



Explicit upper bounds for the Stieltjes constants

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ABSTRACT

Text. Let χ be a primitive Dirichlet character modulo q and let $(-1)^n \gamma_n(\chi)/n!$ (for n larger than 0) be the n -th Laurent coefficient around $z = 1$ of the associated Dirichlet L -series. When χ is non-principal, $(-1)^n \gamma_n(\chi)$ is simply the value of the n -th derivative of $L(z, \chi)$ at $z = 1$. In this paper we give an explicit upper bounds for $|\gamma_n(\chi)|$ for $q \leq \frac{\pi}{2} \frac{e^{(n+1)/2}}{n+1}$. In particular, when $q = 1$ the explicit upper bound we get improves on earlier work. We conclude this paper by showing that we can altogether dispense in these proofs with the functional equation of $L(z, \chi)$.

Video. For a video summary of this paper, please click [here](http://www.youtube.com/watch?v=q340UciEvAA) or visit <http://www.youtube.com/watch?v=q340UciEvAA>.

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1. Introduction and results

Let χ be a Dirichlet character modulo q and let us denote by $\gamma_n(\chi)$ (for n larger than 0) the n -th Laurent coefficient around $z = 1$ of the Dirichlet L -function $L(z, \chi)$. We quote the following relation from [9],

$$\gamma_n(\chi) = \sum_{a=1}^q \chi(a) \gamma_n(a, q), \quad (1)$$

with

$$\gamma_n(a, q) = \lim_{T \rightarrow \infty} \left\{ \sum_{1 \leq m \equiv a \pmod{q}}^T \frac{(\log m)^n}{m} - \frac{(\log T)^{n+1}}{q(n+1)} \right\}. \quad (2)$$

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In particular, $\gamma_0(1, 1)$ is the well-known Euler constant. The constants $\gamma_n(a, q)$ are known as the Stieltjes or the generalized Euler constants. We use the shorter form $\gamma_n(1, 1) = \gamma_n$ (this is also $\gamma_n(\chi_0)$ where χ_0 is the only character modulo 1), so that we have

$$\zeta(z) = \frac{1}{z-1} + \sum_{n=0}^{\infty} (-1)^n \frac{\gamma_n}{n!} (z-1)^n.$$

The problem of finding an explicit upper bound for $|\gamma_n|$ has been addressed by a number of authors. Briggs [2] started this line of investigation by proving that

$$|\gamma_n| < \left(\frac{n}{2e}\right)^n.$$

In 1985, Matsuoka [12] produced an asymptotic expression for γ_n , and he was also able to simplify his method to derive the explicit form:

$$\forall n \geq 10, \quad |\gamma_n| \leq 10^{-4} e^{n \log \log n}. \quad (3)$$

In the paper [10], Kreminski conjectured on numerical evidence (see Table 2) that the above inequality may be considerably strengthened. Recently, Adell [1] proved that, for any $n \geq 4$, we have

$$|\gamma_n| \leq \left(\frac{n! e^m}{m^{n+1}} \left(\frac{n+1}{m} + 1 \right) + \frac{1}{n+1} \right) \log^{n+1}(m+1),$$

where $m = \lfloor n(1 - 1/\log n) \rfloor$ and $\lfloor x \rfloor$ denotes the integer part of x .

Let us turn to the corresponding problem with a character. In 1994, Toyozumi [13] studied the problem of bounding $|\gamma_n(\chi)|$ when n is fixed and q goes to infinity. On using Burgess inequality, he showed that for real non-principal χ , when q is cube-free, for any $\epsilon > 0$, we have

$$|L^{(n)}(1, \chi)| \leq \left(\frac{1}{(n+1)4^{n+1}} \cdot \frac{L(1+\epsilon, \chi)}{\zeta(1+\epsilon)} + \epsilon \right) \log^{n+1} q, \quad (4)$$

when $q > q_0(\epsilon)$, where $q_0(\epsilon)$ is a constant depending only on ϵ .

In another direction and pursuing the groundbreaking result of Matsuoka [12], Ishikawa [6] studied the asymptotic behavior of $L^{(n)}(1, \chi)$ as $n \rightarrow \infty$. He showed that there exists an n_0 such that for all $n \geq n_0$,

$$|L^{(n)}(1, \chi)| \leq q^{n/\log n - 1/2} \exp \left\{ n \log \log n - \frac{n \log \log n}{\log n} \right\}. \quad (5)$$

We note here that the latter result is better than Eq. (4) when q is small with respect to n .

In this paper, we produce an explicit upper bound of $|\gamma_n(\chi)|$ useful for large values of n .

Theorem 1. Let χ be a primitive Dirichlet character to modulus q . Then, for every $1 \leq q \leq \frac{\pi}{2} \frac{e^{(n+1)/2}}{n+1}$, we have

$$\frac{|\gamma_n(\chi)|}{n!} \leq q^{-1/2} C(n, q) (1 + D(n, q))$$

with

$$C(n, q) = 2\sqrt{2} \exp \left\{ -(n+1) \log \theta(n, q) + \theta(n, q) \left(\log \theta(n, q) + \log \frac{2q}{\pi e} \right) \right\},$$

and

$$\theta(n, q) = \frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1, \quad D(n, q) = 2^{-\theta(n, q)-1} \frac{\theta(n, q) + 1}{\theta(n, q) - 1}.$$

This theorem will be a consequence of the two expressions given by Theorem 4. The inequality (5) is a straightforward consequence of the above result. The bound we get is asymptotically as strong as (3) and numerically better as soon as $n \geq 11$. Table 2 displayed later offers many comparative values but we provide here a short form (Table 1):

Table 1

We see that while there is still some room for improvement, our upper bound is much closer to the truth than the previous ones.

