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Some New Congruences for Andrews' Singular Overpartitions

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Abstract:

Recently, Andrews defined combinatorial objects which he called singular overpartitions and proved that these singular overpartitions which depend on two parameters k and i can be enumerated by the function $\overline{C}_{k,i}(n)$, which denotes the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod{k}$ may be overlined. G. E. Andrews, S. C. Chen, M. Hirschhorn, J. A. Sellars, Olivia X. M. Yao, M. S. Mahadeva Naika, D. S. Gireesh, Zakir Ahmed and N. D. Baruah noted numerous congruences modulo $2, 3, 4, 6, 12, 16, 18, 32$ and 64 for $\overline{C}_{3,1}(n)$. In this paper, we prove congruences modulo 128 for $\overline{C}_{3,1}(n)$, and congruences modulo 2 for $\overline{C}_{12,3}(n)$, $\overline{C}_{44,11}(n)$,$\overline{C}_{75,15}(n)$, and $\overline{C}_{92,23}(n)$. We also prove "Mahadeva Naika and Gireesh's conjecture", for $n \geq 0$, $\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}$ is true.

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Keywords: Singular overpartition, Theta function, Congruence, Dissection.

1 INTRODUCTION

A partition of a positive integer n denoted by $p(n)$, is a nonincreasing sequence of positive integers whose sum is n. If ℓ is a positive integer, then a partition is called a ℓ-regular partition denoted by $b_\ell(n)$, if there is no part divisible by ℓ.
The generating function for $b_\ell(n)$, is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty} = \frac{f_\ell}{f_1}, \tag{1.1}$$

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where as customary, we define

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty}(1 - q^{mk}).$$

Several interesting arithmetic properties of ℓ-regular partition are found by many mathematicians, see [4, 9, 10, 14, 18, 20].

In [13], Corteel and Lovejoy introduced overpartitons. An overpartition of n denoted by $\overline{p}(n)$, is a nonincreasing sequence of positive integers whose sum is n in which the first occurrences of number may be overlined. For example, $\overline{p}(3) = 8$. The eight overpartition of 3 are 3, $\overline{3}$, $2+1$, $\overline{2}+1$, $2+\overline{1}$, $\overline{2}+\overline{1}$, $1+1+1$ and $\overline{1}+1+1$. The generating function for $\overline{p}(n)$, is given by

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{f_2}{f_1^2}. \tag{1.2}$$

Recently G. E. Andrews [5] introduced singular overpartition denoted by $\overline{C}_{\delta,i}(n)$, which count the number of overpartitions of n in which no part is divisible by δ and only parts $\equiv i \pmod{\delta}$ may be overlined. For example, $\overline{C}_{3,1}(4) = 10$. The 10 singular overpartitions of 4 are 4, $\bar{4}$, $2+2$, $\bar{2}+2$, $2+1+1$, $\bar{2}+1+1$, $2+\bar{1}+1$, $\bar{1}+\bar{1}+1$, $1+1+1+1$ and $\bar{1}+1+1+1$.
The generating function for $\overline{C}_{\delta,i}(n)$, is given by, $\delta \geq 3$ and $1 \leq i \leq \left\lfloor \frac{\delta}{2} \right\rfloor$,

$$\sum_{n=0}^{\infty}\overline{C}_{\delta,i}(n)q^n = \frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}. \tag{1.3}$$

In his paper [5], G. E. Andrews also proved that for $n \geq 0$,

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}.$$

Chan et al. [11] generalized and found infinite families of congruences modulo 3 for $\overline{C}_{3,1}(n)$, $\overline{C}_{6,1}(n)$, $\overline{C}_{6,2}$ and modulo 2 for $\overline{C}_{4,1}(n)$. For example, they proved that for $n, k \geq 0$,

$$\overline{C}_{3,1}(2^k(6n+5)) \equiv 0 \pmod{8}.$$

Recently, Ahmed and Baruah [2] using simple p-dissections of Ramanujan's theta functions have proved several congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,2}(n)$, $\overline{C}_{12,4}$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$. Subsequently, Naika and Gireesh [19] prove congruence modulo 6, 12, 16, 18 and 24 for $\overline{C}_{3,1}$ and infinite families of congruence modulo 12, 18, 48, and 72 for $\overline{C}_{3,1}(n)$. They conjecture the following congruence for $\overline{C}_{3,1}(n)$ modulo 144,

$$\overline{C}_{3,1}(12n+11) \equiv 0 \pmod{144}. \tag{1.4}$$

The aim of this paper is to prove new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{12,3}(n)$, $\overline{C}_{44,11}(n)$, $\overline{C}_{75,25}(n)$ and $\overline{C}_{92,23}(n)$. The following are our main results:

