



ELSEVIER

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt



# Asymptotic behaviors of means of central values of automorphic $L$ -functions for $GL(2)$



Shingo Sugiyama

*Institute of Mathematics for Industry, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan*

## ARTICLE INFO

### Article history:

Received 10 December 2013  
Received in revised form 10 March 2015

Accepted 19 April 2015  
Available online 6 June 2015  
Communicated by Dipendra Prasad

### Keywords:

Automorphic representations  
Relative trace formulas

## ABSTRACT

Let  $\mathbb{A}$  be the adèle ring of a totally real algebraic number field  $F$ . We push forward an explicit computation of a relative trace formula for periods of automorphic forms along a split torus in  $GL(2)$  from a square-free level case done by Masao Tsuzuki [20], to an arbitrary level case. By using a relative trace formula, we study central values of automorphic  $L$ -functions for cuspidal automorphic representations of  $GL(2, \mathbb{A})$  corresponding to Maass forms with arbitrary level.

© 2015 Elsevier Inc. All rights reserved.

## 0. Introduction

Let  $k \geq 4$  be an even integer. For a prime  $N$ , let  $S_k^{\text{new}}(N)$  be the space of all cuspidal new forms on the Poincaré upper half plane of weight  $k$  for  $\Gamma_0(N)$ . The space  $S_k^{\text{new}}(N)$  has an orthogonal basis  $\mathcal{F}_k^{\text{new}}(N)$  consisting of normalized Hecke eigenforms. For  $\varphi \in S_k^{\text{new}}(N)$ , we denote by  $L(s, \varphi)$  the completed automorphic  $L$ -function for  $\varphi$  satisfying a functional equation that relates the values at  $s$  and  $1 - s$ . Let  $\eta$  be a quadratic Dirichlet character of conductor  $D$  with  $\eta(-1) = -1$ . The Dirichlet  $L$ -series associated to  $\eta$  is denoted by  $L_{\text{fin}}(s, \eta)$ . For a fixed prime  $p \nmid D$ ,  $J_{p, \eta}$  denotes the set of all primes  $N$

*E-mail address:* s-sugiyama@imi.kyushu-u.ac.jp.

satisfying both  $\gcd(p, N) = \gcd(D, N) = 1$  and  $\eta(N) = -1$ . Let  $a_p(\varphi)$  denote the  $p$ -th Fourier coefficient of  $\varphi$  multiplied by  $p^{-(k-1)/2}$ . Ramakrishnan and Rogawski [14] studied a sum of central values of  $L(s, \varphi)$  and proved the following theorem.

**Theorem 1.** (See [14, Theorem A].) *For any interval  $J \subset [-2, 2]$ , we have*

$$\lim_{\substack{N \rightarrow \infty \\ N \in J_{p, \eta}}} \sum_{\substack{\varphi \in \mathcal{F}_k^{\text{new}}(N), \\ a_p(\varphi) \in J}} \frac{L(1/2, \varphi)L(1/2, \varphi \otimes \eta)}{\|\varphi\|^2} = 2^{k-1} \frac{\{(k/2 - 1)!\}^2}{\pi(k-2)!} L_{\text{fin}}(1, \eta) \mu_p^\eta(J),$$

where  $\|\varphi\|$  denotes the Petersson norm of  $\varphi$  and  $\mu_p^\eta$  denotes the probability measure on  $[-2, 2]$  defined by

$$\mu_p^\eta(x) = \begin{cases} \frac{p-1}{(p^{1/2} + p^{-1/2} - x)^2} \mu_{ST}(x) & (\eta(p) = 1), \\ \frac{p+1}{(p^{1/2} + p^{-1/2})^2 - x^2} \mu_{ST}(x) & (\eta(p) = -1). \end{cases}$$

Here,  $\mu_{ST}(x)$  is the Sato–Tate measure  $(2\pi)^{-1} \sqrt{4 - x^2} dx$ .

Feigon and Whitehouse [4] generalized this result to the case of Hilbert modular forms. Tsuzuki [20] gave a similar kind of asymptotic formula for Maass cusp forms with square-free level in terms of automorphic representations of  $GL(2)$  over a totally real algebraic number field. The purpose of this paper is to generalize Tsuzuki’s results of [20] to the case of arbitrary levels.

To state our results in this paper, we prepare some notation. Let  $F$  be a totally real algebraic number field,  $\mathfrak{o}_F$  its integer ring and  $\mathbb{A}$  the adèle ring of  $F$ . We denote by  $\Sigma_F$  (resp.  $\Sigma_\infty$  and  $\Sigma_{\text{fin}}$ ) the set of all places (resp. all infinite places and all finite places) of  $F$ . For each  $v \in \Sigma_{\text{fin}}$ , we fix a uniformizer  $\varpi_v$  of  $F_v$  and denote by  $q_v$  the cardinality of the residue field of  $F_v$ . For an ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ , let  $S(\mathfrak{a})$  denote the set of all  $v \in \Sigma_{\text{fin}}$  such that  $\text{ord}_v(\mathfrak{a}) \geq 1$ . The absolute norm of  $\mathfrak{a}$  is denoted by  $N(\mathfrak{a})$ .