n		10	500	1000
$ \gamma_n $	Matsuoka	0.418944879802	$5.091823280 \cdot 10^{392}$	$2.172418132 \cdot 10^{835}$
	Adell	21622.67050207	$9.480434509 \cdot 10^{393}$	$4.407894620 \cdot 10^{834}$
	Our result	0.714120274361	$3.8204382632 \cdot 10^{277}$	$1.3320458753 \cdot 10^{618}$
γ_n	Kreminski	$2.0533281 \cdot 10^{-4}$	$-1.1655052 \cdot 10^{204}$	$-1.5709538 \cdot 10^{486}$
	Coffey & Knessl	0.000210539	$-1.16551 \cdot 10^{204}$	$-1.570953 \cdot 10^{486}$

Let us mention that in 2011, Coffey and Knessl [7] have proved a variation of Matsuoka's asymptotic expression for these constants, namely:

Theorem 2 (Coffey and Knessl). Let $v = v(n)$ be the unique solution of the equation

$$2\pi \exp[v \tan v] = n \frac{\cos v}{v},$$

in the interval $(0, \pi/2)$, with $v \rightarrow \pi/2$ as $n \rightarrow \infty$. Let $u = v \tan v$ with $u(n) \sim \log n$ as $n \rightarrow \infty$. Then we have for $n \gg 1$,

$$\gamma_n \sim \frac{B}{\sqrt{n}} e^{nA} \cos(an + b),$$

where

$$A = \frac{1}{2} \log(u^2 + v^2) - \frac{u}{u^2 + v^2}, \quad B = \frac{2\sqrt{2\pi} \sqrt{u^2 + v^2}}{[(u+1)^2 + v^2]^{1/4}},$$

and

$$a = \tan^{-1} \left(\frac{v}{u} \right) + \frac{v}{u^2 + v^2}, \quad b = \tan^{-1} \left(\frac{v}{u} \right) - \frac{1}{2} \tan^{-1} \left(\frac{v}{u+1} \right).$$

The \sim -sign above is loosely defined in their paper and the expression “holds as long as we stay bounded away from the zeros of the cosine factor”. They showed that $A \sim \log \log n$ and $B \sim \frac{\pi}{2} (\log n)^{-1}$ and that this form encapsulates both the basic growth rate $\log \log n$ and the oscillations $\cos[\frac{n(\pi/2)}{\log n}]$. By using this form, they gave approximate numerical computations for γ_n from 2 to 35 000 (see Table 2). See also [8] for the case of Dirichlet L -series.

The following two functions will play a prominent role to our work: when $n \geq 1$ and $y > 0$, we define

$$\mathcal{F}_n(y) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z} \cos(\pi z/2) \Gamma(z)}{z^{n+1}} dz \quad (6)$$

and

$$\tilde{\mathcal{F}}_n(y) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z} \sin(\pi z/2) \Gamma(z)}{z^{n+1}} dz. \quad (7)$$

By using these functions, we provide below two formulas for $\gamma_n(\chi)$ that leads to our upper bound. Here, we recall that χ is called odd or even according as $\chi(-1) = -1$ or $\chi(-1) = 1$. In addition, the Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{a \bmod q} \chi(a) e(a/q),$$

where $e(z) = e^{2\pi iz}$.

Theorem 3. Let χ be a primitive Dirichlet character of modulus q . Then we have:

$$\frac{\gamma_n(\chi)}{n!} = \begin{cases} \frac{\tau(\chi)}{q} \sum_{m \geq 1} \bar{\chi}(m) \mathcal{F}_n\left(\frac{2\pi m}{q}\right), & \text{if } \chi(-1) = 1, \\ \frac{\tau(\chi)}{iq} \sum_{m \geq 1} \bar{\chi}(m) \tilde{\mathcal{F}}_n\left(\frac{2\pi m}{q}\right), & \text{if } \chi(-1) = -1, \end{cases}$$

where \mathcal{F}_n and $\tilde{\mathcal{F}}_n$ are defined by Eq. (6) and Eq. (7), respectively.

The case $q = 1$ being of the special importance, we state the following corollary:

Corollary 1. For $n \geq 1$, we have:

$$\frac{\gamma_n}{n!} = \sum_{m \geq 1} \mathcal{F}_n(2\pi m), \quad (8)$$

where \mathcal{F}_n is defined by Eq. (6) above.

We prove Theorem 3 in Section 2 by following the path opened by Matsuoka. In fact, the above expressions for \mathcal{F}_n and $\tilde{\mathcal{F}}_n$ are not efficient for bounding explicitly these functions. In order to bound them, we will have recourse to the following formulas which avoid any complex analysis.

Theorem 4. For any positive real number y , we have:

$$\mathcal{F}_n(y) = \frac{2}{n!} \int_y^\infty \left(\log \frac{u}{y} \right)^n \frac{\cos u}{u} du,$$

and

$$\tilde{J}_n(y) = \frac{2}{n!} \int_y^\infty \left(\log \frac{u}{y} \right)^n \frac{\sin u}{u} du.$$

See also [4]. These integrals do not converge very fast, but this is corrected in Lemma 8 below. In 1972, Liang and Todd [11] calculated the γ_n by use of the Euler–Maclaurin summation formula. They appealed to some recurrence relations for the function $\frac{\log^n x}{x}$ to calculate the derivatives needed in the Bernoulli terms and in the error term. In the final section, we shall use this summation to provide a proof of the Matsuoka formula that avoids any use of the functional equation of $\zeta(z)$ and $L(z, \chi)$. We shall also give another relation recurrence for $\frac{\log^n x}{x}$ in Lemma 10.

