Journal of Number Theory 145 (2014) 126–152

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Galois cohomology of a number field is Koszul

Leonid Positselski [a,b,*]

[a] *Faculty of Mathematics and Laboratory of Algebraic Geometry, National Research University Higher School of Economics, Moscow 117312, Russia*
[b] *Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Moscow 127994, Russia*

A R T I C L E   I N F O

Article history:
Received 22 December 2013
Received in revised form 23 May 2014
Accepted 31 May 2014
Available online 4 July 2014
Communicated by Jean-Louis Colliot-Thélène

Keywords:
Global fields
Local fields
Galois cohomology
Koszul algebras
Koszul modules
Class Field Theory
Chebotarev's density theorem
Filtrations on algebras
Commutative PBW-bases
Commutative Gröbner bases

A B S T R A C T

We prove that the Milnor ring of any (one-dimensional) local or global field K modulo a prime number l is a Koszul algebra over $\mathbb{Z}/l$. Under mild assumptions that are only needed in the case $l = 2$, we also prove various module Koszulity properties of this algebra. This provides evidence in support of Koszulity conjectures for arbitrary fields that were proposed in our previous papers. The proofs are based on the Class Field Theory and computations with quadratic commutative Gröbner bases (commutative PBW-bases).

Contents

* Correspondence to: Sector of Algebra and Number Theory, Institute for Information Transmission Problems, Bolshoy Karetny per. 19, Moscow 127994, Russia.
E-mail address: posic@mccme.ru.

http://dx.doi.org/10.1016/j.jnt.2014.05.024
0022-314X/

0. Introduction

0.0. Let K be a field and $l \neq \operatorname{char} K$ be a prime number. The well-known *Milnor–Bloch–Kato* conjecture claims that the natural morphism of graded $\mathbb{Z}/l$-algebras, called the *Galois symbol*, or the *norm residue homomorphism*,

$$\mathrm{K}^{\mathrm{M}}(K)/l \longrightarrow \bigoplus_n H^n\big(G_K, \mu_l^{\otimes n}\big)$$

is an isomorphism. Here $\mathrm{K}^{\mathrm{M}}(K)$ denotes the Milnor K-theory ring of K, $G_K = \mathrm{Gal}(\overline{K}/K)$ is the absolute Galois group of K, and μ_l is the group of l-roots of unity in $\overline{K}$. For algebraic number fields and their functional analogues, this conjecture was proven by J. Tate in [17]; see also [2]. For arbitrary fields, it was established by A. Merkurjev and A. Suslin [8] in the degree $n = 2$; the prolonged work on a complete proof was recently finished by M. Rost, V. Voevodsky, and collaborators [18].

0.1. Another approach to proving this conjecture was suggested in our paper [14]. There it was shown that the Milnor–Bloch–Kato conjecture would follow from its low-degree part if one knew the quadratic algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ to be *Koszul*. The argument in [14] was only directly applicable to fields K having no algebraic extensions of degree prime to l; it is well-known that it suffices to prove the Milnor–Bloch–Kato conjecture for such fields. The proper scope of the Koszulity conjecture was demonstrated in [10,13], where a motivic interpretation of it was found in the case when K contains a primitive l-root of unity.

A positively graded associative algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ over a field k is called *Koszul* [14, Subsection 2.2] if the groups $H_{ij}(A) = \mathrm{Tor}^A_{ij}(k,k)$ vanish for all $i \neq j$, where the first grading i is the homological grading and the second grading j is the *internal* grading, which is induced from the grading of A. This definition, introduced originally for algebras with (locally) finite-dimensional components by S. Priddy [15] (see [3] or [9] for a detailed treatment), was extended to the infinite-dimensional case in [14]. In particular, the conditions on $H_{ij}(A)$ for $i = 1$ and 2 mean that the algebra A is quadratic, i.e., generated by A_1 with relations of degree 2.

Koszulity Conjecture. *For any field K containing a primitive root of unity of a prime degree l, the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul.*

Another formulation of the Koszulity conjecture is the assertion that the Galois cohomology algebra $\bigoplus_n H^n(G_K, \mu_l^{\otimes n})$ is Koszul, in the same assumptions on K and l. In view of the Merkurjev–Suslin theorem, this formulation clearly implies the Milnor–Bloch–Kato conjecture. By the result of [14], the former formulation also implies the Milnor–Bloch–Kato, provided that one knows just a little bit more than the Merkurjev–Suslin theorem.

0.2. Further and stronger versions of the Koszulity conjecture were proposed in the papers [11,12]. Assume that either l is odd and the field K contains a primitive l-root of unity, or $l = 2$ and the field K contains a square root of -1 (we always presume that $\operatorname{char} K \neq 2$ when speaking of the square root of -1). Then the Milnor algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is the quotient algebra of the exterior algebra $\Lambda_{\mathbb{Z}/l}(K^*/K^{*l})$ by its ideal J_K generated by the Steinberg symbols. It was shown in [11] that if the ideal J_K is a *Koszul module* (in the appropriately shifted grading) over the algebra $\Lambda_{\mathbb{Z}/l}(K^*/K^{*l})$, then both Koszulity of the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ and a certain version of Bogomolov's freeness conjecture for the field K follow.

A positively graded left module $M = M_1 \oplus M_2 \oplus \cdots$ over a Koszul algebra A is called *Koszul* [11, Subsection 3.3] if the groups $H_{ij}(A, M) = \operatorname{Tor}_{ij}^A(k, M)$ vanish for all $i \neq j - 1$ (see 1.4 for comments on the grading conventions). This definition, which first appeared in [3], was studied in detail in [9,11] (see also [1]). In particular, the conditions on $H_{i,j}(A, M)$ for $i = 0$ and 1 mean that the A-module M is quadratic, i.e., generated by M_1 with relations in degree 2. The hypothesis of Koszulity of the ideal J_K can be equivalently restated as follows.

Module Koszulity Conjecture 1. *For any field K and a prime number l such that K contains a primitive l-root of unity if l is odd and K contains a square root of -1 if $l = 2$, the $\Lambda_{\mathbb{Z}/l}(K^*/K^{*l})$-module $\mathrm{K}_+^{\mathrm{M}}(K)/l = \mathrm{K}_1^{\mathrm{M}}(K)/l \oplus \mathrm{K}_2^{\mathrm{M}}(K)/l \oplus \cdots$ is Koszul.*

0.3. Assume that K contains a primitive l-root of unity, and let $c \in K^*$ be an element not belonging to K^{*l}. Let $L = K[\sqrt[l]{c}]$. Assume further that the Milnor–Bloch–Kato conjecture holds for the fields K and L and the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul. Then it was shown in [12, proof of Corollary 15] together with [11, Theorem 6.1] that the algebra $\mathrm{K}^{\mathrm{M}}(L)/l$ is Koszul provided that the annihilator ideal $\operatorname{Ann}(c \bmod l) \subset \mathrm{K}^{\mathrm{M}}(K)/l$ of the element $(c \bmod l) \in \mathrm{K}_1^{\mathrm{M}}(K)/l$ is a Koszul module over $\mathrm{K}^{\mathrm{M}}(K)/l$.

It was conjectured that the annihilator ideal in $\mathrm{K}^{\mathrm{M}}(K)/l$ of any nonzero element in $\mathrm{K}_1^{\mathrm{M}}(K)/l$ is a Koszul module. The silly filtration conjecture for Artin–Tate motives with $\mathbb{Z}/l$-coefficients related to the field extension L/K follows from this module Koszulity conjecture [13, Subsection 9.8]. It can be equivalently restated as follows.

Module Koszulity Conjecture 2. *For any field K containing a root of unity of a prime degree l and any element $c \in \mathrm{K}_1^{\mathrm{M}}(K)/l$, the ideal $(c) = c\mathrm{K}^{\mathrm{M}}(K)/l$ is a Koszul module over the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$.*

Another equivalent formulation of the same conjecture is that the quotient algebra $(\mathrm{K}^{\mathrm{M}}(K)/l)/(c\mathrm{K}^{\mathrm{M}}(K)/l)$ is a Koszul module (in the appropriately shifted grading) over $\mathrm{K}^{\mathrm{M}}(K)/l$.

0.4. In this paper, we prove all these Koszulity conjectures for all local and global fields, i.e., the algebraic extensions of $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{F}_p((z))$, $\mathbb{Q}$, or $\mathbb{F}_p(z)$. The only exception is that our proof of Module Koszulity Conjecture 2 in the case of a global field depends on the assumption that $\{c,c\}=0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$. This assumption always holds when l is odd or K contains a square root of -1. Proving the assertions of these conjectures in the case when K does not contain a primitive l-root of unity is relatively easy, and we do so, even though there are few general reasons to believe in the Koszulity conjectures in such a case. For local fields, we also prove a certain extension of Module Koszulity Conjecture 1 to the case of fields not necessarily containing a square root of -1 when $l=2$.

This paper is a far-reaching extension of the appendix to [14]. Our methods involve the construction of infinite quadratic commutative Gröbner bases (infinite commutative PBW-bases) in the algebras $\mathrm{K}^{\mathrm{M}}(K)/l$. These constructions make heavy use of the descriptions of Galois cohomology provided by the local and global class field theory, some further results from the global class field theory, and Chebotarev's density theorem. These methods also allow to obtain a new proof of the fact that the graded algebra $\bigoplus_n H^n(G_K,\mu_l^{\otimes n})$ is quadratic for a local or global field K.

0.5. The general formalism of filtrations and PBW-bases indexed by well-ordered sets is developed in Section 1. The relevant background facts from the algebraic number theory are recalled and discussed in Section 2. Koszul properties of the Milnor algebra/Galois cohomology of a local field are established in Section 3. Koszulity of the ideal of Steinberg relations J_K for a global field K containing a primitive l-root of unity if l is odd and a square root of -1 if $l=2$ is proven in Section 4. Koszulity of the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ for any global field K containing a primitive l-root of unity is demonstrated in Section 5. Koszulity of the annihilator ideals in Milnor algebras of global fields is shown, under certain assumptions, in Section 6. All the mentioned Koszul properties are proven for a global field K not containing a primitive l-root of unity in Section 7.

1. Preliminaries on PBW-bases

1.1. Well-ordered sets and filtrations

Let k be a field, V be a k-vector space, and $I=\{\alpha\}$ be a well-ordered set. An *increasing filtration F on V with values in I* is a family of subspaces $F_\alpha V\subset V$, $\alpha\in I$, such that $F_\alpha V\subset F_\beta V$ for $\alpha<\beta$ and $V=\bigcup_{\alpha\in I}F_\alpha V$. The *associated quotient* space $\mathrm{gr}^F V=\bigoplus_\alpha \mathrm{gr}^F_\alpha V$ is the I-graded vector space with the components $\mathrm{gr}^F_\alpha V=F_\alpha V/\bigcup_{\beta<\alpha}F_\beta V$.

Lemma. *Let $C^\bullet$ be a complex of I-filtered vector spaces (with the differentials preserving the filtrations). Then one has $H^0(C^\bullet) = 0$ provided that $H^0(\operatorname{gr}^F C^\bullet) = 0$. Conversely, if $H^{-1}(\operatorname{gr}^F C^\bullet) = 0 = H^1(\operatorname{gr}^F C^\bullet)$ and $H^0(C^\bullet) = 0$, then $H^0(\operatorname{gr}^F C^\bullet) = 0$.*

Proof. Straightforward induction on the well-ordering. □

1.2. Graded ordered semigroups

A *graded ordered semigroup* [9, Section 7 of Chapter 4] Γ is a collection of well-ordered sets Γ_n, $n \in \mathbb{Z}_{\geqslant 0}$, endowed with associative multiplication maps $\Gamma_n \times \Gamma_m \longrightarrow \Gamma_{n+m}$ strictly compatible with the orderings, i.e., $\alpha < \beta$ implies $\alpha\gamma < \beta\gamma$ and $\gamma\alpha < \gamma\beta$ for all α, $\beta \in \Gamma_n$ and $\gamma \in \Gamma_m$. In addition, we assume that the only element of Γ_0 is the unit of the semigroup Γ.

