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Cycle integrals of a sesqui-harmonic Maass form of weight zero [☆]

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ABSTRACT

Borchers–Zagier bases of the spaces of weakly holomorphic modular forms of weights $1/2$ and $3/2$ share the Fourier coefficients which are traces of singular moduli. Recently, Duke, Imamoglu, and Tóth have constructed a basis of the space of weight $1/2$ mock modular forms, each member in which has Zagier’s generating series of traces of singular moduli as its shadow. They also showed that Fourier coefficients of their mock modular forms are sums of cycle integrals of the j -function which are real quadratic analogues of singular moduli. In this paper, we prove that the Fourier coefficients of a basis of the space of weight $3/2$ mock modular forms are sums of cycle integrals of a sesqui-harmonic Maass form of weight zero whose image under hyperbolic Laplacian is the j -function. Furthermore, we express these sums as regularized inner products of weakly holomorphic modular forms of weight $1/2$.

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1. Introduction

Fourier coefficients of half-integral weight modular forms carry rich information of number theoretic objects. Classical results include representation numbers of quadratic forms, partition functions and class numbers of imaginary quadratic number fields and more recent results connect the Fourier coefficients of half integral weight cusp forms with central values of quadratic twists of modular L -functions (see [25,26,30]). The new development of the theory of more general automorphic forms has revealed that Fourier coefficients of weakly holomorphic modular forms and harmonic weak Maass forms of half-integral weights also convey various arithmetic properties (see [6,11,14,18,19,32] for example and see [16] for more references).

For instance, weakly holomorphic modular forms of weights $1/2$ and $3/2$ discussed by Borcherds [3] and Zagier [32] are related to traces of singular moduli of the classical j -invariant

$$j(\tau) = q^{-1} + 744 + 196\,884q + 21\,493\,760q^2 + \cdots,$$

where $q := e^{2\pi i\tau} =: e(\tau)$ and $\tau \in \mathbb{H}$, the upper half of the complex plane. It follows from the theory of complex multiplication that if τ is an imaginary quadratic irrational number, then $j(\tau)$ is an algebraic integer, called singular modulus. Let $J = j - 744$, the normalized Hauptmodul for $\Gamma = PSL_2(\mathbb{Z})$ and for $k \in \mathbb{Z} + \frac{1}{2}$, let $M_k^!$ denote the space of weakly holomorphic modular forms of weight k on $\Gamma_0(4)$, in which each form satisfies Kohnen's plus space condition, that is, its Fourier expansion is of the form $\sum a(n)q^n$ where $a(n)$ is non-zero only for integers n satisfying $(-1)^{k-1/2}n \equiv 0, 1 \pmod{4}$. Throughout, $D, d \equiv 0, 1 \pmod{4}$. We also let \mathcal{Q}_d denote the set of integral binary quadratic forms $Q = [a, b, c] = aX^2 + bXY + cY^2$ with discriminant $d = b^2 - 4ac$ that are positive definite if $d < 0$. For each Q of negative discriminant d , there is a corresponding CM point τ_Q , the unique root of $Q(\tau, 1) = 0$ in \mathbb{H} . As $J(\tau_Q)$ depends only on equivalence class of Q under the usual linear fractional action of Γ , we may define the twisted trace of singular moduli, for each fundamental discriminant $D > 0$ and the associated genus character χ , by

$$\mathrm{Tr}_{d,D}(J) = \frac{1}{\sqrt{D}} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \frac{J(\tau_Q)}{|\Gamma_Q|} \quad (Dd < 0), \quad (1.1)$$

where Γ_Q is the group of automorphs of Q . Zagier [32] showed the modularity of the generating series of traces of singular moduli by proving that for each $D > 0$

$$g_D(\tau) = q^{-D} - 2\delta_{D,\square} - \sum_{d < 0} \mathrm{Tr}_{d,D}(J)q^{|d|} \in M_{3/2}^!, \quad (1.2)$$

where $\delta_{D,\square} = 1$ if D is a square and 0 otherwise. He also established a duality relation

$$f_d(\tau) = q^d + \sum_{D>0} \text{Tr}_{d,D}(J)q^D \in M_{1/2}^!, \quad (1.3)$$

for each $d < 0$. Earlier in [3], Borcherds proved that $f_d(\tau)$ ($d < 0$) and $f_0(\tau) := \theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2}$ have an interpretation in terms of infinite product expansions of certain meromorphic modular forms for Γ . In fact, $\{f_d \mid d \leq 0\}$ and $\{g_D \mid D > 0\}$ form bases for $M_{1/2}^!$ and $M_{3/2}^!$, respectively.

This result has inspired many works on Borcherds–Zagier type duality for coefficients of modular forms and harmonic Maass forms as well as connection between coefficients of automorphic forms and traces of singular moduli [6,10,11,13,24]. These papers often use Maass–Poincaré series that was developed earlier in [21]. Recently, Duke, Imamoglu and Tóth [18] have extended Borcherds’ basis $\{f_d \mid d \leq 0\}$ for $M_{\frac{1}{2}}^!$ to a basis $\{f_d\}$ for $\mathbb{M}_{\frac{1}{2}}^!$, where $\mathbb{M}_k^!$ denotes the space of weight k mock modular forms on $\Gamma_0(4)$ satisfying the plus space condition. For each $d > 0$, they constructed a unique mock modular form $f_d(\tau)$ of weight $1/2$ with shadow $g_d(\tau)$ having a Fourier expansion of the form

$$f_d(\tau) = \sum_{D>0} a(D, d)q^D, \quad (1.4)$$

which implies that f_d can be completed to a harmonic weak Maass form by addition of the non-holomorphic Eichler integral of g_d . Furthermore, they showed that for non-square dD with both d and D positive, the Fourier coefficients $a(D, d)$ of q^D in $f_d(\tau)$ are sums of cycle integrals of J -function which are real analogues of traces of singular moduli:

$$\begin{aligned} a(D, d) &= \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{\Gamma_Q \backslash S_Q} J(\tau) \frac{d\tau}{Q(\tau, 1)} \\ &:= \text{Tr}_{d,D}(J) \quad (D > 0, d > 0), \end{aligned} \quad (1.5)$$

where the geodesic S_Q is defined to be the oriented semi-circle $a|\tau|^2 + b\text{Re } \tau + c = 0$, directed counterclockwise if $a > 0$ and clockwise if $a < 0$.

