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One-class genera of maximal integral quadratic forms

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ABSTRACT

Suppose Q is a definite quadratic form on a vector space V over some totally real field $K \neq \mathbb{Q}$. Then the maximal integral $\mathbb{Z}_K$-lattices in (V, Q) are locally isometric everywhere and hence form a single genus. We enumerate all orthogonal spaces (V, Q) of dimension at least 3, where the corresponding genus of maximal integral lattices consists of a single isometry class. It turns out, there are 471 such genera. Moreover, the dimension of V and the degree of K are bounded by 6 and 5 respectively. This classification also yields all maximal quaternion orders of type number one.

1. Introduction

Let K be a totally real number field and $\mathbb{Z}_K$ its maximal order. Two definite quadratic forms over $\mathbb{Z}_K$ are said to be in the same genus if they are locally isometric everywhere. Each genus is the disjoint union of finitely many isometry classes. The genera which consist of a single isometry class are precisely those lattices for which the local-global principle holds. These genera have been under study for many years. In a large series of papers [37–43], Watson classified all such genera in the case $K = \mathbb{Q}$ in three and more than five variables. He also produced partial results in the four and five dimensional cases.

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Assuming the Generalized Riemann Hypothesis, Voight classified the one-class genera in two variables [34, Theorem 8.6]. Recently, Lorch and the author [14] reinvestigated Watson's classification with the help of a computer using the mass formula of Smith, Minkowski and Siegel. We filled in the missing dimensions 4 and 5 and corrected some errors in Watson's tables.

In the case $K \neq \mathbb{Q}$, the local factors in the mass formula of Smith, Minkowski and Siegel [32] are not known in all cases. However, good bounds on these local factors are due to Pfeuffer [22]. Using these bounds, he showed that one-class genera can only occur in at most 32 variables. Moreover Pfeuffer gave explicit upper bounds on the degrees and discriminants of all possible base fields that can afford one-class genera.

If we restrict ourselves to maximal integral lattices, the local factors are known in all cases by the work of Shimura [31]. These results have been recently proved again by Gan, Hanke and Yu in [6] using Bruhat–Tits theory. Their proof is based on results of Gross [7] which builds upon the fundamental work of Prasad [25]. Using this mass formula, Hanke classified the one-class genera of maximal integral lattices over $K = \mathbb{Q}$ (see [8]). The current paper extends this classification to all totally real number fields K. In a future publication, David Lorch and the author will classify all one-class genera over totally real number fields in at least three variables by combining the methods of Watson and Pfeuffer. However, the complete classification will be much more tedious since the local mass factors are not known in all cases. More information of the classification of genera with small class numbers is given by R. Scharlau in Section 2.5 of [27].

The article is organized as follows. Section 2 recalls some basic definitions of quadratic forms over number fields. Section 3 gives the mass formula of Shimura and some consequences for one-class genera. The possible base fields K that can give rise to one-class genera of maximal integral lattices are enumerated in Section 4. Section 5 gives algorithms to perform the enumeration of these genera. In Section 6 we recall some connections between quadratic forms and quaternion algebras. Finally, the last section summarizes the results.

2. Preliminaries

Throughout the paper, let K be an algebraic number field of degree $n \geqslant 2$ and let V be an m-dimensional K-space. Further let $Q : V \to K$ be a quadratic form. The orthogonal group of the quadratic space (V, Q) will be denoted by $\mathrm{O}(V, Q)$ and $\mathrm{SO}(V, Q) = \{\varphi \in \mathrm{O}(V, Q) \mid \det(\varphi) = 1\}$ denotes the special orthogonal group.

The quadratic form Q is isometric to a diagonal form $Q' := \langle a_1, \dots, a_m\rangle$ where $Q'(x) = \sum_{i=1}^{m} a_i x_i^2$. We will always assume that Q is definite, i.e. K is totally real and each a_i is totally positive.

The discriminant of Q is $\operatorname{disc}(Q) = (-1)^{m(m-1)/2} \cdot \prod a_i$. It is unique up to multiplication by $(K^*)^2$. Further, for each prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ let $c_{\mathfrak{p}}(Q) = \prod_{i<j} (\frac{a_i, a_j}{\mathfrak{p}})$ be the Hasse invariant of Q at $\mathfrak{p}$. Here $(\frac{a,b}{\mathfrak{p}}) \in \{\pm 1\}$ denotes the usual Hilbert symbol of

Table 1
Definition of $t_\mathfrak{p}(Q)$.

m	Additional condition	$\omega_\mathfrak{p}(Q)$	$t_\mathfrak{p}(Q)$
odd	$v_\mathfrak{p}(d)$ even	-1	I
odd	$v_\mathfrak{p}(d)$ odd	$+1$	II_+
odd	$v_\mathfrak{p}(d)$ odd	-1	II_-
even	$d \in (K_\mathfrak{p}^*)^2$	-1	I
even	$d \notin (K_\mathfrak{p}^*)^2$ and $\mathfrak{p}$ does not ramify in E_Q/K	-1	II
even	$d \notin (K_\mathfrak{p}^*)^2$ and $\mathfrak{p}$ ramifies in E_Q/K	$+1$	III_+
even	$d \notin (K_\mathfrak{p}^*)^2$ and $\mathfrak{p}$ ramifies in E_Q/K	-1	III_-

$(a,b) \in K^2$ at $\mathfrak{p}$. It takes the value 1 if and only if $ax^2 + by^2 = z^2$ admits a non-trivial solution over the completion $K_\mathfrak{p}$ of K at $\mathfrak{p}$.

It is well known that the isometry class of the definite quadratic space (V,Q) is uniquely determined by m, $\operatorname{disc}(Q)$ and the finite set of prime ideals $\mathfrak{p}$ for which $c_\mathfrak{p}(Q) = -1$ (see for example [19, Remark 66:5]). The same is true if one replaces the Hasse invariants by the Witt invariants $\omega_\mathfrak{p}(Q)$ as defined in [28]:

$$\omega_\mathfrak{p}(Q) = \begin{cases} c_\mathfrak{p}(Q) & \text{if } m \equiv 1,2 \bmod 8,\\ c_\mathfrak{p}(Q)\cdot\left(\frac{-1,-1}{\mathfrak{p}}\right) & \text{if } m \equiv 5,6 \bmod 8,\\ c_\mathfrak{p}(Q)\cdot\left(\frac{-1,\operatorname{disc}(Q)}{\mathfrak{p}}\right) & \text{if } m \equiv 0,3 \bmod 8,\\ c_\mathfrak{p}(Q)\cdot\left(\frac{-1,-\operatorname{disc}(Q)}{\mathfrak{p}}\right) & \text{if } m \equiv 4,7 \bmod 8. \end{cases} \tag{1}$$

If $\operatorname{disc}(Q) \notin (K^*)^2$, let E_Q denote the field $K(\sqrt{\operatorname{disc}(Q)})$. Given any extension of number fields E/K, we denote by $d_{E/K}$ and $N_{E/K}$ its relative discriminant and norm respectively. Further, $d_K = d_{K/\mathbb{Q}}$ denotes the absolute discriminant of K.

Given the dimension m, the discriminant $d := \operatorname{disc}(Q)$ (viewed as an element of $K_\mathfrak{p}^*$) and the Witt invariant $\omega_\mathfrak{p}(Q)$, we define the local type $t_\mathfrak{p}(Q) \in \{0, \mathrm{I}, \mathrm{II}, \mathrm{II}_\pm, \mathrm{III}_\pm\}$ of Q at $\mathfrak{p}$ similar to Hanke in [8]. Let $v_\mathfrak{p}$ denote the usual valuation of $K_\mathfrak{p}$. Then the symbol $t_\mathfrak{p}(Q)$ is nonzero if and only if one of cases from Table 1 holds.