2. Proof of Theorems 3 and 4

In order to prove Theorem 3, we need the two following results:

Proposition 1. *Let χ be an even primitive Dirichlet character of modulus q . Then for any real parameter $a > 1$, we have:*

$$\frac{L^{(n)}(1, \chi)}{n!} = \frac{(-1)^n \tau(\chi)}{2\pi i} \frac{1}{q} \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi}{q} \right)^{-z} \frac{2 \cos(\pi z/2) \Gamma(z) L(z, \bar{\chi})}{z^{n+1}} dz.$$

Proposition 2. *Let χ be an odd primitive Dirichlet character of modulus q . Then for any real parameter $a > 1$, we have:*

$$\frac{L^{(n)}(1, \chi)}{n!} = \frac{(-1)^n \tau(\chi)}{2\pi i} \frac{1}{iq} \int_{a-i\infty}^{a+i\infty} \left(\frac{2\pi}{q} \right)^{-z} \frac{2 \sin(\pi z/2) \Gamma(z) L(z, \bar{\chi})}{z^{n+1}} dz.$$

Proof. The proof of these results is similar to the argument developed in Ishikawa [6, Proposition 1] and relies on the functional equation of $L(z, \chi)$ given by

$$\varepsilon(1-z, \bar{\chi}) = \frac{i^\alpha q^{1/2}}{\tau(\chi)} \varepsilon(z, \chi),$$

where the symbol α is given by

$$\alpha = \begin{cases} 0; & \text{if } \chi(-1) = 1, \\ 1; & \text{if } \chi(-1) = -1, \end{cases}$$

and ε is defined as follows

$$\varepsilon(z, \chi) = \left(\frac{\pi}{q} \right)^{-(z+\alpha)/2} \Gamma\left(\frac{z+\alpha}{2} \right) L(z, \chi). \quad \square$$

Once the propositions are proved, Theorem 3 follow readily, simply by noting that

$$L(z, \chi) = \sum_{m \geq 1} \frac{\chi(m)}{m^z}.$$

2.1. Lemmas for Theorem 4

Lemma 5. When $0 < \Re(z) < 1$, we have:

$$\cos(\pi z/2) \Gamma(z) = \int_0^\infty u^{z-1} \cos u \, du,$$

and

$$\sin(\pi z/2) \Gamma(z) = \int_0^\infty u^{z-1} \sin u \, du.$$

Proof. A proof of this lemma can be found in [5]. \square

Lemma 6. For every $x > 0$, we have:

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{x^{-z}}{z^{n+1}} dz = \begin{cases} \frac{(-1)^n}{n!} \log^n x, & \text{when } x \leq 1, \\ 0, & \text{when } x > 1. \end{cases}$$

Proof. First and foremost, the integral over an infinite path is the limit of the integral on a finite path:

$$\frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{x^{-z}}{z^{n+1}} dz = \lim_{T \rightarrow \infty} \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{x^{-z}}{z^{n+1}} dz. \quad (9)$$

When $x > 1$, we consider a rectangle in the z plane with vertices at the points $1/2 \pm iT$ and $A \pm iT$, where $T > 0$ and $A > 1/2$ going to infinity. By Cauchy's theorem the integral of x^{-z}/z^{n+1} around this rectangle is zero, we get

$$\left(\int_{1/2-iT}^{1/2+iT} + \int_{A+iT}^{A-iT} + \int_{1/2+iT}^{A+iT} + \int_{A-iT}^{1/2-iT} \right) \frac{x^{-z}}{z^{n+1}} dz = 0.$$

The second integral dwindles to zero when A increases. Both integrals on the horizontal segments are bounded by $\frac{x^{-1/2}A}{T^{n+1}}$. This yields

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{x^{-z}}{z^{n+1}} dz = \mathcal{O}\left(\frac{x^{-1/2}A}{T^{n+1}}\right).$$

If we now allow $T \rightarrow \infty$ in the last equality, we find that the limit in Eq. (9) is equal 0. Next, for $x \leq 1$, we remove this time the line of integration to the far left, i.e. we consider the integral of x^{-z}/z^{n+1} taken around a rectangle with vertices of the form $1/2 \pm iT$ and $-A \pm iT$, where $T > 0$ and $A > 1/2$ going to infinity. In this case, the integral around this rectangle is equal to $\frac{(-1)^n}{n!} \log^n x$, since the pole of order n at $z = 0$ lies inside the contour. By the same reasoning used in the case $x > 1$, we find that the limit in Eq. (9) is equal to $\frac{(-1)^n}{n!} \log^n x$, when $x \leq 1$. \square

By these lemmas, we can now prove Theorem 4.

2.2. Proof of Theorem 4

We start with the function:

$$\mathcal{F}_n(y) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{2y^{-z} \cos(\frac{\pi z}{2}) \Gamma(z)}{z^{n+1}} dz.$$

From Lemma 5, we get

$$\mathcal{F}_n(y) = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \int_0^\infty 2 \frac{(y/u)^{-z} \cos u}{z^{n+1} u} du dz.$$

Thanks to Lemma 6, we find that

$$\mathcal{F}_n(y) = \frac{2}{n!} \int_y^\infty \left(\log \frac{u}{y} \right)^n \frac{\cos u}{u} du,$$

and the proof of the first equality in Theorem 4 is complete. Then the proof of the second equality is similar to the previous proof.

3. Lemmas for Theorem 1

Lemma 7. The derivatives of the function $f_n(u) = \frac{(\log u)^n}{u}$ are given by

$$\frac{d^k f_n(u)}{du^k} = \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{(-1)^{k-\ell} k!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell)!} (\log u)^{n-\ell} \frac{1}{u^{k+1}} \quad (10)$$

where $0 \leq k \leq n$.