Theorem 1.1. If p is prime $p \geq 5$, such that $\left(\dfrac{-3}{p}\right) = -1$, than for any nonnegative integer α and n,

$$\overline{C}_{3,1}\left(24p^{2\alpha+1}(pn+j) + 7p^{2\alpha+2}\right) \equiv 0 \pmod{128}.$$

Theorem 1.2. If p is prime $p \geq 5$, such that $\left(\dfrac{-2}{p}\right) = -1$, than for any nonnegative integer α and n,

$$\overline{C}_{3,1}(24p^{2\alpha+1}\left(pn+j\right) + 19p^{2\alpha+2}\right) \equiv 0 \pmod{128}.$$

Theorem 1.3. For $k \geq 0$, we have

$$\overline{C}_{12,3}\left(4^k n + \frac{4^k-1}{3}\right) \equiv \overline{C}_{12,3} \pmod{2}, \tag{1.5}$$

$$\overline{C}_{12,3}\left(4^{k+1} n + \frac{10 \cdot 4^k - 1}{3}\right) \equiv 0 \pmod{2}, \tag{1.6}$$

$$\overline{C}_{12,3}\left(4^{k+1} n + \frac{4^k(6m+1) - 1}{3}\right) \equiv 0 \pmod{2}, \quad 1 \leq m \leq 7. \tag{1.7}$$

Theorem 1.4. For all $n \geq 0$,

$$\overline{C}_{44,11}(16n+2) \equiv 0 \pmod{2}, \tag{1.8}$$

$$\overline{C}_{44,11}(16n+14) \equiv 0 \pmod{2}, \tag{1.9}$$

$$\overline{C}_{44,11}(16n+10) \equiv 0 \pmod{2}, \tag{1.10}$$

$$\overline{C}_{44,11}(176n+16m+6) \equiv 0 \pmod{2}, \quad 1 \leq m \leq 10. \tag{1.11}$$

Theorem 1.5. For all $n \geq 0$,

$$\overline{C}_{75,25}(10n+9) \equiv 0 \pmod{2}, \tag{1.12}$$

$$\overline{C}_{75,25}(80n+20m+14) \equiv 0 \pmod{2}, \quad 1 \leq m \leq 3. \tag{1.13}$$

Theorem 1.6. If $m \in 5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22$, then for all $n \geq 0$,

$$\overline{C}_{92,23}\left(2 \cdot 23^{2k}(23n+m) + \frac{7 \cdot 23^{2k+1} - 73}{88}\right) \equiv 0 \pmod{2}. \tag{1.14}$$

In order to prove our main results, we collect a few definitions and lemmas in section 2. In section 3, we prove Naika and Gireesh's conjecture (1.4) is true. The proofs of Theorems $1.1 - 1.6$ are given in section 4. In the subsequent section we conclude the paper with some interesting congruences for $\overline{C}_{12,3}(n)$, $\overline{C}_{44,11}(n)$ and $b_2(n)$ modulo 2.