Definition 2.1. A lattice $L \subset V$ is a finitely generated $\mathbb{Z}_K$-submodule of V that contains a basis of V. It is said to be maximal integral if $Q(L) \subseteq \mathbb{Z}_K$ and $Q(L') \nsubseteq \mathbb{Z}_K$ for each lattice $L' \supsetneq L$.

Two lattices L, L' in V are said to be isometric if there exists some isometry $g \in \mathrm{O}(V,Q)$ such that $g(L) = L'$. The set of all isometries from L to itself is called the automorphism group $\operatorname{Aut}(L)$ of L.

Given a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ we write $V_\mathfrak{p}$ and $L_\mathfrak{p}$ for the completions $V \otimes_K K_\mathfrak{p}$ and $L \otimes_{\mathbb{Z}_K} \mathbb{Z}_{K_\mathfrak{p}}$ respectively. The lattices L, L' are said to be in the same genus if for each prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ there exists some $g_\mathfrak{p} \in \mathrm{O}(V_\mathfrak{p}, Q)$ such that $g_\mathfrak{p}(L_\mathfrak{p}) = L'_\mathfrak{p}$.

Clearly, each genus Λ decomposes into several isometry classes represented by $L_1, L_2, \ldots, L_h$ say. The number of classes h is always finite and is called the class number of Λ (see for example [19, Theorem 103:4]). Further we define

$$\operatorname{mass}(\Lambda) = \sum_{i=1}^{h} \frac{1}{\# \operatorname{Aut}(L_i)}$$

to be the mass of Λ.

3. The mass formula

Let (V, Q) be a definite quadratic space over some totally real number field K of degree n. Further let m denote the dimension of V and set $r = \lfloor m/2 \rfloor$.

Definition 3.1. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ and $q = N_{K/\mathbb{Q}}(\mathfrak{p}) = \#(\mathbb{Z}_K/\mathfrak{p})$ its norm. Then the local mass factor $\lambda_{\mathfrak{p}}(Q)$ is defined as follows.

m	$t_{\mathfrak{p}}(Q)$	$\lambda_{\mathfrak{p}}(Q)$
–	0	1
$2r+1$	I	$\frac{q^{m-1}-1}{2(q+1)}$
$2r+1$	$\mathrm{II}_{\pm}$	$\frac{q^r+\omega_{\mathfrak{p}}(G)}{2}$

m	$t_{\mathfrak{p}}(Q)$	$\lambda_{\mathfrak{p}}(Q)$
$2r$	I	$\frac{(q^{r-1}-1)(q^r-1)}{2(q+1)}$
$2r$	II	$\frac{(q^{r-1}+1)(q^r+1)}{2(q+1)}$
$2r$	$\mathrm{III}_{\pm}$	$1/2$

Proposition 3.2. *The set Λ of all maximal integral lattices in (V, Q) form a single genus.*

Proof. See for example [19, Theorem 91:2]. □

The following result is an explicit version of the mass formula of Smith, Minkowski and Siegel [32] where the local factors $\lambda_{\mathfrak{p}}$ are given by the work of Shimura [31, Theorem 5.8] and also Gan, Hanke and Yu [6, Proposition 2.13].

Theorem 3.3. *Let Λ be the genus of all maximal integral lattices in (V, Q). If $m \geqslant 3$, then*

$$2 \cdot \operatorname{mass}(\Lambda) = \tau(G) \cdot \gamma_G^n \cdot d_K^{\dim G/2} \cdot L(G) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q)$$

where

$$\tau(G) = 2 \quad \textit{is the Tamagawa number of } G := \mathrm{SO}(V, Q),$$

$$L(G) = \begin{cases} \prod_{i=1}^{r} \zeta_K(2i) & \textit{if } m \textit{ is odd},\\ \zeta_K(r) \cdot \prod_{i=1}^{r-1} \zeta_K(2i) & \textit{if } m \textit{ is even and } \operatorname{disc}(Q) \in (K^*)^2,\\ \frac{\zeta_{E_Q}(r)}{\zeta_K(r)} \cdot N_{K/\mathbb{Q}}(d_{E_Q/K})^{r-1/2} \cdot \prod_{i=1}^{r-1} \zeta_K(2i) & \textit{otherwise}, \end{cases}$$

denotes the L-series attached to G,

$$\dim(G) = r\bigl(2r-(-1)^m\bigr) \quad \textit{is the dimension of } G,$$

$$\gamma_G = \begin{cases} \frac{\prod_{i=1}^{r}(2i-1)!}{(2\pi)^{r(r+1)}} & \textit{if } m \textit{ is odd},\\ \frac{(r-1)!\cdot\prod_{i=1}^{r-1}(2i-1)!}{(2\pi)^{r^2}} & \textit{if } m \textit{ is even}.\end{cases}$$

Note that the formula given in [6] looks much neater than the one above since it uses values of L-series at negative integers. However there are two reasons to state the formula as above. First of all, the L-series involved might have zeros at some negative integers in which case one has to use the first non-vanishing coefficient of some Taylor series expansion. Secondly, we will need to find good lower bounds for the mass and therefore for the product $L(G)\cdot\prod_{\mathfrak p}\lambda_{\mathfrak p}(Q)$. This is much easier when $L(G)$ only depends on values of L-series at positive integers. In fact, the classification of all genera of maximal integral lattices with class number one is based on the following observation.

Proposition 3.4. *Suppose the notation of Theorem 3.3.*

1. *If Λ has class number one then $(2\cdot\mathrm{mass}(\Lambda))^{-1}\in\mathbb{Z}$.*
2. *If $\lambda_{\mathfrak p}(Q)<1$ then $\lambda_{\mathfrak p}(Q)=1/2$ and*
   - *m is even, $\mathrm{disc}(Q)\notin(K^*)^2$ and $\mathfrak p$ ramifies in E_Q or*
   - *$m\leqslant 4$ and $N_{K/\mathbb{Q}}(\mathfrak p)=2$.*
3. *If $\lambda_{\mathfrak p}(Q)\notin\mathbb{Z}$ then $2\lambda_{\mathfrak p}(Q)\in\mathbb{Z}$ and*
   - *m is even, $\mathrm{disc}(Q)\notin(K^*)^2$ and $\mathfrak p$ ramifies in E_Q or*
   - *$2\in\mathfrak p$.*
4. *Let k be the number of prime ideals in $\mathbb{Z}_K$ of norm 2. Then*

$$L(G)\cdot\prod_{\mathfrak p}\lambda_{\mathfrak p}(Q) > \begin{cases}(2/3)^k & \textit{if } m=3,\\ (8/15)^k & \textit{if } m=4,\\ 1 & \textit{if } m\geqslant 5.\end{cases}$$

5. *If m is odd and Λ has class number one then*

$$\bigl(\gamma_G^n\cdot d_K^{\dim G/2}\cdot L(G)\bigr)^{-1}\cdot 2^{\ell-1}\in\mathbb{Z}$$

   where ℓ denotes the number of prime ideals of $\mathbb{Z}_K$ that contain 2.

Proof. Suppose Λ consists of the isometry class of a single lattice L. Then $(2\cdot\mathrm{mass}(\Lambda))^{-1}=\#\mathrm{Aut}(L)/2$ is integral since $-\mathrm{id}_L$ is always an isometry of L. The next two assertions follow from the definition of the local factors $\lambda_{\mathfrak p}(Q)$ and the last statement is a reformulation of the third.

