ARTICLE IN PRESS YJNTH:5979

Journal of Number Theory ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

On perfect powers that are sums of two Fibonacci numbers ☆

Florian Luca[a,b,c], Vandita Patel[d,*]

[a] *School of Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa*
[b] *Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany*
[c] *Department of Mathematics, Faculty of Sciences, University of Ostrava, 30 dubna 22, 701 03 Ostrava 1, Czech Republic*
[d] *Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Room 6290, Toronto, Ontario, M5S 2E4, Canada*

A R T I C L E   I N F O

Article history:
Received 5 October 2017
Received in revised form 9 February 2018
Accepted 14 February 2018
Available online xxxx
Communicated by M. Pohst

MSC:
primary 11D61
secondary 11B39

Keywords:
Exponential equation
Hemachandra numbers
Fibonacci numbers

A B S T R A C T

We study the equation $F_n + F_m = y^p$, where F_n and F_m are respectively the n-th and m-th Fibonacci numbers and $p \geq 2$. We find all solutions under the assumption $n \equiv m \pmod 2$.

☆ The first-named author is supported by grants CPRR160325161141 and an A-rated researcher award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency.
* Corresponding author.
E-mail addresses: florian.luca@wits.ac.za (F. Luca), vandita@math.utoronto.ca (V. Patel).

https://doi.org/10.1016/j.jnt.2018.02.003

1. Introduction

Fibonacci numbers are prominent as well as being ancient. Their first known occurrence dates back to around 200BC, (see [5], [8]) in the earliest known treatise on *Sanskrit prosody* (poetry meters and verse in Sanskrit) entitled *Chandaḥśāstra* and authored by Piṅgala. This work is eight chapters in the late *Sūtra style* and therefore quite complex and not fully comprehensible without commentary. The Fibonacci numbers appear again (much later this time) in the work of Virahāṅka (700AD). Virahāṅka's original work has been lost, but is nevertheless cited clearly in the work of Gopāla (c. 1135); below is a translation of [9, pg. 101];

> "For four, variations of meters of two [and] three being mixed, five happens. For five, variations of two earlier – three [and] four, being mixed, eight is obtained. In this way, for six, [variations] of four [and] of five being mixed, thirteen happens. And like that, variations of two earlier meters being mixed, seven morae [is] twenty-one. In this way, the process should be followed in all mātrā–vṛttas."

The sequence is discussed rigorously and most concisely in the work of Jain scholar Acharya Hemachandra (c. 1150, living in what is known today as Gujarat) about 50 years earlier than Fibonacci's *Liber Abaci* (1202). Hemachandra, just like Piṅgala, Virahāṅka and Gopāla, was in fact studying *Sanskrit prosody* and not mathematics. Given a verse with an ending of n beats to fill, where the choice of beats consists of length 1 (called *short*) and length 2 (called *long*), in how many ways can one finish the verse? The answer lies within the fundamental sequence, defined by the recurrence;

$$H_{n+2} = H_{n+1} + H_n, \qquad H_1 = 1, \quad H_2 = 2, \quad n \geq 1, \tag{$\diamondsuit$}$$

where Hemachandra makes the concise argument that any verse that is to be filled with n beats must end with a long or a short beat. Therefore, this recurrence is enough to answer the question: given a verse with n beats remaining, one has H_n ways of finishing the *prosody*, with H_n satisfying ($\diamondsuit$).

Since the 12th century, the Hemachandra/Fibonacci numbers have sat in the spotlight of modern number theory. They have been vastly studied; intrinsically for their beautiful identities but also for their numerous applications, for example, the golden ratio has a regular appearance in art, architecture and the natural world!

Finding all perfect powers in the Fibonacci sequence was a fascinating long-standing conjecture. In 2006, this problem was completely solved by Y. Bugeaud, M. Mignotte and S. Siksek (see [4]), who innovatively combined the modular approach with classical linear forms in logarithms. In addition to this, Y. Bugeaud, F. Luca, M. Mignotte and S. Siksek also found all of the integer solutions to

$$F_n \pm 1 = y^p \quad p \geq 2, \tag{1}$$

Please cite this article in press as: F. Luca, V. Patel, On perfect powers that are sums of two Fibonacci numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2018.02.003

(see [2]). The authors found a clever factorisation which descended the problem to finding solutions of $F_n = y^p$.

