer points. This explains why we instead characterized in Theorem 2 monomial maps whose orbits *always* contain finitely many integer points.

Example 8. If all entries of the exponent matrix are nonnegative, then starting from $(\alpha, \beta) \in \mathbb{Z}^2$, we get infinitely many integers. So these maps have some orbits containing infinitely many integer points. On the other hand, if α and β are reciprocals of integers, then all orbit points will be reciprocals of integers. So these types of orbits contain infinitely many rational points but only finitely many integral points.

A slightly less trivial example is when the exponent matrix A is diagonalizable with real eigenvalues such that $a > 0$, $c > 0$, and $\lambda_1 > 0$. We have seen in the proof of Theorem 2 that some orbits have infinitely many integer points. However, if we choose (α, β) to satisfy $d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta) < 0$, we conclude from (4) that $\phi^{(n)}(\alpha, \beta)$ is not an integer point for any sufficiently large n. For example, if $\phi = (x^4/y, x^2 y)$, then we can take $\binom{a}{c} = \binom{1}{1}$ with $\lambda_1 = 3$ and $\binom{b}{d} = \binom{1}{2}$ with $\lambda_2 = 2$. Then the point $(3, 27)$ has the property that $2 \operatorname{ord}_3(3) - \operatorname{ord}_3(27) < 0$, so the leading terms in both of (4) have negative coefficients, forcing the orbit to contain only finitely many integer points. On the other hand, the point $(27, 3)$ satisfies $2 \operatorname{ord}_3(27) - \operatorname{ord}_3(3) > 0$, so there are infinitely many distinct integer points in this orbit.

Example 9. We can find a similar situation for the case when the exponent matrix is nondiagonalizable. For example, if $a > 0$ and $c > 0$ then we have seen that there are some orbits with infinitely many integer points. But if $\operatorname{ord}_p(\beta) = 0$, then we cannot satisfy (5) for any sufficiently large n if $ad - bc > 0$ and $\operatorname{ord}_p(\alpha) > 0$. For example, if $\phi = (xy, y^3/x)$, then $\binom{a}{c} = \binom{1}{1}$ with eigenvalue 2 and we can take $\binom{b}{d} = \binom{1}{2}$ as a corresponding generalized eigenvector. Then $ad - bc = 1 > 0$. So the point $(2, 1)$ has only finitely many integers in its orbit, while $(1, 2)$ satisfies $-c \operatorname{ord}_2(\alpha) + a \operatorname{ord}_2(\beta) = 1$, so it has infinitely many integers in its orbit.

4. Size of numerators and denominators in orbits

Here we prove Theorem 4, and then discuss several examples at the end. Let us first deal with the case when the exponent matrix A is diagonalizable with real eigenvalues λ_1, λ_2 satisfying $|\lambda_1| > |\lambda_2|$. As in the proof of Theorem 2, let $\binom{a}{c}$ be an eigenvector for λ_1 and $\binom{b}{d}$ be an eigenvector for λ_2. Without loss of generality, we can assume that $a > 0$, $c < 0$, and $b \geqslant 0$. We will also assume that $d \leqslant 0$: this is guaranteed for case (2) of Theorem 2, and we will see later that the sign of d does not make a difference in case (1). Suppose that $(\alpha, \beta) \in (\mathbb{Q}^*)^2$, and we can assume that $|\alpha|$ and $|\beta|$ are not both 1. We use (4) to write the x-coordinate and the y-coordinate of $\phi^{(n)}(\alpha, \beta)$ in a reduced fraction. One needs to be slightly careful, as $\frac{p^n}{p} = \frac{1/p}{1/p^n}$ as fractions, but $\frac{\log p^n}{\log p} = n$ is completely different from $\frac{\log(1/p)}{\log(1/p^n)} = \frac{1}{n}$. Thus, we have to make sure to distinguish primes appearing in the denominator from those appearing in the numerator to calculate the ratio of logarithms. We define

$$S_1 = \{p \text{ prime: } d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta) > 0\}$$

$$S_2 = \{p \text{ prime: } d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta) < 0\}$$

$$S_3 = \{p \text{ prime: } d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta) = 0 \text{ and } a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha) > 0\}$$

$$S_4 = \{p \text{ prime: } d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta) = 0 \text{ and } a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha) < 0\}.$$