2 Preliminaries

In order to prove the main results of this paper, we collect some definitions and lemmas in this section.
For $\mid ab \mid< 1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b)=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}. \tag{2.1}$$

Using Jacobi's triple product identity [8, Entry 19, p. 35], (2.1) becomes

$$f(a,b)=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{2.2}$$

The most important special cases of $f(a,b)$ are

$$\varphi(q):=f(q,q)=1+2\sum_{n=1}^{\infty}q^{n^2}=(-q;q^2)_\infty^2(q^2;q^2)_\infty=\frac{f_2^5}{f_1^2f_4^2}, \tag{2.3}$$

$$\psi(q):=f(q;q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\frac{f_2^2}{f_1} \tag{2.4}$$

and

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty=f_1, \tag{2.5}$$

where the product representations in (2.5) arise from (2.2) and is Euler's famous pentagonal theorem [3]. After Ramanujan, we also define

$$\chi(q):=(-q;q^2)_\infty=\frac{f_2^2}{f_1f_4}.$$

We also note

$$\psi(-q)=\frac{f_1f_4}{f_2},\quad \varphi(-q)=\frac{f_1^2}{f_2},\quad \chi(-q)=\frac{f_1}{f_2}.$$

By the binomial theorem, for any positive integer k,

$$f^{2^k}\equiv f_2^{2^{k-1}}\pmod{2^k}. \tag{2.6}$$

Lemma 2.1. (Hirschhorn and Sellers [17]) The following 3-dissection holds

$$\frac{f_2}{f_1^2}=\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}. \tag{2.7}$$

Lemma 2.2. (Baruah and Ojah [6, Theorem 4.3]) The following 2-dissection holds

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}. \tag{2.8}$$

Multiplying both sides of (2.8) by f_1^2 and replacing q by q^{11}, we find

$$\frac{f_{11}}{f_{33}} \equiv \frac{f_{22}^5}{f_{132}} + q^{11}\frac{f_{132}}{f_{22}} \pmod 2. \tag{2.9}$$

Lemma 2.3. (Hirschhorn, Garvan and Borwein [15]) The following 2-dissection holds

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.10}$$

Lemma 2.4. (Cui and Gu [12, Theorem 2.2]) If $p \geq 5$ is a prime and

$$\frac{\pm p-1}{6} := \begin{cases} \dfrac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \dfrac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6, \end{cases}$$

then

$$(q;q)_\infty = \sum_{\substack{k=-\frac{p-1}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)$$
$$+ (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2}; q^{p^2})_\infty. \tag{2.11}$$

Furthermore, if $\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2}, k \neq \frac{(\pm p-1)}{6}$, then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Lemma 2.5. (Ahmed and Baruah [1, Lemma 2.3]) If $p \geq 3$ is prime, then

$$(q;q)_\infty^3 = \sum_{\substack{k=0 \\ k \neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn \cdot \frac{pn+2k+1}{2}}$$
$$+ p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2}; q^{p^2})_\infty^3. \tag{2.12}$$

Furthermore, if $k \neq \frac{p-1}{2}, 0 \leq k \leq p-1$, then $\frac{k^2+k}{2} \not\equiv \frac{p^2-1}{8} \pmod p$.

Lemma 2.6. (Hirschhorn [16]) We have,

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(\frac{1}{R^4(q^5)} + \frac{q}{R^3(q^5)} + \frac{2q^2}{R^2(q^5)} + \frac{3q^3}{R(q^5)} + 5q^4 - 3q^5R(q^5) + 2q^6R^2(q^5) - q^7R^3(q^5) + q^8R^4(q^5)\right), \tag{2.13}$$

where $R(q)$ is the Rogers-Ramanujan continued fraction defined, for $\mid q \mid < 1$, by

$$R(q) := \frac{q^{1/5}}{1}\,{}_{+}\frac{q}{1}\,{}_{+}\frac{q^2}{1}\,{}_{+\cdots}.$$

Lemma 2.7. (Hirschhorn and Sellers [18, Theorem 1]) The following 2-dissection holds

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{2.14}$$

Lemma 2.8. (Baruah and Ahmed [7, Eqn. (2.4)])

$$\frac{1}{(q;q)_\infty (q^{11};q^{11})_\infty} \equiv \frac{1}{(q^2;q^2)_\infty^2 (q^{22};q^{22})_\infty^2}\left(\psi(q^{12}) + q^6\frac{\psi(-q^{66})\chi(q^{22})}{\chi(-q^4)} + q\frac{\psi(-q^6)\chi(q^2)}{\chi(-q^{44})} + q^{15}\psi(q^{132})\right) \pmod 2. \tag{2.15}$$

