Journal of Number Theory 228 (2021) 253–275

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

On mod $\mathfrak{p}$ congruences for Drinfeld modular forms of level $\mathfrak{pm}$

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A R T I C L E I N F O

Article history:
Received 29 July 2020
Received in revised form 1 April 2021
Accepted 9 April 2021
Available online 1 June 2021
Communicated by F. Pellarin

MSC:
primary 11F33, 11F52
secondary 11F23, 11G09

Keywords:
Drinfeld modular forms
Θ-operator
Atkin-Lehner involution
Congruences
Hecke operators
Eigenvalue

A B S T R A C T

In [CS04], Calegari and Stein studied the congruences between classical cusp forms $S_k(\Gamma_0(p))$ of prime level and made several conjectures about them. In [AB07] (resp., [BP11]) the authors proved one of those conjectures (resp., their generalizations). In this article, we study the analogous conjecture and its generalizations for Drinfeld modular forms.

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https://doi.org/10.1016/j.jnt.2021.04.020

1. Introduction and statements of the main results

In [CS04], Calegari and Stein studied certain relations between the congruences among classical cusp forms $S_k(\Gamma_0(p))$ of prime level and the integral closures of their associated Hecke algebras. They have made a series of conjectures and established connections between them. One of these conjectures predicts a precise formula for the index of $\mathbb{T}$ in its integral closure, where $\mathbb{T}$ is the algebra of Hecke operators acting on $S_k(\Gamma_0(p), \mathbb{Z})$ generated over $\bar{\mathbb{Z}}_p$.

When $S_k(\Gamma_0(p))$ contains no oldforms (e.g., when $k = 2, 4, 6, 8, 10$, and 14), then $U_p = -p^{\frac{k}{2}-1} w_p$, where w_p is the Fricke involution. Let $S_k^+(\Gamma_0(p))$ (resp., $S_k^-(\Gamma_0(p))$) denote the plus (resp., minus) eigenspace of $S_k(\Gamma_0(p))$ with respect to w_p, and let $\mathbb{T}^+ := \mathbb{T}/(U_p + p^{\frac{k}{2}-1})$ (resp., $\mathbb{T}^- := \mathbb{T}/(U_p - p^{\frac{k}{2}-1})$) be the quotient of the Hecke algebra $\mathbb{T}$. Note that $\mathbb{T}^+$ (resp., $\mathbb{T}^-$) preserves $S_k^+(\Gamma_0(p))$ (resp., $S_k^-(\Gamma_0(p))$). Calegari and Stein (cf. [CS04, Conjecture 3]) conjectured that $\mathbb{T}^+$ and $\mathbb{T}^-$ are integrally closed. Equivalently, any congruences among the Hecke eigenforms in $S_k(\Gamma_0(p), \bar{\mathbb{Z}}_p)$ can occur only between plus and minus eigenforms for w_p. They (cf. [CS04, Conjecture 4]) also conjectured that the eigenvalues of the Fricke involution on $f \in S_2(\Gamma_0(p))$ and $g \in S_4(\Gamma_0(p))$ have opposite signs if there is a mod p congruence between g and the derivative of f. In [AB07], Ahlgren and Barcau settled this conjecture affirmatively.

Theorem 1.1. *Let $p \geq 5$ be a prime. Suppose that $f \in S_2(\Gamma_0(p), \bar{\mathbb{Z}}_p)$ and $g \in S_4(\Gamma_0(p), \bar{\mathbb{Z}}_p)$ are eigenforms for all Hecke operators and satisfy $\Theta f \equiv g \pmod{\mathfrak{p}}$, where $\mathfrak{p}$ is the maximal ideal of $\bar{\mathbb{Z}}_p$. Then the eigenvalues of w_p for f and g have opposite signs.*

Barcau and Paşol (cf. [BP11, §4]) proved that Theorem 1.1 continues to hold for level pN with $p \nmid N$, under an assumption on the weight filtration of f.

Theorem 1.2. *Let $p \geq 5$ be a prime and $N > 4$ be an integer such that $p \nmid N$, and $\mathfrak{p}$ be the maximal ideal of $\overline{\mathbb{Z}}_p$. Let $f \in S_2(\Gamma_0(pN), \overline{\mathbb{Z}}_p)$ and $g \in S_4(\Gamma_0(pN), \overline{\mathbb{Z}}_p)$ be two newforms and satisfy $\Theta f \equiv g \pmod{\mathfrak{p}}$. If $w(f) = p + 1$ then the eigenvalues of $w_p^{(pN)}$ for f and g have opposite signs.*

Our main interest lies in studying the conjectures of Calegari and Stein for Drinfeld modular forms and various connections between them. The present article is a modest first step in this direction where we generalize the results of [AB07] and [BP11] to Drinfeld modular forms of any weight, any type.

1.1. Main results

Let p be an odd prime number and $q = p^r$ for some $r \in \mathbb{N}$. Suppose $\mathbb{F}_q$ denote the finite field of order q. Set $A := \mathbb{F}_q[T]$ and $K := \mathbb{F}_q(T)$. Let $K_\infty = \mathbb{F}_q((\frac{1}{T}))$ be the completion of K with respect to the infinite place ∞ (corresponding to $\frac{1}{T}$-adic valuation), and denote by C the completion of an algebraic closure of K_∞.

Throughout the article, $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial $\pi := \pi(T) \in A$ of degree d and $\mathfrak{m}$ denotes an ideal of A generated by a monic polynomial $m := m(T) \in A$ such that $(\mathfrak{p}, \mathfrak{m}) = 1$ (i.e., $\pi \nmid m$).

For an ideal $\mathfrak{n}$ of A, we define

$$\Gamma_0(\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A) : c \in \mathfrak{n} \right\}$$

to be a congruence subgroup of $\mathrm{GL}_2(A)$. Let $M_{k,l}(\Gamma_0(\mathfrak{n}))$ (resp., $M^1_{k,l}(\Gamma_0(\mathfrak{n}))$) denote the space of Drinfeld modular (resp., cusp) forms of weight k, type l for $\Gamma_0(\mathfrak{n})$. Our first result is the following:

Theorem 1.3 *(Theorem 4.3 in the text). Suppose that $f \in M^1_{k,l}(\Gamma_0(\mathfrak{p}))$ and $g \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{p}))$ have $\mathfrak{p}$-integral u-series expansions at ∞ with $\Theta f \equiv g \pmod{\mathfrak{p}}$. Assume that $w(\overline{F}) = (k-1)(q^d-1)+k$ where F is as in Proposition 3.8 corresponding to kf, and $(k,p) = 1$. If $f|W_\mathfrak{p} = \alpha f$ and $g|W_\mathfrak{p} = \beta g$ with $\alpha, \beta \in \{\pm 1\}$, then $\beta = -\alpha$.*

The Ramanujan's Θ-operator, the weight filtration $w(\overline{F})$ of F, and the Atkin-Lehner involution $W_\mathfrak{p}$ are introduced in §2.2.1, §4.1, and §3.1 respectively. In Proposition 3.8, we establish that for any $f \in M^1_{k,l}(\Gamma_0(\mathfrak{pm}))$, there exists a Drinfeld modular form $F \in M^1_{(k-1)(q^d-1)+k,l}(\Gamma_0(\mathfrak{m}))$ such that $F \equiv f \pmod{\mathfrak{p}}$.

In Theorem 1.3, the condition $w(\overline{F}) = (k-1)(q^d-1)+k$ is automatically satisfied for Drinfeld modular forms of weight 2, type 1. More precisely, we prove

Corollary 1.4. *Suppose that $f \in M^1_{2,1}(\Gamma_0(\mathfrak{p}))$ and $g \in M^1_{4,2}(\Gamma_0(\mathfrak{p}))$ have $\mathfrak{p}$-integral u-series expansions at ∞ with $\Theta f \equiv g \pmod{\mathfrak{p}}$. Assume that $f \not\equiv 0 \pmod{\mathfrak{p}}$. If $f|W_\mathfrak{p} = \alpha f$ and $g|W_\mathfrak{p} = \beta g$ with $\alpha, \beta \in \{\pm 1\}$, then $\beta = -\alpha$.*

Similar to the classical case, Theorem 1.3 can be extended to the level $\mathfrak{pm}$, which is described in the following theorem.

Theorem 1.5 *(Theorem 5.8 in the text). Let $\mathfrak{m}$ be an ideal of A generated by a polynomial in A which has a prime factor of degree prime to $q-1$ and $\mathfrak{p} \nmid \mathfrak{m}$. Suppose that $f \in M^1_{k,l}(\Gamma_0(\mathfrak{pm}))$ and $g \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{pm}))$ have $\mathfrak{p}$-integral u-series expansions at ∞ with $\Theta f \equiv g \pmod{\mathfrak{p}}$. Assume that $w(\overline{F}) = (k-1)(q^d-1)+k$, where F is as in Proposition 3.8 corresponding to kf, and $(k,p) = 1$. If $f|W^{(\mathfrak{pm})}_\mathfrak{p} = \alpha f$ and $g|W^{(\mathfrak{pm})}_\mathfrak{p} = \beta g$ with $\alpha, \beta \in \{\pm 1\}$, then $\beta = -\alpha$.*

The (partial) Atkin-Lehner involution $W^{(\mathfrak{pm})}_\mathfrak{p}$ and the weight filtration $w(\overline{F})$ of F are introduced in §3.1 and §5.5, respectively. We note that in proving Theorem 1.5, we will make use of the recent work of Hattori [Hat20] for which the conditions on $\mathfrak{m}$ are necessary.