These four sets are disjoint and their union is nonempty, as $\operatorname{ord}_p(\alpha) \neq 0$ or $\operatorname{ord}_p(\beta) \neq 0$ for some p and $\binom{a}{c}$ and $\binom{b}{d}$ are linearly independent. Let us assume for now that $\lambda_2 \geqslant 0$. Using (4), for n sufficiently large (and additionally for n even if $\lambda_1 < 0$), the x-coordinate of $\phi^{(n)}(\alpha, \beta)$ in a reduced fraction is

$$\frac{\prod_{p \in S_1} p^{a(d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta))\lambda_1^n + b(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n} \prod_{p \in S_3} p^{b(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n}}{\prod_{p \in S_2} p^{-a(d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta))\lambda_1^n - b(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n} \prod_{p \in S_4} p^{-b(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n}} \tag{7}$$

and the y-coordinate is

$$\frac{\prod_{p \in S_2} p^{c(d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta))\lambda_1^n + d(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n} \prod_{p \in S_4} p^{d(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n}}{\prod_{p \in S_1} p^{-c(d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta))\lambda_1^n - d(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n} \prod_{p \in S_3} p^{-d(a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha))\lambda_2^n}}. \tag{8}$$

For large enough (and if $\lambda_1 < 0$, even) n, the exponents that appear in the numerator and the denominator are guaranteed to be nonnegative. Let $\Theta_p = d \operatorname{ord}_p(\alpha) - b \operatorname{ord}_p(\beta)$ and $\Gamma_p = a \operatorname{ord}_p(\beta) - c \operatorname{ord}_p(\alpha)$. Then the log ratio $\frac{\log N_n}{\log D_n}$ is

$$\frac{\sum_{p\in S_1}(a\Theta_p\lambda_1^n+b\Gamma_p\lambda_2^n)\log p+\sum_{p\in S_2}(c\Theta_p\lambda_1^n+d\Gamma_p\lambda_2^n)\log p+\sum_{p\in S_3}b\Gamma_p\lambda_2^n\log p+\sum_{p\in S_4}d\Gamma_p\lambda_2^n\log p}{\sum_{p\in S_1}(-c\Theta_p\lambda_1^n-d\Gamma_p\lambda_2^n)\log p+\sum_{p\in S_2}(-a\Theta_p\lambda_1^n-b\Gamma_p\lambda_2^n)\log p+\sum_{p\in S_3}-d\Gamma_p\lambda_2^n\log p+\sum_{p\in S_4}-b\Gamma_p\lambda_2^n\log p}. \tag{9}$$

Let us first assume that $S_1\cup S_2\neq\emptyset$. If the eigenvalues are nonrational, then either d or b lie outside of $\mathbb{Q}$, so $d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta)$ is nonzero if $\operatorname{ord}_p(\alpha)$ or $\operatorname{ord}_p(\beta)$ is nonzero. Therefore, as $(\alpha,\beta)\neq(\pm1,\pm1)$, $S_1\cup S_2$ is always nonempty when eigenvalues are nonrational reals. In all situations when $S_1\cup S_2\neq\emptyset$, the coefficients of λ_1^n in the numerator and in the denominator of (9) are strictly positive, since they are nonempty positive linear combinations of logarithms. Hence, dividing the top and the bottom by λ_1^n and letting $n\to\infty$, this fraction tends to

$$\begin{aligned}&\frac{\sum_{p\in S_1}a\Theta_p\log p+\sum_{p\in S_2}c\Theta_p\log p}{\sum_{p\in S_1}-c\Theta_p\log p+\sum_{p\in S_2}-a\Theta_p\log p}\\&\quad=\frac{\sum_{p\in S_1}a(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p+\sum_{p\in S_2}c(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p}{\sum_{p\in S_1}-c(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p+\sum_{p\in S_2}-a(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p}.\end{aligned} \tag{10}$$