Proof. We prove this formula by induction on k . For $k = 0$, Eq. (10) is clearly valid. We prove that it is also valid for $k = 1$. If S the right side of Eq. (10) when $k = 1$ and

$$S = -(\log u)^n \frac{1}{u^2} + n(\log u)^{n-1} \frac{1}{u^2}.$$

Then $S = \frac{d}{du} f_n(u)$ and so Eq. (10) is valid for $k = 1$.

Now, we assume that Eq. (10) is valid for any fixed and nonnegative integer k . Then we have to prove that Eq. (10) is also valid for $k + 1$, i.e.

$$\frac{d^{k+1} f_n(u)}{du^{k+1}} = \sum_{\ell=0}^{k+1} \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k+1} \frac{(-1)^{k-\ell+1} (k+1)!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell)!} (\log u)^{n-\ell} \frac{1}{u^{k+2}}. \quad (11)$$

By induction hypothesis, we get

$$\begin{aligned} \frac{d}{du} \left(\frac{d^k f_n}{du^k} \right) &= \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{(-1)^{k-\ell} k!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell-1)!} (\log u)^{n-\ell-1} \frac{1}{u^{k+2}} \\ &\quad + \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{(-1)^{k-\ell+1} (k+1)!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell)!} (\log u)^{n-\ell} \frac{1}{u^{k+2}}. \end{aligned}$$

By making a simple change of variable $\ell + 1 = \ell'$ in the first sum above and multiplying the resulting equality by $k + 1 = m_{\ell'}$. We obtain

$$\begin{aligned} \frac{d}{du} \left(\frac{d^k f_n(u)}{du^k} \right) &= \sum_{\ell'=1}^{k+1} \sum_{1 \leq m_1 < m_2 < \dots < m_{\ell'-1} \leq k} \frac{(-1)^{k-\ell'+1} (k+1)!}{m_0 \dots m_{\ell'-1} m_{\ell'}} \frac{n!}{(n-\ell')!} (\log u)^{n-\ell'} \frac{1}{u^{k+2}} \\ &\quad + \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{(-1)^{k-\ell+1} (k+1)!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell)!} (\log u)^{n-\ell} \frac{1}{u^{k+2}}. \end{aligned}$$

Noting that the last term (for $\ell' = k + 1$) in the first sum on ℓ' gives us the last term of Eq. (11) for $k + 1$, which equal to

$$(n-1) \dots (n-k) (\log u)^{n-k+1} \frac{1}{u^{k+2}},$$

and that the first term (for $\ell = 0$) in the second sum on ℓ gives the first term of Eq. (11) for $k + 1$, which equal to

$$(-1)^{k+1} (k+1)! (\log u)^n \frac{1}{u^{k+2}}.$$

On the other hand, we have

$$\begin{aligned} &\sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k+1} \frac{1}{m_0 \dots m_\ell} \\ &= \sum_{1 \leq m_1 < m_2 < \dots < m_{\ell-1} \leq k} \frac{1}{m_0 \dots m_{\ell-1} (k+1)} + \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{1}{m_0 \dots m_\ell}. \end{aligned}$$

We conclude from the above that the claimed formula is also valid for $k + 1$. Then it is valid for all values of $k \geq 0$. The lemma is proved. \square

Lemma 8. For all $y > 0$, we have

$$\mathcal{F}_n(y) = \frac{2}{n!} \frac{1}{y^k} \int_1^\infty P_{n,k}(\log u) \frac{S_k(yu)}{u^{k+1}} du,$$

where

$$P_{n,k}(\log u) = \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} (-1)^{k-\ell} \frac{k!}{m_0 \dots m_\ell} \frac{n!}{(n-\ell)!} (\log u)^{n-\ell},$$

with $m_0 = 1$, and

$$S_k(u) = \begin{cases} \mp \cos u & \text{if } k \equiv 0, 2 \pmod{4}, \\ \mp \sin u & \text{if } k \equiv \pm 1 \pmod{4}. \end{cases}$$

Proof. Recall that

$$\mathcal{F}_n(y) = \frac{2}{n!} \int_y^\infty \left(\log \frac{u}{y} \right)^n \frac{\cos u}{u} du.$$

By making a simple change of variable $u/y = t$, we get

$$\mathcal{F}_n(y) = \frac{2}{n!} \int_1^\infty \log^n t \frac{\cos(yt)}{t} dt.$$

By using integration by parts and using Lemma 7, we conclude immediately the proof of this lemma. \square

Lemma 9. For all $y > 0$, we have:

$$\tilde{\mathcal{F}}_n(y) = \frac{2}{n!} \frac{1}{y^k} \int_1^\infty P_{n,k}(\log u) \frac{\tilde{S}_k(yu)}{u^{k+1}} du,$$

where

$$\tilde{S}_k(u) = \begin{cases} \mp \sin u & \text{if } k \equiv 0, 2 \pmod{4}, \\ \pm \cos u & \text{if } k \equiv \pm 1 \pmod{4} \end{cases}$$

and $P_{n,k}(\log u)$ is given by Lemma 8.

Proof. The proof of this lemma is similar to that in the proof of Lemma 8. \square

Lemma 10. For any integer number $k \geq 1$, we have

$$P_{n,k}(X) = P'_{n,k-1}(X) - kP_{n,k-1}(X), \quad (12)$$

where $P_{n,0}(X) = X^n$.