There is a significant difference in our approach to prove Theorem 1.3 and Theorem 1.5. We use the structure of Drinfeld modular forms for $\mathrm{GL}_2(A)$ in the proof of

Theorem 1.3 (cf. §4). We appeal to the geometry of modular curves and use the recent work of Hattori (cf. [Hat20]) to prove Theorem 1.5 (cf. §5).

1.2. Results for $\mathfrak{p}$-new forms

The space of $\mathfrak{p}$-new forms $M_{k,l}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$ for level $\mathfrak{pm}$ was introduced by Bandini and Valentino (cf. [BV20, Definition 2.14]). Now, we state Theorem 1.3 and Theorem 1.5 for $\mathfrak{p}$-new forms. They are natural generalizations of the results of [AB07] and [BP11].

If $f \in M_{k,l}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$, then the relation $f|W_{\mathfrak{p}}^{(\mathfrak{pm})} = -\pi^{1-k/2}(f|U_{\mathfrak{p}})$ (cf. [BV20, Theorem 2.16]) implies that f is an eigenvector for the $W_{\mathfrak{p}}^{(\mathfrak{pm})}$-operator if and only if it is an eigenvector for the $U_{\mathfrak{p}}$-operator. Note that the normalization here differs from that of [BV20]. For such a Drinfeld modular form f, the eigenvalues of f are $\pm\pi^{k/2-1}$ (resp., ∓ 1) with respect to the $U_{\mathfrak{p}}$-operator (resp., the $W_{\mathfrak{p}}^{(\mathfrak{pm})}$-operator). In fact, the above relation also implies that the eigenvalues of f with respect to the $U_{\mathfrak{p}}$-operator and the $W_{\mathfrak{p}}^{(\mathfrak{pm})}$-operator have opposite signs.

Now, we rephrase our main results in terms of the $U_{\mathfrak{p}}$-operator.

Corollary 1.6. *Let $\mathfrak{m} \subseteq A$ be ideal such that either $\mathfrak{m} = (1)$ or as in Theorem 1.5. Suppose $f \in M_{k,l}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$ and $g \in M_{k+2,l+1}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{pm}))$ are two Drinfeld modular forms satisfying the hypothesis of Theorem 1.5. If f and g are eigenforms for the $U_{\mathfrak{p}}$-operator, then the eigenvalues of the $W_{\mathfrak{p}}^{(\mathfrak{pm})}$-operator on f and g have opposite signs.*

For $\mathfrak{m} = (1)$, we get:

Corollary 1.7. *Let $f \in M_{2,1}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{p}))$ and $g \in M_{4,2}^{1,\mathfrak{p}-\mathrm{new}}(\Gamma_0(\mathfrak{p}))$ be two Drinfeld modular forms satisfying the hypothesis of Corollary 1.4. If f and g are eigenforms for the $U_{\mathfrak{p}}$-operator, then the eigenvalues of the $W_{\mathfrak{p}}$-operator on f and g have opposite signs.*

It is natural to wonder what would happen if one drops the assumption on $w(\overline{F})$ in Theorem 1.3 and Theorem 1.5. In §6, we will show that these theorems may not continue to hold if we drop the assumption $w(\overline{F}) = (k-1)(q^d-1)+k$.

Finally, we note that the results of [AB07], [BP11] were proved only for smaller weights and it is unknown whether similar results hold for higher weights. However, our results are valid for Drinfeld modular forms of any weight, any type.

1.3. An overview of the article

The article is organized as follows. In § 2, we recall some basic theory of Drinfeld modular forms. In § 3, we introduce certain operators, study the inter-relations between them, and state two important propositions. In § 4, we give a proof of Theorem 1.3. In § 5, we recall some results from [Hat20], [Hat20a] and use them to prove Theorem 1.5. In the final section, i.e., in § 6, we show that the assumption $w(\overline{F}) = (k-1)(q^d-1)+k$ in Theorem 1.3 and Theorem 1.5 is necessary.

2. Basic theory of Drinfeld modular forms

The theory of Drinfeld modular forms was studied extensively by Goss, Gekeler, and various other authors (cf. [Gos80], [Gos80a], [Gek88], [GR96] for more details). In this section, we recall certain theory of Drinfeld modular forms which are needed to prove our results.

There is an equivalence of categories between the category of Drinfeld modules of rank r over a complete subfield M of C containing K_∞ and the category of M-lattices of rank r (cf. [Gos96, Theorem 4.6.9]). Let $L = \tilde{\pi}A \subseteq C$ be the A-lattice of rank 1 corresponding to the rank 1 Drinfeld module (which is also called the Carlitz module)

$$\rho_T = TX + X^q, \tag{2.1}$$

where $\tilde{\pi} \in K_\infty(\sqrt[q-1]{-T})$ is defined up to a $(q-1)$-th root of unity.

The Drinfeld upper half-plane $\Omega = C - K_\infty$ has a rigid analytic structure. The group $\mathrm{GL}_2(K_\infty)$ acts on Ω via fractional linear transformations. Any $x \in K_\infty^\times$ has the unique expression $x = \zeta_x\left(\frac{1}{T}\right)^{v_\infty(x)} u_x$, where $\zeta_x \in \mathbb{F}_q^\times$, and $v_\infty(u_x - 1) \geq 0$ (v_∞ is the valuation at ∞).

Definition 2.1. Suppose $k \in \mathbb{N}$, $l \in \mathbb{Z}/(q-1)\mathbb{Z}$. Let $f : \Omega \longrightarrow C$ be a rigid holomorphic function. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(K_\infty)$, the slash operator $|_{k,l}\gamma$ on f is defined by

$$f|_{k,l}\gamma := \zeta_{\det\gamma}^l \Big(\frac{\det\gamma}{\zeta_{\det(\gamma)}}\Big)^{k/2} (cz+d)^{-k} f(\gamma z). \tag{2.2}$$

Note that the slash operator has the following property. For $i = 1, 2$, let $k_i \in \mathbb{N}$, $l_i \in \mathbb{Z}/(q-1)\mathbb{Z}$ and f_i be a rigid holomorphic function on Ω. For $\gamma \in \mathrm{GL}_2(K_\infty)$, by (2.2), we have

$$\begin{aligned} f_1|_{k_1,l_1}\gamma \cdot f_2|_{k_2,l_2}\gamma &= \zeta_{\det\gamma}^{l_1}\Big(\frac{\det\gamma}{\zeta_{\det(\gamma)}}\Big)^{\frac{k_1}{2}}(cz+d)^{-k_1} f_1(\gamma z) \\ &\quad \cdot \zeta_{\det\gamma}^{l_2}\Big(\frac{\det\gamma}{\zeta_{\det(\gamma)}}\Big)^{\frac{k_2}{2}}(cz+d)^{-k_2} f_2(\gamma z) \\ &= \zeta_{\det\gamma}^{l_1+l_2}\Big(\frac{\det\gamma}{\zeta_{\det(\gamma)}}\Big)^{\frac{k_1+k_2}{2}}(cz+d)^{-(k_1+k_2)} f(\gamma z)\cdot g(\gamma z) \\ &= (f_1.f_2)|_{k_1+k_2,l_1+l_2}\gamma. \end{aligned} \tag{2.3}$$

We now define the Drinfeld modular forms of weight k, type l for $\Gamma_0(\mathfrak{n})$, as follows:

Definition 2.2. A rigid holomorphic function $f : \Omega \longrightarrow C$ is said to be a Drinfeld modular form of weight k, type l for $\Gamma_0(\mathfrak{n})$ if

(1) $f|_{k,l}\gamma = f$, $\forall \gamma \in \Gamma_0(\mathfrak{n})$,

(2) f is holomorphic at the cusps of $\Gamma_0(\mathfrak{n})$.

The space of Drinfeld modular forms of weight k, type l for $\Gamma_0(\mathfrak{n})$ is denoted by $M_{k,l}(\Gamma_0(\mathfrak{n}))$. Furthermore, if f vanishes at the cusps of $\Gamma_0(\mathfrak{n})$, then we say f is a Drinfeld cusp form of weight k, type l for $\Gamma_0(\mathfrak{n})$ and the space of such forms is denoted by $M_{k,l}^1(\Gamma_0(\mathfrak{n}))$.

If $k \not\equiv 2l \pmod{q-1}$, then $M_{k,l}(\Gamma_0(\mathfrak{n})) = \{0\}$. So, without loss of generality, we can assume that $k \equiv 2l \pmod{q-1}$.

Let $u(z) := \frac{1}{e_L(\tilde{\pi}z)}$, where $e_L(z) := z\prod_{0\neq\lambda\in L}(1-\frac{z}{\lambda})$ be the exponential function attached to the lattice L. Then, each Drinfeld modular form $f \in M_{k,l}(\Gamma_0(\mathfrak{n}))$ has a unique u-series expansion at ∞ given by $f = \sum_{i=0}^{\infty} a_f(i)u^i$. Since $\left(\begin{smallmatrix}\zeta & 0\\ 0 & 1\end{smallmatrix}\right) \in \Gamma_0(\mathfrak{n})$ for $\zeta \in \mathbb{F}_q^{\times}$, condition (1) of Definition 2.2 implies that $a_f(i) = 0$ if $i \not\equiv l \pmod{q-1}$. Hence, the u-series expansion of f at ∞ can be written as

$$\sum_{0\leq\ i\equiv l \mod (q-1)} a_f(i)u^i.$$

Note that any Drinfeld modular form of type > 0 is automatically a cusp form.