This limit is a strictly positive number. Similarly, if $\lambda_1<0$, for n sufficiently large and odd, the x-coordinate becomes

$$\frac{\prod_{p\in S_2}p^{a\Theta_p\lambda_1^n+b\Gamma_p\lambda_2^n}\prod_{p\in S_3}p^{b\Gamma_p\lambda_2^n}}{\prod_{p\in S_1}p^{-a\Theta_p\lambda_1^n-b\Gamma_p\lambda_2^n}\prod_{p\in S_4}p^{-b\Gamma_p\lambda_2^n}}$$

and the y-coordinate becomes

$$\frac{\prod_{p\in S_1}p^{c\Theta_p\lambda_1^n+d\Gamma_p\lambda_2^n}\prod_{p\in S_4}p^{d\Gamma_p\lambda_2^n}}{\prod_{p\in S_2}p^{-c\Theta_p\lambda_1^n-d\Gamma_p\lambda_2^n}\prod_{p\in S_3}p^{-d\Gamma_p\lambda_2^n}}.$$

As before, the exponents in the numerators and the denominators are nonnegative. Thus, the limit of the log ratio is

$$\frac{\sum_{p\in S_1}c(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p+\sum_{p\in S_2}a(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p}{-\sum_{p\in S_1}a(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p-\sum_{p\in S_2}c(d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta))\log p}. \tag{11}$$

Therefore, when $\lambda_1<0$, the limit (11) of the odd iterations is the reciprocal of the limit (10) of the even iterations. So when $\lambda_1<0$, we have a single limit for all iterations if and only if (10) is equal to 1. As the logarithms of primes are linearly independent over number fields, we see that this happens if and only if $a=-c$, that is $j=i-\lambda_1$. When the limits differ, we can formulate a uniform result for all iterations as follows: the ratio of $\max(\log|N_n|,\log|D_n|)$ to $\min(\log|N_n|,\log|D_n|)$ goes to the limit given by (10) (or its reciprocal (11) when (10) is less than 1) as $n\to\infty$.

We remark that if all primes are in S_1 (resp. S_2), then (10) is $-\frac{a}{c}$ (resp. $-\frac{c}{a}$) and (11) is $-\frac{c}{a}$ (resp. $-\frac{a}{c}$). As $\binom{a}{c}$ is an eigenvector for eigenvalue λ_1, $-\frac{a}{c}=\frac{j}{i-\lambda_1}$, so we can write out the limit using the exponents of ϕ. Otherwise, the limit can involve various logarithms, as we will see in Examples 10, 11, and 12.

Note that if $\lambda_2 < 0$, then the products over S_3 and S_4 switch places between the numerator and denominator for odd n. For example, in the case $\lambda_1 > 0$, (7) for odd n becomes

$$\frac{\prod_{p\in S_1} p^{a\Theta_p\lambda_1^n + b\Gamma_p\lambda_2^n} \prod_{p\in S_4} p^{b\Gamma_p\lambda_2^n}}{\prod_{p\in S_2} p^{-a\Theta_p\lambda_1^n - b\Gamma_p\lambda_2^n} \prod_{p\in S_3} p^{-b\Gamma_p\lambda_2^n}}. \tag{12}$$

All exponents are nonnegative for sufficiently large odd n, so this is the reduced fraction expression. When we take logarithms of the numerator and the denominator and then divide by λ_1^n, the placements of S_3 and S_4 terms do not make any differences. Thus, the limit expression will be exactly the same: (10) for $\lambda_1 > 0$ or for even iterations, (11) for $\lambda_1 < 0$ and odd iterations. Similarly, in case (1) of Theorem 2, $S_1 \cup S_2$ is guaranteed to be nonempty, so we can now see that the limit does not change even if d was actually positive, since this also only interchanges S_3 and S_4 in the y-coordinate expression.

If $S_1 \cup S_2$ is empty, then the sign of λ_2 does make a difference. If $\lambda_2 > 0$ or $\lambda_2 < 0$ and n even, we divide the top and the bottom of (9) by λ_2^n and the log ratio is always (there is no dependence on n)

$$\frac{\sum_{p\in S_3} b(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p + \sum_{p\in S_4} d(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p}{\sum_{p\in S_3} -d(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p + \sum_{p\in S_4} -b(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p}. \tag{13}$$

If $\lambda_2 < 0$, then using (12) and the corresponding y-coordinate expression, the log ratio of the odd iterations is always

$$\frac{\sum_{p\in S_3} d(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p + \sum_{p\in S_4} b(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p}{\sum_{p\in S_3} -b(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p + \sum_{p\in S_4} -d(a\operatorname{ord}_p(\beta) - c\operatorname{ord}_p(\alpha))\log p}.$$

So if $d\operatorname{ord}_p(\alpha) - b\operatorname{ord}_p(\beta) = 0$ for all p, then the ratio of $\max(\log|N_n|, \log|D_n|)$ to $\min(\log|N_n|, \log|D_n|)$ is (13) (or its reciprocal if (13) is less than 1). As before, if in addition S_4 is empty, (13) is $-\frac{b}{d} = \frac{j}{i-\lambda_2}$.