In [24], the authors extended Zagier’s basis $\{g_D \mid D > 0\}$ for $M_{\frac{3}{2}}^!$ to a basis $\{g_D\}$ for $\mathbb{M}_{\frac{3}{2}}^!$ satisfying that for each $D \leq 0$, $g_D(\tau)$ is a unique mock modular form of weight $3/2$ with shadow $f_D(\tau)$ having a Fourier expansion of the form

$$g_D(\tau) = \sum_{d \leq 0} b(D, d)q^{|d|}. \quad (1.6)$$

The Fourier coefficients $b(D, d)$ in (1.6) can be interpreted in terms of class numbers and modified traces of cycle integrals of a sesqui-harmonic Maass form. A sesqui-harmonic Maass form may not be annihilated by hyperbolic Laplacian, but transferred to a weakly holomorphic modular form by the operator (see Section 2 or [5] for a precise definition).

We first construct an infinite family of sesqui-harmonic Maass forms of weight 0 whose images under the hyperbolic Laplacian Δ_0 are the modular functions $j_m = q^{-m} + O(1)$, which form a basis for the space of weight 0 weakly holomorphic modular forms.

Theorem 1.1. *For each positive integer m , let $\hat{J}_m(\tau, s)$ be the sesqui-harmonic Maass form defined in (2.10). If we set $\hat{J}_m(\tau) := \hat{J}_m(\tau, 1)$, then we have*

$$\Delta_0(\hat{J}_m) = -j_m - 24\sigma(m),$$

where $\sigma(m)$ denotes the sum of positive divisors of m .

We now represent the Fourier coefficients of $g_D(\tau)$ in terms of traces of cycle integrals of $\hat{J}_1(\tau)$.

Theorem 1.2. *Let d and D be negative discriminants. If d is fundamental and dD is non-square, then the Fourier coefficient $b(D, d)$ of $q^{|d|}$ in the mock modular form $g_D(\tau)$ with shadow $f_D(\tau)$ given in (1.6) satisfies*

$$b(D, d) = 192\pi H(|d|)H(|D|) - 8\sqrt{dD} \operatorname{Tr}_{d,D}^*(\hat{J}(\tau)). \quad (1.7)$$

Here $H(n)$ is the Hurwitz–Kronecker class number and $\operatorname{Tr}_{d,D}^*(\hat{J}(\tau))$ is the modified trace defined in (3.6) with $\hat{J}(\tau) = \hat{J}_1(\tau)$.

Zagier’s result on traces of singular moduli was generalized by Bruinier and Funke [11] and Alfes and Ehlen [2], in which the generating function for CM traces of a harmonic weak Maass form of weight 0 is shown to be a mock modular form of weight $3/2$ whose shadow is a theta series of weight $1/2$. In [12,18], as alluded to earlier, the generating function for traces of cycle integrals of a harmonic weak Maass form of weight 0 is proven to be a mock modular form of weight $1/2$ with shadow a weight $3/2$ weakly holomorphic modular form. In fact, [12] addresses the square case of what was done in [18]. The function $\hat{J}(\tau)$ in Theorem 1.2 is not harmonic, but a sesqui-harmonic Maass form. Hence Theorem 1.2 shows that the generating function for traces of cycle integrals of a sesqui-harmonic Maass form of weight 0 is a mock modular form of weight $3/2$ with shadow a weight $1/2$ weakly holomorphic modular form.

Furthermore, coefficients $b(D, d)$ of mock modular forms $g_D(\tau)$ ($D < 0$) can be expressed as regularized inner products of weakly holomorphic modular forms f_D and f_d . Following [4], we may define the regularized Petersson inner product of two modular forms f and g of weight k for $\Gamma_0(4)$ with singularities only at the cusps by

$$(f, g)^{\text{reg}} = \lim_{Y \rightarrow \infty} \int_{\mathcal{F}_4(Y)} f(\tau) \overline{g(\tau)} y^k \frac{dx dy}{y^2}, \quad (1.8)$$

where $\mathcal{F}_4(Y)$ is the standard truncated fundamental domain for $\Gamma_0(4)$ obtained by removing Y -neighborhoods of the cusps. In [19], Duke, Imamoglu and Tóth showed that for a positive fundamental discriminant D ,

$$(g_0, g_D)^{reg} = -\frac{3}{4} \cdot \frac{\log \varepsilon_D}{\pi \sqrt{D}} \cdot h(D)$$

where ε_D is the smallest unit > 1 and $h(D)$ is the class number of the quadratic field $\mathbb{Q}(\sqrt{D})$ in the narrow sense. They also showed that for positive discriminants d and D with non-square dD ,

$$(g_d, g_D)^{reg} = -\frac{3}{4} \text{Tr}_{d,D}(J).$$

We establish analogous results for inner products of two different f_d 's.

Theorem 1.3. *Let d and D be negative discriminants. If d is fundamental and dD is non-square, then we have:*

- (1) $(f_D, f_d)^{reg} = -12\sqrt{Dd} \text{Tr}_{d,D}^*(\hat{J}(\tau)) + 288\pi H(|D|)H(|d|).$
- (2) $(f_0, f_d)^{reg} = -24\pi H(|d|).$

Theorem 1.3 implies $b(D, d) = \frac{2}{3}(f_D, f_d)^{reg}$ when $Dd > 0$ is not a square. The regularized inner product $(f_0, f_d)^{reg}$ was also discussed by Borcherds in [4, Corollary 9.6] in a vector valued form.