In this paper, we consider the natural generalisation,

$$F_n \pm F_m = y^p, \quad p \geq 2. \tag{2}$$

Theorem 1. *All solutions of the Diophantine equation (2) in integers (n, m, y, p) with $n \equiv m \pmod 2$ either have $\max\{|n|, |m|\} \leq 36$, or $y = 0$ and $|n| = |m|$.*

Since $F_1 = F_2 = 1$, it follows that every solution (n, y, p) of equation (1) can be thought of as a solution (n, m, y, p) of equation (2) with $m = 1$, 2 according to whether n is odd or even. Therefore, Theorem 1 is a genuine generalisation of the main result from [2].

For a complete list of solutions to equation (2) with $\max\{|n|, |m|\} \leq 1000$ without the parity restriction, see Section 5.

Acknowledgments

The authors would like to thank the Max Planck Institute for Mathematics, for their generous hospitality and provisions of a fantastic collaborative working environment. We are grateful to the referee for a careful reading of this paper and for the helpful suggestions and corrections.

2. Preliminaries

Let $(F_n)_{n\geq 0}$ be the Hemachandra/Fibonacci sequence given by;

$$F_{n+2} = F_{n+1} + F_n, \qquad F_0 = 0, \quad F_1 = 1, \quad n \geq 0.$$

Recall that $(F_n)_{n\geq 0}$ can be extended to be defined on the negative indices by using the above recurrence and giving n the values $n = -1, -2, \ldots$. Thus the formula $F_{-n} = (-1)^{n+1} F_n$ holds for all n.

Let $(L_n)_{n\geq 0}$ be the Lucas companion sequence of the Hemachandra/Fibonacci sequence given by;

$$L_{n+2} = L_{n+1} + L_n, \qquad L_0 = 2, \quad L_1 = 1, \quad n \geq 0.$$

Similarly, this can also be extended to negative indices n, and the formula $L_n = (-1)^n L_{-n}$ holds for all n.

The Binet formulas for F_n and L_n are;

$$F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad \text{for all} \quad n \in \mathbb{Z}, \tag{3}$$

Please cite this article in press as: F. Luca, V. Patel, On perfect powers that are sums of two Fibonacci numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2018.02.003

where $(\alpha, \beta) = ((1+\sqrt{5})/2, (1-\sqrt{5})/2)$. There are many formulas relating Hemachandra/Fibonacci numbers and Lucas numbers. Two of which are useful for us are;

$$F_{2n} = F_n L_n \quad \text{and} \quad L_{3n} = L_n(L_n^2 - 3(-1)^n), \tag{4}$$

which hold for all n. They can be proved using Binet's formulae (3).

The following result is well-known and can also be proved using Binet's formulae (3).

Lemma 2.1. *Assume $n \equiv m \pmod 2$. Then*

$$F_n + F_m = \begin{cases} F_{(n+m)/2} L_{(n-m)/2} & \text{if } n \equiv m \pmod 4, \\ F_{(n-m)/2} L_{(n+m)/2} & \text{if } n \equiv m+2 \pmod 4. \end{cases}$$

Similarly,

$$F_n - F_m = \begin{cases} F_{(n-m)/2} L_{(n+m)/2} & \text{if } n \equiv m \pmod 4, \\ F_{(n+m)/2} L_{(n-m)/2} & \text{if } n \equiv m+2 \pmod 4. \end{cases}$$

The following result can be found in [6].

Lemma 2.2. *Let $n = 2^a n_1$ and $m = 2^b m_1$ be positive integers with n_1 and m_1 odd integers and a and b nonnegative integers. Let $d = \gcd(n, m)$. Then*

i) $\gcd(F_n, F_m) = F_d$.
ii) $\gcd(L_n, L_m) = L_d$ *if $a = b$ and it is 1 or 2 otherwise.*
iii) $\gcd(F_n, L_m) = L_d$ *if $a > b$ and it is 1 or 2 otherwise.*

The following results can be extracted from [1], [3] and [4] and will be useful for us.