Now, we deal with case (3) of Theorem 2. We have $|\lambda| > 1$ and $ac < 0$, so we can assume without loss of generality that $a > 0$ and $c < 0$. Then similarly to the diagonalizable case, define

$$\Theta_p = \frac{-c\operatorname{ord}_p(\alpha) + a\operatorname{ord}_p(\beta)}{ad - bc}$$

$$S_1 = \left\{p \text{ prime: } \frac{-c\operatorname{ord}_p(\alpha) + a\operatorname{ord}_p(\beta)}{ad - bc} > 0\right\}$$

$$S_2 = \left\{p \text{ prime: } \frac{-c\operatorname{ord}_p(\alpha) + a\operatorname{ord}_p(\beta)}{ad - bc} < 0\right\}$$

$$S_3 = \left\{p \text{ prime: } \Theta_p = 0,\ \operatorname{ord}_p(\alpha) > 0\right\} = \left\{p \text{ prime: } \Theta_p = 0,\ \operatorname{ord}_p(\beta) < 0\right\}$$

$$S_4 = \left\{p \text{ prime: } \Theta_p = 0,\ \operatorname{ord}_p(\alpha) < 0\right\} = \left\{p \text{ prime: } \Theta_p = 0,\ \operatorname{ord}_p(\beta) > 0\right\}.$$

Let us assume for now that $S_1 \cup S_2$ is not empty and $\lambda > 0$. From (5) and (6), for all n sufficiently large, $\phi^{(n)}(\alpha, \beta)$ in a reduced fraction is

$$\left(\frac{\prod_{p\in S_1} p^{\lambda^{n-1}(a\Theta_p n+\lambda \operatorname{ord}_p(\alpha))} \prod_{p\in S_3} p^{\lambda^n \operatorname{ord}_p(\alpha)}}{\prod_{p\in S_2} p^{\lambda^{n-1}(-a\Theta_p n-\lambda \operatorname{ord}_p(\alpha))} \prod_{p\in S_4} p^{-\lambda^n \operatorname{ord}_p(\alpha)}}, \frac{\prod_{p\in S_2} p^{\lambda^{n-1}(c\Theta_p n+\lambda \operatorname{ord}_p(\beta))} \prod_{p\in S_4} p^{\lambda^n \operatorname{ord}_p(\beta)}}{\prod_{p\in S_1} p^{\lambda^{n-1}(-c\Theta_p n-\lambda \operatorname{ord}_p(\beta))} \prod_{p\in S_3} p^{-\lambda^n \operatorname{ord}_p(\beta)}}\right). \tag{14}$$

Computing the log ratio and multiplying the top and the bottom by $\frac{ad-bc}{n\lambda^{n-1}}$, the limit as $n \to \infty$ is

$$\frac{\sum_{p\in S_1} a(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p + \sum_{p\in S_2} c(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p}{\sum_{p\in S_1} -c(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p + \sum_{p\in S_2} -a(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p}. \tag{15}$$

Again, both the numerator and the denominator are nonempty positive linear combinations of logarithms, so the limit is a strictly positive number. If S_2 (resp. S_1) is empty, then the limit has the form $\frac{a}{-c} = \frac{j}{i-\lambda}$ (resp. $-\frac{c}{a} = \frac{i-\lambda}{j}$).