Let U be a Γ_n-filtered vector space and V be a Γ_m-filtered vector space over k; we will denote both filtrations by F. Define a Γ_{n+m}-valued filtration on the tensor product $U \otimes_k V$ by the rule

$$F_\gamma(U \otimes_k V) = \sum_{\alpha\beta \leqslant \gamma} F_\alpha U \otimes F_\beta V,$$

where $\alpha \in \Gamma_n$, $\beta \in \Gamma_m$, and $\gamma \in \Gamma_{n+m}$. Similarly, if $U = \bigoplus_{\alpha \in \Gamma_n} U_\alpha$ and $V = \bigoplus_{\beta \in \Gamma_m} V_\beta$ are a Γ_n- and Γ_m-graded vector spaces, then the tensor product $U \otimes_k V$ is a Γ_{n+m}-graded vector space with the components

$$(U \otimes_k V)_\gamma = \bigoplus_{\alpha\beta = \gamma} U_\alpha \otimes_k V_\beta.$$

Lemma. *There is a natural isomorphism of Γ_{n+m}-graded vector spaces*

$$\operatorname{gr}^{\Gamma_{n+m}}(U \otimes_k V) \simeq \operatorname{gr}^{\Gamma_n} U \otimes_k \operatorname{gr}^{\Gamma_m} V.$$

Proof. To define a natural map $\operatorname{gr}^{\Gamma_n} U \otimes_k \operatorname{gr}^{\Gamma_m} V \longrightarrow \operatorname{gr}^{\Gamma_{n+m}}(U \otimes_k V)$, choose for any classes $\bar u \in \operatorname{gr}^F_\alpha U$ and $\bar v \in \operatorname{gr}^F_\beta V$ their representatives $u \in F_\alpha U$ and $v \in F_\beta V$, and assign the class $\overline{u \otimes v} \in \operatorname{gr}^F_{\alpha\beta}(U \otimes_k V)$ of the element $u \otimes v \in F_{\alpha\beta}(U \otimes_k V)$ to the tensor product $\bar u \otimes \bar v \in \operatorname{gr}^F_\alpha U \otimes_k \operatorname{gr}^F_\beta V$. Checking that this is a well-defined isomorphism is easy. □

1.3. Filtered algebras and modules

Let $A = \bigoplus_{n=0}^\infty A_n$ be a graded associative k-algebra with $A_0 = k$ and Γ be a graded ordered semigroup. A *Γ-valued filtration* on A is a family of filtrations F with values in Γ_n on the grading components A_n of the algebra A such that the multiplication maps $A_n \otimes_k A_m \longrightarrow A_{n+m}$ are compatible with the Γ_{n+m}-valued filtrations. By Lemma 1.2, the associated graded quotient vector space $\operatorname{gr}^F A = \bigoplus_n \operatorname{gr}^F A_n$ has a natural structure of a graded k-algebra.

Let $M = \bigoplus_{n=1}^{\infty} M_n$ be a graded left module over A. A Γ-valued filtration on M compatible with the given Γ-valued filtration on A is a family of filtrations F with values in Γ_n on the grading components M_n of the module M such that the multiplication maps $A_n \otimes_k M_m \longrightarrow M_{n+m}$ are compatible with the Γ_{n+m}-valued filtrations. The associated graded quotient vector space $\mathrm{gr}^F M = \bigoplus_n \mathrm{gr}^F M_n$ has a natural structure of a graded left module over $\mathrm{gr}^F A$.

Recall the definitions of the homology of positively graded associative algebras and modules from [11, Subsections 2.2–3]: $H_{i,j}(A) = \mathrm{Tor}^A_{i,j}(k,k)$ and $H_{i,j}(A,M) = \mathrm{Tor}^A_{i,j}(k,M)$. Here the second grading j comes from the grading of A and M; the grading i is the *homological* grading and the grading j is called the *internal* grading.

When A and M are endowed with Γ-valued filtrations, the bar-complexes computing $H_*(A)$ and $H_*(A,M)$ acquire the induced filtrations. The components of the bar-complexes of the internal grading n are filtered with values in Γ_n. By Lemma 1.2, the associated quotient complexes of these bar-complexes with respect to these filtrations are the bar-complexes computing $H_*(\mathrm{gr}^F A)$ and $H_*(\mathrm{gr}^F A, \mathrm{gr}^F M)$. By Lemma 1.1, it follows that $H_{i,j}(A) = 0$ provided that $H_{i,j}(\mathrm{gr}^F A) = 0$, and $H_{i,j}(A,M) = 0$ provided that $H_{i,j}(\mathrm{gr}^F A, \mathrm{gr}^F M) = 0$.

1.4. PBW-theorem

A positively graded associative algebra $A = \bigoplus_{n=0}^{\infty} A_n$ is called *quadratic* if it is isomorphic to the quotient algebra of a tensor (free associative) algebra $\bigoplus_{n=0}^{\infty} V^{\otimes n}$ by an ideal generated a vector subspace $R \subset V \otimes_k V$. The *quadratic part* of a positively graded algebra A is the quadratic algebra $\mathrm{q}A$ with the space of generators $V = A_1$ and the subspace of quadratic relations $R = \ker(A_1 \otimes_k A_1 \longrightarrow A_2)$. The natural morphism of graded algebras $\mathrm{q}A \longrightarrow A$ is an isomorphism in degree 1 and a monomorphism in degree 2; it is an isomorphism of graded algebras if and only if the graded algebra A is quadratic.

Similarly, a positively graded left module $M = \bigoplus_{n=1}^{\infty} M_n$ over a quadratic algebra $A = \bigoplus_{n=0}^{\infty} V^{\otimes n}/(R)$ is called *quadratic* if it is isomorphic to the quotient module of a free left A-module $A \otimes_k U$ generated in degree 1 by a submodule generated by a vector subspace $P \subset V \otimes_k U$. The *quadratic part* of a positively graded module M over a positively graded algebra A is the quadratic module $\mathrm{q}_A M$ with the space of generators $U = M_1$ and the subspace of relations $P = \ker(A_1 \otimes M_1 \longrightarrow M_2)$ over the quadratic algebra $\mathrm{q}A$. The natural morphism of graded $\mathrm{q}A$-modules $\mathrm{q}_A M \longrightarrow M$ is an isomorphism in degree 1 and a monomorphism in degree 2; a positively graded module M over a quadratic algebra A is quadratic if and only if this morphism is an isomorphism in all the degrees [11, Subsection 3.1].

Here we use a convention for graded modules slightly different from that in [9,11]: quadratic modules M are generated by M_1 with relations in degree 2. The definition of a Koszul module from [11, Subsection 3.3] is modified accordingly: a positively graded module M over a Koszul algebra A is called Koszul if $H_{i,j}(A,M) = 0$ for all $i \neq j-1$. As it was explained in 1.3, given compatible Γ-valued filtrations F on A and M, the algebra A

is quadratic or Koszul whenever the algebra $\operatorname{gr}^F A$ is, and the A-module M is quadratic or Koszul whenever the $\operatorname{gr}^F A$-module $\operatorname{gr}^F M$ is. The following theorem is a more delicate result in this direction. It is a generalization of the quadratic case of the Diamond Lemma for Gröbner bases [4,5].

Theorem. *Let A be a Γ-filtered graded algebra and M be a Γ-filtered graded A-module. Then*

(1) *if the algebra A is quadratic, the algebra $\operatorname{gr}^F A$ is generated in degree 1, the algebra $\operatorname{q}\operatorname{gr}^F A$ is Koszul, and the natural map $\operatorname{q}\operatorname{gr}^F A \longrightarrow \operatorname{gr}^F A$ is an isomorphism in the degree $n = 3$, then the algebras $\operatorname{gr}^F A$ and A are Koszul;*
(2) *if the algebra $\operatorname{gr}^F A$ is Koszul, the A-module M is quadratic, the $\operatorname{gr}^F A$-module $\operatorname{gr}^F M$ is generated in degree 1, the $\operatorname{gr}^F A$-module $\operatorname{q}_{\operatorname{gr}^F A} \operatorname{gr}^F M$ is Koszul, and the natural map $\operatorname{q}_{\operatorname{gr}^F A} \operatorname{gr}^F M \longrightarrow \operatorname{gr}^F M$ is an isomorphism in the degree $n = 3$, then the $\operatorname{gr}^F A$-module $\operatorname{gr}^F M$ and the A-module M are Koszul.*

Proof. One only has to prove that the algebra $\operatorname{gr}^F A$ or the module $\operatorname{gr}^F M$ is quadratic. Proceed by induction on $n \geqslant 4$ showing the map $\operatorname{q}\operatorname{gr}^F A \longrightarrow \operatorname{gr}^F A$ or $\operatorname{q}_{\operatorname{gr}^F A} \operatorname{gr}^F M \longrightarrow \operatorname{gr}^F M$ is an isomorphism in degree n. For this purpose, consider the component of internal degree n of the initial fragment of the bar-complex $A_+^{\otimes 4} \longrightarrow A_+^{\otimes 3} \longrightarrow A_+^{\otimes 2} \longrightarrow A_+ \longrightarrow k$ or $A_+^{\otimes 3} \otimes_k M \longrightarrow A_+^{\otimes 2} \otimes_k M \longrightarrow A_+ \otimes_k M \longrightarrow M \longrightarrow 0$ (given that $n \geqslant 4$, this fragment is acyclic when the algebra A or the module M are Koszul). Then apply the second assertion of Lemma 1.1 in order to conclude that the associated quotient complex with respect to the Γ_n-valued filtration, which is isomorphic to the similar complex for the algebra $\operatorname{gr}^F A$ or the $\operatorname{gr}^F A$-module $\operatorname{gr}^F M$, has zero cohomology at the middle term (cf. [9, Theorem 7.1 from Chapter 4]). Indeed, the condition $H_{3,n}(\operatorname{q}\operatorname{gr}^F A) = 0$ or $H_{2,n}(\operatorname{q}\operatorname{gr}^F A, \operatorname{q}_{\operatorname{gr}^F A} \operatorname{gr}^F M) = 0$ together with the induction assumption guarantee that the associated graded complex has no cohomology at the second term. The assumption of generation in degree 1 tells that it has no cohomology at the fourth term, and the quadraticity assumption means that the filtered complex has no cohomology at the middle term. $\square$

1.5. Inverse lexicographical ordering

In this paper we will use graded ordered semigroups Γ of the following special form. As a graded semigroup, Γ is isomorphic to the free commutative semigroup generated by the set Γ_1. So elements of Γ_n are the commutative monomials $\alpha_1^{n_1} \cdots \alpha_m^{n_m}$, where $\alpha_1 < \cdots < \alpha_m$ are elements of Γ_1 and $n_1 + \cdots + n_m = n$. The order on Γ_n is the *inverse lexicographical order*: $\alpha_1^{n_1'} \cdots \alpha_m^{n_m'} < \alpha_1^{n_1''} \cdots \alpha_m^{n_m''}$ if there exists $1 \leqslant j \leqslant m$ such that $n_i' = n_i''$ for $i < j$ and $n_j' > n_j''$. For example, $\alpha_1\alpha_4 < \alpha_2\alpha_3$ if $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$.

We will be interested in Γ-valued filtrations F on graded algebras A such that the associated quotient spaces $\operatorname{gr}_\alpha^F A_1$ are one-dimensional for all $\alpha \in \Gamma_1$. Abusing terminology, we will speak of Γ_1-indexed bases $\{x_\alpha\}$ of A_1, presuming that $x_\alpha \in F_\alpha A_1$ has a nonzero image, which will be also denoted by x_α, in $\mathrm{gr}^F_\alpha A_1$. So the basis $\{x_\alpha\}$ in A_1 will be only defined up to an upper-triangular linear transformation. Similarly, we will consider Γ-valued filtrations F on graded modules M such that the associated quotient spaces $\mathrm{gr}^F_\alpha M_1$ are no more than one-dimensional for all $\alpha \in \Gamma_1$.

1.6. Koszulity of algebras

All our graded algebras A will be associative and unital, generated by A_1, and either commutative (when $\operatorname{char} k = 2$) or supercommutative with respect to the parity associated with the grading (when $\operatorname{char} k$ is odd). Here a graded algebra A is called supercommutative if the identity $a^2 = 0$ for all $a \in A_n$ with odd n holds in A, together with the identity $ab = (-1)^{nm} ba$ for $a \in A_n$ and $b \in A_m$. Notice that any supercommutative algebra over a field k of characteristic 2 is commutative, but not the other way.