Fix a quadratic character  $\eta = \prod_{v \in \Sigma_F} \eta_v$  of  $F^\times \backslash \mathbb{A}^\times$  of conductor  $\mathfrak{f}_\eta$  so that  $\eta_v$  is trivial for all  $v \in \Sigma_\infty$ . Fix a finite subset  $S$  of  $\Sigma_F$  such that  $\Sigma_\infty \subset S$  and  $S \cap S(\mathfrak{f}_\eta) = \emptyset$ . Let  $J_{S, \eta}$  be the set of all ideals  $\mathfrak{n}$  of  $\mathfrak{o}_F$  satisfying the following three conditions:

1.  $S(\mathfrak{n}) \cap S(\mathfrak{f}_\eta) = \emptyset$  and  $S(\mathfrak{n}) \cap S = \emptyset$ ,
2.  $\eta_v(\varpi_v) = -1$  for all  $v \in S(\mathfrak{n})$ ,
3.  $\tilde{\eta}(\mathfrak{n}) = \prod_{v \in \Sigma_{\text{fin}}} \eta_v(\varpi_v^{\text{ord}_v(\mathfrak{n})}) = 1$ .

Let  $\mathbf{K}_\infty$  be the standard maximal compact subgroup of  $GL(2, F \otimes_{\mathbb{Q}} \mathbb{R})$ . For an ideal  $\mathfrak{n}$  of  $\mathfrak{o}_F$ ,  $\mathbf{K}_0(\mathfrak{n})$  denotes the Hecke congruence subgroup of  $GL(2, \mathbb{A}_{\text{fin}})$  of level  $\mathfrak{n}$ . Let  $\Pi_{\text{cus}}(\mathfrak{n})$  denote the set of all irreducible cuspidal automorphic representations of  $PGL(2, \mathbb{A})$  having nonzero  $\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})$ -invariant vectors. Let  $\Pi_{\text{cus}}^*(\mathfrak{n})$  be the set of all  $\pi \in \Pi_{\text{cus}}(\mathfrak{n})$  with

conductor  $\mathfrak{f}_\pi = \mathfrak{n}$ . The standard  $L$ -function of  $\pi$  is denoted by  $L(s, \pi)$ . Let  $S_\pi$  denote the set of all  $v \in \Sigma_{\text{fin}}$  satisfying  $\text{ord}_v(\mathfrak{f}_\pi) \geq 2$ .

For  $\mathfrak{n} \in J_{S, \eta}$  and  $\pi = \otimes_v \pi_v \in \Pi_{\text{cus}}(\mathfrak{n})$ ,  $\pi_v$  is isomorphic to a unitarizable spherical principal series representation  $\pi(|\cdot|_v^{\nu_v/2}, |\cdot|_v^{-\nu_v/2})$  of  $GL(2, F_v)$  for all  $v \in S$ . The spectral parameter  $\nu_{\pi, S}$  of  $\pi$  at  $S$  is defined as  $\nu_{\pi, S} = (\nu_v)_{v \in S}$ . It is known that  $\nu_{\pi, S} \in \mathfrak{X}_S^{0+} = \prod_{v \in S} \mathfrak{X}_v^{0+}$ , where  $\mathfrak{X}_v^{0+} = i\mathbb{R}_{\geq 0} \cup (0, 1)$  for  $v \in \Sigma_\infty$  and  $\mathfrak{X}_v^{0+} = i[0, 2\pi(\log q_v)^{-1}] \cup \{x + iy \mid x \in (0, 1), y \in \{0, 2\pi(\log q_v)^{-1}\}\}$  for  $v \in S_{\text{fin}} = S \cap \Sigma_{\text{fin}}$ , respectively.

We define a measure  $\lambda_S^\eta$  on  $\mathfrak{X}_S^{0+}$  by  $4D_F^{3/2}L(1, \eta) \otimes_{v \in S} \lambda_v^{\eta_v}$ , where  $D_F$  denotes the absolute value of the discriminant of  $F$ , and for any  $v \in S$ , the measure  $\lambda_v^{\eta_v}$  on  $\mathfrak{X}_v^{0+}$  with support in  $i\mathbb{R}$  is given by

$$d\lambda_v^{\eta_v}(iy) = \frac{L(1/2, \pi(|\cdot|_v^{iy/2}, |\cdot|_v^{-iy/2}))L(1/2, \pi(|\cdot|_v^{iy/2}, |\cdot|_v^{-iy/2}) \otimes \eta_v)}{L(1, \eta_v)} \\ \times \begin{cases} \frac{1}{4\pi} |\Gamma(iy/2)|^{-2} dy & (v \in \Sigma_\infty), \\ \frac{\log q_v}{4\pi} |1 - q_v^{-iy}|^2 dy & (v \in S_{\text{fin}}). \end{cases}$$

We remark that when  $F = \mathbb{Q}$  and  $v = p < \infty$ ,  $\lambda_v^{\eta_v}(iy)$  is exactly equal to  $\mu_p^\eta(x)$  by the variable change  $x = p^{iy/2} + p^{-iy/2}$ . The main theorem of this paper is stated as follows.