2.1. Examples

We now give some examples of Drinfeld modular forms.

Example 2.3 *([Gos80]).* Let $d \in \mathbb{N}$. For $z \in \Omega$, the function

$$g_d(z) := (-1)^{d+1}\tilde{\pi}^{1-q^d} L_d \sum_{\substack{a,b\in\mathbb{F}_q[T]\\ (a,b)\neq(0,0)}} \frac{1}{(az+b)^{q^d-1}},$$

is a Drinfeld modular form of weight q^d-1, type 0 for $\mathrm{GL}_2(A)$, where $\tilde{\pi}$ is the Carlitz period, and $L_d := (T^q-T)\dots(T^{q^d}-T)$ is the least common multiple of all monic polynomials of degree d. We refer to g_d as the normalized Eisenstein series of weight q^d-1, type 0 for $\mathrm{GL}_2(A)$.

Example 2.4 *([Gos80a]).* For $z \in \Omega$, the function

$$\Delta(z) := (T-T^{q^2})\tilde{\pi}^{1-q^2}E_{q^2-1} + (T^q-T)^q\tilde{\pi}^{1-q^2}(E_{q-1})^{q+1},$$

is a Drinfeld cusp form of weight q^2-1, type 0 for $\mathrm{GL}_2(A)$, where $E_k(z) = \sum_{(0,0)\neq(a,b)\in A^2}\frac{1}{(az+b)^k}$. The u-series expansion of Δ at ∞ is given by $-u^{q-1}+\cdots$.

Example 2.5 *(Poincaré series).* For $z \in \Omega$, define

$$h(z) := \sum_{\gamma \in H \backslash \mathrm{GL}_2(A)} \frac{\det \gamma . u(\gamma z)}{(cz+d)^{q+1}},$$

where $H = \left\{ \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix} \right) \in \mathrm{GL}_2(A) \right\}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A)$. Then h is a Drinfeld cusp form of weight $q+1$, type 1 for $\mathrm{GL}_2(A)$ (cf. [Gek88]). The u-series expansion of h at ∞ is given by $-u - \cdots$.

Example 2.6. In [Gek88], Gekeler defined the function

$$E(z) := \frac{1}{\tilde{\pi}} \sum_{\substack{a \in \mathbb{F}_q[T] \\ a \text{ monic}}} \left(\sum_{b \in \mathbb{F}_q[T]} \frac{a}{az+b} \right)$$

which is analogous to the Eisenstein series of weight 2 over $\mathbb{Q}$. The function E is not modular, but it satisfies the following transformation rule

$$E(\gamma z) = (\det\gamma)^{-1}(cz+d)^2 E(z) - c\tilde{\pi}^{-1}(\det\gamma)^{-1}(cz+d) \tag{2.4}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{GL}_2(A)$. In the proofs of Theorem 1.3 and Theorem 1.5, we use the function $E(z)$ extensively.

2.2. Congruences and Θ-operator

We now define the notion of a congruence between two Drinfeld modular forms.

Definition 2.7. Let $f = \sum_{n \geq 0} a_f(n) u^n$ be a formal u-series in $K[[u]]$. We define

$$v_{\mathfrak{p}}(f) := \inf_n v_{\mathfrak{p}}(a_f(n)),$$

where $v_{\mathfrak{p}}(a_f(n))$ is the $\mathfrak{p}$-adic valuation of $a_f(n)$. We say f has a $\mathfrak{p}$-integral u-series expansion if $v_{\mathfrak{p}}(f) \geq 0$.

Definition 2.8 *(Congruence).* Let $f = \sum_{n \geq 0} a_f(n) u^n$ and $g = \sum_{n \geq 0} a_g(n) u^n$ be two u-formal series in $K[[u]]$. We say that $f \equiv g \pmod{\mathfrak{p}}$ if $v_{\mathfrak{p}}(f-g) \geq 1$.

By [Gek88, Corollary 6.12], we have $g_d \equiv 1 \pmod{\mathfrak{p}}$. A similar congruence holds for the Eisenstein series E_{p-1} in the classical case giving an analogy between g_d and E_{p-1}. Thus, one would expect that g_d plays an essential role in the theory of Drinfeld modular forms.

2.2.1. Θ-operator

For Drinfeld modular forms, there is an analogue of the Ramanujan's Θ-operator, which is defined as

$$\Theta := \frac{1}{\tilde{\pi}} \frac{d}{dz} = -u^2 \frac{d}{du}.$$

The Θ-operator does not preserve modularity, but it preserves quasi-modularity. However, one can perturb the Θ-operator to create an operator which preserves modularity.

Definition 2.9. [Gek88, (8.5)] For $k \in \mathbb{N}$ and $l \in \mathbb{Z}/(q-1)\mathbb{Z}$, we define the operator $\partial_k : M_{k,l}(\Gamma_0(\mathfrak{n})) \to M_{k+2,l+1}(\Gamma_0(\mathfrak{n}))$ by

$$\partial_k f := \Theta f + kEf. \tag{2.5}$$

For simplicity, we write ∂ instead of ∂_k if the weight k is clear from the context. We conclude this section by recalling the following congruence:

Theorem 2.10. *[Vin10, Theorem 1.1]* $E \equiv -\partial_{q^d-1}(g_d) \pmod{\mathfrak{p}}$.

3. Background material for the proofs of the main results

We begin by introducing the (partial) Atkin-Lehner involutions, the modified Drinfeld modular form E^* and the trace operators.

3.1. Atkin-Lehner involutions

Let $\mathfrak{m} = (m)$ and $\mathfrak{n} = (n)$ be two ideals of A, where m and n are non-constant monic polynomials, such that $m || n$, i.e., $m \mid n$ with $(m, n/m) = 1$. The following definition can be found in [Sch96, Page 331].

Definition 3.1. The (partial) Atkin-Lehner involution $W_{\mathfrak{m}}^{(\mathfrak{n})}$ is defined by the action of $\left(\begin{smallmatrix} am & b \\ cn & dm \end{smallmatrix}\right)$ on $M_{k,l}(\Gamma_0(\mathfrak{n}))$, where $a, b, c, d \in A$ are such that $adm^2 - bcn = \zeta \cdot m$ for some $\zeta \in \mathbb{F}_q^\times$.

The following proposition shows that the operator $W_{\mathfrak{m}}^{(\mathfrak{n})}$ is well-defined.

Proposition 3.2. *Let* $W'_{\mathfrak{m}} = \left(\begin{smallmatrix} a'm & b' \\ c'n & d'm \end{smallmatrix}\right)$*, and* $W''_{\mathfrak{m}} = \left(\begin{smallmatrix} a''m & b'' \\ c''n & d''m \end{smallmatrix}\right)$ *be two representatives for the Atkin-Lehner involution* $W_{\mathfrak{m}}^{(\mathfrak{n})}$*. Then,*

$$W'_{\mathfrak{m}} \Gamma_0(\mathfrak{n}) = \Gamma_0(\mathfrak{n}) W''_{\mathfrak{m}}.$$

Proof. A straightforward calculation shows that $W'_{\mathfrak{m}} \Gamma_0(\mathfrak{n}) {W''_{\mathfrak{m}}}^{-1} \subseteq \Gamma_0(\mathfrak{n})$ and ${W'_{\mathfrak{m}}}^{-1} \Gamma_0(\mathfrak{n}) W''_{\mathfrak{m}} \subseteq \Gamma_0(\mathfrak{n})$. Hence the result follows. □

3.2. Action of Atkin-Lehner operator

Recall that $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial $\pi := \pi(T) \in A$ of degree d. Henceforth, $\mathfrak{m} \subseteq A$ denotes an ideal of A generated by a monic polynomial $m := m(T) \in A$ such that $(\mathfrak{p}, \mathfrak{m}) = 1$ (i.e., $\pi \nmid m$).

Since $(\pi, m) = 1$, we take $W_{\mathfrak{p}}^{(\mathfrak{pm})} := \left(\begin{smallmatrix} \pi & b \\ \pi m & d\pi \end{smallmatrix}\right)$ where $b, d \in A$, such that $d\pi^2 - b\pi m = \pi$. An easy verification shows that $W_{\mathfrak{p}}^{(\mathfrak{pm})}.W_{\mathfrak{p}}^{(\mathfrak{pm})} = \left(\begin{smallmatrix} \pi & 0 \\ 0 & \pi \end{smallmatrix}\right)\gamma$ for some $\gamma \in \Gamma_0(\mathfrak{pm})$. This shows that $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ acts as an involution on $M_{k,l}(\Gamma_0(\mathfrak{pm}))$. If $f \in M_{k,l}(\Gamma_0(\mathfrak{pm}))$ such that $f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f$ for $\alpha \in C\backslash\{0\}$, then we must have $\alpha^2 = 1$, i.e., $\alpha \in \{\pm 1\}$.