We will also assume the graded algebra $\mathrm{gr}_F A$ to be generated in degree 1. Once a Γ_1-indexed filtration of A_1 is fixed, this condition defines a unique extension of this filtration to a Γ-valued filtration of A. Given a graded algebra A as above with a Γ-valued filtration F satisfying this condition together with the conditions of 1.5, the associated quotient algebra $\mathrm{gr}^F A$ is a commutative or supercommutative *monomial* algebra. In other words, it is the quotient algebra of the free commutative or supercommutative algebra generated by the elements x_α of degree 1 by an ideal generated by a set of monomials in x_α. This is easy to see; actually, no other relations can be compatible with the Γ-grading of $\mathrm{gr}^F A$.

The quadratic part $\mathrm{q}\,\mathrm{gr}^F A$ of a (super)commutative monomial algebra $\mathrm{gr}^F A$ is always Koszul. Indeed, when the set of generators $\{x_\alpha\}$ is finite, this is the result of the paper [7] (see also [9, Theorem 8.1 of Chapter 4]), and the general case follows by passing to the inductive limit of subalgebras generated by finite subsets of $\{x_\alpha\}$. By Theorem 1.4(1), if A is quadratic and $\mathrm{gr}^F A$ has no relations of degree 3, then $\mathrm{gr}^F A$ is quadratic. By the result of 1.3, if $\mathrm{gr}^F A$ is quadratic, then A is Koszul.

When $\mathrm{gr}^F A$ is quadratic, the basis in A formed by those monomials in x_α that survive in $\mathrm{gr}^F A$ is called a *commutative PBW-basis* of A.

Remark. The commutative PBW-bases of (super)commutative algebras, which are used in this paper, are particular cases of commutative Gröbner bases [5] whose application to Koszulity questions is based on the result of R. Fröberg's paper [7]. These are different from noncommutative PBW-bases, which are particular cases of noncommutative Gröbner bases [4] and whose application to Koszulity was worked out already by Priddy in [15] (see also [9, Sections 1–5 of Chapter 4]). In application to commutative algebras, the commutative PBW-bases are generally more powerful.

1.7. Koszulity of ideals of relations

Let B be a Koszul algebra and $J \subset B$ be a two-sided ideal concentrated in the degrees $n \geqslant 2$. Set $A = B/J$. Then J is a Koszul left B-module (in the grading appropriately shifted by 1) if and only if $A_+ = A_1 \oplus A_2 \oplus A_3 \oplus \cdots$ is a Koszul left B-module. In this case, by [11, Corollary 6.2(c)], the algebra A itself is Koszul.

Now let Λ be the exterior (free supercommutative) algebra over a field k generated by a set of elements x_α of degree 1 and $A = \Lambda/J$ be the quotient algebra of Λ by an ideal of monomials of degree $n \geqslant 2$ in x_α. Let T denote the set of all quadratic monomials $x_\alpha x_\beta$ that are nonzero in A; consider T as the set of edges of an (infinite) graph with the set of vertices $\{x_\alpha\}$.

Proposition. *Assume that $A_n = 0$ for $n \geqslant 3$. Then*

(1) *the monomial algebra A is Koszul if and only if the graph T contains no triangles;*
(2) *the Λ-module A_+ is Koszul if and only if the graph T contains no cycles of any (finite) length.*

Proof. Part (1): it is clear that A is quadratic if and only if T does not contain triangles. Since A is supercommutative monomial, it is Koszul whenever it is quadratic (see 1.6 and [7] or [9, Theorem 8.1 of Chapter 4]).

Part (2), "if": it suffices to consider the case when the set $\{x_\alpha\}$ is finite, since then one can pass to the inductive limit of the similar modules over finitely generated subalgebras of Λ. In the finitely generated case, proceed by induction in the number of vertices. Choose a vertex x_α with a single edge $x_\alpha x_\beta$ coming out of it. The k-vector subspace spanned by x_α and $x_\alpha x_\beta$ is a Λ-submodule in A_+ which is easily seen to be Koszul. Set $(x_\alpha) = \Lambda x_\alpha$. The quotient module $M = A_+/\langle x_\alpha, x_\alpha x_\beta\rangle$ is a Koszul module over $\Lambda/(x_\alpha)$ by the induction assumption. Since the ideal $(x_\alpha) \subset \Lambda$ is a Koszul Λ-module (cf. 1.8), it follows by the way of the spectral sequence $E^2_{p,q} = \operatorname{Tor}_p^{\Lambda/(x_\alpha)}(H_q(\Lambda, \Lambda/(x_\alpha)), M) \Longrightarrow H_{p+q}(\Lambda, M)$ [11, (6.1)] that $A_+/\langle x_\alpha, x_\alpha x_\beta\rangle$ is a Koszul module over Λ, too.

Part (2), "only if": let Γ be the free commutative semigroup generated by the set of indices $\{\alpha\}$; then Λ and A_+ are Γ-graded. Considering subcomplexes of the bar-complex consisting of all the Γ-grading components corresponding to a subsemigroup of Γ spanned by a subset of $\{\alpha\}$, one can see that the Λ-module A_+ corresponding to a graph T is Koszul if and only if the same is true for any full subgraph of T (i.e., any subgraph consisting of all T-edges between a given subset of vertices). So it suffices to consider the case when T is a finite polygon with n vertices and n edges. In this case, the homology exact sequence related to the short exact sequence of Λ-modules $A_2 \longrightarrow A_+ \longrightarrow A_1$ shows that $\dim H_{n-2,n}(\Lambda, A_+) = 1$. □

1.8. Koszulity of annihilator ideals

Let A be a commutative or supercommutative Koszul algebra and $c \in A_1$ be a nonzero element. Then the annihilator ideal $\operatorname{Ann}(c) = \{a \in A \mid ac = 0\} \subset A$ is a Koszul A-module if and only if the ideal $(c) = Ac \subset A$ is a Koszul A-module, and if and only if the quotient algebra $A/(c)$ is a Koszul A-module in the grading shifted by -1.

Assume that there exists a commutative PBW-basis in A corresponding to a well-ordered set of generators $\{x_\alpha\} \in A_1$ such that the minimal element of this set is $x_0 = c$. Then the ideal (c) is a Koszul A-module. This is true due to our particular choice of the inverse lexicographical ordering of monomials in x_α.

Indeed, let the Γ-valued filtration F on $(c) \subset A$ be induced from the Γ-valued filtration F on A. Then the ideal $\mathrm{gr}^F(c) \subset \mathrm{gr}^F A$ is generated by the class $\bar{c} \in \mathrm{gr}^F A_1$ of the element c (because c is the minimal element in the set of generators $\{x_\alpha\}$ and the order of monomials is inverse lexicographical).

This is an ideal in a (super)commutative quadratic monomial algebra generated by a subset of the algebra generators. All such ideals are Koszul. Indeed, for a finitely generated monomial algebra, this is shown in [9, proof of Theorem 8.1 of Chapter 4], and the general case follows by passing to the inductive limit of ideals in finitely generated monomial algebras.

2. Preliminaries on number fields

First of all let us recall that for any field K and a prime number l the multiplication in $\mathrm{K}^{\mathrm{M}}(K)/l$ is supercommutative when l is odd or K contains a square root of -1, and commutative when $l = 2$. More precisely, one has $\{x, x\} = \{-1, x\}$ in $\mathrm{K}_2^{\mathrm{M}}(K)$ for any $x \in \mathrm{K}_1^{\mathrm{M}}(K)$ [2, Section I.1].

2.0. Equal characteristics

For any field K of prime characteristic p such that $[K : K^p] \leqslant p$ one has $\mathrm{K}_n^{\mathrm{M}}(K)/p = 0$ for $n \geqslant 2$ [2, Proposition I.5.13]. This includes any finite extensions of $\mathbb{F}_p((z))$ or $\mathbb{F}_p(z)$. For any field K of characteristic p, one has $H^n(G_K, \mathbb{Z}/p) = 0$ for $n \geqslant 2$ [16, n° II.2.2].

2.1. Finite and archimedean fields

For a finite field $K = \mathbb{F}_q$, one has $\mathrm{K}_n^{\mathrm{M}}(K) = 0 = H^n(G_K, \mu_l^{\otimes n})$ for any $n \geqslant 2$ and any prime l not dividing q [2, Corollary I.5.12].

For the field of complex numbers $K = \mathbb{C}$, one has $\mathrm{K}_n^{\mathrm{M}}(K)/l = 0$ for any $n \geqslant 1$ and any l. For the field of real numbers $K = \mathbb{R}$, one has $\mathrm{K}_n^{\mathrm{M}}(K)/l = 0$ for any $n \geqslant 1$ and any odd l, while $\mathrm{K}^{\mathrm{M}}(K)/2 \simeq \mathbb{Z}/2[\{-1\}] \simeq H(G_K, \mathbb{Z}/2)$ is the polynomial ring with one generator of degree 1 corresponding to the class of the element $-1 \in K^*$.

2.2. Discrete valuation fields

Let K be a Henselian discrete valuation field and k be its residue field. Let $l \neq \mathrm{char}\, k$ be a prime number. Then the $\mathbb{Z}/l$-algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is generated by the $\mathbb{Z}/l$-algebra $\mathrm{K}^{\mathrm{M}}(k)/l$ and an element $\{\pi\} \in \mathrm{K}_1^{\mathrm{M}}(K)/l$, corresponding to any uniformizing element $\pi \in K$, subject to the relations of supercommutativity of $\{\pi\}$ with $\mathrm{K}^{\mathrm{M}}(k)/l$ and $\{\pi,\pi\}=\{-1,\pi\}$ [2, Proposition I.4.3].

The absolute Galois group G_K is an extension of the semidirect product of G_k with the group $\mathbb{Z}_l$, where G_k acts by the cyclotomic character, by a group of order prime to l. This allows to obtain a similar description of the algebra $\bigoplus_n H^n(G_K,\mu_l^{\otimes n})$ in terms of the algebra $\bigoplus_n H^n(G_k,\mu_l^{\otimes n})$.

2.3. Nonarchimedean local fields

Let K be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((z))$ and $l\neq\operatorname{char}K$ be a prime number. The computation of $\mathrm{K}_n^{\mathrm{M}}(K)/l\simeq H^n(G_K,\mu_l^{\otimes n})$ [17, Corollary on p. 268] is provided by the local class field theory.

When K does not contain a primitive l-root of unity, one has $\mathrm{K}_n^{\mathrm{M}}(K)/l=0$ for $n\geqslant 2$. When K contains a primitive l-root of unity, one has $\mathrm{K}_n^{\mathrm{M}}(K)/l=0$ for $n\geqslant 3$, and $\mathrm{K}_2^{\mathrm{M}}(K)/l\simeq\mu_l$. In the latter case, the multiplication map

$$\mathrm{K}_1^{\mathrm{M}}(K)/l\otimes_{\mathbb{Z}/l}\mathrm{K}_1^{\mathrm{M}}(K)/l\longrightarrow\mathrm{K}_2^{\mathrm{M}}(K)/l$$

is a nondegenerate pairing. This pairing provides the comparison between the isomorphism $G_K^{\mathrm{ab}}/l\simeq K^*/K^{*l}$ of the local class field theory and the isomorphism $G_K^{\mathrm{ab}}/l\simeq\operatorname{Hom}_{\mathbb{Z}}(K^*,\mu_l)$ of the Kummer theory; hence the nondegeneracy.

The $\mathbb{Z}/l$-vector space $\mathrm{K}_1^{\mathrm{M}}(K)/l$ is finite-dimensional. Except when K is a finite extension of $\mathbb{Q}_l$, its dimension is equal to 2 when K contains a primitive l-root of unity, and 1 otherwise. When K is a finite extension of $\mathbb{Q}_l$, the dimension is $\geqslant 3$ when K contains a primitive l-root of unity, and $\geqslant 2$ otherwise.