**Theorem 2.** *Let  $\Lambda$  be an infinite subset of  $J_{S, \eta}$ . For any  $f \in C_c(\mathfrak{X}_S^{0+})$ , we have*

$$\text{AL}(\mathfrak{n}; f) = \frac{1}{N(\mathfrak{n})} \sum_{\pi \in \Pi_{\text{cus}}^*(\mathfrak{n})} \frac{L(1/2, \pi)L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \text{Ad})} f(\nu_{\pi, S}) \sim C(\mathfrak{n}) \langle \lambda_S^\eta, f \rangle$$

with

$$C(\mathfrak{n}) = \prod_{v \in S_2(\mathfrak{n})} \{1 - (q_v^2 - q_v)^{-1}\} \prod_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} (1 - q_v^{-2})$$

as  $N(\mathfrak{n}) \rightarrow \infty$  in  $\mathfrak{n} \in \Lambda$ . Here  $S_1(\mathfrak{n})$  (resp.  $S_2(\mathfrak{n})$ ) denotes the set of all  $v \in S(\mathfrak{n})$  such that  $\text{ord}_v(\mathfrak{n}) = 1$  (resp.  $\text{ord}_v(\mathfrak{n}) = 2$ ).

As for equidistribution results for Hecke eigenvalues of Maass forms without weighting central  $L$ -values, there is a work [8] by Knightly and Li when  $F = \mathbb{Q}$ .

The asymptotic formula in Theorem 2 gives the following counterpart of [14, Corollary B].

**Theorem 3.** *Let  $\Lambda$  be an infinite subset of  $J_{S, \eta}$  and let  $\{J_v\}_{v \in S}$  be a family of bounded intervals such that  $J_v$  is contained in  $[1/4, \infty)$  for any  $v \in \Sigma_\infty$  and in  $[-2, 2]$  for any  $v \in S_{\text{fin}}$ . Then, for any  $M > 0$ , there exists an irreducible cuspidal automorphic representation  $\pi$  of  $GL(2, \mathbb{A})$  with trivial central character satisfying the following conditions:*

1. The conductor  $\mathfrak{f}_\pi$  of  $\pi$  belongs to  $\Lambda$  and  $N(\mathfrak{f}_\pi) > M$  holds.
2. Both  $L(1/2, \pi) \neq 0$  and  $L(1/2, \pi \otimes \eta) \neq 0$  hold.
3. The spectral parameter  $\nu_{\pi, S} = (\nu_v)_{v \in S}$  of  $\pi$  at  $S$  satisfies  $(1 - \nu_v^2)/4 \in J_v$  for all  $v \in \Sigma_\infty$  and  $q_v^{-\nu_v/2} + q_v^{\nu_v/2} \in J_v$  for all  $v \in S_{\text{fin}}$ .

We remark that  $L(1/2, \pi)L(1/2, \pi \otimes \eta) > 0$  if  $L(1/2, \pi)L(1/2, \pi \otimes \eta) \neq 0$  by Guo's result [5].

Let  $\{v_j\}_{j \in \mathbb{N}}$  be the set of all places  $v \in \Sigma_{\text{fin}} - (S \cup S(\mathfrak{f}_\eta))$  such that  $\eta_v(\varpi_v) = -1$  and let  $\{\mathfrak{p}_j\}_{j \in \mathbb{N}}$  be the set of all prime ideals of  $\mathfrak{o}_F$  corresponding to  $\{v_j\}_{j \in \mathbb{N}}$ . Here are some examples of  $\Lambda$  in Theorems 2 and 3:

- (1)  $\Lambda = \{\mathfrak{n} = \mathfrak{p}_1 \cdots \mathfrak{p}_{2n} \mid n \in \mathbb{N}\}$ ,
- (2)  $\Lambda = \{\mathfrak{n} = \mathfrak{p}_1^{2n} \mid n \in \mathbb{N}\}$ ,
- (3)  $\Lambda = \{\mathfrak{n} = \mathfrak{p}_n^{2a} \mid n \in \mathbb{N}\}$  for a fixed  $a \in \mathbb{N}$ ,
- (4)  $\Lambda = \{\mathfrak{n} = \mathfrak{p}_1^{an} \mathfrak{p}_2^{bn} \mid n \in \mathbb{N}\}$  for fixed odd integers  $a, b > 0$ .

Case (1) was treated by Tsuzuki [20, Theorem 1.1 and Corollary 1.2].

Motohashi [13] studied the growth of the square mean of central values of automorphic  $L$ -functions attached to Maass forms with full level via Kuznetsov's trace formula. Tsuzuki [20, Theorem 1.3] considered a similar growth in the case where the level is square-free and the base field is totally real. We give a generalization of [20, Theorem 1.3] to the case of arbitrary levels.

**Theorem 4.** *We set  $d_F = [F : \mathbb{Q}]$ . Let  $\mathfrak{n}$  be an arbitrary ideal of  $\mathfrak{o}_F$  and  $\eta : F^\times \backslash \mathbb{A}^\times \rightarrow \{\pm 1\}$  a character of conductor relatively prime to  $\mathfrak{n}$ . Assume that  $\tilde{\eta}(\mathfrak{n}) = 1$  and  $\eta_v(-1) = 1$  for all  $v \in \Sigma_\infty$ . Let  $J$  be a compact subset of  $\prod_{v \in \Sigma_\infty} i\mathbb{R}_{>0}$  with smooth boundary. Then, for any  $\epsilon > 0$ , we have*