For $f \in M_{k,l}(\Gamma_0(\mathfrak{p}))$, the actions of $W_{\mathfrak{p}}^{(\mathfrak{p})}$ and $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ on f are the same. If $\mathfrak{m} = (1)$, then we denote $W_{\mathfrak{p}}^{(\mathfrak{p})}$ by $W_{\mathfrak{p}}$ for simplicity. In order to calculate the action of $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ on some class of modular forms, we need to define the $U_{\mathfrak{p}}$-operator.

3.3. $U_{\mathfrak{p}}$-operator and $V_{\mathfrak{p}}$-operator

For a rigid analytic function $f : \Omega \longrightarrow C$, we define:

$$f|U_{\mathfrak{p}}(z) = \frac{1}{\pi} \sum_{\substack{\lambda \in A \\ \deg(\lambda) < d}} f\left(\frac{z+\lambda}{\pi}\right), \quad f|V_{\mathfrak{p}}(z) = f(\pi z).$$

3.4. Construction of E^* and its properties

We know that E is not a Drinfeld modular form. The following proposition shows how to construct a Drinfeld modular form using the function E.

Proposition 3.3. *The function $E^*(z) := E(z) - \pi E|V_{\mathfrak{p}}(z)$ is a Drinfeld modular form of weight 2, type 1 for $\Gamma_0(\mathfrak{p})$. Moreover, we have $E^*|_{2,1}W_{\mathfrak{p}} = -E^*$.*

Proof. An easy computation using (2.4) shows that $E^*(\gamma z) = (\det \gamma)^{-1}(cz+d)^2 E^*(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(\mathfrak{p})$. Since E and $E|V_{\mathfrak{p}}$ are holomorphic on Ω, the function E^* is also holomorphic on Ω. Now, it remains to check the holomorphicity at the cusps of $\Gamma_0(\mathfrak{p})$.

By [Gek01, Proposition 6.7], we see that 0 and ∞ are the only cusps of $\Gamma_0(\mathfrak{p})$. The function E^* is holomorphic at ∞ since E and $E|V_{\mathfrak{p}}$ are holomorphic at ∞. A straightforward calculation using (2.4) shows that $E^*(z)|_{2,1}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has a power series expansion in u. Since the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ takes the cusp ∞ to the cusp 0, we conclude that E^* is holomorphic at the cusp 0. Thus E^* is a Drinfeld modular form of weight 2, type 1 for $\Gamma_0(\mathfrak{p})$. The last part can be verified easily. $\square$

The following two properties of E^* are of importance to us.

- If $f \in M_{k,l}(\Gamma_0(\mathfrak{pm}))$ such that $f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f$ with $\alpha \in \{\pm 1\}$, then we have $(E^* f)|_{k+2,l+1}W_{\mathfrak{p}}^{(\mathfrak{pm})} = (-\alpha)E^* f$ (cf. (2.3)). So in order to change the sign of the eigenvalue of $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ on f, one can simply multiply f with E^*.
- Since $E(z)$ and $E(\pi z)$ have coefficients in A (cf. [Vin14, Proposition 3.3]), we have the following congruence

$$E^* \equiv E \pmod{\mathfrak{p}}. \tag{3.1}$$

Next, we describe the action of $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ on $\partial_k f$.

Proposition 3.4. *Suppose that $f \in M_{k,l}(\Gamma_0(\mathfrak{pm}))$ and $f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f$ with $\alpha \in \{\pm 1\}$. Then,*

$$(\partial_k f)|_{k+2,l+1}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha(\partial_k f - kE^* f). \tag{3.2}$$

Proof. For $z \in \Omega$, we have

$$\begin{aligned}
&(\partial_k f)|_{k+2,l+1}W_{\mathfrak{p}}^{(\mathfrak{pm})}(z)\\
&= \pi^{\frac{k+2}{2}}(\pi m z + d\pi)^{-(k+2)}(\partial_k f)\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big)\\
&= \pi^{\frac{k+2}{2}}(\pi m z + d\pi)^{-(k+2)}\Big\{\Theta f\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big) + kE\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big) f\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big)\Big\}\\
&= (\Theta f(z))|_{k+2,l+1}W_{\mathfrak{p}}^{(\mathfrak{pm})} + kE\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big).\pi^{\frac{k+2}{2}}(\pi m z + d\pi)^{-(k+2)} f\Big(\frac{\pi z + b}{\pi m z + d\pi}\Big)\\
&= \alpha\Theta(f) + \frac{km\alpha f}{\tilde{\pi}(mz+d)} + k\Big(\pi^2(mz+d)^2E(\pi z) - \frac{m\pi}{\tilde{\pi}}(mz+d)\Big)\frac{1}{\pi(mz+d)^2} f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})}\\
&= \alpha\Theta(f) + k\pi E(\pi z)(\alpha f)\\
&= \alpha\Theta(f) + k\alpha E f - k\alpha E f + k\pi E(\pi z)(\alpha f) = \alpha(\partial_k f - kE^* f).
\end{aligned}$$

Here, we have used the equality $(\Theta f(z))|_{k+2,l+1}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha\Theta(f) + \frac{km\alpha f}{\tilde{\pi}(mz+d)}$. □

3.5. Trace operators

Now, we discuss the trace operators.

Definition 3.5. For any $\mathfrak{a} \mid \mathfrak{n}$, we define the trace operator

$$\mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{a}}} : M_{k,l}(\Gamma_0(\mathfrak{n})) \longrightarrow M_{k,l}\Big(\Gamma_0\Big(\frac{\mathfrak{n}}{\mathfrak{a}}\Big)\Big)$$

by

$$\mathrm{Tr}^{\mathfrak{n}}_{\frac{\mathfrak{n}}{\mathfrak{a}}}(f) = \sum_{\gamma \in \Gamma_0(\mathfrak{n}) \backslash \Gamma_0(\frac{\mathfrak{n}}{\mathfrak{a}})} f|_{k,l}\gamma.$$

We will make use of the following proposition to explicitly compute the action of the trace operator which can be thought of as a generalization of [Vin14, Proposition 3.8] from level $\mathfrak{p}$ to level $\mathfrak{pm}$.

Proposition 3.6. *Let $\mathfrak{p}$, $\mathfrak{m}$ be as before. For any $f \in M_{k,l}(\Gamma_0(\mathfrak{pm}))$, we have*

$$\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(f) = f + \pi^{1-k/2}(f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})|U_{\mathfrak{p}}$$

Proof. By definition, we have

$$\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(f) = \sum_{\gamma\in\Gamma_0(\mathfrak{pm})\backslash\Gamma_0(\mathfrak{m})} f|_{k,l}\gamma.$$

The set $\{\left(\begin{smallmatrix}1 & j\\ m & mj+1\end{smallmatrix}\right)|j \in A, \deg(j) < d\}$, along with the identity matrix, is a complete set of representatives for $\Gamma_0(\mathfrak{pm})\backslash\Gamma_0(\mathfrak{m})$. Using the coset representatives, we obtain

$$\begin{aligned}
\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}} f &= f + \sum_{j\in A,\deg(j)<d} f|_{k,l}\begin{pmatrix}1 & j\\ m & mj+1\end{pmatrix}\\
&= f + \sum_{j\in A,\deg(j)<d} f|_{k,l}\begin{pmatrix}\pi & b\\ \pi m & \pi d\end{pmatrix}\begin{pmatrix}\frac{1}{\pi} & \frac{j-b}{\pi}\\ 0 & 1\end{pmatrix}\\
&= f + \sum_{j\in A,\deg(j)<d} (f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})|\begin{pmatrix}\frac{1}{\pi} & \frac{j-b}{\pi}\\ 0 & 1\end{pmatrix}\\
&= f + \sum_{j\in A,\deg(j)<d} \frac{1}{\pi^{k/2}}(f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})(\frac{z+j-b}{\pi}).
\end{aligned}$$

To complete the proof of Proposition 3.6, we require the following lemma. Its proof is similar to that of [Vin14, Lemma 5.3], and hence, we omit the details.

Lemma 3.7. *For a fixed $z \in \Omega$ and $a \in A$, the set $\{u(\frac{z+j-a}{\pi})|j \in A, \deg(j) < d\}$ is exactly the set of the reciprocal of the roots of the polynomial $\rho_\pi(x) - \frac{1}{u(z)} \in A((u(z)))[x]$ (recall that ρ is the rank one Drinfeld module defined by (2.1)).*

By Lemma 3.7, for a fixed $z \in \Omega$ and $b \in A$, the sets $\{u(\frac{z+j}{\pi})|j \in A, \deg(j) < d\}$ and $\{u(\frac{z+j-b}{\pi})|j \in A, \deg(j) < d\}$ are equal. Therefore, we conclude that

$$\begin{aligned}
\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}} f &= f + \sum_{j\in A,\deg(j)<d} \frac{1}{\pi^{k/2}}(f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})(\frac{z+j-b}{\pi})\\
&= f + \frac{1}{\pi^{k/2}}\sum_{j\in A,\deg(j)<d} (f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})(\frac{z+j}{\pi})\\
&= f + \pi^{1-k/2}(f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})})|U_{\mathfrak{p}}. \quad \square
\end{aligned}$$

3.6. Key propositions

We are now ready to state and prove the main results of this section.