When l is odd and K contains a primitive l-root of unity, or $l=2$ and K contains a square root of -1, $\mathrm{K}_1^{\mathrm{M}}(K)/l$ is a symplectic vector space with respect to the multiplication pairing. In other words, the multiplication pairing is skew-symmetric, i.e., $\{x,x\}=0$ for all $x\in\mathrm{K}_1^{\mathrm{M}}(K)/l$. In particular, the dimension of $\mathrm{K}_1^{\mathrm{M}}(K)/l$ is even.

When $l=2$ and K does not contain a square root of -1, it follows from nondegeneracy and the relation $\{x,x\}=\{-1,x\}$ that $\dim\mathrm{K}_1^{\mathrm{M}}(K)/l$ is even when $\{-1,-1\}=0$ and odd when $\{-1,-1\}\neq 0$. In both cases, the isomorphism class of the (nonskew-symmetric) pairing form is determined by $\dim\mathrm{K}_1^{\mathrm{M}}(K)/l$.

Except when K is a finite extension of $\mathbb{Q}_l$, the product $\{x,y\}$ of the classes of two elements $x,y\in K^*$ with the logarithmic valuations $v(x)=0=v(y)$ is always zero in $\mathrm{K}_2^{\mathrm{M}}(K)/l$, and the product of the classes of two elements $x\in K^*\setminus K^{*l}$ and $y\in K^*$ with $v(x)=0$ and $v(y)$ not divisible by l is always nonzero in $\mathrm{K}_2^{\mathrm{M}}(K)/l$.

2.4. Global fields with root of unity

For any field K and a prime number $l\neq\operatorname{char}K$, the group $H^2(G_K,\mu_l)$ is isomorphic to the subgroup ${}_l\mathrm{Br}\,K$ of the Brauer group $\mathrm{Br}\,K$ consisting of all elements annihilated by l.

For any finite extension K of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ and a prime number $l \neq \operatorname{char} K$ the natural map $\mathrm{K}_2^{\mathrm{M}}(K)/l \longrightarrow H^2(K, \mu_l^{\otimes 2})$ is an isomorphism [17, Theorem 5.1]. Assuming that K contains a primitive l-root of unity and combining this isomorphism with the computation of $\operatorname{Br} K$ provided by the global class field theory, we see that there is a natural short exact sequence

$$0 \longrightarrow \mathrm{K}_2^{\mathrm{M}}(K)/l \longrightarrow \bigoplus_v \mathrm{K}_2^{\mathrm{M}}(K_v)/l \simeq \bigoplus_{v'} \mu_l \longrightarrow \mu_l \longrightarrow 0.$$

Here the direct sum in the second term is over all valuations v of K, and the direct sum in the third term is taken over all the valuations v' not including the complex valuations if $l = 2$, or not including the archimedean valuations if l is odd. The rightmost map is the simple summation over v'. The assertion that the composition of the two maps vanishes is one of the formulations of the reciprocity law.

Given two elements x and y in K^*, or K_v^*, or K_v^*/K_v^{*l}, etc., we will denote by $\{x, y\}_v$ their product in $\mathrm{K}_2^{\mathrm{M}}(K_v)/l$, and identify the latter group with μ_l (assuming that we are not in the case when $\mathrm{K}_2^{\mathrm{M}}(K_v)/l = 0$). So the reciprocity law takes the form $\sum_v \{x, y\}_v = 0$ for any $x, y \in K^*$.

For any $n \geqslant 3$ and any global field K, the natural map

$$\mathrm{K}_n^{\mathrm{M}}(K) \longrightarrow \bigoplus_v \mathrm{K}_n^{\mathrm{M}}(K_v)/2$$

is an isomorphism [2, Theorem II.2.1(3)]. Here the summation is over all the real valuations v of K. One can obtain a compatible description of $H^n(G_K, \mu_l^{\otimes n})$ from the global class field theory, by computing $H^n(G_K, \mu_l)$ in terms of $H^n(G_K, K^*)$ and the latter in terms of $H^n(G_{K_v}, K_v^*)$ and the cohomology of the classes of idèles.

2.5. Exceptional set of valuations

Let K be a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing a primitive l-root of unity. Let S be a finite set of valuations of K containing all the archimedean valuations and all the valuations lying over l, and generating the class group of the field K. Denote by W_S the $\mathbb{Z}/l$-vector space

$$W_S = \bigoplus_{v \in S} K_v^*/K_v^{*l},$$

and let $K_S \subset K^*$ denote the subgroup of all elements b having the logarithmic valuation $p(b) = 0$ (in other words, b is integral and integrally invertible in K_p) for all valuations $p \notin S$.

Lemma 1. *The natural map $K_S/l \longrightarrow W_S$ is injective.*

Proof. This is [6, Lemma VII.9.2 and Remark VII.9.3]. □

Define a μ_l-valued bilinear form on W_S as the orthogonal sum of the bilinear forms on K_v^*/K_v^{*l}, that is $(x,y)_S = \sum_{v\in S}\{x_v, y_v\}_v$.

Lemma 2. *The subspace $K_S/l \subset W_S$ coincides with its own orthogonal complement with respect to the pairing form $(-,-)_S$.*

Proof. The pairing form on W_S is symmetric or skew-symmetric and nondegenerate, since the pairings on K_v^*/K_v^{*l} are. By the reciprocity law, one has $(K_S/l, K_S/l) = 0$. It remains to check that $\dim W_S = 2\dim K_S/l$. We will show that $\dim K_S/l = \#S$ and $\dim W_S = 2\#S$, where $\#S$ is the number of elements in S.

Indeed, by Dirichlet's unit theorem the group K_S is the direct sum of a free abelian group of rank $\#S - 1$ and the finite cyclic group of roots of unity in K, whose order is divisible by l by assumption. This computes $\dim K_S/l$.

To compute $\dim W_S$, consider two cases separately. When K is a finite extension of $\mathbb{F}_q(z)$, one has $\dim K_v^*/K_v^{*l} = 2$ for all $v \in S$. When K is a finite extension of $\mathbb{Q}$, one has

- $\dim K_v^*/K_v^{*l} = 2$ for all nonarchimedean $v \in S$ not lying over l;
- $\dim K_v^*/K_v^{*l} = 2 + [K_v : \mathbb{Q}_l]$ for any nonarchimedean $v \in S$ lying over l;
- $\dim K_v^*/K_v^{*l} = 0$ when $K_v = \mathbb{C}$; and
- $\dim K_v^*/K_v^{*l} = 1$ when $K_v = \mathbb{R}$, since such valuations v can only exist when $l = 2$, as $\mathbb{R}$ does not contain any other l-roots of unity.

Summing this up, one easily obtains $\dim W_S = 2\#S$. □

The following lemma can be thought of as a kind of "approximation theorem for idèles modulo l".

Lemma 3. *Let w be an element of W_S and D be a divisor of K supported outside of S, i.e., a formal linear combination of valuations of K, not belonging to S, with integral coefficients. The pairing with w defines a μ_l-valued linear function on K_S/l, and another such function is provided by the linear combination of Frobenius elements in* $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ *corresponding to the divisor D. Suppose that these two linear functions coincide. Then there exists an element $a \in K^*$ whose image in W_S is equal to w and whose divisor outside S is equal to D. Furthermore, the element a is unique modulo K_S^l.*

Proof. Since S generates the class group of K, one can find an element $b \in K^*$ whose divisor outside of S is equal to D. Let us denote the image of b in W_S also by b. By the reciprocity law, the element $w/b \in W_S$ is orthogonal to K_S/l, hence by Lemma 2 it belongs to K_S/l. Lift it to an element $c \in K_S$ and set $a = bc$. The uniqueness follows immediately from Lemma 1. □

2.6. Symplectic case

The following lemma is useful in the case of a global field K which contains a primitive l-root of unity when l is odd, or contains a square root of -1 when $l = 2$. Recall that the pairing $(-,-)_S$ is skew-symmetric in this case.

Lemma. *Suppose that a (finite-dimensional) symplectic vector space W over a field k is decomposed into an orthogonal direct sum of symplectic vector spaces W_v. Let L be a Lagrangian subspace in W. Then there exist Lagrangian subspaces M_v in W_v such that the direct sum of M_v is complementary to L in W.*

Proof. It suffices to consider the case when all W_v are two-dimensional (otherwise decompose every one of them into an orthogonal direct sum of two-dimensional symplectic vector spaces). Order the set of indices $\{v\}$ and proceed by induction, choosing subspaces $M_{v'} \subset W_{v'}$ such that $\bigoplus_{v' \leqslant v} M_{v'}$ does not intersect L. Assume that $M_{v'}$ have been chosen so that $\bigoplus_{v'<v} M_{v'}$ does not intersect L. Then, since L is Lagrangian, there exists at most one line $N_v \subset W_v$ for which $N_v \oplus \bigoplus_{v'<v} M_{v'}$ intersects L. So a line $M_v \subset W_v$ such that $\bigoplus_{v' \leqslant v} M_{v'}$ does not intersect L can always be chosen. $\square$

2.7. Global fields without root of unity

Let K be a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ that does not contain a primitive l-root of unity. Notice that l is necessarily odd in this case, so $\mathrm{K}^{\mathrm{M}}(K)/l$ is supercommutative and $\mathrm{K}^{\mathrm{M}}_n(K)/l = 0$ for $n \geqslant 3$.

Set $L = K[\sqrt[l]{1}\,]$. The degree $[L : K]$ of this field extension divides $l - 1$ and is prime to l. Passing to $\mathrm{Gal}(L/K)$-invariants in the description of $\mathrm{K}^{\mathrm{M}}_2(L)/l \simeq H^2(G_L, \mu_l^{\otimes 2})$ given in 2.4, one concludes that the $\mathbb{Z}/l$-vector space $\mathrm{K}^{\mathrm{M}}_2(K)/l \simeq H^2(G_K, \mu_l^{\otimes 2})$ is isomorphic to the direct sum of the groups $\mu_{l,v}$ of l-roots of unity in K_v over all the nonarchimedean valuations v of K for which K_v contains a primitive l-root of unity.

So the product $\{x, y\}$ in $\mathrm{K}^{\mathrm{M}}_2(K)/l$ of the classes of two elements $x, y \in K^*$ can be considered as the collection of the local products $\{x, y\}_v \in \mathrm{K}^{\mathrm{M}}_2(K_v)/l \simeq \mu_{l,v}$ indexed by all such valuations v. There are no relations between the local products: any finite collection of elements in $\mu_{l,v}$ corresponds to an element of $\mathrm{K}^{\mathrm{M}}_2(K)/l$.

Let S' and S be finite sets of valuations of K and L, respectively, such that S is the set of all valuations of L lying over the valuations of K belonging to S', the set S' contains all the archimedean valuations of K and all the valuations, lying over l, the set S' generates the class group of K, and the set S generates the class group of L. Set $W_S = \bigoplus_{v \in S} L_v^*/L_v^{*l}$, and let $W'_{S'}$ denote the direct sum of K_v^*/K_v^{*l} over all the valuations $v \in S'$ such that K_v contains a primitive l-root of unity. There is a natural injective map $W'_{S'} \longrightarrow W_S$. Set $M = L[\sqrt[l]{L_S}\,]$.

Lemma. *For any element $w' \in W'_{S'}$ there exist infinitely many valuations p of K outside S' for which K_p does not contain a primitive l-root of unity and there exists* *an element $a_p \in K^*$ whose image in $W'_{S'}$ is equal to w' and whose divisor outside S' is equal to p.*

Proof. Let D' be a divisor of the field K outside of S'; one can naturally assign to it a divisor D of the field L outside of S. For any element $w' \in W'_{S'}$, consider its image $w \in W_S$, the pairing with w in W_S as a linear function $L_S/l \longrightarrow \mu_l$, and this linear function as an element of $\mathrm{Gal}(M/L)$. If the linear combination of Frobenius elements in $\mathrm{Gal}(M/L)$ corresponding to D is equal to this element, then by Lemma 2.5.3 there exists a unique, up to L_S^l, element $a_2 \in L^*$ whose image in W_S is equal to w and whose divisor outside of S is equal to D. Due to the uniqueness, this element defines a $\mathrm{Gal}(L/K)$-invariant class in L^*/L^{*l}.