$$\begin{aligned} & \sum_{\substack{\pi \in \Pi_{\text{cus}}(\mathfrak{n}), \\ \nu_{\pi, \Sigma_\infty} \in tJ}} \frac{w_{\mathfrak{n}}^\eta(\pi)}{[\mathbf{K}_0(\mathfrak{f}_\pi) : \mathbf{K}_0(\mathfrak{n})]} \frac{L(1/2, \pi)L(1/2, \pi \otimes \eta)}{N(\mathfrak{f}_\pi)L^{S_\pi}(1, \pi; \text{Ad})} \\ &= \frac{4D_F^{3/2}}{(2\pi)^{d_F}} (1 + \delta_{\mathfrak{n}, \mathfrak{o}_F}) \text{vol}(J) t^{d_F} \{ \text{Res}_{s=1} L(s, \eta) (d_F \log(t/4) + V(J)) + \mathbf{C}^\eta(F, \mathfrak{n}) \} \\ &+ \mathcal{O}(t^{d_F-1}(\log t)^3) + \mathcal{O}(t^{d_F(1+4\theta)+\epsilon}), \quad t \rightarrow \infty, \end{aligned}$$

where  $V(J) = \text{vol}(J)^{-1} \int_J (\sum_{v \in \Sigma_\infty} \log |x_v|) dx$ ,  $\delta_{\mathfrak{n}, \mathfrak{o}_F}$  is the Kronecker delta,  $w_{\mathfrak{n}}^\eta(\mathfrak{n})$  is a constant explicitly defined in Lemma 12,

$$\begin{aligned} \mathbf{C}^\eta(F, \mathfrak{n}) &= \text{CT}_{s=1} L(s, \eta) \\ &+ \text{Res}_{s=1} L(s, \eta) \left\{ \frac{d_F}{2} (C_{\text{Euler}} + 2 \log 2 - \log \pi) + \log(D_F N(\mathfrak{n})^{1/2}) \right\} \end{aligned}$$

and  $\theta \in \mathbb{R}$  is a constant such that

$$|L_{\text{fin}}(1/2 + it, \chi)| \ll \mathfrak{q}(\chi \cdot \cdot |_{\mathbb{A}}^{it})^{1/4+\theta}, \quad t \in \mathbb{R}$$

holds uniformly for any character  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$ . Here  $\mathfrak{q}(\chi \cdot \cdot |_{\mathbb{A}}^{it})$  is the analytic conductor of  $\chi \cdot \cdot |_{\mathbb{A}}^{it}$ .

We remark that the main term of the formula in [20, Theorem 1.3] is not correct.

Moreover, we obtain the following results on subconvexity bounds depending on  $\theta < 0$ .

**Theorem 5.** *Let  $\mathfrak{n}$  be an arbitrary ideal of  $\mathfrak{o}_F$ . Let  $J \subset \prod_{v \in \Sigma_\infty} i\mathbb{R}$  be a closed cone such that  $J - \{0\} \subset \prod_{v \in \Sigma_\infty} i\mathbb{R}_{>0}$ . Then, for any  $\epsilon > 0$ , we have*

$$|L_{\text{fin}}(1/2, \pi)| \ll (1 + \|\nu_{\pi, \Sigma_\infty}\|)^{d_F/2 + \sup(2d_F\theta, -1/2) + \epsilon}$$

for  $\pi \in \Pi_{\text{cus}}(\mathfrak{n})_J = \{\pi \in \Pi_{\text{cus}}(\mathfrak{n}) \mid \nu_{\pi, \Sigma_\infty} \in J\}$  with the implied constant may depend on  $J$  and  $\mathfrak{n}$ . Here we set  $\|\nu\| = (\sum_{v \in \Sigma_\infty} |\nu_v|^2)^{1/2}$  for  $\nu \in \mathfrak{X}_{\Sigma_\infty}^{0+}$ .

We remark that Theorem 5 was proved by Tsuzuki [20, Corollary 1.4] when  $\mathfrak{n}$  is square-free. When  $F = \mathbb{Q}$ , Jutila and Motohashi [7] gave a sharper estimate  $L_{\text{fin}}(1/2, \pi) \ll_\epsilon (1 + \|\nu_{\pi, \Sigma_\infty}\|)^{1/3+\epsilon}$  uniformly for  $\pi \in \Pi_{\text{cus}}(\mathbb{Z})$ . Recently, Michel and Venkatesh [10] gave subconvexity bounds for automorphic  $L$ -functions for  $GL(1)$  and  $GL(2)$  in a more general case. According to [10, Theorem 1.1], we have the estimate

$$|L_{\text{fin}}(1/2, \pi)| \ll (1 + \|\nu_{\pi, \Sigma_\infty}\|)^{d_F/2 - 2d_F\delta} N(\mathfrak{n})^{1/4-\delta}, \quad \pi \in \Pi_{\text{cus}}(\mathfrak{n})$$

with implicit  $\delta > 0$ . Since we may take  $\theta$  such that  $\theta < 0$  by [10, Theorem 1.1], Theorem 5 gives an explicit subconvex exponent depending on  $\theta$  in the Laplacian eigenvalue aspect. In particular, if  $d_F > 1/4|\theta|$  then we have an explicit subconvex exponent  $d_F/2 - 1/2 + \epsilon$ .