Proposition 3.8. *If $f \in M^1_{k,l}(\Gamma_0(\mathfrak{pm}))$ has $\mathfrak{p}$-integral u-series expansion at ∞ such that $f|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f$ with $\alpha \in \{\pm 1\}$, then there exists $F \in M^1_{(k-1)(q^d-1)+k,l}(\Gamma_0(\mathfrak{m}))$ with $\mathfrak{p}$-integral u-series expansion at ∞ such that $f \equiv F \pmod{\mathfrak{p}}$.*

Proof. For an integer $k \geq 2$, we define

$$g_{(k)} := (g_d - \pi^{(q^d-1)/2} g_d|_{q^d-1,0} W_{\mathfrak{p}})^{k-1},$$

where g_d is the Eisenstein series of weight $q^d - 1$, type 0 for $\mathrm{GL}_2(A)$ (cf. Example 2.3). Then $g_{(k)} \in M_{(k-1)(q^d-1),0}(\Gamma_0(\mathfrak{p}))$ and it satisfies the following congruences

$$g_{(k)} \equiv 1 \pmod{\mathfrak{p}} \tag{3.3}$$

and

$$g_{(k)}|_{(k-1)(q^d-1),0} W_{\mathfrak{p}} \equiv 0 \pmod{\mathfrak{p}^{\frac{(k-1)(q^d-1)}{2}+k-1}}, \tag{3.4}$$

(cf. [Vin14, Page 32] for more details).

Since $f \in M^1_{k,l}(\Gamma_0(\mathfrak{pm}))$ has $\mathfrak{p}$-integral u-series expansion at ∞, we have $v_{\mathfrak{p}}(f) \geq 0$. The function $fg_{(k)}$ is a Drinfeld cusp form of weight $(k-1)(q^d-1)+k$, type l for $\Gamma_0(\mathfrak{pm})$ with $\mathfrak{p}$-integral u-series expansion at ∞. Thus, $\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}} fg_{(k)}$ is a Drinfeld cusp form of weight $(k-1)(q^d-1)+k$, type l for $\Gamma_0(\mathfrak{m})$.

From Proposition 3.6, we obtain

$$\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(fg_{(k)}) - fg_{(k)} = \pi^{1-\frac{k+(k-1)(q^d-1)}{2}} (fg_{(k)}|_{(k-1)(q^d-1)+k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})})|U_{\mathfrak{p}}.$$

Since $v_{\mathfrak{p}}(f|U_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f)$ (cf. [Vin14, Corollary 3.2]), it follows that:

$$\begin{aligned}
& v_{\mathfrak{p}}(\mathrm{Tr}^{\mathfrak{pm}}_{\mathfrak{m}}(fg_{(k)}) - fg_{(k)}) \\
& \qquad \geq 1 - \frac{(k-1)(q^d-1)+k}{2} + v_{\mathfrak{p}}(fg_{(k)}|_{(k-1)(q^d-1)+k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})}) \\
& \qquad = 1 - \frac{(k-1)(q^d-1)+k}{2} + v_{\mathfrak{p}}(f|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})}) + v_{\mathfrak{p}}(g_{(k)}|_{(k-1)(q^d-1),0} W_{\mathfrak{p}}) \\
& \qquad \underset{(3.4)}{=} 1 - \frac{(k-1)(q^d-1)+k}{2} + v_{\mathfrak{p}}(f|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})}) + \frac{(k-1)(q^d-1)}{2} + k - 1 \\
& \qquad = \frac{k}{2} + v_{\mathfrak{p}}(f|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})}) \\
& \qquad = \frac{k}{2} + v_{\mathfrak{p}}(f) \geq \frac{k}{2} \geq 1 \qquad (\text{since } f|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f \text{ and } v_{\mathfrak{p}}(f) \geq 0).
\end{aligned}$$

We thus get

$$\mathrm{Tr}_{\mathfrak{m}}^{\mathfrak{pm}} f g_{(k)} \equiv f g_{(k)} \pmod{\mathfrak{p}}. \tag{3.5}$$

Combining (3.3) with (3.5), we conclude that

$$\mathrm{Tr}_{\mathfrak{m}}^{\mathfrak{pm}} f g_{(k)} \equiv f \pmod{\mathfrak{p}}. \tag{3.6}$$

Thus, the Drinfeld modular form $F := \mathrm{Tr}_{\mathfrak{m}}^{\mathfrak{pm}} f g_{(k)} \in M^1_{(k-1)(q^d-1)+k,l}(\Gamma_0(\mathfrak{m}))$ has $\mathfrak{p}$-integral u-series expansion at ∞ and it satisfies the conclusion of the proposition. □

Proposition 3.9. *Suppose that $h \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{pm}))$ has a $\mathfrak{p}$-integral u-series expansion at ∞ and $\alpha \in \{\pm 1\}$. Then there exists $H \in M^1_{(k-1)q^d+3,l+1}(\Gamma_0(\mathfrak{m}))$ such that $H \equiv \alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \pmod{\mathfrak{p}}$.*

Proof. Since $\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)} \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{pm}))$, by the definition of the trace operator, we get that $\mathrm{Tr}_{\mathfrak{m}}^{(\mathfrak{pm})}(\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) \in M^1_{(k-1)q^d+3,l+1}(\Gamma_0(\mathfrak{m}))$.

By Proposition 3.6 and (2.3), we obtain

$$\begin{aligned}
& v_{\mathfrak{p}}(\mathrm{Tr}_{\mathfrak{m}}^{(\mathfrak{pm})}(\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) - \alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) \\
&= v_{\mathfrak{p}}(\pi^{1-\frac{(k-1)q^d+3}{2}} \alpha\pi(((h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}).g_{(k)})|_{(k-1)(q^d-1)+k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})})|U_{\mathfrak{p}}) \\
&= v_{\mathfrak{p}}(\alpha\pi^{2-\frac{(k-1)q^d+3}{2}}(((h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})})|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}).(g_{(k)}|_{(k-1)(q^d-1),l+1} W_{\mathfrak{p}}))|U_{\mathfrak{p}}) \\
&= v_{\mathfrak{p}}(\alpha\pi^{2-\frac{(k-1)q^d+3}{2}}(h.(g_{(k)}|_{(k-1)(q^d-1),0} W_{\mathfrak{p}}))|U_{\mathfrak{p}}) \\
&\geq v_{\mathfrak{p}}(\alpha\pi^{2-\frac{(k-1)q^d+3}{2}} g_{(k)}|_{(k-1)(q^d-1),0} W_{\mathfrak{p}}) \qquad (\text{since } v_{\mathfrak{p}}(f|U_{\mathfrak{p}}) \geq v_{\mathfrak{p}}(f) \text{ and } v_p(h) \geq 0) \\
&\geq \frac{(k-1)(q^d-1)}{2} + k - 1 + 2 - \frac{(k-1)q^d+3}{2} = \frac{k}{2} \geq 1 \qquad (\text{using } (3.4)).
\end{aligned}$$

We thus get

$$\mathrm{Tr}_{\mathfrak{m}}^{(\mathfrak{pm})}(\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) \equiv \alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)} \pmod{\mathfrak{p}}. \tag{3.7}$$

Combining (3.3) with (3.7), we conclude that

$$\mathrm{Tr}_{\mathfrak{m}}^{(\mathfrak{pm})}(\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) \equiv \alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \pmod{\mathfrak{p}}.$$

Thus, the Drinfeld modular form $H := \mathrm{Tr}_{\mathfrak{m}}^{(\mathfrak{pm})}(\alpha\pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})}.g_{(k)}) \in M^1_{(k-1)q^d+3,l+1}(\Gamma_0(\mathfrak{m}))$ satisfies the conclusion of the proposition. □

Remark 3.10. The above result is true for any $\alpha \in K$ with $v_{\mathfrak{p}}(\alpha) \geq 0$. Throughout the article, we work with $\alpha \in \{\pm 1\}$, so we restrict ourselves in deviating from it.

Now, we are ready to prove Theorem 1.3 and Theorem 1.5.

4. Proof of Theorem 1.3

Before going into the proof of Theorem 1.3, we recall the notion of weight filtration for Drinfeld modular forms for $\mathrm{GL}_2(A)$ and list some of its properties. Let M_k denote the space of Drinfeld modular forms of weight k (any type) for $\mathrm{GL}_2(A)$.

4.1. Filtration for level 1 case

Recall that $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial $\pi := \pi(T)$ of degree d. Let f be a Drinfeld modular form of weight k, type l for $\mathrm{GL}_2(A)$ with $\mathfrak{p}$-integral u-series expansion at ∞.

Definition 4.1. If $f \not\equiv 0 \pmod{\mathfrak{p}}$, then we define the weight filtration $w(\overline{f})$ of f as

$$w(\overline{f}) := \inf\{k_0 | \exists f' \in M_{k_0} \text{ with } f \equiv f' \pmod{\mathfrak{p}}\}.$$

If $f \equiv 0 \pmod{\mathfrak{p}}$, then we define $w(\overline{f}) = -\infty$. Since the weight filtration of f is defined mod $\mathfrak{p}$, we choose to write $w(\overline{f})$ rather than $w(f)$.