From the short exact sequence $\mu_l \longrightarrow L^* \longrightarrow L^{*l}$, Hilbert's Theorem 90, and the order of the group $\mathrm{Gal}(L/K)$ being prime to l, one can see that $H^1(\mathrm{Gal}(L/K), L^{*l}) = 0$. Hence there exists a $\mathrm{Gal}(L/K)$-invariant element $a_1 \in L^*$ which differs from a_2 by an element of L^{*l}. The element a_1 belongs to K^*, and its image in $W'_{S'}$ is equal to w', since its image in W_S is equal to w. The divisor of $a_1 \in K^*$ is congruent to D' modulo l, since the divisor of $a_1 \in L^*$ is congruent to D modulo l and the ramification indices in the extension L/K are prime to l. Since S' generates the class group of K, one can multiply a_1 with an element of K^{*l} so that the resulting element $a \in K$ has the divisor D' outside of S'; clearly, the image of a in $W'_{S'}$ is equal to w'.

The linear combination of Frobenius elements in $\mathrm{Gal}(M/L)$ corresponding to D is the image of the linear combination of Frobenius elements in $\mathrm{Gal}(M/K)^{\mathrm{ab}}$ corresponding to D' under the transfer map $\mathrm{tr}\colon \mathrm{Gal}(M/K)^{\mathrm{ab}} \longrightarrow \mathrm{Gal}(M/L)$. Since the element $h \in \mathrm{Gal}(M/L)$ corresponding to w is invariant under $\mathrm{Gal}(L/K)$ and the order of the latter group is prime to l, the element h is equal to the transfer of the element $g \in \mathrm{Gal}(M/K)^{\mathrm{ab}}$ obtained as the image of $h/[L:K]$ under the map $\mathrm{Gal}(M/L) \longrightarrow \mathrm{Gal}(M/K)^{\mathrm{ab}}$.

Being an extension of the abelian groups $\mathrm{Gal}(L/K)$ and $\mathrm{Gal}(M/L)$ of coprime orders, the group $\mathrm{Gal}(M/K)$ is their semidirect product. So the group $\mathrm{Gal}(L/K)$ can be embedded into $\mathrm{Gal}(M/K)$. Choose a nontrivial element $f \in \mathrm{Gal}(L/K)$ and consider the product $q = (h/[L:K])f \in \mathrm{Gal}(M/K)$. Its image in $\mathrm{Gal}(M/K)^{\mathrm{ab}}$ is the product of g with the image of f, which we will denote also by f. For the reasons of orders of the elements, or commutation of transfer with the corestriction in group homology, it is clear that $\mathrm{tr}(f) = 1$ and $\mathrm{tr}(gf) = \mathrm{tr}(g) = h$ in $\mathrm{Gal}(L/K)$.

By Chebotarev's density theorem, there exist infinitely many valuations p of the field K outside of S' whose Frobenius elements in $\mathrm{Gal}(M/K)$ are conjugate to q. In this case, the Frobenius element of p in $\mathrm{Gal}(L/K)$ is nontrivial, so K_p does not contain a primitive l-root of unity. Furthermore, let D be the divisor of L outside S equal to the image of p (which is considered as a divisor of K). Then the linear combination of Frobenius elements in $\mathrm{Gal}(M/L)$ corresponding to D is equal to the pairing with w in W_S as a linear function $L_S/l \longrightarrow \mu_l$. Hence the element $a_p = a \in K^*$ constructed above has the desired properties. $\square$

3. Koszulity for local fields

For any field K and a prime number l denote by $\Lambda(K,l)$ the graded algebra over $\mathbb{Z}/l$ generated by $\Lambda_1(K,l) = K^*/K^{*l}$ with the relations $\{x,-x\} = 0$ for $x \in K^*$. The algebra $\Lambda(K,l)$ is always Koszul.

Indeed, when l is odd or K contains a square root of -1, this algebra is simply the exterior algebra generated by K^*/K^{*l}. Otherwise, choose any well-ordered basis $\{x_\alpha\}$ of the $\mathbb{Z}/l$-vector space K^*/K^{*l} such that the first basis vector is $x_0 = \{-1\}$, and consider the related Γ-valued filtration F of $\Lambda(K,l)$ (see 1.5–1.6). Then the algebra $\operatorname{gr}^F \Lambda(K,l)$ is isomorphic to the tensor product of the symmetric algebra with one generator $\{-1\}$ and the exterior algebra generated by $\Lambda_1(K,l)/\langle\{-1\}\rangle$.

There is a natural morphism of graded $\mathbb{Z}/l$-algebras $\Lambda(K,l) \longrightarrow \mathrm{K}^{\mathrm{M}}(K)/l$. Let J_K denote its kernel; it is the ideal generated by the Steinberg symbols.

Theorem 1.

(1) *Let K be an algebraic extension of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((z))$, and l be a prime number. Then the ideal $J_K \subset \Lambda(K,l)$ is a Koszul $\Lambda(K,l)$-module (in the grading shifted by 1). In particular, the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul.*

(2) *Let K be a Henselian discrete valuation field with the residue field k and $l \neq \operatorname{char} k$ be a prime number. Then the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul whenever the algebra $\mathrm{K}^{\mathrm{M}}(k)/l$ is. The ideal J_K is a Koszul $\Lambda(K,l)$-module whenever the ideal J_k is a Koszul $\Lambda(k,l)$-module.*

Proof. Part (1): the cases $K \supset \mathbb{R}$ and $l = \operatorname{char} K$ are trivial in view of 2.1 and 2.0, respectively. Indeed, in the former case one has $J_K = 0$, and in the latter case one can use the fact that the left A-module $A_{\geqslant 2} = A_2 \oplus A_3 \oplus \cdots$ is Koszul for any Koszul algebra A. Passing to the inductive limit, one reduces the problem to the case when K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((z))$ (and $l \neq \operatorname{char} K$).

The case when K does not contain a primitive l-root of unity is similar to the above; see 2.3. When l is odd and K contains a primitive l-root of unity, or $l = 2$ and K contains a square roof of -1, one can choose any ordered basis of K^*/K^{*l}; the corresponding supercommutative monomial algebra $\operatorname{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$ obviously satisfies the condition of Proposition 1.7(2), since the graph T contains only one edge.

When $l = 2$, the class of -1 is nontrivial in $\mathrm{K}^{\mathrm{M}}_1(K)/2$, but $\{-1,-1\} = 0$ in $\mathrm{K}^{\mathrm{M}}_2(K)/2$, choose any ordered basis $\{x_\alpha\}$ of K^*/K^{*2} with the minimal element $x_0 = \{-1\}$ and the second minimal element x_1 such that $x_0x_1 \neq 0$. Consider the related Γ-valued filtrations F of $\Lambda(K,2)$ and $\mathrm{K}^{\mathrm{M}}(K)/2$. The corresponding algebra $\operatorname{gr}^F \Lambda(K,2)$ is the tensor product of the symmetric algebra generated by x_0 and the exterior algebra in other variables. The algebra $\operatorname{gr}^F \mathrm{K}^{\mathrm{M}}(K)/2$ is the commutative monomial algebra with the only nonzero monomial x_0x_1 in the degrees $\geqslant 2$. One easily checks that both the $\operatorname{gr}^F \Lambda(K,2)$-module spanned by x_0 and x_0x_1 and the quotient module of $\operatorname{gr}^F \mathrm{K}^{\mathrm{M}}_+(K)/2$ by this submodule are Koszul.

When $l = 2$ and $\{-1,-1\} \neq 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/2$, choose any ordered basis $\{x_\alpha\}$ of K^*/K^{*2} in which the minimal three elements x_0, x_1, x_2 are such that $x_1 = \{-1\}$ and $x_0^2 = x_0x_1 = 0 \neq x_0x_2$ in $\mathrm{K}_2^{\mathrm{M}}(K)/2$. Consider the related Γ-valued filtrations F of $\Lambda(K,2)$ and $\mathrm{K}^{\mathrm{M}}(K)/2$. The corresponding algebra $\mathrm{gr}^F\,\mathrm{K}^{\mathrm{M}}(K)/2$ is the commutative monomial algebra with the only nonzero monomial x_0x_2 in the degrees $\geqslant 2$. The algebra $\mathrm{gr}^F\,\Lambda(K,2)$ is the quadratic commutative monomial algebra with the defining relations $x_0x_1 = 0$ and $x_\alpha^2 = 0$ for $\alpha \geqslant 2$. As above, one easily checks that the graded $\mathrm{gr}^F\,\Lambda(K,2)$-module $\mathrm{gr}^F\,\mathrm{K}_+^{\mathrm{M}}(K)/2$ is Koszul.

Part (2): the argument is based on the description of $\mathrm{K}^{\mathrm{M}}(K)/l$ given in 2.2. This time, the increasing filtrations on graded algebras that we need to use are indexed by the conventional integers. Set $F_0\mathrm{K}^{\mathrm{M}}(K)/l = \mathrm{K}^{\mathrm{M}}(k)/l$ and $F_1\mathrm{K}^{\mathrm{M}}(K)/l = \mathrm{K}^{\mathrm{M}}(K)/l$. Then the graded algebra $\mathrm{gr}^F\,\mathrm{K}^{\mathrm{M}}(K)/l$ is the supertensor product of $\mathrm{K}^{\mathrm{M}}(k)/l$ and the exterior algebra with one generator $\{\pi\}$ in degree 1, hence it is Koszul provided that $\mathrm{K}^{\mathrm{M}}(k)/l$ is [9, Corollary 1.2 of Chapter 3].

To prove the second assertion, define also a compatible increasing filtration F on $\Lambda(K,l)$ by the rule $F_0\Lambda(K,l) = \Lambda(k,l)$ and $F_1\Lambda(K,l) = \Lambda(K,l)$. Then the graded algebra $\mathrm{gr}^F\,\Lambda(K,l)$ is the supertensor product of $\Lambda(k,l)$ and the exterior algebra with one generator $\{\pi\}$, so it remains to apply the module part of the same corollary from [9] (or more precisely, its straightforward generalization to the infinite-dimensional setting). $\square$

Theorem 2.

(1) *Let K be an algebraic extension of $\mathbb{R}$, $\mathbb{Q}_p$, or $\mathbb{F}_p((z))$, and l be a prime number. Let c be an element of $\mathrm{K}_1^{\mathrm{M}}(K)/l$. Then the ideal $(c) \subset \mathrm{K}^{\mathrm{M}}(K)/l$ is a Koszul module over $\mathrm{K}^{\mathrm{M}}(K)/l$.*

(2) *Let K be a Henselian discrete valuation field with the residue field k and $l \neq \operatorname{char} k$ be a prime number. Assume that the graded algebra $\mathrm{K}^{\mathrm{M}}(k)/l$ is Koszul and for any element $c \in \mathrm{K}_1^{\mathrm{M}}(k)/l$ the ideal $(c) \subset \mathrm{K}^{\mathrm{M}}(k)/l$ is a Koszul module over $\mathrm{K}^{\mathrm{M}}(k)/l$. Then the graded algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ has the same properties.*

Proof. Part (1): the case of an infinite algebraic extension is deduced from that of a finite extension by passing to an inductive limit. When $K \supset \mathbb{R}$, $l = \operatorname{char} K$, or K does not contain a primitive l-root of unity, the assertion is trivial. The assertion is also trivial when $c = 0$. So let us assume that K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((z))$ containing a primitive l-root of unity and $c \neq 0$.

When $\{c,c\} = 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$, choose any ordered basis of K^*/K^{*l} starting with $x_0 = c$ (for simplicity, one can also pick the next basis vector x_1 so that $x_0x_1 \neq 0$) and use the result of 1.8. This covers the cases when l is odd or K contains a square root of -1. When $\{c,c\} \neq 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/2$ but $c \neq \{-1\}$ in $\mathrm{K}_1^{\mathrm{M}}(K)/2$, choose an ordered basis of K^*/K^{*2} starting with x_0, x_1 such that $x_0^2 = 0$, $x_1 = c$, and $x_0x_1 \neq 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/2$. Then for the corresponding Γ-valued filtration F on $\mathrm{K}^{\mathrm{M}}(K)/2$ the ideal $\mathrm{gr}^F(c) \subset \mathrm{gr}^F\,\mathrm{K}^{\mathrm{M}}(K)/2$ is generated by the element c, so one can argue as in 1.8.