If $\lambda < 0$, for sufficiently large odd n, S_3 and S_4 switch places between the numerator and denominator in (14), but the limit remains the same as (15). For sufficiently large even n, $\phi^{(n)}(\alpha, \beta)$ in a reduced fraction is

$$\left(\frac{\prod_{p\in S_2} p^{\lambda^{n-1}(a\Theta_p n+\lambda \operatorname{ord}_p(\alpha))} \prod_{p\in S_3} p^{\lambda^n \operatorname{ord}_p(\alpha)}}{\prod_{p\in S_1} p^{\lambda^{n-1}(-a\Theta_p n-\lambda \operatorname{ord}_p(\alpha))} \prod_{p\in S_4} p^{-\lambda^n \operatorname{ord}_p(\alpha)}}, \frac{\prod_{p\in S_1} p^{\lambda^{n-1}(c\Theta_p n+\lambda \operatorname{ord}_p(\beta))} \prod_{p\in S_4} p^{\lambda^n \operatorname{ord}_p(\beta)}}{\prod_{p\in S_2} p^{\lambda^{n-1}(-c\Theta_p n-\lambda \operatorname{ord}_p(\beta))} \prod_{p\in S_3} p^{-\lambda^n \operatorname{ord}_p(\beta)}}\right).$$

Hence the limit of the log ratio for the even iterations will be

$$\frac{\sum_{p\in S_1} c(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p + \sum_{p\in S_2} a(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p}{-\sum_{p\in S_1} a(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p - \sum_{p\in S_2} c(-c \operatorname{ord}_p(\alpha) + a \operatorname{ord}_p(\beta)) \log p},$$

precisely the reciprocal of (15). As before, these two limits will be identical if and only if the limits happen to be both 1, i.e. when $a = -c$, or equivalently $j = i - \lambda$.

If $S_1 \cup S_2$ is empty, then for all n if $\lambda > 0$ and for all even n if $\lambda < 0$, $\phi^{(n)}(\alpha, \beta)$ in a reduced fraction is

$$\left(\frac{\prod_{p\in S_3} p^{\lambda^n \operatorname{ord}_p(\alpha)}}{\prod_{p\in S_4} p^{-\lambda^n \operatorname{ord}_p(\alpha)}}, \frac{\prod_{p\in S_4} p^{\lambda^n \operatorname{ord}_p(\beta)}}{\prod_{p\in S_3} p^{-\lambda^n \operatorname{ord}_p(\beta)}}\right).$$

So the log ratio is always (independent of n) equal to

$$\frac{\sum_{p\in S_3} \operatorname{ord}_p(\alpha) \log p + \sum_{p\in S_4} \operatorname{ord}_p(\beta) \log p}{-\sum_{p\in S_3} \operatorname{ord}_p(\beta) \log p - \sum_{p\in S_4} \operatorname{ord}_p(\alpha) \log p}$$

As before, for $\lambda < 0$ and n odd, we get the reciprocal of above. This finishes the proof of Theorem 4. $\square$

Remark. As mentioned in Section 1, a more natural generalization of Silverman's result and a direct parallel of [7] would be looking at the ratio of the logarithm of the homogeneous Z-coordinate to the logarithmic height. For example, suppose A has two real eigenvalues and $\lambda_1 > 1$. If we write $\phi^{(n)}(\alpha, \beta) = [X_n : Y_n : Z_n]$ with integers without common divisors, then using (7) and (8), we have

$$\begin{aligned}
\log|X_n| &= \sum_{p\in S_1}\bigl((a-c)\Theta_p\lambda_1^n + (b-d)\Gamma_p\lambda_2^n\bigr)\log p + \sum_{p\in S_3}(b-d)\Gamma_p\lambda_2^n\log p\\
\log|Y_n| &= \sum_{p\in S_2}\bigl((c-a)\Theta_p\lambda_1^n + (d-b)\Gamma_p\lambda_2^n\bigr)\log p + \sum_{p\in S_4}(d-b)\Gamma_p\lambda_2^n\log p\\
\log|Z_n| &= \sum_{p\in S_1}\bigl(-c\Theta_p\lambda_1^n - d\Gamma_p\lambda_2^n\bigr)\log p + \sum_{p\in S_2}\bigl(-a\Theta_p\lambda_1^n - b\Gamma_p\lambda_2^n\bigr)\log p\\
&\quad - \sum_{p\in S_3} d\Gamma_p\lambda_2^n\log p - \sum_{p\in S_4} b\Gamma_p\lambda_2^n\log p.
\end{aligned}$$