Theorem 2. *If*

$$F_n = 2^s \cdot y^b$$

for some integers $n \geq 1$, $y \geq 1$, $b \geq 2$ and $s \geq 0$ then $n \in \{1, 2, 3, 6, 12\}$. The solutions of the similar equation with F_n replaced by L_n have $n \in \{1, 3, 6\}$.

Theorem 3. *If*

$$F_n = 3^s \cdot y^b$$

for some integers $n \geq 1$, $y \geq 1$, $b \geq 2$ and $s \geq 0$ then $n \in \{1, 2, 4, 6, 12\}$. The solutions of the similar equation with F_n replaced by L_n have $n \in \{1, 2, 3\}$.

The following result is due to McDaniel and Ribenboim (see [7]).

Theorem 4.

i) *Assume $u \mid v$ are positive integers such that $F_v/F_u = y^2$. Then, either $u = v$ or $(v, u) \in \{(12, 1), (12, 2), (2, 1), (6, 3)\}$.*
ii) *Assume that $u \mid v$, v/u is odd and $L_v/L_u = y^2$. Then, $u = v$ or $(v, u) = (3, 1)$.*

3. Perfect powers from products of a Fibonacci and a Lucas number

Theorem 5. *The only solutions to*

$$F_N \cdot L_M = 2^s \cdot y^p$$

with N, M, y positive integers, $s \geq 0$ and $p \geq 2$ satisfy

$$(N, M) = (1, 1), \quad (1, 3), \quad (1, 6), \quad (2, 1), \quad (2, 3), \quad (2, 6), \quad (3, 1), \quad (3, 3),$$
$$(3, 6), \quad (4, 2), \quad (4, 6), \quad (6, 1), \quad (6, 3), \quad (6, 6), \quad (12, 1), \quad (12, 2),$$
$$(12, 3), \quad (12, 6), \quad (24, 12).$$

Proof. We shall in fact show that $N \leq 24$ and $M \leq 12$. The proof is then completed by a simple program. Write

$$N = 2^a N_1, \qquad M = 2^b M_1,$$

where N_1, M_1 are odd. If $a \leq b$, then by Lemma 2.2, we know $\gcd(F_N, L_M) = 1$ or 2, so $F_N = 2^u y_1^p$ and $L_M = 2^v y_2^p$. By Theorem 2, we deduce that $N \leq 12$ and $M \leq 6$.

Thus, we may assume that $a > b$. Let $r = a - b \geq 1$ and $d = \gcd(N, M)$. Therefore, $d = 2^b \gcd(N_1, M_1)$. Write $N = 2^r kd$ where k is odd. Then we obtain;

$$2^s y^p = F_N \cdot L_M = F_{2^r kd} \cdot L_M = F_{kd} \cdot L_{kd} \cdot L_{2kd} \cdots L_{2^{r-1}kd} \cdot L_M,$$

by repeated application of (4). Note that

$$\upsilon_2(kd) = \upsilon_2(M), \qquad \upsilon_2(kd) \leq \upsilon_2(2^i kd) \quad \text{for } i \geq 0.$$

Thus, by Lemma 2.2, the greatest common divisor of F_{kd} and $L_{kd} \cdot L_{2kd} \cdots L_{2^{r-1}kd} \cdot L_M$ is a power of 2. Hence,

$$F_{kd} = 2^u y_1^p.$$

By Theorem 2, since $kd \geq 1, u \geq 0, y_1 \geq 1$ and $p \geq 2$, with kd, u, y_1 and p all integers, we deduce that $kd \in \{1, 2, 3, 6, 12\}$. Moreover,

$$L_{kd} \cdot L_{2kd} \cdots L_{2^{r-1}kd} \cdot L_M = 2^v y_2^p.$$

Please cite this article in press as: F. Luca, V. Patel, On perfect powers that are sums of two Fibonacci numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2018.02.003

Suppose $r \geq 2$. Then $\upsilon_2(2^{r-1}kd) > \upsilon_2(M)$. Once more, we use Lemma 2.2 to see that the greatest common divisor of $L_{2^{r-1}kd}$ and $L_{kd} \cdot L_{2kd} \cdots L_{2^{r-2}kd} \cdot L_M$ is a power of 2. Hence,

$$L_{2^{r-1}kd} = 2^w y_3^p.$$

By Theorem 2, we conclude that $2^{r-1}kd \in \{1, 3, 6\}$. Therefore, $r = 2$ and $kd = 3$. This then tells us that $N = 2^r kd = 12$. Otherwise, $r = 1$ and $N \in \{2, 4, 6, 12, 24\}$.