Proof. We prove this lemma by induction on k . In order to show that Eq. (12) is valid for $k = 1$, we start with

$$\begin{aligned}\mathcal{F}_n(y) &= \frac{2}{n!} \int_1^{\infty} \log^n u \frac{\cos(yu)}{u} du, \\ &= \frac{2}{n!} \int_1^{\infty} P_{n,0}(\log u) \frac{S_0(yu)}{u} du,\end{aligned}$$

where $S_k(u)$ are defined in Lemma 8. By using integration by parts, we get

$$\begin{aligned}\mathcal{F}_n(y) &= \frac{2}{n!} \frac{1}{y} \int_1^{\infty} \left(\frac{P'_{n,0}(\log u)}{u^2} - \frac{P_{n,0}(\log u)}{u^2} \right) S_1(yu) du \\ &= \frac{2}{n!} \frac{1}{y} \int_1^{\infty} P_{n,1}(\log u) \frac{S_1(yu)}{u^2} du.\end{aligned}$$

Then

$$P_{n,1}(\log u) = P'_{n,0}(\log u) - P_{n,0}(\log u)$$

and so Eq. (12) is valid for $k = 1$. Now, we assume that Eq. (12) is valid for any fixed and nonnegative integer k . Then we have to prove that it is also valid for $k + 1$. By induction hypothesis, we have

$$\mathcal{F}_n(y) = \frac{2}{n!} \frac{1}{y^k} \int_1^{\infty} P_{n,k}(\log u) \frac{S_k(yu)}{u^{k+1}} du.$$

By using integration by parts, and selecting $v = P_{n,k}(\log u) u^{-k-1}$ and $dw = S_k(yu) du$, we obtain

$$\begin{aligned}\mathcal{F}_n(y) &= \frac{2}{n!} \frac{1}{y^{k+1}} \int_1^{\infty} (P'_{n,k}(\log u) - (k+1)P_{n,k}(\log u)) \frac{S_{k+1}(yu)}{u^{k+2}} du, \\ &= \frac{2}{n!} \frac{1}{y^{k+1}} \int_1^{\infty} P_{n,k+1}(\log u) \frac{S_{k+1}(yu)}{u^{k+2}} du,\end{aligned}$$

which gives us

$$P_{n,k+1}(\log u) = P'_{n,k}(\log u) - kP_{n,k}(\log u).$$

We conclude from the above that Eq. (12) is valid for $k + 1$. Then it is valid for all $k \geq 1$. This completes the proof. \square

Lemma 11. For $k \geq 1$, we have:

$$\sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{t^\ell}{m_0 \cdots m_\ell} = \binom{k+t}{k}. \quad (13)$$

Proof. A proof of this lemma can be found in [3, formula (33)]. \square

Lemma 12. For all $k \geq 1$, we have:

$$\frac{(2k)!}{k!} \leq \sqrt{2} \left(\frac{4k}{e} \right)^k. \quad (14)$$

Proof. We prove this lemma by using the fact that

$$\left(\frac{k}{e} \right)^k \sqrt{2\pi k} e^{1/(12k+1)} \leq k! \leq \left(\frac{k}{e} \right)^k \sqrt{2\pi k} e^{1/(12k)},$$

which gives us

$$\frac{(2k)!}{k!} \leq \sqrt{2} \left(\frac{4k}{e} \right)^k e^{\frac{1}{24k} - \frac{1}{12k+1}} \leq \sqrt{2} \left(\frac{4k}{e} \right)^k,$$

as required. \square

With the aid of these lemmas, we can now prove our theorem.

4. Proof of Theorem 1

Recall that our function is

$$\mathcal{F}_n(y) = \frac{2}{n!} \int_y^\infty \left(\log \frac{u}{y} \right)^n \frac{\cos u}{u} du.$$

By Lemma 7 and an integration by parts, we obtain

$$\mathcal{F}_n(y) = \frac{2}{n!} \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{(-1)^{k-\ell} k!}{m_0 \cdots m_\ell} \frac{n!}{(n-\ell)!} \int_y^\infty \left(\log \frac{u}{y} \right)^{n-\ell} \frac{S_k(u)}{u^{k+1}} du,$$

where

$$S_k(u) = \begin{cases} \mp \cos u & \text{if } k \equiv 0, 2 \pmod{4}, \\ \mp \sin u & \text{if } k \equiv \pm 1 \pmod{4}, \end{cases}$$

and $m_0 = 1$. It follows that

$$|\mathcal{F}_n(y)| \leq 2 \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{k!}{m_0 \cdots m_\ell} \frac{1}{(n-\ell)!} \int_y^\infty \left(\log \frac{u}{y} \right)^{n-\ell} \frac{1}{u^{k+1}} du.$$

After a change of variable in the integral $\frac{y}{t} = t$, we get

$$|\mathcal{F}_n(y)| \leq \frac{2}{y^k} \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{k!}{m_0 \dots m_\ell} \frac{1}{(n-\ell)!} \int_1^\infty (\log t)^{n-\ell} \frac{1}{t^{k+1}} dt.$$

Since

$$\int_1^\infty (\log t)^{n-\ell} \frac{dt}{t^{k+1}} = \frac{(n-\ell)!}{k^{n-\ell+1}},$$

we get

$$|\mathcal{F}_n(y)| \leq 2k! \frac{y^{-k}}{k^{n+1}} \sum_{\ell=0}^k \sum_{1 \leq m_1 < m_2 < \dots < m_\ell \leq k} \frac{k^\ell}{m_0 \dots m_\ell}.$$

Using Lemma 11 yields

$$|\mathcal{F}_n(y)| \leq 2k! \frac{y^{-k}}{k^{n+1}} \binom{2k}{k}.$$

Using Lemma 12, we get

$$|\mathcal{F}_n(y)| \leq 2\sqrt{2} \frac{y^{-k}}{k^{n+1}} \left(\frac{4k}{e}\right)^k.$$

This inequality may be written as

$$|\mathcal{F}_n(y)| \leq 2\sqrt{2} y^{-k} \exp\{-(n+1)\log k + k\log k + (-1 + \log 4)k\}. \quad (15)$$