Our method to prove Theorems 2, 3, 4 and 5 is based on that of [20]. We introduce adelic Green functions on  $GL(2, \mathbb{A})$  for ideals of  $\mathfrak{o}_F$  and then we give a relative trace formula by computing the regularized period of a Poincaré series of an adelic Green function in two different ways. Regularized periods used in this paper are toral period integrals regularized by Tsuzuki in order to define periods for nonrapidly decreasing functions on  $GL(2, \mathbb{A})$ . Since regularized periods of automorphic forms on  $GL(2, \mathbb{A})$  with arbitrary level were studied by a previous paper [18], we can compute the spectral side of our relative trace formula.

We explain the structure of this paper. In Section 1, we introduce notation used throughout this paper. In Section 2, we review results of [18] and prepare several lemmas. In Section 3, we introduce adelic Green functions on  $GL(2, \mathbb{A})$ . In Section 4, we regularize a Poincaré series of an adelic Green function. The regularized Poincaré series is called the regularized automorphic smoothed kernel. In Section 5, we compute the regularized period of the regularized automorphic smoothed kernel by using the spectral

expansion. Contrary to [20], the term  $\mathbb{I}_{\text{eis}}^\eta(\mathbf{n}|\alpha)$  in the spectral side is described in terms of regularized periods of Eisenstein series induced from ramified and unramified characters (see Theorem 26). Furthermore, a generalized Siegel theorem for Hecke  $L$ -functions is used to prove Lemma 20. The above phenomenon requiring hard calculation did not occur in the case of square-free levels (see also Lemmas 18, 19, 22, 23 and 24). In Section 6, we decompose the regularized period of the regularized automorphic smoothed kernel into the sum of terms derived from the double coset space  $H(F)\backslash GL(2, F)/H(F)$ , where  $H$  is the diagonal maximal split torus of  $GL(2)$ . In Section 7, we prove Theorems 2 and 3. The average over  $\Pi_{\text{cus}}^*(\mathbf{n})$  is extracted from that over  $\Pi_{\text{cus}}(\mathbf{n})$  in Section 7.1; this is a key step which did not appear in the case of square-free levels. We mention that Theorems 4 and 5 are obtained by the same method as in [20, §14]. In Section 7.3, we explain why the main term of the formula in [20, Theorem 1.3] should be modified by using factors  $\log(t/4)$  and  $V(J)$ . Two propositions are given in Appendix A. These compensate for inaccuracy of [20].

## Notation

We write  $\mathbb{N}$  for the set of natural numbers and put  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For sets  $A$  and  $B$ , the set  $\text{Map}(A, B)$  denotes the set of mappings from  $A$  to  $B$ . For  $f, g \in \text{Map}(A, \mathbb{R}_{\geq 0})$ , let us denote by  $f(x) \ll g(x), x \in A$  an inequality  $f(x) \leq Cg(x)$  with some constant  $C > 0$  for all  $x \in A$ . For a given condition  $P$ ,  $\delta(P) \in \{0, 1\}$  is defined by  $\delta(P) = 1$  (resp.  $\delta(P) = 0$ ) if  $P$  is true (resp. false).

Let  $F$  be a totally real field with its degree  $d_F$  and  $\mathfrak{o}_F$  its integer ring. Let  $\mathbb{A}$  and  $\mathbb{A}_{\text{fin}}$  be the adèle ring and the finite adèle ring of  $F$ , respectively. The symbols  $\Sigma_\infty$  and  $\Sigma_{\text{fin}}$  denote the set of all infinite places and the set of all finite places of  $F$ , respectively. For a place  $v \in \Sigma_F = \Sigma_\infty \cup \Sigma_{\text{fin}}$ , let  $|\cdot|_v$  denote the normalized valuation of the completion  $F_v$  of  $F$  at  $v$ . For each  $v \in \Sigma_{\text{fin}}$ , let  $\varpi_v$  be a uniformizer of  $F_v$ . Then,  $\mathfrak{p}_v = \varpi_v \mathfrak{o}_v$  for  $v \in \Sigma_{\text{fin}}$  is a maximal ideal of the integer ring  $\mathfrak{o}_v$  of  $F_v$  and we have  $|\varpi_v|_v = q_v^{-1}$ , where  $q_v$  is the cardinality of the residue field  $\mathfrak{o}_v/\mathfrak{p}_v$ . For an ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ , let  $S(\mathfrak{a})$  denote the set of all  $v \in \Sigma_{\text{fin}}$  such that  $v$  divides  $\mathfrak{a}$ . For any  $k \in \mathbb{N}$ , we write  $S_k(\mathfrak{a})$  for the set of all  $v \in S(\mathfrak{a})$  with  $\text{ord}_v(\mathfrak{a}) = k$ , where  $\text{ord}_v(\mathfrak{a})$  is the order of  $\mathfrak{a}$  at  $v$ . Let  $N(\mathfrak{a})$  denote the absolute norm of  $\mathfrak{a}$ .