To discuss some properties of $w(\overline{f})$, we recall the structure of the ring $M(\mathrm{GL}_2(A)) = \bigoplus_{k,l} M_{k,l}(\mathrm{GL}_2(A))$. By [Gek88, Theorem 5.13], we have $M(\mathrm{GL}_2(A)) = C[g_1, h]$. In particular, every Drinfeld modular form corresponds to a unique isobaric polynomial in g_1 and h over C. Let $A_d(X,Y)$ and $B_d(X,Y)$ be the isobaric polynomials attached to g_d and $\partial(g_d)$ respectively, i.e., $A_d(g_1,h) = g_d$ and $B_d(g_1,h) = \partial(g_d)$.

In [Gek88], [Vin10], the authors proved the following properties of $w(\overline{f})$.

Theorem 4.2. *Let $f \in M_{k,l}(\mathrm{GL}_2(A))$ and $f = \phi(g_1,h)$ where $\phi(X,Y)$ is the isobaric polynomial attached to f. Then the following hold.*

(1) If $f \not\equiv 0 \pmod{\mathfrak{p}}$, then $w(\overline{f}) \equiv k \pmod{q^d-1}$,
(2) $w(\overline{f}) < k$ if and only if $\overline{A}_d | \overline{\phi}$, where $\overline{U}$ denotes the reduction of $U \pmod{\mathfrak{p}}$.
(3) $\overline{A_d}(X,Y)$ shares no common factor with $\overline{B_d}(X,Y)$.

4.2. Proof of Theorem 1.3

Let us recall the statement of this theorem for the convenience of the reader.

Theorem 4.3. *Suppose that $f \in M^1_{k,l}(\Gamma_0(\mathfrak{p}))$ and $g \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{p}))$ have $\mathfrak{p}$-integral u-series expansions at ∞ with $\Theta f \equiv g \pmod{\mathfrak{p}}$. Assume that $w(\overline{F}) = (k-1)(q^d-1)+k$ where F is as in Proposition 3.8 corresponding to kf, and $(k,p) = 1$. If $f|W_{\mathfrak{p}} = \alpha f$ and $g|W_{\mathfrak{p}} = \beta g$ with $\alpha, \beta \in \{\pm 1\}$, then $\beta = -\alpha$.*

Proof. We shall prove this theorem by contradiction. Suppose that $\beta = \alpha$.

Since $E^* \equiv E \pmod{\mathfrak{p}}$ (cf. (3.1)) and $\Theta f \equiv g \pmod{\mathfrak{p}}$, we have

$$\partial f \equiv g + kE^* f \pmod{\mathfrak{p}} \qquad \text{(cf. (2.5))}.$$

Hence, there exists $h \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{p}))$ with $v_{\mathfrak{p}}(h) \geq 0$ such that

$$g - \partial f + kE^* f = \pi h. \tag{4.1}$$

Applying $W_{\mathfrak{p}}$ on both sides of (4.1), we obtain

$$\alpha(g - \partial f) = \pi h|_{k+2,l+1} W_{\mathfrak{p}} \qquad \text{(cf. Proposition 3.4)}. \tag{4.2}$$

Combining this with (4.1), we get

$$kE^* f = \pi h - \alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}} \tag{4.3}$$

and hence

$$kE^* f \equiv -\alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}} \pmod{\mathfrak{p}}.$$

Since $v_{\mathfrak{p}}(E^*) \geq 0$ and $v_{\mathfrak{p}}(f) \geq 0$, setting $\mathfrak{m} = (1)$ in Proposition 3.8, there exists $F \in M^1_{(k-1)q^d+1,l}(\mathrm{GL}_2(A))$ such that $kf \equiv F \pmod{\mathfrak{p}}$. Hence

$$E^* F \equiv -\alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}} \pmod{\mathfrak{p}}. \tag{4.4}$$

By (4.4) and Proposition 3.9 with $\mathfrak{m} = (1)$, we obtain

$$E^* F \equiv -H \pmod{\mathfrak{p}},$$

where $H \in M^1_{(k-1)q^d+3,l+1}(\mathrm{GL}_2(A))$ such that $H \equiv \alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}} \pmod{\mathfrak{p}}$.

Recalling that $E^* \equiv E \equiv -\partial(g_d) \pmod{\mathfrak{p}}$, we get

$$H \equiv \partial(g_d) F \pmod{\mathfrak{p}}. \tag{4.5}$$

The last congruence implies that H has $\mathfrak{p}$-integral u-series expansion at ∞. Since both sides of (4.5) are congruent mod $\mathfrak{p}$, we have

$$w(\overline{H}) = w(\overline{\partial(g_d)F}). \tag{4.6}$$

We now calculate the weight filtration $w(\overline{\partial(g_d)F})$.

Let ϕ be the unique isobaric polynomial attached to F. Consequently, the unique isobaric polynomial attached to $\partial(g_d)F$ is $B_d\phi$. The weights of F and $\partial(g_d)F$ are $(k-1)(q^d-1)+k$ and kq^d+2, respectively. The assumption $w(\overline{F}) = (k-1)(q^d-1)+k$ implies that $\overline{A_d} \nmid \overline{\phi}$ (cf. Theorem 4.2(2)). Combining this with Theorem 4.2(3), we obtain

$$\overline{A_d} \nmid \overline{B_d \phi}. \tag{4.7}$$

Finally, (4.7) and Theorem 4.2(2) together yield $w(\overline{\partial(g_d)F}) = kq^d + 2$.

Since the weight of H is $(k-1)q^d + 3$, we conclude that

$$w(\overline{H}) \leq (k-1)q^d + 3 < kq^d + 2 = w(\overline{\partial(g_d)F}),$$

which contradicts (4.6). Therefore, we must have $\beta = -\alpha$. $\square$

4.3. Proof of Corollary 1.4

Now, arguing as in the proof of Theorem 1.3, we get $F \in M^1_{q^d+1,1}(\mathrm{GL}_2(A))$ and $H \in M^1_{q^d+3,2}(\mathrm{GL}_2(A))$ when $k = 2$ and $l = 1$. Since $f \not\equiv 0 \pmod{\mathfrak{p}}$, the congruence $w(\overline{F}) \equiv q^d + 1 \pmod{q^d - 1}$ implies that the possible values of $w(\overline{F})$ are 2 or $q^d + 1$.

The space $M(\mathrm{GL}_2(A))$ is generated by g_1 and h, where g_1 is of weight $q-1$ and h is of weight $q+1$. For $q > 3$, the weight of g_1 is $q - 1 > 2$. Therefore, there is no modular form of weight 2. For $q = 3$, we have $M_{2,l}(\mathrm{GL}_2(A)) = \{0\}$ whenever $l \not\equiv 0 \pmod 2$ and $M^1_{2,0}(\mathrm{GL}_2(A)) = \{0\}$. Thus, $w(\overline{F})$ cannot be 2. Hence we obtain

$$w(\overline{F}) = q^d + 1.$$

Now, the desired result follows from Theorem 1.3.

5. Proof of Theorem 1.5

Before going into the proof of Theorem 1.5, let us introduce some notations and recall the relevant results from [Hat20] and [Hat20a]. Using them, we shall prove an important proposition about the weight filtration.

5.1. Geometry of the Drinfeld modular curves

Let $\mathfrak{m} = (m)$ be as in Theorem 1.5, where $m \in A$ is a non-constant monic polynomial. The conditions on $\mathfrak{m}$ allow us to choose a subgroup $\Delta \subseteq (A/\mathfrak{m})^\times$ such that the natural inclusion $\mathbb{F}_q^\times \hookrightarrow (A/\mathfrak{m})^\times$ gives $\Delta \oplus \mathbb{F}_q^\times = (A/\mathfrak{m})^\times$.

The fine moduli scheme $Y_1^\Delta(\mathfrak{m})$ classifies the tuples $(E, \lambda, [\mu])$, where E is a Drinfeld module of rank 2 over an $A[1/m]$-scheme S, λ is a $\Gamma_1(\mathfrak{m})$-structure on E, and $[\mu]$ is a Δ-structure on E (cf. [Hat20, Page 20] for more details).

Let E^Δ_{un} be the universal Drinfeld module over $Y_1^\Delta(\mathfrak{m})$ and $\omega^\Delta_{\mathrm{un}}$ be the sheaf of invariant differential forms on E^Δ_{un}. Let $X_1^\Delta(\mathfrak{m})$ be the compactification of $Y_1^\Delta(\mathfrak{m})$. Suppose that R_0 is a flat $A[1/m]$-algebra which is an excellent regular domain. The invertible sheaf $\omega^\Delta_{\mathrm{un}}$ on $Y_1^\Delta(\mathfrak{m})_{R_0}$ extends to an invertible sheaf $\overline{\omega}^\Delta_{\mathrm{un}}$ on $X_1^\Delta(\mathfrak{m})_{R_0}$ (cf. [Hat20a, Theorem 5.3]).

Following [Hat20, Page 26], we define $\Gamma_1^\Delta(\mathfrak{m}) := \{\gamma \in \mathrm{SL}_2(A) | \gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\mathfrak{m}}\}$.

Definition 5.1. [Hat20, Definition 4.7] Let k be an integer and M be an $A[1/m]$-module. The space of Drinfeld modular forms of level $\Gamma_1^{\Delta}(\mathfrak{m})$ and weight k with coefficients in M is defined by

$$M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_M := H^0(X_1^{\Delta}(\mathfrak{m})_{A[1/\mathfrak{m}]}, (\overline{\omega}_{\mathrm{un}}^{\Delta})^{\otimes k} \otimes_{A[1/\mathfrak{m}]} M).$$

Consider the map $x_{\infty}^{\Delta} : \operatorname{Spec}(A[1/m][[x]]) \to X_1^{\Delta}(\mathfrak{m})_{A[1/\mathfrak{m}]}$ as in [Hat20a, Theorem 5.3].