Lemma 2.9. (Berndt [8, Entry 31, p. 48])
Let $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ and $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ for an integer n. Then

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{2.16}$$

3 Proof of M. S. M. Naika and D. S. Gireesh's Conjecture (1.4)

From [19, Eq. 3.19], we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}(4n+3)q^n = 6\frac{f_2^3 f_6^3}{f_1^6}. \tag{3.1}$$

Substituting (2.7) in (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{3,1}(4n+3)q^n &= 6f_6^3\left(\frac{f_6^4f_9^6}{f_3^8f_{18}^3}+2q\frac{f_6^3f_9^3}{f_3^7}+4q^2\frac{f_6^2f_{18}^3}{f_3^6}\right)^3\\ &= 6\frac{f_6^{15}f_9^{18}}{f_3^{24}f_{18}^9}+36q\frac{f_6^{14}f_9^{15}}{f_3^{23}f_{18}^6}+144q^2\frac{f_6^{13}f_9^{12}}{f_3^{22}f_{18}^3}+336q^3\frac{f_6^{12}f_9^9}{f_3^{21}}\\ &\quad+576q^4\frac{f_6^{11}f_9^6f_{18}^3}{f_3^{20}}+576q^5\frac{f_6^{10}f_9^3f_{18}^6}{f_3^{19}}+384q^6\frac{f_6^9f_{18}^9}{f_3^{18}}. \qquad (3.2)\end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+11)q^n = 144\frac{f_2^{13}f_3^{12}}{f_1^{22}f_6^3}+576q\frac{f^{10}f_3^3f_6^6}{f_1^{19}}. \qquad (3.3)$$

Conjecture (1.4) follow from (3.3).

4 Proof of Theorems 1.1-1.6

Theorem 4.1. If p is prime with $p\equiv 5 \pmod 6$ and $\alpha\geq 0$, then

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}n+7p^{2\alpha}\right)q^n \equiv 36p^{\alpha}(-1)^{\alpha\cdot\frac{p-2}{3}}(q;q)_\infty^3(q^4;q^4)_\infty \pmod{128}. \qquad (4.1)$$

Proof. It follows from (3.2) that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+7)q^n = 36\frac{f_2^{14}f_3^{15}}{f_1^{23}f_6^6}+576q\frac{f_2^{11}f_3^6f_6^3}{f_1^{20}}. \qquad (4.2)$$

Using (2.6) in (4.2), we have

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+7)q^n \equiv 36\frac{f_2^3f_{12}}{f_1f_3}+64qf_2f_{12}^3 \pmod{128}. \qquad (4.3)$$

Substituting (2.10) into (4.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{3,1}(12n+7)q^n &\equiv 36f_2^3f_{12}\left(\frac{f_8}{f_{12}}+q\frac{f_{24}}{f_4}\right)+64qf_2f_{12}^3 \pmod{128},\\ &\equiv 36f_2^3f_8+100qf_2f_{12}^3 \pmod{128}. \qquad (4.4)\end{aligned}$$

From (4.4), we have

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(24n+7)q^n \equiv 36f_1^3f_4 \pmod{128}, \qquad (4.5)$$

which is the $\alpha = 0$ case of (4.1). Now suppose that (4.1) holds for some $\alpha \geq 0$. Substituting (2.11) and (2.12) in (4.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha}n + 7p^{2\alpha}\right) q^n \\
&\quad \equiv 36p^{\alpha}(-1)^{\alpha\left(\frac{\pm p-1}{6}+\frac{p-1}{2}\right)} \left[\sum_{\substack{k=0 \\ k\neq \frac{p-1}{2}}}^{p-1} (-1)^k q^{\frac{k(k+1)}{2}} \sum_{n=0}^{\infty} (-1)^n (2pn+2k+1) q^{pn\cdot\frac{pn+2k+1}{2}} \right. \\
&\quad \left. + p(-1)^{\frac{p-1}{2}} q^{\frac{p^2-1}{8}} (q^{p^2};q^{p^2})_\infty^3 \right] \\
&\quad \times \left[\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{4\frac{3k^2+k}{2}} f\left(-q^{4\frac{3p^2+(6k+1)p}{2}}, -q^{4\frac{3p^2-(6k+1)p}{2}}\right) \right. \\
&\quad \left. + (-1)^{\frac{\pm p-1}{6}} q^{4\frac{p^2-1}{24}} (q^{4p^2};q^{4p^2})_\infty \right] \pmod{128}. \qquad (4.6)
\end{aligned}$$