Note that the denominators of (7) and (8) have no primes in common, so there will be no common divisors when we clear denominators. Assuming $S_1 \cup S_2 \neq \emptyset$, as $n \to \infty$, $\frac{\log\max(|X_n|,|Y_n|,|Z_n|)}{\log|Z_n|}$ goes to

$$\frac{\max\bigl(\sum_{p\in S_1}(a-c)\Theta_p\log p,\ \sum_{p\in S_2}(c-a)\Theta_p\log p, -\sum_{p\in S_1}c\Theta_p\log p - \sum_{p\in S_2}a\Theta_p\log p\bigr)}{-\sum_{p\in S_1}c\Theta_p\log p - \sum_{p\in S_2}a\Theta_p\log p}.$$

So this natural generalization also works, but the expression is more complicated and the limit is certainly not equal to 1 in general. Other cases can be treated similarly. Note that case (4) of Theorem 2 still does not have a nonzero limit here: for $\phi = (1/x, x/y)$, $\phi^{(n)} = (1/x, x^n/y)$ for odd n, and so $\phi^{(n)}(2,1) = (1/2,\ 2^n) = [1 : 2^{n+1} : 2]$.

We finish this article with several examples of Theorem 4.

Example 10. First, we take another look at the Fibonacci case (Example 3): $\phi = (y/x, x)$. Then $\lambda_1 = \frac{-1-\sqrt{5}}{2}$, $\lambda_2 = \frac{-1+\sqrt{5}}{2}$, so $a = 1$, $c = \frac{1-\sqrt{5}}{2}$, and $b = 1$, $d = \frac{1+\sqrt{5}}{2}$. Since $\lambda_1 < 0$, the limit for the odd iterations should be the reciprocal of the limit for the even iterations. We calculate that

$$\phi^{(10)}(1,2) = \left(\frac{1}{36028797018963968}, 17179869184\right)$$

so the log ratio is $\risingdotseq 0.618182$. In this case, S_1 is empty and $S_2 = \{2\}$, so the theoretical limit is $c/(-a) = \frac{\sqrt{5}-1}{2} \risingdotseq 0.618034$. We can also check that $\phi^{(9)}(1,2) = (17179869184, 1/2097152)$, and the log ratio is 1.61905, whose reciprocal is 0.617647. Similarly, for the initial point $(6,2)$, $\Theta_p = \frac{1+\sqrt{5}}{2}\operatorname{ord}_p(6) - \operatorname{ord}_p(2)$ is positive for both $p = 2$ and $p = 3$. And we have

$$\phi^{(8)}(6,2) = \left(1366194724836685332448, \frac{1}{2677850419968}\right) \quad \text{log ratio} \risingdotseq 1.6202$$

$$\text{while the expected limit} = \frac{a}{-c} = \frac{2}{\sqrt{5}-1} \risingdotseq 1.61803.$$

For the start point $(2,6)$, $S_1=\{2\}$ and $S_2=\{3\}$. Therefore, the expected limit for even iterations is

$$\frac{(\frac{1+\sqrt{5}}{2}-1)\log 2+\frac{1-\sqrt{5}}{2}\cdot(-1)\cdot\log 3}{\frac{\sqrt{5}-1}{2}(\frac{1+\sqrt{5}}{2}-1)\log 2+(-1)(-1)\log 3}=\frac{(\sqrt{5}-1)\log 6}{\log 72-\sqrt{5}\log 2}\risingdotseq 0.812228.$$

We compare this with

$$\phi^{(10)}(2,6)=\left(\frac{17179869184}{174449211009120179071170507},\frac{16677181699666569}{2097152}\right),$$

which gives the log ratio $\risingdotseq 0.812483$.