If $N = 2$ or 6 then $F_N = 1$ or 8 so $L_M = 2^\sigma y^p$. Theorem 2 allows us to readily conclude that $M \leq 6$. The cases $N = 4$, 12 and 24 remain and require delicate treatment. First, we deal with the cases $N = 4$ and $N = 12$. Since $F_4 = 3$ and $F_{12} = 2^4 \times 3^2$ we have $L_M = 2^\alpha 3^\beta y_0^p$ where y_0 is odd. If $\alpha = 0$ then by Theorem 3 we know that $M \leq 3$ and so we may suppose that $s \geq 1$. Thus, $2 \mid L_M$. Note that $2 \mid\mid M$ (as $r = 1$ and $a = 2$). As $6 \mid L_M$, we have $3 \mid M$. Thus, we can write $M = 2 \cdot 3^t \cdot \ell$, where ℓ is coprime to 6. Now, observe that

$$L_{3\delta} = L_\delta(L_\delta^2 - 3)$$

for $\delta = 2 \cdot 3^i \cdot \ell$ with $i = t-1, t-2, \ldots, 0$. Hence, we obtain inductively $L_{2 \cdot 3^i \cdot \ell} = 2^{s_i} 3^{w_i} y_i^p$. For $i = 0$, we have $3 \nmid 2\ell$ and so $s_0 = 0$. Using Theorem 3, we infer that $\ell = 1$. Hence, $M = 2 \cdot 3^t$. Note that $107 \mid\mid L_{18} \mid L_M$ and $107^2 \nmid L_n$ unless $(18 \times 107) \mid n$. We conclude that $t \leq 1$ and so $M \leq 6$.

Finally, let $N = 24$, whence $F_{24} = 2^5 \cdot 3^2 \cdot 7 \cdot 23$. Moreover, $M = 2^2 \cdot M_1$ where M_1 is odd. Thus, $d = 2^2 \cdot \gcd(3, M_1) = 4$ or 12. However, $\gcd(F_N, L_M) = L_d$. We may rewrite the equation $F_N \cdot L_M = 2^s y^p$ as

$$L_d^2 \cdot \frac{F_N}{L_d} \cdot \frac{L_M}{L_d} = 2^s y^p.$$

If $d = 4$ then $L_d = 7$ and we see that 23 divides the left-hand side exactly once, giving a contradiction. Thus, $d = 12$ and so $L_d = 2 \times 7 \times 23$. In this case the left-hand side is divisible by 3 exactly twice and therefore $p = 2$. Thus, $L_M/L_{12} = 2^\alpha y_1^2$. Since M is an odd multiple of 12, we can easily see that L_M/L_{12} is odd and so $\alpha = 0$. Hence, we can apply Theorem 4 to draw the inference that $M = 12$. $\square$

4. Proof of Theorem 1

If either $n = 0$ or $m = 0$, then the theorem follows from [4]. Via the identity $F_{-n} = (-1)^{n+1} F_n$, we can suppose that $n \geq m > 0$. Note that changing signs does not change parities, so we maintain the assumption $n \equiv m \pmod 2$.

If $n = m$, then we need to solve $2F_n = y^p$, which is equivalent to solving $F_n = 2^{p-1} y_1^p$. By Theorem 2, we have that $n \leq 12$.

Please cite this article in press as: F. Luca, V. Patel, On perfect powers that are sums of two Fibonacci numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2018.02.003

Thus, we may suppose that $n > m > 0$. By Lemma 2.1, there is some $\varepsilon = \pm 1$ such that letting $N = (n+\varepsilon m)/2$ and $M = (n-\varepsilon m)/2$, we have $F_n \pm F_m = F_N \cdot L_M$. Observe that N and M are both positive. By Theorem 5, we know that $N \leq 24$ and $M \leq 12$. We finally conclude that $n = N + M \leq 36$ and $m = |N - M| < 24$. This completes the proof.