Let  $G$  be the algebraic group  $GL(2)$  with unit element  $e$ . For any  $F$ -algebraic subgroup  $M$  of  $G$ , we set  $M_F = M(F)$ ,  $M_v = M(F_v)$  (for  $v \in \Sigma_F$ ),  $M_{\mathbb{A}} = M(\mathbb{A})$  and  $M_{\text{fin}} = M(\mathbb{A}_{\text{fin}})$ , respectively. The diagonal maximal split torus of  $G$  is denoted by  $H$ . Then, the Borel subgroup  $B = HN$  of  $G$  consists of all upper triangular matrices, where  $N$  is the subgroup of  $G$  consisting of all unipotent matrices. The center of  $G$  is denoted by  $Z$ . We put  $\mathbf{K}_v = O(2, \mathbb{R})$  (resp.  $\mathbf{K}_v = GL(2, \mathfrak{o}_v)$ ) for  $v \in \Sigma_\infty$  (resp.  $v \in \Sigma_{\text{fin}}$ ). Then,  $\mathbf{K} = \prod_{v \in \Sigma_F} \mathbf{K}_v$  is a maximal compact subgroup of  $G_{\mathbb{A}}$ . Set  $\mathbf{K}_\infty = \prod_{v \in \Sigma_\infty} \mathbf{K}_v$  and  $\mathbf{K}_0(\mathfrak{p}_v^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_v \mid c \equiv 0 \pmod{\mathfrak{p}_v^n} \right\}$  for any  $n \in \mathbb{N}_0$ . For an ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ , we put  $\mathbf{K}_0(\mathfrak{a}) = \prod_{v \in \Sigma_{\text{fin}}} \mathbf{K}_0(\mathfrak{a}\mathfrak{o}_v)$ .

## 1. Preliminaries

Let  $\mathbb{A}_{\mathbb{Q}}$  be the adèle ring of  $\mathbb{Q}$  and  $\psi_{\mathbb{Q}} = \prod_p \psi_p$  the additive character of  $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$  with archimedean component  $\psi_{\infty}(x) = \exp(2\pi i x)$  for  $x \in \mathbb{R}$ . Then,  $\psi_F = \psi \circ \text{tr}_{F/\mathbb{Q}} = \prod_{v \in \Sigma_F} \psi_{F_v}$  is a nontrivial additive character of  $F \backslash \mathbb{A}$ . Let  $\mathfrak{D}_{F/\mathbb{Q}}$  be the global different of  $F/\mathbb{Q}$  and set  $d_v = \text{ord}_v \mathfrak{D}_{F/\mathbb{Q}}$  for any  $v \in \Sigma_{\text{fin}}$ .

For  $v \in \Sigma_F$ , let  $dx_v$  be the self-dual Haar measure on  $F_v$  with respect to  $\psi_{F_v}$ . We set  $d^{\times}x_v = (1 - q_v^{-1})^{-1} dx_v / |x_v|_v$  for  $v \in \Sigma_{\text{fin}}$  and  $d^{\times}x_v = d^{\times}x_v / |x_v|_v$  for  $v \in \Sigma_{\infty}$ , respectively. Then,  $d^{\times}x_v$  is a Haar measure on  $F_v^{\times}$  and the product measure  $d^{\times}x = \prod_{v \in \Sigma_F} d^{\times}x_v$  gives a Haar measure on  $\mathbb{A}^{\times}$ . For each  $v \in \Sigma_F$ , we take a Haar measure  $dk_v$  on  $\mathbf{K}_v$  such that total volume is one, and take a Haar measure  $dg_v$  on  $G_v$  in the following way. Let  $dh_v$  (resp.  $dn_v$ ) denote the Haar measure on  $H_v$  (resp.  $N_v$ ) induced via the isomorphism  $H_v \cong F_v^{\times} \times F_v^{\times}$  (resp.  $N_v \cong F_v$ ). Then,  $dg_v = dh_v dn_v dk_v$  gives a Haar measure on  $G_v$  via the Iwasawa decomposition  $g_v = h_v n_v k_v \in H_v N_v \mathbf{K}_v$ . We remark that the volume  $\text{vol}(\mathbf{K}_v, dg_v)$  equals  $q_v^{-3d_v/2}$  for any  $v \in \Sigma_{\text{fin}}$ . We denote the Haar measure  $\prod_{v \in \Sigma_F} dk_v$  on  $\mathbf{K}$  by  $dk$ .

Let  $|\cdot|_{\mathbb{A}} = \prod_{v \in \Sigma_F} |\cdot|_v$  be the idele norm of  $\mathbb{A}^{\times}$  and set  $\mathbb{A}^1 = \{x \in \mathbb{A}^{\times} \mid |x|_{\mathbb{A}} = 1\}$ . For  $y \in \mathbb{R}_{>0}$ ,  $\underline{y}$  denotes the idele such that the  $v$ -th component of  $\underline{y}$  satisfies  $\underline{y}_v = y^{1/d_F}$  (resp.  $\underline{y}_v = 1$ ) for  $v \in \Sigma_{\infty}$  (resp.  $v \in \Sigma_{\text{fin}}$ ). Set  $G_{\mathbb{A}}^1 = \{g \in G_{\mathbb{A}} \mid |\det g|_{\mathbb{A}} = 1\}$  and  $\mathfrak{A} = \left\{ \begin{pmatrix} \underline{y} & 0 \\ 0 & \underline{y} \end{pmatrix} \mid y > 0 \right\}$ . Then,  $G_{\mathbb{A}} = \mathfrak{A} G_{\mathbb{A}}^1$  holds.