For a prime $p \geq 5$, $0 \leq k \leq p-1$ and $\dfrac{-(p-1)}{2} \leq m \leq \dfrac{(p-1)}{2}$, now consider the congruence

$$\frac{k^2+k}{2} + 4\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p}, \qquad (4.7)$$

which is equivalent to

$$3(2k+1)^2 + (12m+2)^2 \equiv 0 \pmod{p}.$$

Since $\left(\dfrac{-3}{p}\right) = -1$ as $p \equiv 5 \pmod 6$ the solution (4.7) is $k = \dfrac{p-1}{2}$ and $m = \dfrac{p-1}{6}$.

Therefore, extracting the terms involving $q^{pn+\frac{7p^2-7}{24}}$ from both sides of (4.6) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+1}n + 7p^{2\alpha+2}\right) q^n \equiv 36p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}} (q^p;q^p)_\infty^3 (q^{4p};q^{4p})_\infty \pmod{128}. \qquad (4.8)$$

Extracting the terms containing q^{pn} from both sides of identity (4.8) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha+2}n+7p^{2\alpha+2}\right)q^n \equiv 36p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}}(q;q)_\infty^3(q^4;q^4)_\infty \pmod{128} \tag{4.9}$$

This completes the proof by induction of (4.1). ☐

We can now prove Theorem 1.1.
Proof of Theorem 1.1
Empolying (2.11) and (2.12) and then comparing the coefficients of q^{pn+j}, $1 \le j \le p-1$, on both side of (4.8), we deduce Theorem 1.1

Theorem 4.2. If p is prime with $p \equiv 5$ or $7 \pmod 8$ and $\alpha \ge 0$, then

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}n+19p^{2\alpha}\right)q^n \equiv 100p^{\alpha}(-1)^{\alpha\cdot\frac{p-2}{3}}(q^6;q^6)_\infty^3(q;q)_\infty \pmod{128} \tag{4.10}$$

Proof. From (3.2) we have

$$\sum_{n=0}^{\infty}\overline{C}_{3,1}(24n+19)q^n \equiv 100f_6^3f_1 \pmod{128} \tag{4.11}$$

which is the $\alpha = 0$ case of (4.10). Now suppose that (4.10) holds for some $\alpha \ge 0$. Substituting (2.11) and (2.12) in (4.10), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\overline{C}_{3,1}\left(24p^{2\alpha}n+19p^{2\alpha}\right)q^n\\
&\equiv 100p^{\alpha}(-1)^{\alpha\left(\frac{\pm p-1}{6}+\frac{p-1}{2}\right)}\left[\sum_{\substack{k=0\\k\neq\frac{p-1}{2}}}^{p-1}(-1)^k q^{6\frac{k(k+1)}{2}}\sum_{n=0}^{\infty}(-1)^n(2pn+2k+1)q^{6pn\cdot\frac{pn+2k+1}{2}}\right.\\
&\left.+\,p(-1)^{\frac{p-1}{2}}q^{6\frac{p^2-1}{8}}(q^{6p^2};q^{6p^2})_\infty^3\right]\\
&\times\left[\sum_{\substack{k=-\frac{p-1}{2}\\k\neq\frac{\pm p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}}f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)\right.
\end{aligned}$$

$$\left. + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} (q^{p^2}; q^{p^2})_\infty \right] \pmod{128}. \tag{4.12}$$