This paper is organized as follows. In Section 2, we present definitions of harmonic weak Maass forms, sesqui-harmonic Maass forms and mock modular forms along with examples in terms of Niebur Poincaré series and prove Theorem 1.1. In Section 3, we use Niebur Poincaré series to prove Theorem 1.2. In Section 4, we compare the family of harmonic weak Maass forms of weight 2 having modular function j_m 's as their shadows with a subset of the basis for the space of weight 2 harmonic weak Maass forms found in [20]. Finally, in Section 5, we prove Theorem 1.3.

2. Maass forms and mock modular forms

Throughout, $\tau = x + iy$ with $y > 0$. Let $k \in \frac{1}{2}\mathbb{Z}$ and N be a positive integer with $4|N$ when k is not an integer. In case k is an integer, we assume k is even. A *harmonic weak Maass form* h of weight k for $\Gamma_0(N)$ is as defined in [11], a smooth function on \mathbb{H} which satisfies:

- (i) $h|_k \gamma = h$ for all $\gamma \in \Gamma_0(N)$, where $|_k$ is the weight k slash operator,
- (ii) $\Delta_k(h) = 0$, where $\Delta_k = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) + ik y(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$,
- (iii) h has at most exponential growth at all cusps.

Let $H_k(N)$ denote the space of weight k harmonic weak Maass forms for $\Gamma_0(N)$. There is an antilinear differential operator $\xi_k := 2iy^k \frac{\bar{\partial}}{\partial \bar{\tau}}$ which plays important roles in the theory of harmonic weak Maass forms and more general automorphic forms. Considering

$$\Delta_k = -\xi_{2-k} \circ \xi_k, \quad (2.1)$$

we find that if $h \in H_k(N)$, then $\xi_k(h)$ is a weight $2-k$ weakly holomorphic modular form. In general, a weight k harmonic weak Maass form has a Fourier expansion at infinity of the form

$$h(\tau) = \sum_{n \gg -\infty} c_h^+(n) q^n + c_h^-(0) y^{1-k} + \sum_{0 \neq n \ll \infty} c_h^-(n) \Gamma(1-k, -4\pi n y) q^n$$

so that

$$\xi_k(h) = (1-k) \overline{c_h^-(0)} - \sum_{0 \neq n \ll \infty} \overline{c_h^-(n) (-4\pi n)^{1-k}} q^{-n}, \quad (2.2)$$

where $\Gamma(a, x)$ is the incomplete gamma function. Following Zagier [33], we call the holomorphic part $h^+(\tau) := \sum_{n \gg -\infty} c_h^+(n) q^n$ a *mock modular form of weight k* and $g(\tau) := \xi_k(h)$ the *shadow* of the mock modular form f . The non-holomorphic part $h^- := h - h^+$ is then the Eichler integral of the weakly holomorphic modular form $g(\tau)$.

Now, we define another family of automorphic forms that includes harmonic weak Maass forms. A *sesqui-harmonic Maass form* \mathcal{F} of weight k for $\Gamma_0(N)$ is a smooth function on \mathbb{H} which satisfies:

- (i) $\mathcal{F}|_k \gamma = \mathcal{F}$ for all $\gamma \in \Gamma_0(N)$,
- (ii) $\Delta_{k,2}(\mathcal{F}) = 0$, where $\Delta_{k,2} = \xi_k \circ \Delta_k = -\xi_k \circ \xi_{2-k} \circ \xi_k = \Delta_{2-k} \circ \xi_k$,
- (iii) \mathcal{F} has at most exponential growth at all cusps.

A sesqui-harmonic Maass form was first defined in [5], while first examples had appeared in [17,18] and the underlying method had been utilized earlier in [7–9]. See [5] for more information.

It is well-known that the space of weight 0 weakly holomorphic modular forms on Γ has a unique basis $\{j_m \mid m \geq 0\}$ where j_m is uniquely determined by having the form $j_m = q^{-m} + O(q)$. For example, $j_0 = 1$ and $j_1 = j - 744 = J$, the Hauptmodul for Γ . This basis can be extended to a basis for the space of weight 0 harmonic weak Maass forms using Poincaré series. If $\phi : \mathbb{R}^+ \rightarrow \mathbb{C}$ is a smooth function satisfying $\phi(y) = O_\varepsilon(y^{1+\varepsilon})$ for any $\varepsilon > 0$ and Γ_∞ is the subgroup of translations of Γ , then the general Poincaré series

$$G_m(\tau, \phi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m \operatorname{Re}(\gamma \tau)) \phi(\operatorname{Im}(\gamma \tau)) \quad (m \in \mathbb{Z}) \quad (2.3)$$

is a smooth Γ -invariant function on \mathbb{H} . Let

$$\phi_{m,s}(y) = \begin{cases} 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y), & m \neq 0, \\ y^s, & m = 0, \end{cases} \quad (2.4)$$

with I_ν the usual I -Bessel function. Then the Niebur Poincaré series $G_m(\tau, s) := G_m(\tau, \phi_{m,s})$ is defined for $\operatorname{Re} s > 1$ [29] and satisfies

$$\Delta_0 G_m(\tau, s) = (s - s^2)G_m(\tau, s). \quad (2.5)$$

As each $G_m(\tau, s)$ when $m \neq 0$ has an analytic continuation to $\operatorname{Re}(s) > 1/2$, we obtain an infinite family of weight 0 harmonic weak Maass forms $G_m(\tau, 1)$ ($m \in \mathbb{Z}$). The Fourier coefficients of $G_m(\tau, s)$ can be written in terms of I , J , K -Bessel functions and generalized Kloosterman sum which is defined by

$$K_k(m, n; c) := \begin{cases} \sum_{v(c)^*} e\left(\frac{m\bar{v}+nv}{c}\right), & \text{if } k \in \mathbb{Z}, \\ \sum_{v(c)^*} \left(\frac{c}{v}\right)^{2k} \varepsilon_v^{2k} e\left(\frac{m\bar{v}+nv}{c}\right), & \text{if } k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}, \end{cases} \quad (2.6)$$