For  $v \in \Sigma_{\text{fin}}$  and a quasi-character  $\chi_v$  of  $F_v^{\times}$ ,  $\mathfrak{p}_v^{f(\chi_v)}$  denotes the conductor of  $\chi_v$ . We define the Gauss sum associated with  $\chi_v$  by

$$\mathcal{G}(\chi_v) = \int_{\mathfrak{o}_v^{\times}} \chi_v(u \varpi_v^{-d_v - f(\chi_v)}) \psi_{F_v}(u \varpi_v^{-d_v - f(\chi_v)}) d^{\times}u.$$

For any quasi-character  $\chi = \prod_{v \in \Sigma_F} \chi_v$  of  $F^{\times} \backslash \mathbb{A}^{\times}$ , we define the conductor of  $\chi$  by the ideal  $\mathfrak{f}_{\chi}$  of  $\mathfrak{o}_F$  such that  $\mathfrak{f}_{\chi} \mathfrak{o}_v = \mathfrak{p}_v^{f(\chi_v)}$  for all  $v \in \Sigma_{\text{fin}}$ . We write  $\chi_{\text{fin}}$  for  $\prod_{v \in \Sigma_{\text{fin}}} \chi_v$ . The Gauss sum  $\mathcal{G}(\chi)$  associated with  $\chi$  is defined by the product of  $\mathcal{G}(\chi_v)$  over all  $v \in \Sigma_{\text{fin}}$ . We set  $\tilde{\chi}(\mathfrak{a}) = \prod_{v \in \Sigma_{\text{fin}}} \chi_v(\varpi_v^{\text{ord}_v(\mathfrak{a})})$  for any ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ . For  $v \in \Sigma_F$ , we denote the trivial character of  $F_v^{\times}$  by  $\mathbf{1}_v$ , and the trivial character of  $\mathbb{A}^{\times}$  by  $\mathbf{1}$ . Throughout this paper, any quasi-character  $\chi$  of  $F^{\times} \backslash \mathbb{A}^{\times}$  is assumed to satisfy  $\chi(\underline{y}) = 1$  for all  $y \in \mathbb{R}_{>0}$ . Such a quasi-character is a character. For any  $v \in \Sigma_{\infty}$  (resp.  $v \in \Sigma_{\text{fin}}$ ) and any character  $\chi_v$  of  $F_v^{\times}$ , let  $b(\chi_v)$  denote  $b_v \in \mathbb{R}$  (resp.  $b_v \in [0, 2\pi(\log q_v)^{-1})$ ) such that the restriction of  $\chi_v$  to  $\mathbb{R}_{>0}$  (resp.  $\varpi_v^{\mathbb{Z}}$ ) is of the form  $|\cdot|^{ib_v}$ . For any character  $\chi$  of  $F^{\times} \backslash \mathbb{A}^{\times}$ , the analytic conductor  $\mathfrak{q}(\chi)$  of  $\chi$  is defined to be

$$\mathfrak{q}(\chi) = \left\{ \prod_{v \in \Sigma_{\infty}} (3 + |b(\chi_v)|) \right\} N(\mathfrak{f}_{\chi}).$$

Let  $\mathfrak{n}$  be an ideal of  $\mathfrak{o}_F$ . For an ideal  $\mathfrak{c}$  of  $\mathfrak{o}_F$ , let  $\Xi_0(\mathfrak{c})$  be the set of all characters  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$  such that  $\mathfrak{f}_\chi = \mathfrak{c}$  and  $\chi_v(-1) = 1$  for all  $v \in \Sigma_\infty$ . We write  $\Xi(\mathfrak{n})$  for  $\bigcup_{\mathfrak{c}^2 | \mathfrak{n}} \Xi_0(\mathfrak{c})$ . Let  $U_F^+$  be the set of all totally positive units of  $\mathfrak{o}_F$  and set

$$\log U_F^+ = \{(\log u_v)_{v \in \Sigma_\infty} \mid (u_v)_{v \in \Sigma_\infty} \in U_F^+\}.$$

Then,  $\log U_F^+$  is a lattice of  $\mathbb{Z}$ -rank  $d_F - 1$  in  $V$ , where  $V = \{(x_v)_{v \in \Sigma_\infty} \in \mathbb{R}^{d_F} \mid \sum_{v \in \Sigma_\infty} x_v = 0\}$ . Set

$$L_0 = \{(b_v)_{v \in \Sigma_\infty} \in V \mid \sum_{v \in \Sigma_\infty} b_v l_v \in \mathbb{Z} \text{ for all } (l_v)_{v \in \Sigma_\infty} \in \log U_F^+\}.$$

Then,  $L_0$  is also a  $\mathbb{Z}$ -lattice in  $V$ . Let  $\chi$  be a character of  $F^\times \backslash \mathbb{A}^\times$ . Since  $\chi(y) = 1$  for any  $y \in \mathbb{R}_{>0}$ , we have  $\sum_{v \in \Sigma_\infty} b(\chi_v) = 0$ . Thus, if we denote by  $b(\chi)$  the element  $(b(\chi_v))_{v \in \Sigma_\infty}$  of  $\mathbb{R}^{d_F}$ , then  $b(\chi) \in L_0$  holds. Therefore the mapping  $\chi \mapsto b(\chi)$  is a surjection from  $\Xi(\mathfrak{n})$  onto  $L_0$  and the kernel  $\Xi_{\ker}(\mathfrak{n})$  of this mapping is a finite abelian group.