For a prime $p \geq 5$, $0 \leq k \leq p-1$ and $\frac{-(p-1)}{2} \leq m \leq \frac{(p-1)}{2}$, now consider the congruence

$$6 \cdot \frac{k^2+k}{2} + \frac{3m^2+m}{2} \equiv \frac{19p^2-19}{24} \pmod{p}, \tag{4.13}$$

which is equivalent to

$$2(6k+3)^2 + (6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-2}{p}\right) = -1$ as $p \equiv 5$ or $7 \pmod{8}$ the solution to (4.13) is $k = \frac{p-1}{2}$ and $m = \frac{p-1}{6}$. Therefore, extracting the terms involving $q^{pn+\frac{19p^2-19}{24}}$ from both sides of (4.12) and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+1}n + 19p^{2\alpha+2}\right) q^n \equiv 100p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}} (q^{6p}; q^{6p})_\infty^3 (q^p; q^p)_\infty \pmod{128}. \tag{4.14}$$

Extracting the terms containing q^{pn} from both sides of the above and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,1}\left(24p^{2\alpha+2}n + 19p^{2\alpha+2}\right) q^n \equiv 100p^{\alpha+1}(-1)^{(\alpha+1)\cdot\frac{p-2}{3}} (q^6; q^6)_\infty^3 (q; q)_\infty \pmod{128}, \tag{4.15}$$

This completes the proof by induction of (4.10). □

We can now prove Theorem 1.2.
Proof of Theorem 1.2
Empolying (2.11) and (2.12) and then comparing the coefficients of q^{pn+j}, $1 \leq j \leq p-1$, from both side of (4.14) we deduce Theorem 1.2
Proof of Theorem 1.3
From (1.3) we have

$$\sum_{n=0}^{\infty} \overline{C}_{12,3}(n) q^n \equiv \frac{f_3^3}{f_1} \pmod{2}. \tag{4.16}$$

Using (2.10) in (4.16), we found

$$\sum_{n=0}^{\infty} \overline{C}_{12,3}(2n+1) q^n \equiv \frac{f_6^3}{f_2} \pmod{2}. \tag{4.17}$$

It follows that

$$\overline{C}_{12,3}(4n+1) \equiv \overline{C}_{12,3}(n) \pmod 2 \tag{4.18}$$

and

$$\overline{C}_{12,3}(4n+3) \equiv 0 \pmod 2. \tag{4.19}$$

The results (1.5) and (1.6) follow by induction, using (4.18) and (4.19) respectively. Again from (4.16) we have

$$\overline{C}_{12,3}(2n) \equiv f_8 \pmod 2. \tag{4.20}$$

It follow that

$$\overline{C}_{12,3}(16n) \equiv f_1 \pmod 2 \tag{4.21}$$

and

$$\overline{C}_{12,3}(16n+2m) \equiv 0 \pmod 2, \tag{4.22}$$

for $1 \leq m \leq 7$ using (1.5) in (4.22) we have the result (1.7).
Proof of Theorem 1.4
Again from (1.3) we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(n)q^n \equiv \frac{f_{22}^2}{f_1 f_{11}} \pmod 2. \tag{4.23}$$

Substituting (2.15) in (4.23) and extracting the terms involving q^{2n} from both sides of the congruence and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(2n)q^n \equiv \frac{1}{f_2}\left(\psi(q^6) + q^3 \frac{f_{66}^2 f_4 f_{11}}{f_{22} f_2 f_{33}}\right) \pmod 2. \tag{4.24}$$

Using (2.9) in (4.24) and extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(4n+2)q^n \equiv q\frac{f_{33}^2 f_{11}^5}{f_{11} f_{66}}, \equiv q f_{44} \pmod 2. \tag{4.25}$$

Extracting the terms involving q^{4n+1} from both sides of the congruence, dividing both sides by q and then replacing q^4 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(16n+6)q^n \equiv f_{11} \pmod 2 \tag{4.26}$$

The results (1.8)-(1.10), follow from (4.25). The result (1.11) follows from (4.26).
Proof of Theorem 1.5
Again from (1.3) we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(n)q^n \equiv \frac{f_{25}}{f_1} \pmod 2. \tag{4.27}$$

Substituting (2.13) in (4.27), extracting the terms involving q^{5n+4} from both sides of the congruence, dividing both sides by 4 and then replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(5n+4)q^n \equiv \frac{f_5^6}{f_1^6} \equiv \frac{f_{10}^3}{f_2^3} \pmod{2}. \tag{4.28}$$