Replace y by $2\pi m/q$ and sum both sides of Eq. (15) from $m = 1$ to ∞ . We find that

$$\sum_{m \geq 1} \left| \mathcal{F}_n\left(\frac{2\pi m}{q}\right) \right| \leq C(n, k, q) \sum_{m \geq 1} \frac{1}{m^k} \quad (16)$$

where

$$C(n, k, q) = 2\sqrt{2} \exp\{-(n+1)\log k + k\log k + k\log q + (-1 - \log 2\pi + \log 4)k\}.$$

Now, let us determine such an admissible value of k . To do so, we put

$$\frac{d}{dk} C(n, k, q) = C'(n, k, q).$$

We see that the derivative of $C(n, k, q)$ with respect to k verifies

$$\begin{aligned}\frac{C'(n, k, q)}{C(n, k, q)} &= -\frac{(n+1)}{k} + \log k + \log q - \log 2\pi + \log 4 \\ &= -\frac{(n+1)}{k} + \log\left(\frac{2qk}{\pi}\right).\end{aligned}$$

The quantity $C(n, k, q)$ is maximal when

$$\frac{(n+1)}{k} = \log\left(\frac{2qk}{\pi}\right).$$

We multiply both sides of this equation by $2qk/\pi$ and put $K = 2qk/\pi$, to get

$$\frac{2q}{\pi}(n+1) = K \log K.$$

By setting $\frac{2q}{\pi}(n+1) = x$, we find that

$$f(K) = K \log K = x.$$

This equation has a unique solution K_0 given by

$$K_0 = \frac{x}{W(x)},$$

where W denotes the Lambert W -function. In addition, K_0 corresponds to a parameter k_0 given by

$$k_0 = \frac{n+1}{W(2q(n+1)/\pi)}.$$

The above analysis shows that $C(n, k, q)$ decreases when $k \leq k_0$ and increases afterwards. Letting K_1 denote nonnegative real number given by

$$K_1 = \frac{x}{\log x},$$

which corresponds to a parameter k_1 given by $\frac{(n+1)}{\log \frac{2q(n+1)}{\pi}}$. We stress that k_1 is not necessarily an integer. Since

$$f\left(\frac{x}{\log x}\right) = \frac{x}{\log x} \log \frac{x}{\log x} < x,$$

it follows that $k_1 \leq k_0$. It follows further that

$$1 \leq \left\lfloor \frac{n+1}{\log \frac{2q(n+1)}{\pi}} \right\rfloor \leq k_0.$$

Since $C' < 0$ for any integer number less than k_0 and putting

$$k_2 = \frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1. \quad (17)$$

Table 2
This table compares our upper bound for $|\gamma_n|$ with the ones given by Matsuoka and Adell and also with the exact value computed by Kreminski, Coffey and Knessl.

$ \gamma_n $				γ_n	
n	Matsuoka	Adell	Our result	Kreminski	Coffey, Knessl
2	–	–	0.070245078183852	–	–
3	–	–	0.067719699218508	–	0.00190188
4	–	62.66226985729476870185	0.079906995076743	–	0.00231644
5	–	253.2551193757444924660	0.101671067958902	–	0.000812965
10	0.418944879802952047	21622.67050207189979471	0.714120274361758	$2.0533281 \cdot 10^{-4}$	0.000210539
20	338890.9439428998678	8108016580.566186017960	340.4220669628279	$4.6634356 \cdot 10^{-4}$	0.000471981
30	889109155106.7848077	13920583854580676.03702	858791.0355985352	$3.5577288 \cdot 10^{-3}$	0.00359535
40	4740002165959979407.4	53544339684077132669735.2	5972285271.236439	$2.4873155 \cdot 10^{-1}$	0.251108
100	$2.111389883 \cdot 10^{62}$	$6.131935463 \cdot 10^{65}$	$3.3048214111 \cdot 10^{37}$	$-4.2534015 \cdot 10^{17}$	$-4.2534 \cdot 10^{17}$
150	$9.636492256 \cdot 10^{100}$	$1.201425843 \cdot 10^{104}$	$7.3910426765 \cdot 10^{63}$	$8.0288537 \cdot 10^{35}$	$8.02885 \cdot 10^{35}$
200	$6.723474349 \cdot 10^{140}$	$4.028139634 \cdot 10^{143}$	$7.4716821273 \cdot 10^{91}$	$-6.9746497 \cdot 10^{55}$	$-6.97465 \cdot 10^{55}$
250	$3.262125160 \cdot 10^{181}$	$1.005492856 \cdot 10^{184}$	$1.1136521652 \cdot 10^{121}$	$3.0592128 \cdot 10^{79}$	$3.05921 \cdot 10^{79}$
300	$7.061327217 \cdot 10^{222}$	$1.171684058 \cdot 10^{2251}$	$1.3071668576 \cdot 10^{151}$	$-5.5567282 \cdot 10^{102}$	$-5.55673 \cdot 10^{102}$
400	$1.030854890 \cdot 10^{307}$	$5.475491836 \cdot 10^{308}$	$2.0198164612 \cdot 10^{213}$	$-1.7616421 \cdot 10^{152}$	$-1.76164 \cdot 10^{152}$
500	$5.091823280 \cdot 10^{392}$	$9.480434509 \cdot 10^{393}$	$3.8204382632 \cdot 10^{277}$	$-1.1655052 \cdot 10^{204}$	$-1.16551 \cdot 10^{204}$
600	$3.827762436 \cdot 10^{479}$	$2.665372229 \cdot 10^{480}$	$2.9158215571 \cdot 10^{343}$	$3.5627462 \cdot 10^{257}$	$3.56275 \cdot 10^{257}$
700	$2.624506258 \cdot 10^{567}$	$7.173560533 \cdot 10^{567}$	$4.4254777108 \cdot 10^{410}$	$-3.5494521 \cdot 10^{312}$	$-3.549452 \cdot 10^{312}$
800	$1.150665897 \cdot 10^{656}$	$1.282553972 \cdot 10^{656}$	$8.1932108605 \cdot 10^{478}$	$4.9135405 \cdot 10^{369}$	$4.9135405 \cdot 10^{369}$
1000	$2.172418132 \cdot 10^{835}$	$4.407894620 \cdot 10^{834}$	$1.3320458753 \cdot 10^{618}$	$-1.5709538 \cdot 10^{486}$	$-1.570953 \cdot 10^{486}$
1200	$6.041294880 \cdot 10^{1016}$	$2.443852795 \cdot 10^{1015}$	$1.7273464190 \cdot 10^{760}$	$8.6840299 \cdot 10^{605}$	$8.6840303 \cdot 10^{605}$
1400	$9.747968498 \cdot 10^{1199}$	$8.441567295 \cdot 10^{1197}$	$5.0329835559 \cdot 10^{904}$	$-4.0972873 \cdot 10^{728}$	$-4.097289 \cdot 10^{728}$