It remains to consider the case when $c = \{-1\}$ in $\mathrm{K}_1^{\mathrm{M}}(K)/2$ and $\{-1,-1\} \neq 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/2$. Choose an ordered basis of K^*/K^{*2} starting from x_0, x_1, x_2 such that $x_0^2 = 0 = x_1^2$, $x_0x_1 \neq 0$, and $x_2 = \{-1\}$ (or $x_0^2 = 0$, $x_1 = \{-1\}$, and $x_0x_2 \neq 0$). Consider the related Γ-valued filtration on $\mathrm{K}^{\mathrm{M}}(K)/2$. Let us define a Γ-valued filtration on the ideal $(\{-1\})$ that is compatible with the action of $\mathrm{K}^{\mathrm{M}}(K)/2$ on $(\{-1\})$ but is not induced by the embedding $(\{-1\}) \subset \mathrm{K}^{\mathrm{M}}(K)/2$.

Namely, choose any Γ-valued filtration F on the degree 1 component of the ideal $(\{-1\})$ and extend it to the degree 2 component in such a way that the $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/2$-module $\mathrm{gr}^F(c)$ be generated by its degree 1 component (cf. 1.6). Then the $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/2$-module $\mathrm{gr}^F(c)$ is isomorphic to the quotient module of the quadratic commutative monomial algebra $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/2$ by the ideal generated by all the x_α except x_2 (resp., x_1), so it remains to use the result of [9, proof of Theorem 8.1 from Chapter 4]. It is essential here that $x_2^2 = 0$ (resp., $x_1^2 = 0$) in $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/2$.

Part (2): recall that $\mathrm{K}^{\mathrm{M}}(k)/l$ can be naturally considered as a subalgebra of $\mathrm{K}^{\mathrm{M}}(K)/l$. If $c \in \mathrm{K}_1^{\mathrm{M}}(k)/l$, consider the filtration F on $\mathrm{K}^{\mathrm{M}}(K)/l$ defined in the proof of part (2) of Theorem 1 and the induced filtration on the ideal $c\mathrm{K}^{\mathrm{M}}(K)/l$. Then $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$ is the supertensor product of $\mathrm{K}^{\mathrm{M}}(k)/l$ with the exterior algebra with one generator in degree 1, and the $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$-module $\mathrm{gr}^F c\mathrm{K}^{\mathrm{M}}(K)/l$ is the supertensor product of $\mathrm{gr}^F c\mathrm{K}^{\mathrm{M}}(k)/l$ with the same exterior algebra. So it remains to apply [9, Corollary 1.2 of Chapter 3].

If $c \in \mathrm{K}_1^{\mathrm{M}}(K)/l$ but $c \notin \mathrm{K}_1^{\mathrm{M}}(k)/l$, one can assume that $c = \{\pi\}$ in the notation of 2.2. In this case the ideal $(c) \subset \mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul whenever the algebra $\mathrm{K}^{\mathrm{M}}(k)/l$ is Koszul. It suffices to consider the same filtration F on $\mathrm{K}^{\mathrm{M}}(K)/l$ and the induced filtration on the ideal $(\{\pi\})$. The $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$-module $\mathrm{gr}^F(\{\pi\})$ is the supertensor product of the $\mathrm{K}^{\mathrm{M}}(k)/l$-module $\mathrm{K}^{\mathrm{M}}(k)/l$ and the trivial one-dimensional module over the exterior algebra with one generator. $\square$

4. Module Koszulity in symplectic case

Let l be a prime number, and K be an algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing a primitive l-root of unity if l is odd, or containing a square root of -1 if $l = 2$. Let $\Lambda(K,l)$ denote the exterior algebra generated by the $\mathbb{Z}/l$-vector space K^*/K^{*l}, and J_K denote the kernel of the morphism of graded algebras $\Lambda(K,l) \longrightarrow \mathrm{K}^{\mathrm{M}}(K)/l$.

Theorem. *The ideal J_K is a Koszul module over $\Lambda(K,l)$ (in the grading shifted by 1). In other words, the $\Lambda(K,l)$-module $\mathrm{K}_+^{\mathrm{M}}(K)/l$ is Koszul.*

Proof. Passing to the inductive limit of finite extensions of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing the needed root of unity, one reduces the problem to the case when K is such a finite extension. So let K be a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing a primitive l-root of unity if l is odd, or containing a square root of -1 if $l = 2$.

Apply Lemma 2.6 to the case of the symplectic vector space W_S decomposed into the orthogonal direct sum of symplectic subspaces K_v^*/K_v^{*l}, $v \in S$, and the Lagrangian subspace $K_S/l \subset W_S$ (see 2.5). Let $M_v \subset K_v^*/K_v^{*l}$ be the Lagrangian subspaces so obtained. The restriction of the form $(-,-)_S$ defines a nondegenerate pairing between K_S/l and $\bigoplus_{v\in S} M_v$, which allows to identify K_S/l with the direct sum of the dual spaces $\bigoplus_{v\in S} M_v^*$. So we have constructed a direct sum decomposition of K_S/l indexed by $v \in S$; let $\{b_i : i = 0, \dots, \#S-1\}$ be a basis in K_S/l whose elements belong to the direct summands of this decomposition. Let b_i^* denote the dual basis in $\bigoplus M_v$; introduce the notation $b_i^* \in M_{v(i)}$. Notice that the image of b_i in K_v^*/K_v^{*l} belongs to M_v for all $v \neq v(i)$.

For any divisor D of K outside S there exists a unique element $a_D \in K^*/K_S^l$ whose divisor outside S is equal to D and whose image in W_S belongs to $\bigoplus M_v$. In particular, for any valuation p of K outside S there exists a unique element $a_p \in K^*/K_S^l$ with this property, whose divisor outside S is equal to p. The pairing with the image of a_p in $\bigoplus M_v$, as a linear function $K_S/l \longrightarrow \mu_l$, coincides with the Frobenius element of p in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ (see Lemma 2.5.3).

Choose a well-ordered basis of K^*/K^{*l} consisting of the elements b_i and a_p (in any order). Consider the related Γ-valued filtration F on $\mathrm{K}^{\mathrm{M}}(K)/l$ and pass to the associated quotient monomial algebra $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$. The graph T of nonzero quadratic monomials in the latter algebra contains no cycles, so the assertion of theorem follows from Proposition 1.7(2). Indeed, it suffices to notice that one can assign a valuation to every basis element in this basis so that the product of any two basis elements can only have nonzero components (see 2.4) at the two valuations corresponding to the two basis vectors being multiplied. Thus for any elements $x_1, x_2, \dots, x_n$ in this basis the products x_1x_2, $x_2x_3, \dots, x_{n-1}x_n, x_nx_1$ cannot be linearly independent in $\mathrm{K}_2^{\mathrm{M}}(K)$ (one also has to take into account the reciprocity law).

To obtain a more explicit PBW-basis, choose for each $i = 1, \dots, \#S-1$ a valuation p_i of K outside S such that the Frobenius element of p_i in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ is equal to the pairing with $b_0^* + b_i^*$, while its Frobenius element in $\mathrm{Gal}(K[\sqrt[l]{a_{p_1}}, \dots, \sqrt[l]{a_{p_{i-1}}}\,]/K)$ is trivial. Denote by q those valuations of K outside of S and $\{p_i\}$ whose Frobenius elements in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ are equal to the pairing with b_0^*, while the Frobenius elements in $\mathrm{Gal}(K[\sqrt[l]{a_{p_1}}, \dots, \sqrt[l]{a_{p_{\#S-1}}}\,]/K)$ are trivial. Denote by r the remaining valuations. Notice that for each valuation r there exists a valuation q whose Frobenius element in $\mathrm{Gal}(K[\sqrt[l]{a_r}\,])$ is nontrivial.

Choose the following well-ordered basis of K^*/K^{*l}

$$b_0, a_{p_1}, \dots, a_{p_{\#S-1}}, a_q, b_1, \dots, b_{\#S-1}, a_r,$$

where the ordering between a_q and between a_r is arbitrary. Then the set of surviving quadratic monomials T will consist of all the monomials $b_0a_{p_i}$ and b_0a_q, some of the monomials b_0b_i or $a_{p_i}b_i$ (exactly one monomial of one of these forms for every nonarchimedean valuation $v(i) \neq v(0)$ in S), and some of the monomials b_0a_r, $a_{p_i}a_r$, or a_qa_r (exactly one monomial of one of these forms for every valuation r). $\square$

Remark. One may wish to extend the above result to the global fields not necessarily containing a square root of -1 when $l = 2$ in the way suggested by Theorem 3.1. There is the following obstacle, however. If one tries to argue as in the proof of Theorem 3.1, one has to find a well-ordered basis of K^*/K^{*2} that defines a PBW-basis for *both* algebras $\Lambda(K,2)$ and $\mathrm{K}^{\mathrm{M}}(K)/2$. But constructing a PBW-basis for $\Lambda(K,2)$ requires putting the element -1 near the bottom of the Γ_1-valued filtration on K^*/K^{*2}, while constructing a PBW-basis for $\mathrm{K}^{\mathrm{M}}(K)/2$ requires putting all elements of K^*/K^{*2} that are negative at some real valuations near the top of that filtration (cf. the construction in Section 5).

5. Algebra Koszulity in general case

Let l be a prime number, and K be an algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing a primitive l-root of unity.

Theorem. *The graded algebra* $\mathrm{K}^{\mathrm{M}}(K)/l$ *is Koszul.*

Proof. As above, one can assume that K is a finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$. The case when l is odd follows from the result of Section 4 and [11, Corollary 6.2(c)], so we will implicitly assume that $l = 2$ (though this is not necessary).

Choose a set of exceptional valuations S for the field K satisfying a slightly stronger condition than in 2.5: namely, let it be additionally required that the nonarchimedean valuations in S generate the extended class group of K (i.e., the class group defined taking into account the signs of elements of K^* at the real valuations). Let $K_S^+ \subset K_S$ and $K^+ \subset K^*$ denote the subgroups of all elements that are positive at all the real valuations. Then one has $K^* = K^+K_S$.

For each nonarchimedean valuation $s \in S$ pick an element $w_s \in K_s^*/K_s^{*l}$ orthogonal to the class of -1 in K_s^*/K_s^{*l} with respect to the pairing $\{-,-\}_s$, and consider w_s as an element of W_S. We need the pairings with the elements w_s to define nonzero linear functions $K_S^+/K_S^l \longrightarrow \mu_l$. Enlarging, if it be necessary, the set S, one can always choose such elements w_s.

Indeed, one only has to use the weak approximation theorem [6, Section II.6] in order to find a finite set of valuations $S' \supset S$ such that the pairings with the given elements w_s define nonzero linear functions $K_{S'}^+/K_{S'}^l \longrightarrow \mu_l$ for all $s \in S$. Now for any $u \in S' \setminus S$ there is an element $d \in K_{S'}^+$ with the logarithmic valuation $u(d)$ not divisible by l, because nonarchimedean valuations in S generate the extended class group. Since u is a nonarchimedean valuation not lying over l, choosing w_u to be the class of an element $b \in K_u^*$ with $u(b) = 0$ and $b \notin K_u^{*l}$ guarantees $\{-1, w_u\}_u = 0$ and $\{w_u, d\}_u \neq 0$, as desired.

Let p be a valuation of K outside S such that the Frobenius element of p in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ is trivial. Then by Lemma 2.5.3 there exists an element $a_p \in K^*$ whose divisor outside S is equal to p and whose image in W_S is zero. For each nonarchimedean valuation $u \in S$, pick a valuation q_u outside S whose Frobenius in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$

as a linear function $K_S \longrightarrow \mu_l$ is equal to the pairing with w_u, while the Frobenius in $\mathrm{Gal}(K[\sqrt[l]{a_p}]/K)$ is nontrivial. All the valuations q_u must be different. By the same lemma, there exists an element $a_{q_u} \in K^*$ whose divisor outside S is equal to q_u and whose image in W_S is equal to w_u. By the definition, the elements a_p and a_{q_u} belong to K^+, and one has $\{a_p, a_p\} = 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$.