The fourth claim is clearly true if either $m\geqslant 5$ is odd or $m>4$ is even and $\mathrm{disc}(Q)$ is a square since then $L(G)>1$ and $\prod_{\mathfrak p}\lambda_{\mathfrak p}(Q)\geqslant 1$. If either $m=3$ or $m=4$ and

$\operatorname{disc}(Q) \in (K^*)^2$ then

$$L(G) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q) \geqslant \zeta_K(2) \cdot (1/2)^k > \left(1 - 2^{-2}\right)^{-k} \cdot (1/2)^k = (2/3)^k.$$

So we may now assume that $m \geqslant 4$ is even and $\operatorname{disc}(Q) \notin (K^*)^2$. Further let t denote the number of prime ideals that are ramified in E_Q/K. If $m = 4$ then

$$\begin{aligned} L(G) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q) &\geqslant \zeta_E(2) \cdot \left(N_{K/\mathbb{Q}}(d_{E_Q/K})^{3/2} \cdot 2^{-t}\right) \cdot 2^{-k} \\ &> \left(1 - 2^{-4}\right)^{-k} \cdot 2^{-k} = (8/15)^k. \end{aligned}$$

Similarly, if $m \geqslant 6$ then

$$L(G) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q) \geqslant \frac{\zeta_K(2)}{\zeta_K(m/2)} \cdot \zeta_E(m/2) \cdot \frac{N_{K/\mathbb{Q}}(d_{E_Q/K})^{(m-1)/2}}{2^t} > 1. \qquad \square$$

Note that if m is odd and fixed, the last statement of the previous proposition is a very strong restriction on the field K.

4. Restricting the possible base fields

The enumeration of all base fields K that can possibly give rise to one-class genera of all maximal integral lattices is based on the following Odlyzko type bounds.

Theorem 4.1. *Let K be a totally real number field of degree $n \geqslant 2$. Let $B(n)$ and $B'(n)$ be defined by*

n	2	3	4	5	6	7	8	9	10	$\geqslant 11$
$B(n)$	2.236	3.659	5.189	6.809	8.182	11.051	11.385	12.869	12.985	14.083
$B'(n)$	2.828	5.289	6.727	9.599	11.098	12.460	13.779	15.000	15.093	16.204

Then $d_K^{1/n} \geqslant B(n)$. Moreover, if $\mathbb{Z}_K$ contains an ideal of norm 2, then $d_K^{1/n} \geqslant B'(n)$. The bounds $B(n)$ and $B'(n)$ are sharp for $n \leqslant 9$ and $n \leqslant 8$ respectively.

Proof. The bounds for $n \leqslant 9$ follow from Voight's tables [35]. The other values for $B(n)$ have been computed by Martinet in [15]. The values for $B'(n)$ for $n \geqslant 10$ are given by Brueggeman and Doud in [3]. $\square$

Let Λ be the genus of all maximal integral lattices in a definite quadratic space (V, Q) of dimension m over some totally real number field K of degree n.

Proposition 4.2. *Suppose Λ has class number one and $K \neq \mathbb{Q}$. If $m = 2r + 1 \geqslant 3$ is odd, then $m \in \{3, 5\}$. Moreover:*

1. *If $m = 3$ then $K = \mathbb{Q}(\sqrt{d}\,)$ with*

$$d \in \{2, 3, 5, 6, 7, 13, 15, 17, 21, 29, 33, 41, 65, 69, 77, 137\}$$

or $K = \mathbb{Q}(\theta_\ell)$ is the maximal totally real subfield of the ℓ-th cyclotomic field $\mathbb{Q}(\zeta_\ell)$ with $\ell \in \{7, 9, 15, 20, 21, 24\}$ or $K \cong \mathbb{Q}[x]/(f(x))$ where $f(x)$ is one of

$$\begin{array}{cccc}
x^3 - x^2 - 3x + 1 & x^4 - x^3 - 3x^2 + x + 1 & x^4 - 5x^2 - x + 1 & x^5 - 5x^3 - x^2 + 3x + 1 \\
x^3 - x^2 - 4x - 1 & x^4 - x^3 - 5x^2 + 2x + 4 & x^4 - 5x^2 + 1 & x^5 - 2x^4 - 3x^3 + 5x^2 + x - 1 \\
x^3 - 4x - 1 & x^4 - 2x^3 - 3x^2 + 2x + 1 & x^4 - x^3 - 9x^2 + 4x + 16 & x^5 - 5x^3 + 4x - 1 \\
x^3 - x^2 - 4x + 3 & x^4 - x^3 - 4x^2 + x + 2 & x^4 - x^3 - 5x^2 + 5x + 1 & x^5 - x^4 - 5x^3 + 3x^2 + 5x - 2 \\
x^3 - x^2 - 4x + 2 & x^4 + 2x^3 - 7x^2 - 8x + 1 & x^4 - 5x^2 + 2 & x^5 - 6x^3 + 8x - 1 \\
x^3 - x^2 - 4x + 1 & x^4 - x^3 - 4x^2 + 2x + 1 & x^4 - 6x^2 - 3x + 3 & x^5 - 6x^3 - x^2 + 8x + 3 \\
x^3 - x^2 - 6x + 7 & x^4 - x^3 - 5x^2 - x + 1 & x^4 - 2x^3 - 5x^2 + x + 2 & x^5 - 2x^4 - 4x^3 + 7x^2 + 3x - 4 \\
x^3 - x^2 - 5x + 4 & x^4 - 9x^2 + 4 & x^4 - 7x^2 - 6x + 1 & x^5 - 2x^4 - 4x^3 + 4x^2 + 3x - 1 \\
x^3 - 7x - 5 & x^4 - 6x^2 - 4x + 2 & x^4 - x^3 - 5x^2 + 2x + 1 & x^6 - x^5 - 5x^4 + 4x^3 + 5x^2 - 2x - 1 \\
x^3 - x^2 - 7x + 8 & x^4 - 2x^3 - 3x^2 + 4x + 1 & x^4 - x^3 - 6x^2 - x + 1 & x^6 - 2x^5 - 4x^4 + 6x^3 + 4x^2 - 3x - 1 \\
x^3 - 6x - 1 & x^4 - x^3 - 6x^2 + x + 1 & x^4 - 7x^2 + 2 & x^6 - 3x^5 - 2x^4 + 9x^3 - x^2 - 4x + 1 \\
x^4 - 4x^2 - x + 1 & x^4 - 2x^3 - 6x^2 + 7x + 11 & x^4 - x^3 - 6x^2 + 7x + 1 & x^6 - 3x^5 - 3x^4 + 10x^3 + 3x^2 - 6x + 1
\end{array}$$

2. *If $m = 5$ then $K = \mathbb{Q}(\sqrt{5}\,)$.*

Proof. If $m = 3$ let k be the number of prime ideals of $\mathbb{Z}_K$ with norm 2. Otherwise set $k = 0$. By Proposition 3.4 and the assumption that Λ has class number one, it follows that

$$1 \geqslant 2 \cdot \operatorname{mass}(\Lambda) > 2 \cdot \gamma_G^n \cdot d_K^{r(2r+1)/2} \cdot (2/3)^k$$

and therefore the root discriminant $d_K^{1/n}$ is bounded above by

$$d_K^{1/n} < \left(\gamma_G \cdot 2^{1/n} \cdot (2/3)^{k/n}\right)^{-2/(r(2r+1))}. \tag{2}$$

Let us first assume that $k \geqslant 1$. Then $m = 3$ and $d_K^{1/n} < (4\pi^2 \cdot 3/2)^{2/3} < 15.20$. By Theorem 4.1, this implies that $n \leqslant 10$. Thus Eq. (2) shows that in fact $d_K^{1/n} < (4\pi^2 \cdot (1/2)^{1/10} \cdot 3/2)^{2/3} < 14.51$ and thus $n \leqslant 8$.