The result (1.12) follow from (4.28). Also from (4.28) we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(10n+4)q^n \equiv \frac{f_5^3}{f_1^3} \equiv \frac{f_{10}f_5}{f_2 f_1} \pmod{2}. \tag{4.29}$$

Substituting (2.14) in (4.29), extracting the terms involving q^{2n+1} from both sides of the congruence, dividing both sides by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{75,25}(20n+14)q^n \equiv \frac{f_4 f_{40}}{f_8} \pmod{2}. \tag{4.30}$$

The result (1.13) follows from (4.30).

Theorem 4.3. For any non-negative integer k, we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}\left(2\cdot 23^{2k}n + \frac{7\cdot 23^{2k+1}-73}{88}\right)q^n \equiv f_{23}^2 + q f_1 f_{23}^3 \pmod{2}. \tag{4.31}$$

Proof. Again from (1.3) we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}(n)q^n \equiv \frac{f_{46}^2}{f_1 f_{23}} \pmod{2}. \tag{4.32}$$

Now, from [6, Eq. (1.9)], we have

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(2n+1)q^n = \frac{f_2 f_{46}}{f_1^2 f_{23}^2} + q\frac{f_2^2 f_{46}^2}{f_1^3 f_{23}^3}, \tag{4.33}$$

where $p_{[1^1 23^1]}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{[1^1 23^1]}(n)q^n := \frac{1}{f_1 f_{23}}. \tag{4.34}$$

Extracting the terms involving q^{2n+1} from both sides of (4.32), replacing q^2 by q and then employing (4.33), we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}(2n+1)q^n \equiv f_{23}^2 + q f_1 f_{23}^3 \pmod{2}, \tag{4.35}$$

which is the $k = 0$ case of (4.31). Now suppose (4.31) holds for some $k \geq 0$. Setting $U_1 = a = -q$, $V_1 = b = -q^2$ and $n = 23$ in (2.16) and using the identity $f(a, b) = af(a^{-1}, a^2 b)$, we find the following 23-dissection of $f(-q, -q^2) = f_1$.

$$\begin{aligned} f_1 &= f(-q^{782}, -q^{805}) - qf(-q^{851}, -q^{736}) - q^2 f(-q^{713}, -q^{874}) - q^5 f(-q^{920}, -q^{667}) \\ &+ q^7 f(-q^{644}, -q^{943}) - q^{12} f(-q^{989}. - q^{598}) - q^{15} f(-q^{575}, -q^{1012}) \\ &+ q^{22} f(-q^{1058}, -q^{529}) + q^{26} f(-q^{506}, -q^{1081}) - q^{35} f(-q^{1127}, -q^{460}) \\ &- q^{40} f(-q^{437}, -q^{1150}) + q^{51} f(-q^{1196}, -q^{391}) + q^{57} f(-q^{368}, -q^{1219}) \\ &- q^{70} f(-q^{1265}, -q^{322}) - q^{77} f(-q^{299}, -q^{1288}) + q^{92} f(-q^{1334}, -q^{253}) \\ &+ q^{100} f(-q^{230}, -q^{1357}) - q^{117} f(-q^{1403}, -q^{184}) - q^{126} f(-q^{161}, -q^{1426}) \\ &+ q^{145} f(-q^{1472}, -q^{115}) + q^{155} f(-q^{92}, -q^{1495}) - q^{176} f(-q^{1541}, -q^{46}) \\ &- q^{187} f(-q^{23}, -q^{1564}). \end{aligned} \tag{4.36}$$

Employing (4.36) in (4.31) extracting the terms involving q^{23n} from both sides of the resulting congruence, replacing q^{23} by q, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{C}_{92,23}\left(2 \cdot 23^{2k} n + \frac{7 \cdot 23^{2k+1} - 73}{88}\right) q^n &\equiv f_1^2 + q f_1^3 f_{23} \\ &\equiv f_2 + q f_1 f_2 f_{23} \pmod 2 \end{aligned} \tag{4.37}$$