where the sum runs through the primitive residue classes modulo c and $v\bar{v} \equiv 1 \pmod{c}$. The function $G_0(\tau, s)$ is the usual Eisenstein series whose Fourier expansion is given by

$$\begin{aligned} G_0(\tau, s) &= y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \\ &\quad + \sum_{n \neq 0} \frac{2y^{1/2}\pi^s}{\Gamma(s)\zeta(s)} |n|^{s-1/2} \sigma_{1-2s}(|n|) K_{s-\frac{1}{2}}(2\pi|n|y) e(nx), \end{aligned} \quad (2.7)$$

where $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ and $\sigma_a(n)$ is the sum of a th powers of positive divisors of n (for example, see [23]). For $m \neq 0$, the Fourier expansion of $G_m(\tau, s)$ is given in [21] by

$$\begin{aligned} G_m(\tau, s) &= 2\pi|m|^{1/2}y^{1/2}I_{s-1/2}(2\pi|m|y)e(mx) + \frac{4\pi|m|^{1-s}\sigma_{2s-1}(|m|)}{(2s-1)\xi(2s)}y^{1-s} \\ &\quad + 4\pi|m|^{1/2}y^{1/2} \sum_{n \neq 0} |n|^{1/2} c_m(n, s) K_{s-\frac{1}{2}}(2\pi|n|y) e(nx), \end{aligned} \quad (2.8)$$

where

$$c_m(n, s) = \sum_{c>0} c^{-1} K_0(m, n, c) \begin{cases} I_{2s-1}(4\pi\sqrt{|mn|}c^{-1}), & mn < 0, \\ J_{2s-1}(4\pi\sqrt{|mn|}c^{-1}), & mn > 0. \end{cases}$$

Following [18], we define for positive integer m

$$j_m(\tau, s) := G_{-m}(\tau, s) - \frac{2m^{1-s}\sigma_{2s-1}(m)}{\pi^{-(s+1/2)}\Gamma(s+1/2)\zeta(2s-1)} G_0(\tau, s) \quad (2.9)$$

and apply Weil's bound to its Fourier coefficients so that we have an analytic continuation for $j_m(\tau, s)$ to $\operatorname{Re}(s) > 3/4$. As $s \rightarrow 1$, the pole in $G_0(\tau, s)$ cancels with the zero in the quotient multiplied to $G_0(\tau, s)$. Moreover, it follows from (2.8) and $j_m(\tau) = q^{-m} + O(q)$ that

$$j_m(\tau, 1) = G_{-m}(\tau, 1) - 24\sigma(m) = j_m(\tau),$$

because a bounded harmonic function is constant. (See [18,21,28,29] for more details and references for Niebur Poincaré series.)

Following ideas of [5] and [18], we construct sesqui-harmonic Maass forms $\hat{J}_m(\tau)$ satisfying $\Delta_0(\hat{J}_m(\tau)) = -j_m - 24\sigma(m)$. For positive integer m and $\operatorname{Re} s > 1$, define

$$\hat{J}_m(\tau, s) := \frac{\partial}{\partial s} G_{-m}(\tau, s) = G_{-m}\left(\tau, \frac{\partial}{\partial s} \phi_{-m,s}\right). \quad (2.10)$$

Then $\hat{J}_m(\tau, s)$ has an analytic continuation to $\operatorname{Re}(s) > 1/2$ as G_m has such a continuation. Hence it follows from (2.5) and $\Delta_0 \frac{\partial}{\partial s} = \frac{\partial}{\partial s} \Delta_0$ that

$$\Delta_0(\hat{J}_m(\tau)) = \Delta_0\left(\frac{\partial}{\partial s} G_{-m}(\tau, s)\right)\Bigg|_{s=1} = -G_{-m}(\tau, 1) = -j_m(\tau) - 24\sigma(m),$$

which proves Theorem 1.1.

By means of (2.1), we observe that $h_m^*(\tau) := \xi_0(\hat{J}_m(\tau))$ is a harmonic weak Maass form of weight 2 satisfying $\xi_2(h_m^*(\tau)) = j_m(\tau) + 24\sigma(m)$. In other words, the holomorphic part of $h_m^*(\tau)$ is a mock modular form of weight 2 with shadow $j_m(\tau) + 24\sigma(m)$. Recently, Duke, Imamoglu and Tóth [20] have found harmonic weak Maass forms $h_m(\tau)$ which have bounded holomorphic parts and satisfy $\xi_2(h_m) = \frac{1}{4\pi}(j_m + 24\sigma(m))$. This implies that for each positive integer m , $h_m^*(\tau)$ and $4\pi h_m(\tau)$ differ by a weakly holomorphic modular form. In fact, $h_m^*(\tau) = 4\pi h_m(\tau)$ and we confirm this in Section 4.

3. Proof of Theorem 1.2

For negative dD , $\operatorname{Tr}_{d,D}(J)$ is a Fourier coefficient of a weakly holomorphic modular form while, for positive discriminants D and d with dD non-square, $\operatorname{Tr}_{d,D}(J)$ is a Fourier coefficient of a harmonic weak Maass form. But when both D and d are negative discriminants, $\operatorname{Tr}_{d,D}(J)$ is simply zero, as seen below.