We conclude that

$$C(k_0, n, q) \leq C(k_1, n, q) \leq C(\lfloor k_1 \rfloor, n, q) \leq C(k_2, n, q),$$

when $k_2 \geq 1$, i.e. when

$$q \leq \frac{\pi}{2} \frac{e^{(n+1)/2}}{n+1}.$$

We now return to Eq. (16) and replaces k_2 by $(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1)$ to deduce that

$$\sum_{m \geq 1} \left| \mathcal{F}_n \left(\frac{2\pi m}{q} \right) \right| \leq C(n, q) \sum_{m \geq 1} m^{-\frac{n+1}{\log(\frac{2q(n+1)}{\pi})} + 1},$$

where

$$\begin{aligned} C(n, q) = 2\sqrt{2} \exp \left\{ -(n+1) \log \left(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1 \right) \right. \\ \left. + \left(\frac{(n+1)}{\log \frac{2q(n+1)}{\pi}} - 1 \right) \left(\log \left(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1 \right) + \log \frac{2q}{\pi e} \right) \right\}. \end{aligned}$$

For the last sum of this inequality, we write

$$\sum_{m \geq 1} \frac{1}{m^{k_2}} \leq 1 + \frac{1}{2^{k_2}} + \int_2^\infty \frac{dt}{t^{k_2}} \leq 1 + \frac{1}{2^{k_2}} \left(1 + \frac{2}{k_2 - 1} \right) \leq 1 + \frac{1}{2^{k_2}} \left(\frac{k_2 + 1}{k_2 - 1} \right).$$

Replacing k_2 by $(\frac{n+1}{\log \frac{2q(n+1)}{\pi}} - 1)$, we find that

$$\sum_{m \geq 1} \left| \mathcal{F}_n \left(\frac{2\pi m}{q} \right) \right| \leq C(n, q) (1 + D(n, q))$$

where

$$D(n, q) = 2^{-\frac{n+1}{\log \frac{2q(n+1)}{\pi}}} \left(\frac{n+1}{n - 2 \log \frac{2q(n+1)}{\pi} + 1} \right).$$

From Theorem 3 and the fact that $|\tau(\chi)| = \sqrt{q}$, Theorem 1 now readily follows.

5. The Matsuoka formula via Euler–Maclaurin summation formula

In this section, we prove Theorem 3 without using the functional equation of $\zeta(z)$ and $L(z, \chi)$. To do so, we begin with a slight modification of Eq. (2) when $1 \leq a \leq q$

$$\gamma_n(a, q) = \lim_{T \rightarrow \infty} \left\{ \sum_{t=0}^T \frac{\log^n(a + qt)}{a + qt} - \frac{\log^{n+1}(a + qT)}{q(n+1)} \right\}.$$

Recall that the Euler–Maclaurin summation formula can be written as:

$$\begin{aligned} \sum_{t=0}^T h_n(a+qt) &= \frac{1}{q} \int_a^{a+qT} h_n(t) dt + \frac{1}{2} (h_n(a) + h_n(a+qT)) \\ &\quad + \sum_{r=1}^k \frac{B_{2r}}{(2r)!} q^{2r-1} (h_n^{(2r-1)}(a+qT) - h_n^{(2r-1)}(a)) \\ &\quad + \frac{q^{2k}}{(2k+1)!} \int_a^{a+qT} B_{2k+1}\left(\frac{t-a}{q}\right) h_n^{(2k+1)}(t) dt, \end{aligned}$$

where $B_{k+1}(t)$ are the Bernoulli polynomials and B_k are the Bernoulli numbers, such that $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6} \dots$ and that $h_n^{(r)}(t)$ being the r -th derivative of the function $h_n(t) = \log^n t/t$. We use this summation in the following way. Taking $k = 1$, we get

$$\begin{aligned} \sum_{t=0}^T \frac{\log^n(a+qt)}{a+qt} - \frac{\log^{n+1}(a+qT)}{q(n+1)} \\ = -\frac{\log^{n+1}a}{q(n+1)} + \frac{\log^n a}{2a} + \frac{\log^n(a+qT)}{2(a+qT)} + \frac{q}{12} \left(n \frac{\log^{n-1}(a+qT)}{(a+qT)^2} - \frac{\log^n(a+qT)}{(a+qT)^2} \right) \\ - \frac{q}{12} \left(n \frac{\log^{n-1}a}{a^2} - \frac{\log^n a}{a^2} \right) + \frac{q^2}{6} \left(\int_1^{a+qT} B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt - \int_1^a B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt \right). \end{aligned}$$