Definition 5.2 *(x-expansion).* For any $f \in M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_M$, we define the x-expansion of f at the ∞-cusp as the unique power series $f_{\infty}(x) \in A[1/m][[x]] \otimes_{A[1/m]} M$, satisfying $(x_{\infty}^{\Delta})^*(f) = f_{\infty}(x)(dX)^{\otimes k}$.

By [Hat20, Proposition 4.8(ii)], if $f_{\infty}(x) = 0$, then $f = 0$, which we refer to as the x-expansion principle.

Remark 5.3. The x-expansion principle implies that we can consider a modular form not only as a global section but also in terms of its x-expansion.

This definition of Drinfeld modular forms is compatible with the classical Drinfeld modular forms over C, which are described in [Gek86], [Gek88]. In fact, one can show that the x-expansion $f_{\infty}(x)$ of f at the ∞-cusp agrees with the u-series expansion at ∞ of the associated classical Drinfeld modular form to f (cf. [Hat20, Page 26] and references therein for more details).

By [Hat20, Proposition 4.8(ii)], for any $k \geq 2$, and any $A[1/m]$-module M, we have an isomorphism

$$M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A[1/m]} \otimes_{A[1/m]} M \cong M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_M. \tag{5.1}$$

Let $A_{\mathfrak{p}}$ be the completion of A at $\mathfrak{p}$. By (5.1), we obtain an isomorphism

$$M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A[1/m]} \otimes_{A[1/m]} A_{\mathfrak{p}} \cong M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A_{\mathfrak{p}}},$$

tensoring with $A/\mathfrak{p}$, we obtain the following isomorphism

$$\begin{aligned} M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A_{\mathfrak{p}}} \otimes_{A[1/m]} A/\mathfrak{p} &\cong M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A[1/m]} \otimes_{A[1/m]} (A_{\mathfrak{p}} \otimes_{A[1/m]} A/\mathfrak{p}) \\ &\cong M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A/\mathfrak{p}}. \end{aligned} \tag{5.2}$$

Let $\widetilde{f}$ denote the image of $f \in M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A_{\mathfrak{p}}}$ under the isomorphism (5.2). By [Har77, Corollary 9.4 of Chapter III], the element $\widetilde{f}$ can also be treated as an element of $H^0(X_1^{\Delta}(\mathfrak{m})_{A/\mathfrak{p}}, (\overline{\omega}_{\mathrm{un}}^{\Delta}|_{A/\mathfrak{p}})^{\otimes k})$.

Remark 5.4. For $f \in M_k(\Gamma_1^{\Delta}(\mathfrak{m}))_{A_{\mathfrak{p}}}$, the x-expansion of $\widetilde{f}$ at the ∞-cusp is same as the mod $\mathfrak{p}$-reduction of the u-series expansion of f at ∞.

5.2. Weight filtration

We are now in a position to define the weight filtration for any $f \in M_k(\Gamma_1^\Delta(\mathfrak{m}))_{A_\mathfrak{p}}$.

Definition 5.5. If $f \not\equiv 0 \pmod{\mathfrak{p}}$, then we define the weight filtration $w(\overline{f})$ of f as

$$w(\overline{f}) := \inf\{k_0 | \exists\ f' \in M_{k_0}(\Gamma_1^\Delta(\mathfrak{m}))_{A_\mathfrak{p}} \text{ with } f \equiv f' \pmod{\mathfrak{p}}\}. \tag{5.3}$$

By $f \equiv f' \pmod{\mathfrak{p}}$, we mean that the corresponding x-expansions of f and f' at the ∞-cusp are congruent modulo $\mathfrak{p}$, i.e., $f_\infty(x) \equiv f'_\infty(x) \pmod{\mathfrak{p}}$. If $f \equiv 0 \pmod{\mathfrak{p}}$, then define $w(\overline{f}) = -\infty$.

By [Hat20, Theorem 4.16], we have $w(\overline{f}) \equiv k \pmod{q^d - 1}$.

In the proof of Theorem 1.5, the following proposition about the weight filtration of f is useful. We follow the approach of Gross in [Gro90, Page 459] to prove the proposition.

Proposition 5.6. *If $f \in M_k(\Gamma_1^\Delta(\mathfrak{m}))_{A_\mathfrak{p}}$, then $w(\overline{f}) < k$ if and only if $\widetilde{f}$ vanishes at all supersingular points of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$.*

Proof. Suppose $w(\overline{f}) = k' < k$. Then there exists $f' \in M_{k'}(\Gamma_1^\Delta(\mathfrak{m}))_{A_\mathfrak{p}}$ such that $f \equiv f'$ $\pmod{\mathfrak{p}}$. Let $\widetilde{f}, \widetilde{f'}$ and $\widetilde{g_d}$ be the images of f, f' and g_d respectively under the isomorphism (5.2). By line 5 in the proof of Proposition 4.22 in [Hat20] together with the injectivity of (4.15) in [Hat20], we get that $\widetilde{g_d}$ divides $\widetilde{f}$. Since g_d is a lift of the Hasse invariant, we conclude that $\widetilde{f}$ vanishes at all supersingular points of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$.

Conversely, suppose that $\widetilde{f}$ vanishes at all supersingular points of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$. Since $\widetilde{g_d}$ vanishes at all supersingular points exactly once and remains non-zero elsewhere on $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$, $\widetilde{f}/\widetilde{g_d}$ defines a holomorphic global section in $M_{k-(q^d-1)}(\Gamma_1^\Delta(\mathfrak{m}))_{A/\mathfrak{p}}$. Let $f' \in M_{k-(q^d-1)}(\Gamma_1^\Delta(\mathfrak{m}))_{A_\mathfrak{p}}$ be a lift of $\widetilde{f}/\widetilde{g_d}$ under (5.2). Thus $f \equiv f' \pmod{\mathfrak{p}}$ since $(g_d)_\infty(x) \equiv 1 \pmod{\mathfrak{p}}$. This implies $w(\overline{f}) < k$. □

Remark 5.7. Observing that $\Gamma_1^\Delta(\mathfrak{m}) \subset \Gamma_0(\mathfrak{m})$, we get $M_{k,l}(\Gamma_0(\mathfrak{m})) \subset M_k(\Gamma_1^\Delta(\mathfrak{m}))$. Since the order of the determinant group of $\Gamma_1^\Delta(\mathfrak{m})$ is 1, the type does not play any role for Drinfeld modular forms of level $\Gamma_1^\Delta(\mathfrak{m})$. In fact, for a fixed k, all $M_{k,l}(\Gamma_1^\Delta(\mathfrak{m}))$ are isomorphic (cf. [Boc, Page 49]).

5.3. Proof of Theorem 1.5

Let us recall the statement of this theorem for the convenience of the reader.

Theorem 5.8. *Let $\mathfrak{m}$ be an ideal of A generated by a polynomial in A which has a prime factor of degree prime to $q-1$ and $\mathfrak{p} \nmid \mathfrak{m}$. Suppose that $f \in M_{k,l}^1(\Gamma_0(\mathfrak{pm}))$ and $g \in M_{k+2,l+1}^1(\Gamma_0(\mathfrak{pm}))$ have $\mathfrak{p}$-integral u-series expansions at ∞ with $\Theta f \equiv g \pmod{\mathfrak{p}}$. Assume that $w(\overline{F}) = (k-1)(q^d-1)+k$, where F is as in Proposition 3.8 corresponding* *to* kf*, and* $(k,p) = 1$*. If* $f|W_{\mathfrak{p}}^{(\mathfrak{pm})} = \alpha f$ *and* $g|W_{\mathfrak{p}}^{(\mathfrak{pm})} = \beta g$ *with* $\alpha, \beta \in \{\pm 1\}$*, then* $\beta = -\alpha$*.*

Proof. We shall prove this theorem by contradiction. Suppose that $\beta = \alpha$. We follow the argument as in the proof of Theorem 4.3.

Since $E^* \equiv E \pmod{\mathfrak{p}}$ (cf. (3.1)) and $\Theta f \equiv g \pmod{\mathfrak{p}}$, we have

$$\partial f \equiv g + kE^* f \pmod{\mathfrak{p}} \qquad \text{(cf. (2.5))}.$$

Hence, there exists $h \in M^1_{k+2,l+1}(\Gamma_0(\mathfrak{pm}))$ with $v_{\mathfrak{p}}(h) \geq 0$, such that

$$g - \partial f + kE^* f = \pi h. \tag{5.4}$$

Applying $W_{\mathfrak{p}}^{(\mathfrak{pm})}$ on both sides, we obtain

$$\alpha(g - \partial f) = \pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \qquad \text{(cf. Proposition 3.4)}. \tag{5.5}$$

Combining this with (5.4), we get

$$kE^* f = \pi h - \alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \tag{5.6}$$

and hence

$$kE^* f \equiv -\alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \pmod{\mathfrak{p}}. \tag{5.7}$$

Since $v_{\mathfrak{p}}(E^*) \geq 0$ and $v_{\mathfrak{p}}(f) \geq 0$, by Proposition 3.8, there exists $F \in M^1_{(k-1)q^d+1,l}(\Gamma_0(\mathfrak{m}))$ such that $kf \equiv F \pmod{\mathfrak{p}}$, hence

$$E^* F \equiv -\alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \pmod{\mathfrak{p}}. \tag{5.8}$$

By Proposition 3.9 and (5.8), we obtain

$$E^* F \equiv -H \pmod{\mathfrak{p}},$$

where $H \in M^1_{(k-1)q^d+3,l+1}(\Gamma_0(\mathfrak{m}))$ such that $H \equiv \alpha \pi h|_{k+2,l+1} W_{\mathfrak{p}}^{(\mathfrak{pm})} \pmod{\mathfrak{p}}$.