Suppose that D and d are discriminants and d is fundamental. For a quadratic form $Q = [a, b, c]$ with discriminant dD , define a genus character χ on $\Gamma \backslash Q_{dD}$ by [22]

$$\chi_d(Q) = \begin{cases} \left(\frac{d}{r}\right), & (a, b, c, d) = 1 \text{ and } (r, d) = 1 \text{ where } Q \text{ represents } r, \\ 0, & (a, b, c, d) > 1. \end{cases}$$

Then for a non-square dD , the twisted trace of a general Poincaré series defined in (2.3) is given by

$$\mathrm{Tr}_{d,D}(G_m(\tau, \phi)) = \frac{1}{2\pi} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{\Gamma_Q \backslash S_Q} G_m(\tau, \phi) \frac{d\tau}{Q(\tau, 1)}.$$

If we denote the subset of \mathcal{Q}_{dD} consisting of quadratic forms with $a > 0$ by \mathcal{Q}_{dD}^+ , then it follows from the proof of [18, Lemma 7] that

$$\begin{aligned} 2\pi \mathrm{Tr}_{d,D}(G_m(\tau, \phi)) &= \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) \int_{S_Q} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) \frac{d\tau}{Q(\tau, 1)} \\ &\quad + \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(-Q) \int_{S_{-Q}} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) \frac{d\tau}{-Q(\tau, 1)}. \end{aligned}$$

Since the genus character χ satisfies $\chi(-Q) = \chi_d(-Q) = \operatorname{sgn}(d)\chi_d(Q)$, we have

$$\begin{aligned} 2\pi \mathrm{Tr}_{d,D}(G_m(\tau, \phi)) &= \begin{cases} 2 \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) \int_{S_Q} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) \frac{d\tau}{Q(\tau, 1)}, & \text{if } d > 0, D > 0, \\ 0, & \text{if } d < 0, D < 0. \end{cases} \end{aligned}$$

Thus for negative discriminants D and d , $\mathrm{Tr}_{d,D}(G_m(\tau, \phi)) = 0$, and hence $\mathrm{Tr}_{d,D}(J(\tau)) = \mathrm{Tr}_{d,D}(G_{-1}(\tau, 1) - 24) = 0$.

It is then reasonable to define a modified trace $\mathrm{Tr}_{d,D}^*$ for each Poincaré series $G_m(\tau, \phi)$ by

$$\mathrm{Tr}_{d,D}^*(G_m(\tau, \phi)) = \frac{1}{\pi} \sum_{Q \in \Gamma_\infty \backslash \mathcal{Q}_{dD}^+} \chi(Q) \int_{S_Q} e(m \operatorname{Re} \tau) \phi(\operatorname{Im} \tau) \frac{d\tau}{Q(\tau, 1)} \quad (3.1)$$

so that $\mathrm{Tr}_{d,D}^*(G_m(\tau, \phi)) = \mathrm{Tr}_{d,D}(G_m(\tau, \phi))$ when both D and d are positive. From now to the end of this section, we assume both d and D are negative discriminants and d is fundamental with dD non-square. Using [18, Lemma 8], we obtain that

$$2\pi\sqrt{dD}\mathrm{Tr}_{d,D}^*(G_m(\tau, \phi)) = \sum_{0 < c \equiv 0 \pmod{4}} S_m(D, d; c) \Phi_m\left(\frac{2\sqrt{dD}}{c}\right),$$

where $\Phi_m(t) = \int_0^\pi \cos(2\pi m t \cos \theta) \phi(t \sin \theta) \frac{d\theta}{\sin \theta}$ for $t > 0$ and

$$S_m(D, d; c) = \sum_{\substack{b \pmod{c} \\ b^2 \equiv Dd \pmod{c}}} \chi\left(\left[\frac{c}{4}, b, \frac{b^2 - Dd}{c}\right]\right) e\left(\frac{2mb}{c}\right).$$

If we take $\phi = \phi_{m,s}$ defined in (2.4) for nonzero integer m , we then have

$$\begin{aligned}\Phi_m(t) &= \int_0^\pi \cos(2\pi m t \cos \theta) 2\pi \sqrt{|m|} (t \sin \theta)^{\frac{1}{2}} I_{s-\frac{1}{2}}(2\pi |m| t \sin \theta) \frac{d\theta}{\sin \theta} \\ &= 2\pi \sqrt{|m|} t \int_0^\pi \cos(2\pi m t \cos \theta) I_{s-\frac{1}{2}}(2\pi |m| t \sin \theta) \frac{d\theta}{(\sin \theta)^{\frac{1}{2}}} \\ &= \pi |m|^{\frac{1}{2}} t^{\frac{1}{2}} \frac{2^s \Gamma(\frac{s}{2})^2}{\Gamma(s)} J_{s-\frac{1}{2}}(2\pi |m| t),\end{aligned}$$

where the last equality follows from [31, p. 47 and p. 143, Eq. (1)] for $\operatorname{Re}(s) > 0$. Applying [25, Proposition 5] in the second equality below with $K^+(m, n; c) = (1-i)(1+(\frac{4}{c/4}))K_{1/2}(m, n; c)$ and making a suitable change of variables in the last, we find that

$$\begin{aligned}& \frac{\Gamma(s)}{2^s \Gamma(\frac{s}{2})^2} 2\pi \sqrt{dD} \operatorname{Tr}_{d,D}^*(G_{-m}(\tau, s)) \\ &= \pi \sqrt{2|m|} (dD)^{\frac{1}{4}} \sum_{0 < c \equiv 0 \pmod{4}} \frac{S_{-m}(D, d; c)}{\sqrt{c}} J_{s-\frac{1}{2}}\left(\frac{4\pi |m| \sqrt{dD}}{c}\right) \\ &= \pi \sqrt{2|m|} (dD)^{\frac{1}{4}} \sum_{0 < c \equiv 0 \pmod{4}} \frac{1}{\sqrt{c}} \sum_{n|(m, \frac{c}{4})} \left(\frac{d}{n}\right) \sqrt{\frac{n}{c}} K^+\left(D, \frac{m^2 d}{n^2}; \frac{c}{n}\right) J_{s-\frac{1}{2}}\left(\frac{4\pi |m| \sqrt{dD}}{c}\right) \\ &= \pi \sqrt{2|m|} (dD)^{\frac{1}{4}} \sum_{n|m} \left(\frac{d}{n}\right) n^{-\frac{1}{2}} \sum_{0 < c \equiv 0 \pmod{4}} \frac{1}{c} K^+\left(D, \frac{m^2 d}{n^2}; c\right) J_{s-\frac{1}{2}}\left(\frac{4\pi}{c} \sqrt{\frac{m^2}{n^2} dD}\right).\end{aligned}\tag{3.2}$$