For each valuation r of the field K outside of S, p, and q_u, choose an element $a_r \in K^+$ whose divisor outside of S is equal to r. Let us denote by r' those valuations r whose Frobenius element in $\mathrm{Gal}(K[\sqrt[l]{a_p}]/K)$ is nontrivial and by r'' the remaining ones. For each real valuation v pick an element $a_v \in K_S$ that is negative at v and positive at all the other real valuations. Choose any basis k_j in K_S^+/K_S^l.

Consider the following well-ordered basis of K^*/K^{*l}

$$a_p, a_{q_u}, k_j, a_{r'}, a_{r''}, a_v,$$

where the order within each group can be arbitrary. Consider the related Γ-valued filtration F on $\mathrm{K}^{\mathrm{M}}(K)/l$ and the associated quotient algebra $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$. The set of surviving quadratic monomials T consists of all the monomials $a_p a_{q_u}$ and $a_p a_{r'}$, some monomials of the form $a_{q_u} k_j$ (exactly one such monomial for every u), some monomials of the forms $a_{q_u} a_{r''}$, $k_j a_{r''}$, or $a_{r'} a_{r''}$ (exactly one monomial of one of these forms for every r''), and all the monomials a_v^2.

To prove these assertions, introduce the notion of the *support* of an element $\alpha \in \mathrm{K}_2^{\mathrm{M}}(K)/l$, defined as the set of all valuations y such that the image of α in $\mathrm{K}_2^{\mathrm{M}}(K_y)/l$ is nontrivial. The subspace of $\mathrm{K}_2^{\mathrm{M}}(K)/l$ consisting of all the elements supported inside a set of valuations Y has the dimension equal to the number of noncomplex valuations in Y minus one (see 2.4).

Let us discuss all the quadratic monomials in our basis in the order of their increase. The product $\{a_p, a_{q_u}\}$ is nonzero in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ and supported in p and q_u. Likewise, the product $\{a_p, a_{r'}\}$ is nontrivial and supported in p and r'. Taken together, these products generate the whole subspace of all elements supported inside the set of valuations p, q_u, and r'. Every element divisible by a_p in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ is supported inside this set, hence the products $\{a_p, a_v\}$ are linear combinations of smaller monomials with respect to our ordering. The products $\{a_p, k_j\}$ and $\{a_p, a_{r''}\}$ vanish in $\mathrm{K}_2^{\mathrm{M}}(K)/l$.

A product of the form $\{a_{q_{u_1}}, a_{q_{u_2}}\}$, where u_1 and u_2 belong to the set of valuations u, is supported inside the set of two valuations q_{u_1} and q_{u_2}, so it is a linear combination of smaller monomials. Indeed, this holds for $u_1 \neq u_2$, since $\{w_{u_1}, w_{u_2}\}_s = 0$ for all $s \in S$, and one actually has $\{a_{q_u}, a_{q_u}\} = 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ for $u_1 = u = u_2$, because $\{w_u, w_u\}_u = \{-1, w_u\}_u = 0$.

A product of the form $\{a_{q_u}, k_j\}$ either vanishes or is supported in u and q_u, and there exists at least one nonvanishing product of such form for every u. Taken together with the monomials containing a_p, this product generates the subspace of all elements supported inside the set of valuations in the above list together with the valuation u. The products $\{a_{q_u}, a_{r'}\}$ and $\{a_{q_u}, a_v\}$ are contained in this subspace, so they are linear combinations of smaller monomials. Taken together for all u, the products we have mentioned up to this point generate the subspace of all elements supported inside the set of valuations p, q_u, r', and all the nonarchimedean valuations from S. The products $\{k_{j_1}, k_{j_2}\}$, $\{k_j, a_{r'}\}$, $\{k_j, a_v\}$, $\{a_{r'_1}, a_{r'_2}\}$, and $\{a_{r'}, a_v\}$ are contained in this subspace, so they are also linear combinations of smaller monomials.

The product $\{a_{q_u}, a_{r''}\}$ is supported inside the set of three valuations u, q_u, and r'', so one can easily see that at most one such product belongs to the set of surviving monomials T, and this can only happen if the support of this product contains r''. On the other hand, recall that it is only the Γ_1-valued filtration on $\mathrm{K}_1^{\mathrm{M}}(K)/l$ rather than the basis itself that determines the set T. The filtration does not change if we assume that for those valuations r' whose Frobenius element is trivial in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ the element $a_{r'}$ is chosen in such a way that its image in W_S is trivial.

By Chebotarev's density theorem applied to the field extension $K[\sqrt[l]{K_S, a_p, a_{r''}}\,]/K$, for every valuation r'' there exists a valuation r' with the above property such that the Frobenius element of r' in $\mathrm{Gal}(K[\sqrt[l]{a_{r''}}\,]/K)$ is nontrivial. Then the product $\{a_{r'}, a_{r''}\}$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ is supported in r' and r'', and nonzero. For every r'', the set T contains the minimal of the products $\{a_{q_u}, a_{r''}\}$, $\{k_j, a_{r''}\}$, and $\{a_{r'}, a_{r''}\}$ whose support contains r''; the above argument shows that such a monomial exists.

Taken together, the products listed up to this point generated the whole subspace of all elements in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ supported outside of the real valuations v. The products $\{a_{r''_1}, a_{r''_2}\}$, $\{a_{r''}, a_v\}$, and $\{a_{v_1}, a_{v_2}\}$ for $v_1 \neq v_2$ belong to this subspace, so they are linear combinations of smaller monomials. The support of the product $\{a_v, a_v\}$ contains v and does not contain any other real valuations, so all the monomials of this type belong to T.

The set/graph T contains no triangles and no monomials divisible by a_v except a_v^2. The elements a_v^n are obviously linearly independent in $\mathrm{K}_n^{\mathrm{M}}(K)/l$ for all $n \geqslant 1$, so one readily checks that the algebra $\mathrm{gr}^F \mathrm{K}^{\mathrm{M}}(K)/l$ is quadratic. Consequently the algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul (see 1.3 and 1.6).

Alternatively, one can write after the elements a_{q_u} in the above well-ordering the elements $a_{q'}$ with zero images in W_S corresponding to the valuations q' outside of S and p whose Frobenius elements in $\mathrm{Gal}(K[\sqrt[l]{K_S}\,]/K)$ are trivial and in $\mathrm{Gal}(K[\sqrt[l]{a_p}\,]/K)$ are nontrivial. In this approach, one does not introduce the distinction between r' and r'', but instead excludes the valuations q' from the list of valuations r. Then the set of surviving quadratic monomials T will consist of all the monomials $a_p a_{q_u}$ and $a_p a_{q'}$, some monomials of the form $a_{q_u} k_j$ (exactly one such monomial for every u), some monomials of the forms $a_p a_r$, $a_{q_u} a_r$, or $a_{q'} a_r$ (exactly one monomial of one of these forms for every r), and all the monomials a_v^2. $\square$

6. Koszulity of annihilator ideals

Let l be a prime number, K be an algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ containing a primitive l-root of unity, and $c \in K^*/K^{*l}$ be an element such that $\{c, c\} = 0$ in $\mathrm{K}_2^{\mathrm{M}}(K)/l$.

In particular, when l is odd, or $l = 2$ and K contains a square root of -1, the element c can be arbitrary.

Theorem. *The ideal* $(c) = c\mathrm{K}^{\mathrm{M}}(K)/l \subset \mathrm{K}^{\mathrm{M}}(K)/l$ *is a Koszul module over the Koszul algebra* $\mathrm{K}^{\mathrm{M}}(K)/l$*.*

Proof. The argument below is a variation of the proof in Section 5. As above, we can assume that K is finite over $\mathbb{Q}$ or $\mathbb{F}_q(z)$; we can also assume that $c \neq 0$ in K^*/K^{*l}.

Choose a set of exceptional valuations S for the field K satisfying the conditions of Section 5 and containing the divisor of the element c. Choose an element $a_p \in K^*$ whose divisor outside S is equal to a certain valuation p and whose image in W_S is zero. For each nonarchimedean valuation $u \in S$ such that the image of c in K_u^*/K_u^{*l} is zero, choose an element $w_u \in K_u^*/K_u^{*l}$ such that $\{w_u, -1\}_u = 0$ and the pairing with w_u is a nonzero linear function $K_S^+/K_S^l \longrightarrow \mu_l$. Pick an element $a_{q_u} \in K^*$ whose divisor outside S is equal to a certain valuation q_u and whose image in W_S is equal to w_u. We also need the Frobenius element of q_u to be nontrivial in $\mathrm{Gal}(K[\sqrt[l]{a_p}\,]/K)$ and all the valuations q_u to be different. Clearly, one has a_p and $a_{q_u} \in K^+$ and $\{c, a_p\} = \{c, a_{q_u}\} = \{a_p, a_p\} = 0$ in $\mathrm{K}^{\mathrm{M}}(K)/l$.

For each valuation r outside of S, p and q_u, choose an element $a_r \in K^+$ whose divisor outside of S is equal to r. Denote by r' those valuations r whose Frobenius element in $\mathrm{Gal}(K[\sqrt[l]{a_p}\,]/K)$ is nontrivial and by r'' the remaining ones. For each real valuation v pick an element $a_v \in K_S$ that is negative at v and positive at all the other real valuations. Notice that $c \in K_S^+/K_S^l$; let elements $k_j \in K_S^+$ complement the element c to a basis of K_S^+/K_S^l.

Consider the following well-ordered basis of K^*/K^{*l}

$$c, a_p, a_{q_u}, k_j, a_{r'}, a_{r''}, a_v,$$

where the ordering within each group can be arbitrary. The related set T of surviving quadratic monomials consists of all the monomials $a_p a_{q_u}$, some monomials of the forms ck_j and $ca_{r'}$, some monomials of the form $a_{q_u} k_j$ (exactly one such monomial for every valuation u), some monomials $a_p a_{r'}$, some monomials of the forms $ca_{r''}$, $a_{q_u} a_{r''}$, $k_j a_{r''}$, or $a_{r'} a_{r''}$ (exactly one monomial of one of these forms for every r''), and all the monomials a_v^2.

Let us prove these assertions. Denote by Y the set of all valuations y of K for which $c \in K_y^* \setminus K_y^{*l}$. First we will have to show that the products $\{c, k_j\}$ and $\{c, a_{r'}\}$ generate the subgroup of all elements in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ supported inside the set of those valuations $y \in S$ or $y = r'$ that belong to Y. Specifically, for every element $w \in W_S$ such that $(c, w)_S = 0$ let us consider a valuation r' such that the image of $a_{r'}$ in W_S belongs to $w + K_S/l$. Then $\{c, a_{r'}\}_{r'} = 0$, hence $r' \notin Y$, and the support of $\{c, a_{r'}\}$ is contained in S.

The products $\{c, a_{r'}\}$ for such valuations r' generate the subgroup of all elements in $\mathrm{K}_2^{\mathrm{M}}(K)/l$ supported inside the set of valuations $S \cap Y$. Indeed, the subspace of vectors of the form $(\{c_s, w_s\}_s)_{s\in S}$ in $\bigoplus_{s\in S}\mu_l$, where $w = (w_s)_{s\in S}$ runs over all the elements in W_S for which $\sum_{s\in S}\{c_s, w_s\}_s = 0$, consists precisely of those vectors that belong to the kernel of the summation map $\bigoplus_{s\in S}\mu_l \longrightarrow \mu_l$ and are supported inside the set of all places $s\in S$ at which the component $c_s \in K_s^*/K_v^{*l}$ is nonzero. On the other hand, for a valuation r' belonging to Y the support of the product $\{c, a_{r'}\}$ is contained in $S\cup\{r'\}$ and contains r'.