Suppose now $k = 0$. Then by Eq. (2), we have

$$d_K^{1/n} < \gamma_G^{-2/(r(2r+1))}. \tag{3}$$

Since $K \neq \mathbb{Q}$ we have $d_K^{1/n} \geqslant \sqrt{5}$. The right hand side of Eq. (3) is strictly decreasing and the only cases where it is at most $\sqrt{5}$ are

r	1	2	3	4	5	6
$d_K^{1/n} >$	11.60	6.35	4.37	3.33	2.70	2.26

In particular, $n \leqslant 8$ by Theorem 4.1.

Since Voight's tables [35] list all totally real number fields of degree $n \leqslant 8$ with root discriminant $\leqslant 15$, we can now simply enumerate all pairs (r, K) such that inequality (2) holds. As it turns out, there are only 218 such pairs. Among those, only the 71 pairs given above satisfy the fourth condition of Proposition 3.4. $\square$

If m is even, the factor $L(G)$ does not solely depend on K but also on E_Q and therefore on the discriminant of Q. Thus we cannot get as sharp bounds on the base field K as in Proposition 4.2. But we still can enumerate a finite set of fields K that needs to be checked explicitly.

Proposition 4.3. *Suppose Λ has class number one and $K \neq \mathbb{Q}$. If $m = 2r \geqslant 4$ is even, then $m \leqslant 14$. Further:*

1. *If $m = 4$ then $d_K^3 \leqslant \frac{1}{2} \cdot (2\pi)^{4n} \cdot (15/8)^k$ where k denotes the number of prime ideals of $\mathbb{Z}_K$ with norm 2. There are 249 such fields and the largest one has degree 7.*
2. *If $m = 6$ then $K = \mathbb{Q}(\sqrt{d}\,)$ with $\{d \in 2, 3, 5, 6, 7, 13, 17, 21, 29, 33, 37\}$ or $K = \mathbb{Q}(\theta_\ell)$ with $\ell \in \{7, 9, 15\}$ or $K \cong \mathbb{Q}[x]/(f(x))$ where $f(x)$ is one of $x^3 - x^2 - 3x + 1$, $x^3 - x^2 - 4x - 1$, $x^3 - 4x - 1$, $x^4 - x^3 - 3x^2 + x + 1$ or $x^4 - 6x^2 + 4$.*
3. *If $m = 8$ then $K = \mathbb{Q}(\sqrt{d}\,)$ with $d \in \{2, 3, 5, 13, 17\}$ or $K = \mathbb{Q}(\theta_\ell)$ with $\ell \in \{7, 9\}$.*
4. *If $m = 10$ then $K = \mathbb{Q}(\sqrt{2}\,)$ or $K = \mathbb{Q}(\sqrt{5}\,)$.*
5. *If $m \in \{12, 14\}$ then $K = \mathbb{Q}(\sqrt{5}\,)$.*

Proof. If $m = 4$ let k be the number of prime ideals of $\mathbb{Z}_K$ with norm 2, otherwise set $k = 0$. As in the proof of Theorem 4.2 we have

$$d_K^{1/n} < \left(\gamma_G \cdot 2^{1/n} \cdot (8/15)^{k/n}\right)^{-2/(r(2r-1))}. \tag{4}$$

Suppose first that $k \neq 0$. Then $m = 4$ and the above inequality implies that $d_K^{1/n} < (30\pi^4)^{1/3} < 14.30$. Thus Theorem 4.1 shows that $n \leqslant 8$. Suppose now that $k = 0$. Then by Eq. (4), we have

$$d_K^{1/n} < \gamma_G^{-2/(r(2r-1))}.$$

The right hand side of this equation is strictly decreasing. It is greater than $\sqrt{5}$ if and only if $r \leqslant 7$. Further, if $m = 4$ it takes the value $(2\pi)^{4/3} < 11.60$. Therefore $n \leqslant 8$. Since [35] lists all totally real number fields of degree $\leqslant 8$ and root discriminant $\leqslant 15$, we can now simply enumerate all pairs (r, K) such that inequality (4) holds. The result follows. $\square$

5. Enumerating the one-class genera

5.1. Odd dimensions

Suppose $m = 2r + 1 \geqslant 3$ is odd. Then the enumeration of all one-class genera of maximal integral forms is straightforward. For each of the possible pairs (m, K) from Proposition 4.2, we apply the following algorithm.

Algorithm 5.1.
Input: Let K be a totally real number field of degree n and let $m \geqslant 3$ be odd.
Output: A set $\mathcal{L}$ of representatives for the one-class genera of maximal integral lattices in definite orthogonal K-spaces of dimension m.

1. Evaluate $s(m,K) := 2 \cdot \gamma(G)^n \cdot L(G)$ and set $\mathcal{L} = \emptyset$.
2. Compute all possible combinations S of local symbols such that
$$\Bigl(s(m,K) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q)\Bigr)^{-1} \in \mathbb{Z}.$$
3. For each such combination S compute the set D_S of all possible values for $\operatorname{disc}(Q)$ (up to squares).
4. For each such combination S and each $d \in D_S$ do
   (a) Turn the set S into the Hasse invariants using Table 1 and Eq. (1).
   (b) Check if there exists an m-dimensional quadratic K-space with discriminant d and the requested local Hasse invariants.
   (c) If such a space exists, construct a maximal integral lattice L in it.
   (d) Let Λ be the genus of L. If $\#\operatorname{Aut}(L) = 1/\operatorname{mass}(\Lambda)$, include L into $\mathcal{L}$.
5. Return $\mathcal{L}$.

We give some comments and hints how to do the above steps.

1. If $t_{\mathfrak{p}}(Q) \neq 0$ then $\lambda_{\mathfrak{p}}(Q) \geqslant (N_{K/\mathbb{Q}}(\mathfrak{p})^r - 1)/2 > 1$ except if $r = 1$ and $N_{K/\mathbb{Q}}(\mathfrak{p}) \in \{2,3\}$. So we only have to consider finitely many prime ideals $\mathfrak{p}$ for which $t_{\mathfrak{p}}(Q) \neq 0$.
2. The set D_S can be computed as follows. By Dirichlet's unit theorem, the quotient $\{u \in \mathbb{Z}_K^* \mid u \text{ totally positive}\}/(\mathbb{Z}_K^*)^2$ is finite. Let $u_1, \ldots, u_s$ be a transversal and let $J = \prod_{t_{\mathfrak{p}}(Q)=\mathrm{II}_\pm} \mathfrak{p}$. Further, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ represent the ideal classes of $\mathbb{Z}_K$. We start with $D_S = \emptyset$. For each $1 \leqslant i \leqslant h$ we then check if $J\mathfrak{a}_i^2 = \alpha_i \mathbb{Z}_K$ for some $\alpha_i \in \mathbb{Z}_K$ such that $(-1)^r \cdot \alpha_i$ is totally positive. If such an α_i exists, we include $\{\alpha_i \cdot u_j \mid 1 \leqslant j \leqslant s\}$ into D_S.
3. In step 4(a) one has to evaluate several Hilbert symbols. A computationally efficient way to evaluate these symbols has been given in [36].
4. Step 4(b) is done as follows. By [19, 72:1] there exists a definite orthogonal K-space with discriminant d and given Hasse invariants if and only if the Hasse invariants are -1 at an even number of prime ideals.
5. Constructing a global space with the given invariants can be done by trial and error. Let $\mathcal{T}$ be a set of prime ideals of $\mathbb{Z}_K$ that includes the ideals for which $t_{\mathfrak{p}}(Q) \neq 0$. Then one tests quadratic forms
$$\langle a_1, \ldots, a_{m-1}, (-1)^r \cdot a_1 \cdots a_{m-1} \cdot d \rangle$$

where the a_i are totally positive generators of products of ideals in $\mathcal{T}$. If the set $\mathcal{T}$ is large enough, this will quickly produce a form Q that has the correct local Hasse invariants. The computation of a maximal integral lattice with respect to Q is then straightforward. Finally, the computation of the automorphism group of this lattice is done using the algorithm of Plesken and Souvignier [23].