Example 11. Next we look at an example where the exponent matrix is diagonalizable with rational eigenvalues. In this case, $\Theta_p=d\operatorname{ord}_p(\alpha)-b\operatorname{ord}_p(\beta)$ can be zero, so the limit could be of the form in (13). Let $\phi(x,y)=(x^7y^{10},1/x^5y^8)$. Then the exponent matrix has an eigenvector $\binom{1}{-1}$ with eigenvalue -3 and an eigenvector $\binom{2}{-1}$ with eigenvalue 2. So $\lambda_1=-3$, $\lambda_2=2$, $a=1$, $b=2$, $c=-1$, $d=-1$. If $\alpha=1/9$ and $\beta=3$, then $\Theta_3=-(-2)-2\cdot 1=0$. On the other hand, $\Gamma_3=\operatorname{ord}_3(\beta)+\operatorname{ord}_3(\alpha)=-1$, so $S_4=\{3\}$. Hence, according to the proof of Theorem 4, we expect the log ratio to be *always* $d/(-b)=\frac{1}{2}$, independent of n (note that $\lambda_2>0$ so even and odd iterations give the same behavior). Indeed, the first several points in the orbit are $(1/81,9)$, $(1/6561,81)$, $(1/43046721,6561)$. If we instead let $(\alpha,\beta)=(4/9,3/2)$, then $\Theta_2=-2-2(-1)=0$ and $\Theta_3=0$ as before, so we have $S_3=\{2\}$ and $S_4=\{3\}$. Therefore, from (13), we expect *all* points in the orbit to satisfy the log ratio of

$$\frac{2\cdot 1\cdot\log 2+(-1)(-1)\log 3}{1\cdot 1\cdot\log 2+(-2)(-1)\log 3}=\frac{\log 12}{\log 18}\risingdotseq 0.859719.$$

Indeed, we have

$$\phi^{(3)}\left(\frac{4}{9},\frac{3}{2}\right)=\left(\frac{65536}{43046721},\frac{6561}{256}\right),\qquad\frac{65536\cdot 6561}{43046721\cdot 256}=\frac{12^8}{18^8}.$$
$$\phi^{(4)}\left(\frac{4}{9},\frac{3}{2}\right)=\left(\frac{4294967296}{1853020188851841},\frac{43046721}{65536}\right),\qquad\frac{4294967296\cdot 43046721}{1853020188851841\cdot 65536}=\frac{12^{16}}{18^{16}}.$$

Example 12. We end the paper with an example of a nondiagonalizable exponent matrix. Let $\phi(x,y)=(xy^4,1/xy^3)$. Then $\lambda=-1$ is the unique eigenvalue, with $\binom{2}{-1}$ as an eigenvector and $\binom{-1}{1}$ as a generalized eigenvector. So we can let $a=2$, $b=-1$, $c=-1$, and $d=1$, and so $\Theta_p=\operatorname{ord}_p(\alpha)+2\operatorname{ord}_p(\beta)$. If $(\alpha,\beta)=(2,6)$, then $\Theta_2=3$ and $\Theta_3=2$, so $S_1=\{2,3\}$. As the eigenvalue is negative, using (15), the limit of odd iterations should be $-\frac{a}{c}=2$ and the limit of the even iterations should be $-\frac{c}{a}=\frac{1}{2}$. Indeed we can compute that $\phi^{(10)}(2,6)$ is

$$\left(\frac{1}{7008416976781303646959092879367077888},\ 22463437455746924544\right)$$

giving the log ratio $\risingdotseq 0.525204$, while $\phi^{(11)}(2,6)$ is

$$\left(3633163360763427810583593748663893l771392,\ \frac{1}{16173674968137785671 68}\right)$$

giving the log ratio $\fallingdotseq 1.91243$. If $(\alpha, \beta) = (2, 2/3)$, then $\Theta_2 = 3$ and $\Theta_3 = -2$, so $S_1 = \{2\}$ and $S_2 = \{3\}$. Hence the limit of the odd iterations is expected to be $\frac{2\cdot 3\cdot \log 2+(-1)(-2)\log 3}{1\cdot 3\log 2+(-2)(-2)\log 3} \fallingdotseq 0.981806$, and the limit of the even iterations is its reciprocal $\fallingdotseq 1.01853$. We check $\phi^{(21)}(2, 2/3)$ is

$$\left(\frac{4253529586511730793292182592897102643 2}{11972515182562019788602740026717047105681}, \frac{32825696739453707762 7}{18446744073709551616}\right)$$

whose log ratio is 0.979793, while $\phi^{(20)}(2, 2/3)$ is

$$\left(\frac{1478088294143459233160832102063832976 01}{664613997892457936451903530140172288}, \frac{2305843009213693952}{36472996377170786403}\right)$$

whose log ratio is 1.02073. In the nondiagonalizable case, the dominant term in the numerator and in the denominator is bigger than the smaller term by just a factor of n/λ (in contrast to $(\lambda_1/\lambda_2)^n$ in the diagonalizable case), so the convergence is much slower than other examples.

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