The product $\{c, a_{r''}\}$ is supported inside S and r'', so it belongs to the set of surviving monomials T if and only if its support contains r'', that is $r''\in Y$. The products $\{c, k_j\}$, $\{c, a_{r'}\}$, and $\{c, a_{r''}\}$ generate the subgroup of all elements supported inside the set Y, so the products $\{c, a_v\}$ are linear combinations of smaller monomials in the ordering. The products $\{a_p, k_j\}$ and $\{a_p, a_{r''}\}$ vanish in $\mathrm{K}_2^{\mathrm{M}}(K)/l$. The product $\{a_p, a_{q_u}\}$ is nonzero and supported in the two valuations p and q_u, which do not belong to Y; hence this monomial belongs to T. Likewise, the product $\{a_p, a_{r'}\}$ is nontrivial and supported in p and r', hence it belongs to T whenever $r'\notin Y$; of all the products $\{a_p, a_{r'}\}$ with $r'\in Y$, it is only the smallest one that belongs to T.

Taken together, the products mentioned up to this point generate the subgroup of $\mathrm{K}_2^{\mathrm{M}}(K)/l$ supported inside the set of valuations p, q_u, r', and all valuations from Y. The rest of the argument is very similar to the one in Section 5, the only difference being that all the valuations from Y have been already “covered”.

The set/graph T contains no triangles and no monomials divisible by a_v except a_v^2, so one readily checks that the algebra $\mathrm{gr}^F\,\mathrm{K}^{\mathrm{M}}(K)/l$ is quadratic. By the result of 1.8, the ideal $(c)\subset \mathrm{K}^{\mathrm{M}}(K)/l$ is a Koszul module over $\mathrm{K}^{\mathrm{M}}(K)/l$. □

7. Fields without the root of unity

Let K be an algebraic extension of $\mathbb{Q}$ or $\mathbb{F}_q(z)$ and l be a prime number such that either $l = \operatorname{char} K$ or K contains no primitive l-root of unity. Let $\Lambda(K,l)$ be the exterior algebra generated by $\Lambda_1(K,l) = K^*/K^{*l}$ and $J_K\subset \Lambda(K,l)$ be the kernel of the map of graded algebras $\Lambda(K,l)\longrightarrow \mathrm{K}^{\mathrm{M}}(K)/l$. Let $c\in \mathrm{K}_1^{\mathrm{M}}(K)/l$ be an element and $(c)\subset \mathrm{K}^{\mathrm{M}}(K)/l$ be the ideal generated by c.

Theorem. *The ideal J_K is a Koszul $\Lambda(K,l)$-module (in the grading shifted by 1). The ideal (c) is a Koszul module over a Koszul algebra $\mathrm{K}^{\mathrm{M}}(K)/l$.*

Proof. The case $l = \operatorname{char} K$ is trivial (see 2.0 and the beginning of the proof of Theorem 3.1). It also suffices to consider the case when K is finite over $\mathbb{Q}$ or $\mathbb{F}_q(z)$.

Let S' be an exceptional set of valuations of K satisfying the conditions of 2.7. To prove the first assertion of theorem, for each nonarchimedean valuation $u\in S'$ such that K_u contains a primitive l-root of unity choose a pair of elements $w_u', w_u''\in K_u^*/K_u^{*l}$ such that $\{w_u', w_u''\}_u\neq 0$ in $\mathrm{K}_2^{\mathrm{M}}(K_u)/l$. Using Lemma 2.7, choose valuations p_u' and p_u'' of K such that all of them are different, do not belong to S', the completions $K_{p_u'}$ and $K_{p_u''}$ do not contain a primitive l-root of unity, and there exist elements $a_{p_u'}$ and $a_{p_u''}\in K^*$

whose images in $W'_{S'}$ are equal to w'_u and w''_u, and whose divisors outside S' are equal to p'_u and p''_u.

Choose a numbering by nonnegative integers for all the valuations r of K outside S' such that K_r contains a primitive l-root of unity, and order them according to this numbering. By induction on this order, choose for each such valuation r a valuation $q = q(r)$ outside of S' and all p'_u, p''_u such that K_q does not contain a primitive l-root of unity, the valuations $q(r)$ are different for different r, and there exists an element $a_q \in K^*$ whose divisor outside S' is equal to q, whose image in $W'_{S'}$ and in $K_{r'}/K_{r'}^{*l}$ is zero for all $r' < r$, and whose image in K_r/K_r^{*l} is nontrivial. The existence of such a valuation $q(r)$ follows from Lemma 2.7 applied to the set $S' \cup \{r' \mid r' < r\} \cup \{r\}$ in place of S'.

Finally, for all the valuations v of K outside of the sets S', $\{p'_u\}$, $\{p''_u\}$, and $\{q(r)\}$ choose elements $a_v \in K^*$ whose divisors outside S' are equal to v. Let k_j be any basis of $K_{S'}/K_{S'}^l$. Consider the following well-ordered basis of K^*/K^{*l}

$$a_{q(r)}, a_{p'_u}, a_{p''_s}, k_j, a_v,$$

where the ordering of $a_{q(r)}$ is according to the ordering of r, while the ordering within each of the other groups is arbitrary. The related set T of surviving quadratic monomials consists of the monomials $a_{q(r)}a_r$ and $a_{p'_u}a_{p''_u}$. Not only this graph does not contain any cycles, but there is even no vertex adjacent to more than one edge. By Proposition 1.7(2), the ideal J_K is Koszul. It follows that the graded algebra $\mathrm{K}^{\mathrm{M}}(K)/l$ is Koszul, too.

Now let us prove the second assertion. We can assume that the element c is nonzero in K^*/K^{*l}. Let S' be an exceptional set of valuations satisfying the conditions of 2.7 and containing the divisor of c. For each nonarchimedean valuation $u' \in S'$ such that $K_{u'}$ contains a primitive l-root of unity and the image of c in $K_{u'}^*/K_{u'}^{*l}$ is nonzero, choose an element $w_{u'} \in K_{u'}^*/K_{u'}^{*l}$ such that $\{w_{u'}, c\}_{u'} \neq 0$. For each of the remaining nonarchimedean valuations $u'' \in S'$ such that $K_{u''}$ contains a primitive l-root of unity, choose a pair of elements $w'_{u''}, w''_{u''} \in K_{u''}$ such that $\{w'_{u''}, w''_{u''}\}_{u''} \neq 0$.

Choose valuations $p_{u'}$, $p'_{u''}$, and $p''_{u''}$ outside of S' such that all of them are different, the corresponding completions do not contain a primitive l-root of unity, and there exist elements $a_{p_{u'}}$, $a_{p'_{u''}}$, and $a_{p''_{u''}} \in K^*$ whose divisors outside S' are equal to these valuations and whose images in $W'_{S'}$ are equal to $w_{u'}$, $w'_{u''}$, and $w''_{u''}$. For each valuation r outside S' such that K_r contains a primitive l-root of unity, choose a valuation $q(r)$ outside of S', $p_{u'}$, $p'_{u''}$, and $p''_{u''}$ such that the valuations $q(r)$ are different for different r, the completion $K_{q(r)}$ does not contain a primitive l-root of unity, and there exists an element $a_{q(r)} \in K^*$ whose divisor outside S' is equal to $q(r)$ and whose image in K_r^*/K_r^{*l} is nonzero.

Finally, for all the valuations v of K outside of the sets S', $\{p_{u'}\}$, $\{p'_{u''}\}$, $\{p''_{u''}\}$, and $\{q(r)\}$ choose elements $a_v \in K^*$ whose divisors outside S' are equal to v. Let elements $k_j \in K_{S'}/K_{S'}^l$ complement c to a basis of $K_{S'}/K_{S'}^l$. Consider the following well-ordered basis of K^*/K^{*l}

$$c, a_{p_{u'}}, a_{p'_{u''}}, a_{p''_{u''}}, a_{q(r)}, k_j, a_v,$$

where the ordering within each group can be arbitrary. Then the related set T of surviving quadratic monomials consists of all the monomials $ca_{p_{u'}}$ and $a_{p'_{u''}}a_{p''_{u''}}$ and some monomials of the forms ca_v, $a_{p_{u'}}a_v$, $a_{p'_{u''}}a_v$, $a_{p''_{u''}}a_v$, or $a_{q(r)}a_v$ (exactly one monomial of one of these forms for each valuation v outside S' such that K_v contains a primitive l-root of unity, and no such monomial for all other valuations v). The graph T contains no triangles (and not even any cycles), so the desired assertion follows from Proposition 1.7(1) and the result of 1.8. □

Acknowledgments

The author is grateful to Vladimir Voevodsky for posing the problem. This project would never have a chance to succeed without the invaluable participation of Alexander Vishik in its early stages. The work was largely done when I was a graduate student at Harvard University in the Fall of 1995, and I want to thank Harvard for its hospitality. The author was partially supported by a 2008–2010 grant from P. Deligne 2004 Balzan prize and RFBR grants 10-01-93113-NTsNIL_a, 11-01-00393-a, 12-01-92697-IND_a while finalizing the arguments and preparing the manuscript. I was also visiting Weizmann Institute of Science during the later part of this time in 2014. Finally, I want to thank the referee for reading the manuscript carefully and making a number of helpful suggestions.

References

[1] J. Backelin, R. Fröberg, Koszul algebras, Veronese subrings and rings with linear resolutions, Rev. Roumaine Math. Pures Appl. 30 (2) (1985) 85–97.
[2] H. Bass, J. Tate, The Milnor ring of a global field, in: K-Theory II, in: Lecture Notes in Math., vol. 342, 1973, pp. 349–446.
[3] A. Beilinson, V. Ginzburg, W. Soergel, Koszul duality patterns in representation theory, J. Amer. Math. Soc. 9 (2) (1996) 473–527.
[4] G. Bergman, The diamond lemma for ring theory, Adv. Math. 29 (2) (1978) 178–218.
[5] B. Buchberger, Gröbner bases: an algorithmic method in polynomial ideal theory, in: N.K. Bose (Ed.), Multidimensional Systems Theory, Reidel, Dordrecht, 1985, pp. 184–232.
[6] J.W.S. Cassels, A. Fröhlich (Eds.), Algebraic Number Theory, Proceedings of an Instructional Conference Organized by the London Mathematical Society (a NATO Advanced Study Institute) with the Support of the International Mathematical Union, Academic Press, 1967.
[7] R. Fröberg, Determination of a class of Poincaré series, Math. Scand. 37 (1) (1975) 29–39.
[8] A.S. Merkurjev, A.A. Suslin, K-cohomology of Severi–Brauer varieties and the norm residue homomorphism, Math. USSR Izv. 21 (2) (1983) 307–340.
[9] A. Polishchuk, L. Positselski, Quadratic Algebras, Univ. Lecture Ser., vol. 37, Amer. Math. Soc., Providence, RI, 2005.
[10] L. Positselski, Mixed Tate motives with finite coefficients and conjectures about the Galois groups of fields, abstracts of talks at the conference "Algebraische K-Theorie", Tagungsbericht 39/1999, September–October 1999, Oberwolfach, Germany, pp. 8–9, available from http://www.mfo.de/document/9939/Report_39_99.ps or http://www.math.uiuc.edu/K-theory/0375/.
[11] L. Positselski, Koszul property and Bogomolov's conjecture, Int. Math. Res. Not. IMRN 2005 (31) (2005) 1901–1936, arXiv:1405.0965 [math.KT].
[12] L. Positselski, Galois cohomology of certain field extensions and the divisible case of Milnor–Kato conjecture, K-Theory 36 (1–2) (2005) 33–50, arXiv:math.KT/0209037.
[13] L. Positselski, Mixed Artin–Tate motives with finite coefficients, Mosc. Math. J. 11 (2) (2011) 317–402, arXiv:1006.4343 [math.KT].

[14] L. Positselski, A. Vishik, Koszul duality and Galois cohomology, Math. Res. Lett. 2 (6) (1995) 771–781, arXiv:alg-geom/9507010.
[15] S. Priddy, Koszul resolutions, Trans. Amer. Math. Soc. 152 (1) (1970) 39–60.
[16] J.-P. Serre, Cohomologie Galoisienne, Lecture Notes in Math., vol. 5, Springer, 1964–1994.
[17] J. Tate, Relations between K_2 and Galois cohomology, Invent. Math. 36 (1) (1976) 257–274.
[18] V. Voevodsky, On motivic cohomology with $\mathbf{Z}/l$-coefficients, Ann. of Math. 174 (1) (2011) 401–438, arXiv:0805.4430 [math.AG].