If in step 4(d) equality did not hold, the genus Λ has been enumerated completely with Kneser's neighbor method. An explanation of this method is given by Schulze-Pillot in [29] as well as Hemkemeier and Scharlau in [9]. This cautionary check assures that we have evaluated the mass correctly and that we have constructed a maximal lattice in the correct orthogonal space.

5.2. Even dimensions

Suppose $m = 2r \geqslant 4$ is even. In the odd dimensional cases, one can reconstruct the possible values for $\operatorname{disc}(Q)$ by the local types $t_{\mathfrak{p}}(Q)$. If m is even however, we first have to compute all possible values for $\operatorname{disc}(Q)$. For this we compute all possible quadratic extensions E_Q/K.

Lemma 5.2. *Suppose $m \geqslant 4$ is even and the genus of all maximal integral lattices Λ of (V, Q) has class number one. If $m = 4$, let k denote the number of ideals of $\mathbb{Z}_K$ of norm 2 otherwise set $k = 0$. Further, for each prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ set $e_{\mathfrak{p}} = 2$ if $2 \in \mathfrak{p}$ and set $e_{\mathfrak{p}} = 1$ otherwise. If $\operatorname{disc}(Q) \notin (K^*)^2$ then*

$$\prod_{\mathfrak{p} \mid d_{E_Q/K}} \frac{N_{K/\mathbb{Q}}(\mathfrak{p})^{e_{\mathfrak{p}}(r-1/2)}}{2} \leqslant 2^{k-1} \cdot \gamma(G)^{-n} \cdot \zeta_K(r) \cdot d_K^{r(1-2r)/2} \cdot \prod_{i=1}^{r} \zeta_K(2i)^{-1}.$$

Proof. If $\mathfrak{p}$ ramifies in E_Q/K then $\mathfrak{p}^{e_{\mathfrak{p}}}$ divides $d_{E_Q/K}$ (see for example [30, Proposition III.13]). Thus it follows from Proposition 3.4 that

$$\begin{aligned} 1 &\geqslant 2 \cdot \operatorname{mass}(\Lambda) \\ &\geqslant 2 \cdot \gamma(G)^n \cdot d_K^{r(2r-1)/2} \cdot \prod_{i=1}^{r-1} \zeta_K(2i) \cdot \frac{\zeta_{E_Q}(r)}{\zeta_K(r)} \cdot 2^{-k} \prod_{\mathfrak{p} \mid d_{E_Q/K}} \frac{N_{K/\mathbb{Q}}(\mathfrak{p})^{e_{\mathfrak{p}}(r-1/2)}}{2}. \end{aligned}$$

The result follows since $\zeta_{E_Q}(r) \geqslant \zeta_K(2r)$. $\square$

The above lemma restricts the prime ideals of K that could possibly be ramified in E_Q/K to a finite set. By Class Field Theory, we can now construct all quadratic extensions E/K such that $d_{E/K}$ satisfies the inequality in Lemma 5.2. There are only finitely many such fields. Now if $E = E_Q$ then $E = K(\sqrt{\alpha})$ for some $\alpha \in K$ such that $(-1)^r \alpha$ is totally positive since $\alpha / \operatorname{disc}(Q) \in (K^*)^2$. By listing all such fields E, we have then effectively enumerated all possible discriminants $\operatorname{disc}(Q) \in K^*/(K^*)^2$ that can give rise to one-class genera of maximal integral lattices.

The computation of all one-class genera is now similar to the odd dimensional case.

Algorithm 5.3.
Input: Let K be a totally real number field of degree n and let $m \geqslant 4$ be even.
Output: A set $\mathcal{L}$ of representatives for the one-class genera of maximal integral lattices in definite orthogonal K-spaces of dimension m.

1. Set $\mathcal{L} = \emptyset$.
2. Compute the set D of possible nonsquare discriminants $\operatorname{disc}(Q)$ with Lemma 5.2. If $m \in 4\mathbb{Z}$, include 1 to D.
3. For all $d \in D$ do:
   (a) Compute all possible combinations S of local symbols such that

   $$\left(2 \cdot \gamma(G)^n \cdot L(G) \cdot \prod_{\mathfrak{p}} \lambda_{\mathfrak{p}}(Q)\right)^{-1} \in \mathbb{Z}.$$

   Note that $t_{\mathfrak{p}}(Q) \in \{\mathrm{III}_+, \mathrm{III}_-\}$ if and only if $d \neq 1$ and $\mathfrak{p} \mid d_{K(\sqrt{d})/K}$.
   (b) For each such set S do:
      i. Turn the set S into the Hasse invariants using Table 1 and Eq. (1).
      ii. Check if there exists an m-dimensional quadratic K-space with discriminant d and the requested local Hasse invariants.
      iii. If such a space exists, find a maximal integral lattice L in it.
      iv. Let Λ be the genus of L. If $\#\operatorname{Aut}(L) = 1/\operatorname{mass}(\Lambda)$ then include L into $\mathcal{L}$.
4. Return $\mathcal{L}$.

6. Quaternion orders

We first recall some basic properties of quaternion orders. More details can be found in the book of Vignéras [33] for example.

Let K be a number field. A quaternion algebra $\mathcal{Q}$ over K is a central simple K-algebra of dimension four. Every quaternion algebra $\mathcal{Q}$ admits a unique involution $^- : \mathcal{Q} \to \mathcal{Q}$ such that the reduced norm $\operatorname{nr}_{\mathcal{Q}/K}(x) := x\bar{x}$ and reduced trace $\operatorname{tr}_{\mathcal{Q}/K}(x) := x + \bar{x}$ are contained in K for all $x \in \mathcal{Q}$. The reduced norm is a quadratic form on $\mathcal{Q}$ with corresponding bilinear form

$$\mathcal{Q} \times \mathcal{Q} \to K, (x, y) \mapsto \operatorname{nr}_{\mathcal{Q}/K}(x + y) - \operatorname{nr}_{\mathcal{Q}/K}(x) - \operatorname{nr}_{\mathcal{Q}/K}(y) = \operatorname{tr}_{\mathcal{Q}/K}(x\bar{y}).$$

We say that $\mathcal{Q}$ is ramified at some place v of K if and only if the completion $\mathcal{Q}_v := \mathcal{Q} \otimes_K K_v$ of $\mathcal{Q}$ at v is a skew-field. The algebra $\mathcal{Q}$ is determined by its ramified places up to isomorphism and $\mathcal{Q}$ is said to be (totally) definite, if it is ramified at all infinite places of K. This is equivalent to say that $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$ is totally positive definite.

An order $O \subset \mathcal{Q}$ is a subring of $\mathcal{Q}$ that is also a $\mathbb{Z}_K$-lattice in $\mathcal{Q}$. The order O is said to be maximal if it is not contained in a larger one.

Finally, given a subset $S \subset \mathcal{Q}$ we denote by $S^0 = \{s \in S \mid \operatorname{tr}_{\mathcal{Q}/K}(s) = 0\}$ the set of all elements in S that have trace 0.