Next, squaring (4.36), we have

$$\begin{aligned} f_2 &\equiv f^2(-q^{782}, -q^{805}) + q^2 f^2(-q^{851}, -q^{736}) + q^4 f^2(-q^{713}, -q^{874}) + q^{10} f^2(-q^{920}, -q^{667}) \\ &+ q^{14} f^2(-q^{644}, -q^{943}) + q^{24} f^2(-q^{989}. - q^{598}) + q^{30} f^2(-q^{575}, -q^{1012}) \\ &+ q^{44} f^2(-q^{1058}, -q^{529}) + q^{52} f^2(-q^{506}, -q^{1081}) + q^{70} f^2(-q^{1127}, -q^{460}) \\ &+ q^{80} f^2(-q^{437}, -q^{1150}) + q^{102} f^2(-q^{1196}, -q^{391}) + q^{114} f^2(-q^{368}, -q^{1219}) \\ &+ q^{140} f^2(-q^{1265}, -q^{322}) + q^{154} f^2(-q^{299}, -q^{1288}) + q^{184} f^2(-q^{1334}, -q^{253}) \\ &+ q^{200} f^2(-q^{230}, -q^{1357}) + q^{234} f^2(-q^{1403}, -q^{184}) + q^{252} f^2(-q^{161}, -q^{1426}) \\ &+ q^{290} f^2(-q^{1472}, -q^{115}) + q^{310} f^2(-q^{92}, -q^{1495}) + q^{352} f^2(-q^{1541}, -q^{46}) \\ &+ q^{374} f^2(-q^{23}, -q^{1564}) \pmod 2. \end{aligned} \tag{4.38}$$

Employing (4.36) and (4.38) in (4.37), extracting the terms involving q^{23n+21} from both sides of the congruence, dividing both sides by q^{21} and then replacing q^{23} by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{92,23}\left(2 \cdot 23^{2k+1} n + \frac{7 \cdot 23^{2k+2} - 73}{88}\right) q^n \equiv f_{23}^2 + q f_1 f_{23}^3, \tag{4.39}$$

This completes the proof by induction of (4.31). □

Next we prove Theorem 1.6.
Proof of Theorem 1.6
Employing (4.36) in (4.31) and then equating the coefficients of q^{23n+m} from both sides we deduce Theorem 1.6.

5 More congruences for Andrews' singular overpartitions:

From (4.26), we have

$$\sum_{n=0}^{\infty} \overline{C}_{44,11}(176n+6)q^n \equiv f_1 \pmod 2 \tag{5.1}$$

Theorem 5.1. For any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$,

$$\overline{C}_{12,3}\left(16p^{2\alpha}n + \frac{2(24i+p)p^{2\alpha-1}-2}{3}\right) \equiv 0 \pmod 2, \tag{5.2}$$

For $i = 1, 2, \cdots, p-1$. For any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,

$$\overline{C}_{12,3}\left(16p^{2\alpha+1}n + \frac{2(24i+1)p^{2\alpha}-2}{3}\right) \equiv 0 \pmod 2, \tag{5.3}$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{24j+1}{p} = -1\right)$.

Proof. We note for 2-regular partitions modulo 2

$$\sum_{n=0}^{\infty} b_2(n)q^n \equiv f_1 \pmod 2.$$

In [12] Cui and Gu have proved several interesting results, for example, for any prime $p \geq 5$, $\alpha \geq 1$, and $n \geq 0$,

$$b_2\left(p^{2\alpha}n + \frac{(24i+p)p^{2\alpha-1}-1}{24}\right) \equiv 0 \pmod 2. \tag{5.4}$$

And for any prime $p \geq 5$, $\alpha \geq 0$, and $n \geq 0$,

$$b_2\left(p^{2\alpha+1}n + \frac{(24j+1)p^{2\alpha}-1}{24}\right) \equiv 0 \pmod 2, \tag{5.5}$$

where j is an integer with $0 \leq j \leq p-1$ such that $\left(\frac{24j+1}{p} = -1\right)$. Theorem 5.1 follows from (4.21) and the results (5.2) and (5.3). $\square$

Remark: Similar results can be obtained for (5.1).

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