On the other hand, in the continuation of preceding work such as [6–9, 13, 18, 21], the authors ([24]) constructed general Maass–Poincaré series using spherical Whittaker functions $\mathcal{M}_n(y, s)$ and $\mathcal{W}_n(y, s)$ which are defined by

$$\begin{aligned}\mathcal{M}_n(y, s) &= \begin{cases} \Gamma(2s)^{-1} (4\pi |n| y)^{-k/2} M_{\frac{k}{2} \operatorname{sgn}(n), s-1/2}(4\pi |n| y), & \text{if } n \neq 0, \\ y^{s-k/2}, & \text{if } n = 0, \end{cases} \\ \mathcal{W}_n(y, s) &= \begin{cases} \Gamma(s + \frac{k}{2} \operatorname{sgn}(n))^{-1} |n|^{k/2-1} (4\pi y)^{-k/2} W_{\frac{k}{2} \operatorname{sgn}(n), s-1/2}(4\pi |n| y), & \text{if } n \neq 0, \\ \frac{(4\pi)^{1-k} y^{1-s-k/2}}{(2s-1)\Gamma(s-k/2)\Gamma(s+k/2)}, & \text{if } n = 0, \end{cases}\end{aligned}$$

where $M_{\mu, \nu}(y)$ and $W_{\mu, \nu}(y)$ are Whittaker functions. (See [24, Section 2] or (4.2) for definition.) In particular, families of Maass–Poincaré series of weight $3/2$ on $\Gamma_0(4)$ satisfying the plus space condition are given by

$$F_m^+(\tau, s) = \mathcal{M}_m(y, s)e(mx) + \sum_{n \equiv 0,3 \pmod{4}} b_m(n, s) \mathcal{W}_n(y, s)e(nx) \quad (3.3)$$

for each $m \equiv 0, 3 \pmod{4}$ in [24, Theorem 4.4]. In case both m and n are positive, it follows from [24, Theorem 4.4] and the property of Kloosterman sum

$$K_{\frac{3}{2}}(m, n; c) = -iK_{\frac{1}{2}}(-m, -n; c)$$

that

$$b_m(n, s) = -\sqrt{2}\pi \sum_{0 < c \equiv 0 \pmod{4}} \frac{K^+(-m, -n; c)}{c} |mn|^{-\frac{1}{4}} J_{2s-1} \left(\frac{4\pi\sqrt{|mn|}}{c} \right). \quad (3.4)$$

Comparing (3.4) with (3.2), we find that

$$\frac{\Gamma(s)}{2^s \Gamma(\frac{s}{2})^2} 2\pi \operatorname{Tr}_{d,D}^*(G_{-m}(\tau, s)) = - \sum_{n|m} \sqrt{\frac{m^2}{n^2}} \left(\frac{d}{n} \right) b_{|D|} \left(\frac{m^2}{n^2} |d|, \frac{s}{2} + \frac{1}{4} \right). \quad (3.5)$$

This implies that $\operatorname{Tr}_{d,D}^*(G_{-m}(\tau, s))$ should vanish at $s = 1$ as $F_m^+(\tau, 3/4) = \{0\}$ for positive integer m , according to [24, Proposition 5.1].

Now we differentiate both sides of (3.5) with respect to s at $s = 1$ so that we have

$$\begin{aligned} \operatorname{Tr}_{d,D}^*(\hat{J}_m(\tau, s))|_{s=1} &= - \sum_{n|m} \left| \frac{m}{n} \right| \left(\frac{d}{n} \right) \frac{\partial}{\partial s} \left[b_{|D|} \left(\frac{m^2}{n^2} |d|, \frac{s}{2} + \frac{1}{4} \right) \right]_{s=1} \\ &= - \frac{1}{2} \sum_{n|m} \left| \frac{m}{n} \right| \left(\frac{d}{n} \right) \frac{\partial}{\partial s} \left[b_{|D|} \left(\frac{m^2}{n^2} |d|, s \right) \right]_{s=\frac{3}{4}}. \end{aligned}$$

For simplicity, if we define

$$\operatorname{Tr}_{d,D}^*(\hat{J}_m(\tau)) = \operatorname{Tr}_{d,D}^*(\hat{J}_m(\tau, s))|_{s=1}, \quad (3.6)$$

then we have

$$\operatorname{Tr}_{d,D}^*(\hat{J}(\tau)) = \operatorname{Tr}_{d,D}^*(\hat{J}_1(\tau)) = -\frac{1}{2} \frac{\partial}{\partial s} [b_{|D|}(|d|, s)]_{s=\frac{3}{4}}. \quad (3.7)$$

For each negative discriminant $D < 0$, the mock modular form $g_D(\tau)$ in (1.6) and [24, Theorem 1.1] is given by $g_D(\tau) = 2\sqrt{\pi|D|}k_D^+(\tau)$, where $k_D^+(\tau)$ denotes the holomorphic part of a harmonic weak Maass form $h_{-D,3/2}(\tau)$ found in [24, Theorem 5.3]. More precisely, the harmonic weak Maass form of weight $3/2$ satisfying the plus space condition was constructed via

$$k_D(\tau) := h_{-D,3/2}(\tau) = \frac{\partial}{\partial s} F_{|D|}^+(\tau, s) \Big|_{s=\frac{3}{4}} - 8\sqrt{\frac{\pi}{|D|}} H(|D|) F_0^+(\tau, 3/4) \quad (3.8)$$

and it follows from [24, Theorem 5.2 and Theorem 5.3] that the Fourier expansion of k_D^+ is given by

$$k_D^+ = -2\sqrt{\pi}iq^{|D|} - 8\sqrt{\frac{\pi}{|D|}}H(|D|) + \sum_{\substack{0 < n \equiv 0,3 \pmod{4} \\ n \neq |D|}} \left(\frac{\partial}{\partial s} [b_{|D|}(n, s)]_{s=\frac{3}{4}} \frac{2\sqrt{n}}{\sqrt{\pi}} + 96\sqrt{\frac{\pi}{|D|}}H(|D|)H(n) \right) q^n. \quad (3.9)$$

Theorem 1.2 then follows from (3.7) and (3.9).