Lemma 6.1. *Let (V, Q) be a four-dimensional definite quadratic space over K such that $\operatorname{disc}(Q) \in (K^*)^2$.*

1. *There exists a definite quaternion algebra $\mathcal{Q}$ over K such that (V, Q) is isometric to $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$. Further, $\mathcal{Q}$ is unique up to isomorphism.*
2. *Each maximal order M in $\mathcal{Q}$ is a maximal integral lattice in $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$.*
3. *Two maximal orders in $\mathcal{Q}$ are conjugate in $\mathcal{Q}$ if and only if they are isometric lattices in $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$.*
4. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the prime ideals at which $\mathcal{Q}$ ramifies and let M be some maximal order in $\mathcal{Q}$. The genus Λ of all maximal integral lattices in $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$ has class number one if and only if every maximal order of $\mathcal{Q}$ is conjugate to M and*

$$\frac{2^n}{|\zeta_K(-1)|} = \#M^1 \cdot \prod_{i=1}^{s} \big(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1\big)$$

where $M^1 = \{x \in M \mid \operatorname{nr}_{\mathcal{Q}/K}(x) = 1\}$ denotes the norm one group of M.

Proof. A proof for the first assertion is for example given in [24, Propositions 1 and 4]. The second statement is clear from the local descriptions of maximal orders. For a proof of the third assertion, see for example [16, Corollary 4.4].

For the proof of the last statement, let H denote the number of isomorphism classes of finitely generated nonzero M-bimodules in $\mathcal{Q}$. Under the assumption that M is unique up to conjugacy, Eichler's mass formula (see for example [33, Corollaire V.2.3]) states that

$$\frac{H}{[M^* : \mathbb{Z}_K^*]} = 2^{1-n} \cdot \big|\zeta_K(-1)\big| \cdot h_K \cdot \prod_{i=1}^{s} \big(N_{K/\mathbb{Q}}(\mathfrak{p}) - 1\big)$$

where h_K denotes the class number of $\mathbb{Z}_K$. The quotient $\frac{H}{[M^* : \mathbb{Z}_K^*]}$ is related to the automorphism group of the lattice M in $(\mathcal{Q}, \operatorname{nr}_{\mathcal{Q}/K})$ by [16, Corollary 4.5] as follows

$$\frac{H}{[M^* : \mathbb{Z}_K^*]} = \frac{2^{s+1} \cdot h_K \cdot \#M^1}{\#\operatorname{Aut}(M)}.$$

The last two equations show that

$$\frac{1}{\#\operatorname{Aut}(M)} = 2^{-n} \cdot \big|\zeta_K(-1)\big| \cdot \frac{1}{\#M^1} \cdot \prod_{i=1}^{s} \frac{N_{K/\mathbb{Q}}(\mathfrak{p}) - 1}{2}.$$

Theorem 3.3 implies that Λ has class number one if and only if

$$\frac{2}{\#\operatorname{Aut}(M)} = 2^{1-2n} \cdot \zeta_K(-1)^2 \cdot \prod_{i=1}^{s} \frac{(N_{K/\mathbb{Q}}(\mathfrak{p})-1)^2}{2}$$

since the local type of the norm form $t_\mathfrak{p}(\mathrm{nr}_{\mathcal{Q}/K}) \in \{0, \mathrm{I}\}$ and it takes the value I if and only if $\mathcal{Q}$ ramifies at $\mathfrak{p}$. Combining the two equations for $\#\operatorname{Aut}(M)$ gives the result. $\square$

There is a similar correspondence for ternary lattices.

Lemma 6.2. *Let M, N be maximal orders in a definite quaternion algebra $\mathcal{Q}$.*

1. *Every isometry $\varphi : M^0 \to N^0$ (with respect to $\mathrm{nr}_{\mathcal{Q}^0/K}$) extends to an isometry $\psi : M \to N$ (with respect to $\mathrm{nr}_{\mathcal{Q}/K}$).*
2. *The orders of the automorphism groups satisfy $\#\operatorname{Aut}(M) = \#M^1 \cdot \#\operatorname{Aut}(M^0)$.*

Proof. Since the canonical involution is an isometry on M^0 with determinant -1, we may assume that $\det(\varphi) = 1$. By extension of scalars, φ is an isometry on $\mathcal{Q}^0$. Then $\psi : \mathcal{Q} \to \mathcal{Q}$, $\lambda + x \mapsto \lambda + \varphi(x)$ for all $\lambda \in K$ and $x \in \mathcal{Q}^0$ is the only isometry of determinant 1 that extends φ. It remains to show that $\psi(M) = N$. By [5, Appendix IV, Proposition 3], ψ is simply conjugation by some element in $\mathcal{Q}^*$. In particular, $\psi(M)$ is a maximal order that contains N^0. If $\mathcal{Q}$ ramifies at $\mathfrak{p}$ then $\mathcal{Q}_\mathfrak{p}$ has a unique maximal order which implies that $\varphi(M)_\mathfrak{p} = N_\mathfrak{p}$. If $\mathcal{Q}$ does not ramify at $\mathfrak{p}$ then without loss of generality $N_\mathfrak{p} = \mathbb{Z}_{K_\mathfrak{p}}^{2\times 2}$. Then $\varphi(M)_\mathfrak{p}$ contains $e := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $f := \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Hence the order $\varphi(M)_\mathfrak{p}$ must also contain the $\mathbb{Z}_{K_\mathfrak{p}}$-span of $\{e, f, ef, fe\}$ which is $N_\mathfrak{p}$. This proves the first claim.

For a proof of the second, let $\operatorname{Aut}^+(M) = \operatorname{Aut}(M) \cap \mathrm{SO}(\mathcal{Q}, \mathrm{nr}_{\mathcal{Q}/K})$ and $\operatorname{Aut}^+(M^0) = \operatorname{Aut}(M^0) \cap \mathrm{SO}(\mathcal{Q}^0, \mathrm{nr}_{\mathcal{Q}^0/K})$. Since the canonical involution induces isometries of determinant -1 on M and M^0, it suffices to show that $[\operatorname{Aut}^+(M) : \operatorname{Aut}^+(M^0)] = \#M^1$. From the first part of the proof we know that $\operatorname{Aut}^+(M^0)$ can be identified with the subgroup $S = \{\psi \in \operatorname{Aut}^+(M) \mid \psi(1) = 1\}$ of $\operatorname{Aut}^+(M)$. But S has index $\#M^1$ in $\operatorname{Aut}^+(M)$ by [16, Corollary 4.5]. $\square$

Lemma 6.3. *Let (V, Q) be a three-dimensional definite quadratic space over K such that $-\operatorname{disc}(Q) \in (K^*)^2$.*

1. *There exists a definite quaternion algebra $\mathcal{Q}$ over K such that (V, Q) is isometric to $(\mathcal{Q}^0, \mathrm{nr}_{\mathcal{Q}^0/K})$. Further, $\mathcal{Q}$ is unique up to isomorphism.*
2. *The trace zero submodule M^0 of any maximal order M in $\mathcal{Q}$ is a maximal integral lattice in $(\mathcal{Q}^0, \mathrm{nr}_{\mathcal{Q}^0/K})$.*
3. *Two maximal orders in $\mathcal{Q}$ are conjugate in $\mathcal{Q}$ if and only if their trace zero submodules are isometric lattices in $(\mathcal{Q}^0, \mathrm{nr}_{\mathcal{Q}^0/K})$.*

Proof. For the first claim, see for example [12, (6.20) and (6.21)]. The second assertion follows again from the local description of maximal orders. The third statement is an immediate consequence of Lemmas 6.1 and 6.2. □

Theorem 6.4. *Let $\mathcal{Q}$ be a definite quaternion algebra over K. Then the genus Λ of all maximal integral lattices in $(\mathcal{Q}^0, \mathrm{nr}_{\mathcal{Q}^0/K})$ has class number one if and only if all maximal orders in $\mathcal{Q}$ are conjugate.*

Proof. If Λ has class number one then all maximal orders in $\mathcal{Q}$ must be conjugate by the previous lemma. The converse follows again from comparing the mass formulas of Eichler and Shimura while using the identity $\#\operatorname{Aut}(M) = \#M^1 \cdot \#\operatorname{Aut}(M^0)$. □

Note that there are other maps between ternary quadratic forms and quaternion orders such that a similar statement as Theorem 6.4 holds. There is the correspondence of Brzeziński [4] which goes back to work of Peters [21], Brandt [2] and Latimer [13]. There is also the correspondence of Nipp [18] which generalizes a result of Pall [20]. These maps are in general not onto. Further, they do not map maximal integral forms to maximal orders. Hence we do not pursue these connections further.