Letting $T \rightarrow \infty$ gives us

$$\begin{aligned} \gamma_n(a, q) &= -\frac{\log^{n+1}a}{q(n+1)} + \frac{\log^n a}{2a} - \frac{q}{12} \left(n \frac{\log^{n-1}a}{a^2} - \frac{\log^n a}{a^2} \right) \\ &\quad + \frac{q^2}{6} \int_1^\infty B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt - \frac{q^2}{6} \int_1^a B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt. \end{aligned}$$

Using integration by parts the last integral above, selecting $dv = h_n^{(3)}(t) dt$ and $w = B_3(\frac{t-a}{q})$. We obtain

$$\begin{aligned} \gamma_n(a, q) &= -\frac{\log^{n+1}a}{q(n+1)} + \frac{\log^n a}{2a} - \frac{q}{12} \left(n \frac{\log^{n-1}a}{a^2} - \frac{\log^n a}{a^2} \right) \\ &\quad + \frac{q^2}{6} \int_1^\infty B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt + \frac{q}{2} \int_1^a B_2\left(\frac{t-a}{q}\right) h_n^{(2)}(t) dt. \end{aligned}$$

We integrate by parts one more time to get

$$\gamma_n(a, q) = -\frac{\log^{n+1}a}{q(n+1)} + \frac{\log^n a}{2a} + \frac{q^2}{6} \int_1^\infty B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt - \int_1^a B_1\left(\frac{t-a}{q}\right) h_n^{(1)}(t) dt.$$

Since $B_1(t) = t - 1/2$ and $h_n^{(1)}(t) = (n \log^{n-1} t - \log^n t)/t^2$, then

$$\int_1^a B_1\left(\frac{t-a}{q}\right) h_n^{(1)}(t) dt = \frac{\log^{n+1} a}{q(n+1)} - \frac{\log^n a}{2a}.$$

It follows that

$$\gamma_n(a, q) = \frac{q^2}{6} \int_1^\infty B_3\left(\frac{t-a}{q}\right) h_n^{(3)}(t) dt \quad (18)$$

where For $0 \leq t \leq 1$, we recall that:

$$B_{2k+1}(t) = \frac{(-1)^{k-1} 2(2k+1)!}{(2\pi)^{2k+1}} \sum_{m=1}^\infty \frac{\sin(2\pi mt)}{m^{2k+1}}$$

and that

$$\sin(u-a) = \sin u \cos a - \cos u \sin a.$$

Thus, the right-hand side of Eq. (18) may be written as

$$\begin{aligned} \gamma_n(a, q) &= \frac{3!q^2}{3(2\pi)^3} \int_1^\infty \sum_{m \geq 1} \frac{\cos\left(\frac{2\pi ma}{q}\right) \sin\left(\frac{2\pi mt}{q}\right)}{m^3} h_n^{(3)}(t) dt \\ &\quad - \frac{3!q^2}{3(2\pi)^3} \int_1^\infty \sum_{m \geq 1} \frac{\sin\left(\frac{2\pi ma}{q}\right) \cos\left(\frac{2\pi mt}{q}\right)}{m^3} h_n^{(3)}(t) dt \\ &= \frac{2}{q} \sum_{m \geq 1} \frac{\cos\left(\frac{2\pi ma}{q}\right)}{\left(\frac{2\pi m}{q}\right)^3} \int_1^\infty \sin\left(\frac{2\pi mt}{q}\right) h_n^{(3)}(t) dt \\ &\quad - \frac{2}{q} \sum_{m \geq 1} \frac{\sin\left(\frac{2\pi ma}{q}\right)}{\left(\frac{2\pi m}{q}\right)^3} \int_1^\infty \cos\left(\frac{2\pi mt}{q}\right) h_n^{(3)}(t) dt. \end{aligned}$$

It follows that

$$\gamma_n(a, q) = \frac{1}{q} \sum_{m \geq 1} \left\{ \cos\left(\frac{2\pi ma}{q}\right) n! \mathcal{F}_n\left(\frac{2\pi m}{q}\right) + \sin\left(\frac{2\pi ma}{q}\right) n! \tilde{\mathcal{F}}_n\left(\frac{2\pi m}{q}\right) \right\}$$

where \mathcal{F}_n and $\tilde{\mathcal{F}}_n$ are defined by Lemmas 8 and 9 (with $k=3$). Now notice that:

$$\sum_{a=1}^q \chi(a) \cos\left(\frac{2\pi ma}{q}\right) = \bar{\chi}(m) \frac{\tau(\chi) + \chi(-1)\tau(\chi)}{2},$$

and

$$\sum_{a=1}^q \chi(a) \sin\left(\frac{2\pi ma}{q}\right) = \bar{\chi}(m) \frac{\tau(\chi) - \chi(-1)\tau(\chi)}{2i}.$$

Recalling Eq. (1), we get

$$\begin{aligned} \frac{\gamma(\chi)}{n!} &= \frac{\tau(\chi) + \chi(-1)\tau(\chi)}{2q} \sum_{m \geq 1} \bar{\chi}(m) \mathcal{F}_n\left(\frac{2\pi m}{q}\right) \\ &\quad + \frac{\tau(\chi) - \chi(-1)\tau(\chi)}{2iq} \sum_{m \geq 1} \bar{\chi}(m) \tilde{\mathcal{F}}_n\left(\frac{2\pi m}{q}\right). \end{aligned}$$

In the case χ even, we see that this equation equal to

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{q} \sum_{m \geq 1} \bar{\chi}(m) \mathcal{F}_n\left(\frac{2\pi m}{q}\right).$$

In the case χ odd, we find that

$$\frac{\gamma_n(\chi)}{n!} = \frac{\tau(\chi)}{iq} \sum_{m \geq 1} \bar{\chi}(m) \tilde{\mathcal{F}}_n\left(\frac{2\pi m}{q}\right),$$

which completes the proof.

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Supplementary material

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