4. Weight 2 harmonic weak Maass form

In Section 2, for positive integers m , we established weight 0 sesqui-harmonic Maass forms $\hat{J}_m(\tau)$ and made an observation that $h_m^*(\tau) := \xi_0(\hat{J}_m(\tau))$ are harmonic weak Maass forms of weight 2 satisfying $\xi_2(h_m^*(\tau)) = j_m(\tau) + 24\sigma(m)$. The members h_m ($m > 0$) of the basis for the space of weight 2 harmonic weak Maass forms found by Duke, Imamoglu and Tóth in [20] have the same property. In this section, we prove h_m and h_m^* are equal up to a constant multiple.

By its definition, $h_m^*(\tau) = \xi_0(\hat{J}_m(\tau))$ and

$$\begin{aligned} \xi_0(\hat{J}_m(\tau, s)) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\xi_0 \left(\frac{\partial}{\partial s} \phi_{-m,s}(y) e(-mx) \right) \right] |_2 \gamma \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{\partial}{\partial s} \xi_0(\phi_{-m,s}(y) e(-mx)) \right) |_2 \gamma. \end{aligned}$$

It follows from [1, (13.6.3)], [15, p. 10] or [18, Appendix A] that

$$\phi_{-m,s}(y) = 2\pi\sqrt{m}\sqrt{y}I_{s-\frac{1}{2}}(2\pi my) = 2^{1-2s}\Gamma\left(s + \frac{1}{2}\right)^{-1} \sqrt{\pi}M_{0,s-\frac{1}{2}}(4\pi my),$$

where the $M_{\mu,\nu}(y)$ is the M -Whittaker function given by

$$M_{\mu,\nu}(y) = e^{-y/2}y^{\nu+1/2}M\left(\nu - \mu + \frac{1}{2}, 1 + 2\nu, y\right) \quad (4.1)$$

and

$$M(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)(a+1)(a+2)\cdots(a+n-1)}{(b)(b+1)(b+2)\cdots(b+n-1)} \frac{x^n}{n!}. \quad (4.2)$$

If we set $A(s) = 2^{1-2s} \Gamma(s + \frac{1}{2})^{-1} \sqrt{\pi}$, then we have

$$\begin{aligned}
 & \xi_0(\phi_{-m,s}(y)e(-mx)) \\
 &= \overline{A(s)} \xi_0(M_{0,s-\frac{1}{2}}(4\pi my)e(-mx)) \\
 &= \overline{A(s)} \xi_0(M_{0,s-\frac{1}{2}}(4\pi my)e^{-2\pi my}) \bar{q}^{-m} \\
 &= \overline{A(s)} \frac{\partial}{\partial y} (M_{0,s-\frac{1}{2}}(4\pi my)e^{-2\pi my}) \bar{q}^{-m} \\
 &= \overline{A(s)} (4\pi m) \left(sY^{-1} M_{0,s-\frac{1}{2}}(Y)e^{-\frac{Y}{2}} - \frac{1}{2} \frac{1}{\sqrt{Y}} M_{\frac{1}{2},s}(Y)e^{-\frac{Y}{2}} \right) \bar{q}^{-m} \\
 &= \overline{A(s)} (4\pi m) \left(se^{-Y} Y^{s-1} M(s, 2s; Y) - \frac{1}{2} e^{-Y} Y^s M(s, 2s+1; Y) \right) \bar{q}^{-m}
 \end{aligned}$$

where the penultimate equality follows from [5, p. 127] with $Y = 4\pi my$ and the last equality follows from (4.2). Using the transformation formula for confluent hypergeometric function $M(\alpha, \gamma; Y) = M(\alpha + 1, \gamma; Y) - \frac{Y}{\gamma} M(\alpha + 1, \gamma + 1; Y)$ from [27, (9.9.12)] and (4.2), we obtain

$$\xi_0(\phi_{-m,s}(y)e(-mx)) = \overline{A(s)} (4\pi m) \frac{s}{Y} M_{1,s-\frac{1}{2}}(Y)e(mx).$$

Now, let $\varphi_{m,s}(y) := (4\pi y)^{-1} M_{1,s-\frac{1}{2}}(4\pi my)$. Then

$$\xi_0(\phi_{-m,s}(y)e(-mx)) = (\overline{A(s)} 4\pi s) \varphi_{m,s}(y)e(mx). \quad (4.3)$$

Differentiating both sides with respect to s and summing over all $\gamma \in \Gamma_\infty \backslash \Gamma$ after applying the weight 2 slash operator, we derive

$$h_m^*(\tau) = 4\pi \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left(\frac{\partial}{\partial s} (\varphi_{m,s}(y)e(mx)) \right) |_2 \gamma,$$

because $\varphi_{m,1}(y)e(mx) = 0$. As $\frac{\partial}{\partial s} \varphi_{m,s}(y)$ is the spherical function used to construct h_m in [20], $h_m^*(\tau) = 4\pi h_m(\tau)$.

Remark. As the referee suggested, the argument in this section can be replaced by showing that $4\pi h_m$ and h_m^* have the same principal parts. As in such a case, the difference is mapped to a cusp form under ξ_2 , which implies $4\pi h_m = h_m^*$, because there are no weight 2 holomorphic modular forms in $SL_2(\mathbb{Z})$.