7. Results

The enumeration of all one-class genera has been implemented by the author in Magma (see [1]). A summary of the results is given here.

Theorem 7.1. *Let (V, Q) be a definite quadratic space of dimension $m \geqslant 3$ over some totally real number field $K \neq \mathbb{Q}$. If the genus of all maximal integral lattices in (V, Q) has class number one then $m \leqslant 6$. Moreover:*

1. *If $m = 3$ then there are 402 such genera over 29 different fields. In 96 cases, $-\operatorname{disc}(Q)$ is a square.*
2. *If $m = 4$ then there are 67 such genera over 19 different fields. In 51 cases the discriminant of Q is a square.*
3. *If $m = 5$ then $K = \mathbb{Q}(\sqrt{5})$ and $(V, Q) \cong \langle 1, 1, 1, 1, 1\rangle$.*
4. *If $m = 6$ then $K = \mathbb{Q}(\sqrt{5})$ and $(V, Q) \cong \langle 1, 1, 1, 1, 1, (5 + \sqrt{5})/2\rangle$.*

The complete list of these genera can be obtained electronically from [10].

Almost all lattices in Theorem 7.1 are free. More precisely, the following is true.

Remark 7.2. The field $K = \mathbb{Q}(\sqrt{15})$ is the only base field with non-trivial class group that affords maximal integral lattices with class number one and rank at least 3. In fact, up to isometry there are four such lattices and all of them have exactly rank 3. These lattices can be constructed as follows.

Let $\mathcal{Q}$ be the definite quaternion algebra over K ramified at the prime ideals $\mathfrak{p}_2$ and $\mathfrak{p}_3$ of norm 2 and 3 respectively. Let M be a maximal order in $\mathcal{Q}$. It is unique up to isometry. Hence $(M^0, \mathrm{nr}_{\mathcal{Q}^0/K})$ is a maximal integral lattice with class number one by Lemma 6.4.

Let $u = \sqrt{15} + 4$ be a fundamental unit of $\mathbb{Z}[\sqrt{15}\,]$. Then $(\mathfrak{p}_3^a M, u^b/3^a \cdot \mathrm{nr}_{\mathcal{Q}^0/K})$ is maximal integral for all $a, b \in \{0, 1\}$. By comparing discriminants, we see that this gives 4 pairwise non-isomorphic lattices. Further, rescaling forms and lattices does not change class numbers. Since M^0 is free and $\mathfrak{p}_3$ generates the class group of $\mathbb{Z}[\sqrt{15}\,]$ we see that $\mathfrak{p}_3 M^0$ cannot be free by Steinitz' theorem.

Thus the classification in Theorem 7.1 contains only two non-free lattices.

By [14] or [8], there exist 9 one-class genera of maximal integral ternary lattices in rational orthogonal spaces (V, Q) such that $-\operatorname{disc}(Q) \in \mathbb{Q}^2$. Together with Theorems 6.4 and 7.1 we have thus just proven the following.

Theorem 7.3. *Let $\mathcal{Q}$ be a definite quaternion algebra over some number field K (possibly $\mathbb{Q}$) such that $\mathcal{Q}$ contains up to conjugacy only one maximal order M. Let c denote the ideal class number of M, i.e. the number of finitely generated nonzero M-left modules in $\mathcal{Q}$. Then $c \in \{1, 2, 4, 8\}$. More precisely*:

1. *There are 49 algebras with $c = 1$ and 53 algebras with $c = 2$. They have been enumerated by Voight and the author in [11].*
2. *If $c = 4$ then either $K = \mathbb{Q}(\sqrt{7}\,)$ and $\mathcal{Q}$ ramifies at the two prime ideals over 2 and 7 or $K = \mathbb{Q}(\sqrt{21}\,)$ and $\mathcal{Q}$ ramifies at the two prime ideals over 3 and 7.*
3. *If $c = 8$ then $K = \mathbb{Q}(\sqrt{15}\,)$ and $\mathcal{Q}$ ramifies at the two prime ideals over 2 and 3.*

We now give some more details for the one-class genera for base fields $K \neq \mathbb{Q}$. Let Λ be a one-class genus of maximal integral lattices in (V, Q). Further let L be a representative of Λ.

It is clear that for a Galois extension $K/\mathbb{Q}$, the Galois group acts on the set of definite quadratic spaces (V, Q) over K and thus on the set of genera of maximal integral lattices. Further, the action preserves class numbers. Thus it suffices to give only one representative for each orbit.

Dimension 4

Among the 102 quaternion algebras in Theorem 7.3, only 56 satisfy the condition of Lemma 6.1 part 4. In 51 cases, the center of these algebras is a proper extension of $\mathbb{Q}$ which agrees with Theorem 7.1. They are listed in the following table. For each algebra $\mathcal{Q}$ we give its ramified prime ideals where $\mathfrak{p}_q$ denotes some prime ideal of $\mathbb{Z}_K$ over the rational prime q. Further we give the isomorphism type of $\mathrm{Aut}(M)$ for some maximal order M in $\mathcal{Q}$ as well as the length of the Galois orbit.

K	Ram. primes	#orbit	Aut(M)
$\mathbb{Q}(\sqrt{5})$	–	1	$\mathrm{Aut}(\mathbb{H}_4)$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_2, \mathfrak{p}_5$	1	$(\pm D_{10})^2$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_2, \mathfrak{p}_{11}$	2	D_8^2
$\mathbb{Q}(\sqrt{2})$	–	1	$\mathrm{Aut}(\mathbb{D}_4(\sqrt{2}))$
$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_2, \mathfrak{p}_7$	2	D_{16}^2
$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	$(\pm S_3)^2$
$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_2, \mathfrak{p}_5$	1	C_2^4
$\mathbb{Q}(\sqrt{3})$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	$(D_{12}.2)^2$
$\mathbb{Q}(\sqrt{3})$	$\mathfrak{p}_2, \mathfrak{p}_{13}$	2	C_2^4
$\mathbb{Q}(\sqrt{13})$	–	1	$\mathrm{Aut}(\mathbb{D}_4^{\sim})$
$\mathbb{Q}(\sqrt{13})$	$\mathfrak{p}_2, \mathfrak{p}_3$	2	D_8^2
$\mathbb{Q}(\sqrt{17})$	–	1	$\mathrm{Aut}((2\mathbb{A}_2)^{\sim})$
$\mathbb{Q}(\sqrt{6})$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	D_8^2
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_2$	1	$\mathrm{Aut}(\mathbb{D}_4)$
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_7$	1	$(\pm D_{14}) \wr C_2$
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_{13}$	3	$\pm D_{14}^2$
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_{29}$	3	$\pm D_6^2$
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_{43}$	3	$C_2^4.C_2$
$\mathbb{Q}(\theta_9)$	$\mathfrak{p}_3$	1	D_{36}^2
$\mathbb{Q}(\theta_9)$	$\mathfrak{p}_{19}$	3	$C_2^4.C_2$
$\mathbb{Q}(\theta_9)$	$\mathfrak{p}_{37}$	3	C_2^3
$x^3 - x^2 - 3x + 1$	$\mathfrak{p}_2$	1	$\mathrm{Aut}(\mathbb{D}_4)$
$x^3 - x^2 - 3x + 1$	$\mathfrak{p}_5$	1	$\pm S_3^2$
$x^3 - x^2 - 3x + 1$	$\mathfrak{p}_{13}$	1	C_2^3
$x^3 - x^2 - 4x - 1$	$\mathfrak{p}_5$	3	$\pm S_3^2$
$x^3 - x^2 - 4x - 1$	$\mathfrak{p}_{13}$	1	C_2^3
$x^3 - 4x - 1$	$\mathfrak{p}_2$	1	$C_2^3.C_2$
$x^3 - x^2 - 4x + 2$	$\mathfrak{p}_2$	1	$\pm S_3^2$
$x^3 - x^2 - 4x + 1$	$\mathfrak{p}_3$	1	$C_2^4.C_2$
$x^4 - x^3 - 3x^2 + x + 1$	–	1	$\mathrm{Aut}(\mathbb{H}_4)$
$x^4 - 4x^2 - x + 1$	–	1	$\mathrm{Aut}(\mathbb{D}_4^{\sim})$
$\mathbb{Q}(\theta_{20})$	$\mathfrak{p}_2, \mathfrak{p}_5$	1	C_2^4
$\mathbb{Q}(\theta_{24})$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	C_2^4
$x^4 - x^3 - 4x^2 + x + 2$	–	1	$\mathrm{Aut}((2\mathbb{A}_2)^{\sim})$
$x^5 - 5x^3 - x^2 + 3x + 1$	$\mathfrak{p}_5$	1	$\pm S_3^2$