Recalling that $E^* \equiv E \equiv -\partial(g_d) \pmod{\mathfrak{p}}$, we get

$$H \equiv \partial(g_d) F \pmod{\mathfrak{p}}. \tag{5.9}$$

In particular, the last congruence implies that H has $\mathfrak{p}$-integral u-series expansion at ∞.

Since both sides of (5.9) are congruent mod $\mathfrak{p}$, we have

$$w(\overline{H}) = w(\overline{\partial(g_d) F}). \tag{5.10}$$

Note that the weight of $\partial(g_d)F$ is kq^d+2. We claim that $w(\overline{\partial(g_d)F}) = kq^d+2$. By Proposition 5.6, it suffices to show that $\widetilde{\partial(g_d)F}$ does not vanish at least at one supersingular point of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$.

The vanishing of the function $\widetilde{g_d}$ at all supersingular points of $X(1)_{A/\mathfrak{p}}$ implies that the function $\widetilde{\partial(g_d)}$ does not vanish at any supersingular point of $X(1)_{A/\mathfrak{p}}$ (cf. Theorem 4.2(3)). Since all the supersingular points of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$ lie above the supersingular points of $X(1)_{A/\mathfrak{p}}$, the function $\widetilde{\partial(g_d)}$ does not vanish at any supersingular point of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$. On the other hand, since $w(\overline{F}) = (k-1)(q^d-1)+k$, the function $\widetilde{F}$ does not vanish at least at one supersingular point of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$ (cf. Proposition 5.6). Thus $\widetilde{\partial(g_d)} \cdot \widetilde{F}$ does not vanish at least at one supersingular point of $X_1^\Delta(\mathfrak{m})_{A/\mathfrak{p}}$, and the claim $w(\overline{\partial(g_d)F}) = kq^d+2$ follows.

Since the weight of H is $(k-1)q^d+3$, we conclude that

$$w(\overline{H}) \le (k-1)q^d+3 < kq^d+2 = w(\overline{\partial(g_d)F}),$$

which contradicts (5.10). Therefore, we must have $\beta = -\alpha$. $\square$

Remark 5.9. In the above argument, we have used the equality $\widetilde{\partial(g_d)F} = \widetilde{\partial(g_d)} \cdot \widetilde{F}$. This follows from Remark 5.4 and the equality $\overline{\partial(g_d)F} = \overline{\partial(g_d)} \cdot \overline{F}$ (where $\overline{X}$ refers to the reduction of the u-series expansion of X modulo $\mathfrak{p}$).

6. Counterexamples

In this section, we shall show that the assumption $w(\bar{F}) = (k-1)(q^d-1)+k$ is necessary in Theorem 1.3 and Theorem 1.5.

6.1. Eigenforms for $W_{\mathfrak{p}}^{(\mathfrak{pm})}$

Recall that $\mathfrak{p}$ denotes a prime ideal of A generated by a monic irreducible polynomial $\pi := \pi(T) \in A$ of degree d and $\mathfrak{m}$ denotes an ideal of A generated by a monic polynomial $m := m(T) \in A$ such that $(\mathfrak{p}, \mathfrak{m}) = 1$ (i.e., $\pi \nmid m$). We now discuss the existence of eigenforms for the $W_{\mathfrak{p}}^{(\mathfrak{pm})}$-operator.

For $f \in M_{k,l}^1(\Gamma_0(\mathfrak{m}))$, we have $f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) = \pi^{k/2} f(\pi z) \in M_{k,l}^1(\Gamma_0(\mathfrak{pm}))$. By [Vin14, Proposition 3.3], we get $v_{\mathfrak{p}}(f(\pi z)) = v_{\mathfrak{p}}(f|V_{\mathfrak{p}}) \ge v_{\mathfrak{p}}(f)$. This implies that $f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \equiv 0 \pmod{\mathfrak{p}}$ when f has $\mathfrak{p}$-integral u-series expansion at ∞.

Lemma 6.1. *If $f \in M_{k,l}^1(\Gamma_0(\mathfrak{m}))$, then*

(1) $(f + f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})} = f + f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$,
(2) $(f - f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))|_{k,l} W_{\mathfrak{p}}^{(\mathfrak{pm})} = -(f - f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))$.

Proof.

$$\begin{aligned}(f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))|_{k,l}W_{\mathfrak{p}}^{(\mathfrak{pm})} &= (f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))|_{k,l}\left(\begin{smallmatrix}\pi & b\\ \pi m & d\pi\end{smallmatrix}\right)\\ &= f|_{k,l}\left(\begin{smallmatrix}\pi & b\\ \pi m & d\pi\end{smallmatrix}\right) \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\pi & b\\ \pi m & d\pi\end{smallmatrix}\right)\\ &= f|_{k,l}\left(\begin{smallmatrix}1 & b\\ m & d\pi\end{smallmatrix}\right)\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \pm f|_{k,l}\left(\begin{smallmatrix}\pi & b\\ m & d\end{smallmatrix}\right)\left(\begin{smallmatrix}\pi & 0\\ 0 & \pi\end{smallmatrix}\right)\\ &= f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \pm f \quad \square\end{aligned}$$

Note that the eigenvectors $f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$ are oldforms.

6.2. *Prototype for a counterexample*

Suppose there exists $f \in M_{k,l}^1(\Gamma_0(\mathfrak{m}))$ with $\mathfrak{p}$-integral u-series expansion at ∞ such that $\Theta f \equiv fE \pmod{\mathfrak{p}}$. By definition, we have $f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \in M_{k,l}^1(\Gamma_0(\mathfrak{pm}))$. Clearly,

$$f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right) \equiv f \pmod{\mathfrak{p}}. \tag{6.1}$$

The above congruence shows that $w(\bar{F}) < (k-1)(q^d-1)+k$, where F is as in Proposition 3.8, corresponding to $f \pm f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$.

By (6.1), we obtain

$$\Theta(f + f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)) \equiv \Theta f \equiv fE \equiv fE^* \equiv (f \mp f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))E^* \pmod{\mathfrak{p}}. \tag{6.2}$$

According to Lemma 6.1 and Proposition 3.3, the modular forms $f + f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$ and $(f \mp f|_{k,l}\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right))E^*$ have the same (resp., opposite) sign under the action of $W_{\mathfrak{p}}^{(\mathfrak{pm})}$. So, the existence of such a function f implies that the assumption on the weight filtration of F is necessary in Theorem 1.3 and Theorem 1.5.

6.3. *Counterexamples*

Here, we shall produce some Drinfeld modular forms f satisfying $\Theta f \equiv fE \pmod{\mathfrak{p}}$, so that we can apply the above recipe to produce counterexamples.

- An easy computation shows that $\partial_{q^2-1}\Delta = 0$, i.e., $\Theta\Delta + (q^2-1)E\Delta = 0$. Hence, $\Theta\Delta \equiv \Delta E \pmod{\mathfrak{p}}$. Taking $f = \Delta$ in the previous section, we conclude that the assumption $w(\overline{F}) = (k-1)(q^d-1)+k$ in Theorem 1.3 is necessary.
  Note that the weight of Δ is q^2-1 and the type is 0. Since $q > 2$, q^2-1 can never be 2. So, this example does not contradict Corollary 1.4.
- Let $\mathfrak{m}$ be as in Theorem 1.5. Consider any non-zero Drinfeld modular form $g \in M_{k,l}(\Gamma_0(\mathfrak{m}))$ with $\mathfrak{p}$-integral u-series expansion at ∞. A straightforward calculation shows that $g^{q^i}\Delta \in M^1_{kq^i+q^2-1,l}(\Gamma_0(\mathfrak{m}))$ and $\partial(g^{q^i}\Delta) = 0$, for $i \geq 1$. Taking $f = g^{q^i}\Delta$ in the previous section, we conclude that the assumption $w(\overline{F}) = (k-1)(q^d-1)+k$ in Theorem 1.5 is necessary.

In [BP11], the authors have produced an example to demonstrate the necessity of the assumption on the weight filtration in their theorem. In the function field case, we are able to produce infinitely many examples.

Acknowledgments

The authors are grateful to the anonymous referee for his/her valuable mathematical comments, suggestions which improved the readability of this article. The authors thank Prof. Dr. Gebhard Böckle for informing us about the article [Hat20] and Prof. Shin Hattori, Prof. Cécile Armana for helpful discussions. The authors also thank Dr. Pradipto Banerjee for his valuable suggestions.

The first author thanks University Grants Commission, India for the financial support provided in the form of Research Fellowship to carry out this research work at IIT Hyderabad. The second author's research was partially supported by the SERB grant MTR/2019/000137.

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