5. An open problem

It is still an open problem to find all solutions to equation (2) in the case $n \not\equiv m \pmod 2$. Under the condition, $n \not\equiv m \pmod 2$, no factorisation is known for the left-hand side. We searched for solutions with $0 \leq m \leq n \leq 1000$ and found the following:

$$\begin{gathered}
F_0 + F_0 = 0, \quad F_1 + F_0 = 1, \quad F_2 + F_0 = 1, \quad F_3 + F_3 = 2^2, \quad F_4 + F_1 = 2^2, \\
F_4 + F_2 = 2^2, \quad F_5 + F_4 = 2^3, \quad F_6 + F_0 = 2^3, \quad F_6 + F_1 = 3^2, \quad F_6 + F_2 = 3^2, \\
F_6 + F_6 = 2^4 = 4^2, \quad F_7 + F_4 = 2^4 = 4^2, \quad F_9 + F_3 = 6^2, \quad F_{11} + F_{10} = 12^2, \\
F_{12} + F_0 = 12^2, \quad F_{16} + F_7 = 10^3, \quad F_{17} + F_4 = 40^2, \quad F_{36} + F_{12} = 3864^2.
\end{gathered}$$

$$\begin{gathered}
F_n - F_n = 0, \quad F_1 - F_0 = 1, \quad F_2 - F_0 = 1, \quad F_2 - F_1 = 0, \quad F_3 - F_1 = 1, \\
F_3 - F_2 = 1, \quad F_4 - F_3 = 1, \quad F_5 - F_1 = 2^2, \quad F_5 - F_2 = 2^2, \quad F_6 - F_0 = 2^3, \\
F_7 - F_5 = 2^3, \quad F_8 - F_5 = 2^4 = 4^2, \quad F_8 - F_7 = 2^3, \quad F_9 - F_3 = 2^5, \\
F_{11} - F_6 = 3^4 = 9^2, \quad F_{12} - F_0 = 12^2, \quad F_{13} - F_6 = 15^2, \quad F_{13} - F_{11} = 12^2, \\
F_{14} - F_9 = 7^3, \quad F_{14} - F_{13} = 12^2, \quad F_{15} - F_9 = 24^2.
\end{gathered}$$

We conjecture that the above lists all of the solutions to equation (2) with the restriction $n \geq m \geq 0$.

References

[1] Y. Bugeaud, F. Luca, M. Mignotte, S. Siksek, Perfect powers from products of terms in Lucas sequences, J. Reine Angew. Math. 611 (2007) 109–129.

[2] Y. Bugeaud, F. Luca, M. Mignotte, S. Siksek, Fibonacci numbers at most one away from a perfect power, Elem. Math. 63 (2008) 65–75.

[3] Y. Bugeaud, F. Luca, M. Mignotte, S. Siksek, Almost powers in the Lucas sequence, J. Théor. Nombres Bordeaux 20 (2008) 555–600.

[4] Y. Bugeaud, M. Mignotte, S. Siksek, Classical and modular approaches to exponential Diophantine equations I, Fibonacci and Lucas perfect powers, Ann. of Math. 163 (2006) 969–1018.

[5] S. Goonatilake, Toward a Global Science, Indiana University Press, 1998, p. 126.

[6] W. McDaniel, The g.c.d. in Lucas sequences and Lehmer number sequences, Fibonacci Quart. 29 (1991) 24–29.

[7] W. McDaniel, P. Ribenboim, Square-classes in Lucas sequences having odd parameters, J. Number Theory 73 (1998) 14–27.

[8] P. Singh, The so-called Fibonacci numbers in ancient and medieval India, Historia Math. 12 (1985) 229–244.

[9] H.D. Velankar, Vṛttajātisamuccaya, Rājasthāna purātana granthamālā, Jodhpur Rajasthana Pracyavidya Pratisthana, 1962 (in Sanskrit).

Please cite this article in press as: F. Luca, V. Patel, On perfect powers that are sums of two Fibonacci numbers, J. Number Theory (2018), https://doi.org/10.1016/j.jnt.2018.02.003