Here S_i, C_i and D_i denote the symmetric, cyclic and dihedral groups of order i respectively. Further $\mathbb{A}$ and $\mathbb{D}$ stand for the root lattices of the corresponding type. The four one-class genera of unimodular lattices over totally real quadratic fields have been found by Scharlau in [26]. In this paper, he gives explicit constructions for these lattices and the corresponding automorphism groups $\mathrm{Aut}(\mathbb{H}_4) = (\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)) : 2$ in the notation of [17], $\mathrm{Aut}(\mathbb{D}_4(\sqrt{2})) = \mathrm{Aut}(\mathbb{D}_4).2$, $\mathrm{Aut}(\mathbb{D}_4^{\sim})$ and $\mathrm{Aut}((2\mathbb{A}_2)^{\sim})$.

The 16 quaternary one-class genera for which $\mathrm{disc}(Q)$ is not a square, are organized in 11 Galois orbits. For each such orbit, the following table lists the base field K, the determinant of L (which is a free $\mathbb{Z}_K$-module), the set of primes for which the Hasse invariant $c_{\mathfrak{p}}(Q)$ is -1, the orbit length and finally the isomorphism type of $\mathrm{Aut}(L)$.

K	$\det(L)$	$c_\mathfrak{p} = -1$	#orbit	$\mathrm{Aut}(L)$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_2^2\mathfrak{p}_5$	$\mathfrak{p}_2, \mathfrak{p}_5$	1	$(C_2 \wr S_3) \times C_2$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_2^2\mathfrak{p}_3$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	$D_{10} \times C_2^2$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_3\mathfrak{p}_5$	–	1	$(\pm A_5) \times \{\pm 1\}$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_3\mathfrak{p}_5$	$\mathfrak{p}_3, \mathfrak{p}_5$	1	$(\pm D_{10}) \times (\pm S_3)$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_{29}$	–	2	$(\pm A_5) \times \{\pm 1\}$
$\mathbb{Q}(\sqrt{5})$	$\mathfrak{p}_5\mathfrak{p}_{41}$	$\mathfrak{p}_5, \mathfrak{p}_{41}$	2	$S_3 \times C_2^2$
$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_2^2\mathfrak{p}_3$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	$D_{16} \times C_2^2$
$\mathbb{Q}(\sqrt{2})$	$\mathfrak{p}_2^2\mathfrak{p}_{17}$	$\mathfrak{p}_2, \mathfrak{p}_{17}$	2	$S_3 \times C_2^2$
$\mathbb{Q}(\sqrt{3})$	$\mathfrak{p}_2^2\mathfrak{p}_5$	$\mathfrak{p}_2, \mathfrak{p}_5$	1	C_2^3
$\mathbb{Q}(\theta_7)$	$\mathfrak{p}_3\mathfrak{p}_7$	$\mathfrak{p}_2, \mathfrak{p}_3$	1	$C_2 \wr S_3$
$\mathbb{Q}(\theta_9)$	$\mathfrak{p}_3\mathfrak{p}_{71}$	$\mathfrak{p}_2, \mathfrak{p}_{71}$	3	$\pm S_4$

Table 2
Distribution of the 402 ternary one-class genera among the 29 different base fields.

K	n_K	K	n_K	K	n_K
x^2-5	64	x^3-x^2-2x+1	38	x^4-4x^2-x+1	6
x^2-2	48	x^3-3x-1	28	x^4-5x^2+5	4
x^2-3	34	x^3-x^2-3x+1	12	x^4-4x^2+1	4
x^2-13	31	x^3-x^2-4x-1	14	$x^4-x^3-4x^2+x+2$	2
x^2-17	15	x^3-4x-1	6	x^4-6x^2-4x+2	8
x^2-21	16	x^3-x^2-4x+3	6	$x^4-2x^3-3x^2+4x+1$	4
x^2-6	12	x^3-x^2-4x+2	4	$x^5-5x^3-x^2+3x+1$	2
x^2-7	12	x^3-x^2-4x+1	4	$x^5-2x^4-3x^3+5x^2+x-1$	2
x^2-33	8	$x^4-x^3-3x^2+x+1$	6	x^5-5x^3+4x-1	4
x^2-15	4	$x^4-x^3-4x^2+4x+1$	4		

Dimensions 5 and 6

In dimensions 5 and 6 we have $K = \mathbb{Q}(\sqrt{5})$. Let M be a maximal order in $\mathcal{Q} = (\frac{-1,-1}{K})$, the quaternion algebra over K ramified only at the infinite places. By Theorem 7.1, the two quadratic spaces of dimension 5 or 6 over K that admit one-class genera of maximal integral lattices are $(\mathcal{Q}, \mathrm{nr}_{\mathcal{Q}/K}) \perp \langle 1\rangle$ and $(\mathcal{Q}, \mathrm{nr}_{\mathcal{Q}/K}) \perp (\mathbb{Q}(\zeta_5), N_{\mathbb{Q}(\zeta_5)/K})$.

The lattices $L_1 = M \perp \langle 1\rangle$ and $L_2 = M \perp \mathbb{Z}[\zeta_5]$ are maximal integral in these spaces respectively. The corresponding automorphism groups are

$$\begin{aligned}\mathrm{Aut}(L_1) &= \mathrm{Aut}(\mathbb{H}_4) \times \{\pm 1\} = \bigl(\bigl(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)\bigr) : 2\bigr) \times \{\pm 1\} \quad \text{and}\\ \mathrm{Aut}(L_2) &= \mathrm{Aut}(\mathbb{H}_4) \times \{\pm D_{10}\} = \bigl(\bigl(\mathrm{SL}_2(5) \circ \mathrm{SL}_2(5)\bigr) : 2\bigr) \times \{\pm D_{10}\}.\end{aligned}$$

Dimension 3

Instead of listing all 402 one-class genera of maximal integral ternary lattices, we only list the number n_K of genera for each base field K in Table 2.

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