5. Proof of Theorem 1.3

For a smooth function f of weight k for $\Gamma_0(4)$ with the Fourier expansion

$$f(\tau) = \sum_n a(n, y) e(nx),$$

we set

$$f^e(\tau) = \sum_{n \equiv 0 \pmod{2}} a\left(n, \frac{y}{4}\right) e\left(\frac{nx}{4}\right) \quad \text{and} \quad f^o(\tau) = \sum_{n \equiv 1 \pmod{2}} a\left(n, \frac{y}{4}\right) e\left(\frac{n}{8}\right) e\left(\frac{nx}{4}\right).$$

For negative discriminants D and d , the function f_D in (1.3) is holomorphic of weight $1/2$ and k_d in (3.8) is smooth of weight $3/2$ for $\Gamma_0(4)$ on \mathbb{H} , both of which have Fourier expansions satisfying the plus space condition. Let $\mathcal{F}_4(Y)$ be the truncated domain used in [19], namely, the domain obtained from the fundamental domain for $\Gamma_0(4)$ given in [19, Fig. 1] by truncating at cusp $i\infty$ by the line $\text{Im}(\tau) = Y$, at the cusp $1/2$ by the circle $|\tau - (\frac{1}{2} + \frac{i}{8Y})| = \frac{1}{8Y}$, and at cusp 0 by the circle $|\tau - \frac{i}{8Y}| = \frac{1}{8Y}$. Then [19, Lemma 2] implies that for $Y \geq 2$,

$$\begin{aligned} (f_D, \xi_{3/2} k_d)^{reg} &= \lim_{Y \rightarrow \infty} \int_{\mathcal{F}_4(Y)} f_D(\tau) \overline{\xi_{3/2}(k_d(\tau))} y^{1/2} \frac{dx dy}{y^2} \\ &= \lim_{Y \rightarrow \infty} \int_{-\frac{1}{2} + iY}^{\frac{1}{2} + iY} \left(f_D(\tau) k_d(\tau) + \frac{1}{2} f_D^e(\tau) k_d^e(\tau) + \frac{1}{2} f_D^o(\tau) k_d^o(\tau) \right) d\tau. \end{aligned}$$

Letting $\tau_Y = x + iY$, we have

$$\begin{aligned} (f_D, \xi_{3/2} k_d)^{reg} &= \lim_{Y \rightarrow \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(f_D(\tau_Y) k_d^-(\tau_Y) + \frac{1}{2} f_D^e(\tau_Y) (k_d^-)^e(\tau_Y) + \frac{1}{2} f_D^o(\tau_Y) (k_d^-)^o(\tau_Y) \right) dx \\ &\quad + \lim_{Y \rightarrow \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(f_D(\tau_Y) k_d^+(\tau_Y) + \frac{1}{2} f_D^e(\tau_Y) (k_d^+)^e(\tau_Y) + \frac{1}{2} f_D^o(\tau_Y) (k_d^+)^o(\tau_Y) \right) dx. \end{aligned}$$

According to [24, Proposition 5.2, Theorem 5.3 and Eq. (5.12)], the Fourier expansion of the non-holomorphic part $k_d^-(\tau)$ of $k_d(\tau)$ is given by

$$\begin{aligned}
k_d^-(\tau) &= (-i)\Gamma\left(-\frac{1}{2}, 4\pi dy\right)q^{-d} + \sum_{\substack{n < 0 \\ n \equiv 0, 3 \pmod{4}}} b_{-d}\left(n, \frac{3}{4}\right)\sqrt{|n|}\Gamma\left(-\frac{1}{2}, 4\pi|n|y\right)q^n \\
&\quad + \frac{24H(|d|)}{\sqrt{d}} \sum_{0 < n = \square} \sqrt{|n|}\Gamma\left(-\frac{1}{2}, 4\pi|n|y\right)q^{-n},
\end{aligned}$$

where $b_D(n, s)$ is given by (3.4). Since $\Gamma(-\frac{1}{2}, 4\pi|n|y) \sim e^{-4\pi|n|y}(4\pi|n|y)^{-\frac{3}{2}}$ as $y \rightarrow \infty$,

$$\begin{aligned}
&(f_D, \xi_{3/2}k_d)^{reg} \\
&= \lim_{Y \rightarrow \infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(f_D(\tau_Y)k_d^+(\tau_Y) + \frac{1}{2}f_D^e(\tau_Y)(k_d^+)^e(\tau_Y) + \frac{1}{2}f_D^o(\tau_Y)(k_d^+)^o(\tau_Y) \right) dx \\
&= \text{constant term of } (f_D(\tau)k_d^+(\tau) + f_D^e(\tau)(k_d^+)^e(\tau) + f_D^o(\tau)(k_d^+)^o(\tau)).
\end{aligned}$$

Hence it follows from $f_D(\tau) = q^D + O(1)$ that

$$(f_D, \xi_{3/2}k_d)^{reg} = \frac{3}{2} \times \text{coefficient of } q^{|D|} \text{ in } k_d^+(\tau).$$

Recall from Section 3 that the mock modular form $g_d(\tau) = 2\sqrt{\pi|d|}k_d^+(\tau)$ has shadow f_d . Now, Theorem 1.3 (i) follows from Theorem 1.2. Using similar arguments and (3.9), we obtain

$$\begin{aligned}
(f_0, f_d)^{reg} &= (f_0, \xi_{\frac{3}{2}}(2\sqrt{\pi|d|}k_d))^{reg} = \frac{3}{2} \times \text{constant term in } 2\sqrt{\pi|d|}k_d^+(\tau) \\
&= -24\pi H(|d|),
\end{aligned}$$

which proves Theorem 1.3 (ii).

Alternatively, we have

$$\begin{aligned}
(f_0, f_d)^{reg} &= (f_d, f_0)^{reg} = (f_d, \xi_{\frac{3}{2}}(-16\pi E))^{reg} \\
&= \frac{3}{2} \times (-16\pi) \times \text{coefficient of } q^{|D|} \text{ in } E^+(\tau) \\
&= -24\pi H(|d|).
\end{aligned}$$

Here $E(\tau) := -\frac{1}{12}k_0(\tau)$ denotes Zagier's Eisenstein series of weight $3/2$.

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