**Lemma 6.** *Let  $X(\mathfrak{n})$  be the order of  $\Xi_{\ker}(\mathfrak{n})$ . Then, for any  $\epsilon > 0$ , the estimate*

$$X(\mathfrak{n}) \ll N(\mathfrak{n})^{1/2+\epsilon}$$

*holds with the implied constant independent of  $\mathfrak{n}$ .*

**Proof.** For any ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ , we set  $I_F(\mathfrak{a}) = \prod_{v \in \Sigma_\infty} \mathbb{R}^\times \times \prod_{v \in \Sigma_{\text{fin}}} (1 + \mathfrak{a}\mathfrak{o}_v)$ . Then, the ray class group  $C_F(\mathfrak{a})$  modulo  $\mathfrak{a}$  is defined by  $C_F(\mathfrak{a}) = F^\times \backslash F^\times I_F(\mathfrak{a})$ . For any fixed  $\mathfrak{c}$  satisfying  $\mathfrak{c}^2 | \mathfrak{n}$ , the group  $\Xi_0(\mathfrak{c}) \cap \Xi_{\ker}(\mathfrak{n})$  is equal to the set of all characters of  $F^\times \backslash \mathbb{A}^\times$  of finite order contained in  $\Xi_0(\mathfrak{c})$ . Hence

$$\#(\Xi_0(\mathfrak{c}) \cap \Xi_{\ker}(\mathfrak{n})) \leq \#(F^\times \backslash \mathbb{A}^\times / I_F(\mathfrak{c})) = h_F \#(C_F(\mathfrak{o}_F) / C_F(\mathfrak{c})) \leq h_F N(\mathfrak{c}) \leq h_F N(\mathfrak{n})^{1/2}$$

holds, where  $h_F$  is the class number of  $F$ . By  $\sum_{\mathfrak{c}^2 | \mathfrak{n}} 1 \ll N(\mathfrak{n})^\epsilon$  for any  $\epsilon > 0$ , we obtain the assertion.  $\square$

## 2. Regularized periods of automorphic forms

In this section, we recall explicit formulas in [18] of the regularized periods of automorphic forms on  $G_{\mathbb{A}}$ .

### 2.1. Zeta integrals of cusp forms

Let  $\pi$  be a  $\mathbf{K}_\infty$ -spherical irreducible cuspidal automorphic representation of  $G_{\mathbb{A}}$  with trivial central character, where the representation space  $V_\pi$  is realized in the space of



cuspidal forms. For any quasi-character  $\eta$  of  $F^\times \backslash \mathbb{A}^\times$  and  $\varphi \in V_\pi$ , we define the global zeta integral by

$$Z(s, \eta, \varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \varphi \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \eta(t) |t|_{\mathbb{A}}^{s-1/2} d^\times t, \quad s \in \mathbb{C}.$$

The defining integral converges absolutely for any  $s \in \mathbb{C}$ , and hence  $Z(s, \eta, \varphi)$  is an entire function in  $s$ .

We fix a family  $\{\pi_v\}_{v \in \Sigma_F}$  consisting of irreducible admissible representations such that  $\pi \cong \bigotimes_{v \in \Sigma_F} \pi_v$ . The conductor of  $\pi$  is denoted by  $\mathfrak{f}_\pi$ , which is the ideal of  $\mathfrak{o}_F$  defined by  $\mathfrak{f}_\pi \mathfrak{o}_v = \mathfrak{p}_v^{c(\pi_v)}$  for all  $v \in \Sigma_{\text{fin}}$ , where  $\mathfrak{p}_v^{c(\pi_v)}$  is the conductor of  $\pi_v$ . Let  $\mathfrak{n}$  be an ideal of  $\mathfrak{o}_F$  which is divided by  $\mathfrak{f}_\pi$ .

Let  $n$  be the maximal non-negative integer  $m$  such that  $S_m(\mathfrak{n} \mathfrak{f}_\pi^{-1}) \neq \emptyset$ . For  $\rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\pi^0(\mathfrak{n}) = \prod_{k=1}^n \text{Map}(S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1}), \{0, \dots, k\})$ , let  $\varphi_{\pi, \rho}$  denote the cuspidal form in  $V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$  corresponding to

$$\bigotimes_{v \in \Sigma_\infty} \phi_{0,v} \otimes \bigotimes_{v \in S_1(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \phi_{\rho_1(v), v} \otimes \cdots \otimes \bigotimes_{v \in S_n(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \phi_{\rho_n(v), v} \otimes \bigotimes_{v \in \Sigma_{\text{fin}} - S(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \phi_{0,v}$$

by the isomorphism  $V_\pi \cong \bigotimes_{v \in \Sigma_F} V_{\pi_v}$ . Here,  $V_{\pi_v}$  denotes the Whittaker model of  $\pi_v$  with respect to  $\psi_{F_v}$ ,  $\phi_{0,v}$  is the spherical vector in  $V_{\pi_v}$  for  $v \in \Sigma_\infty$  given in [18, 1.4], and the function  $\phi_{k,v}$  is the  $\mathbf{K}_0(\mathfrak{n} \mathfrak{o}_v)$ -invariant vector for  $v \in \Sigma_{\text{fin}}$ , which is constructed in [18, §2 and §3]. Then, the finite set  $\{\varphi_{\pi, \rho}\}_{\rho \in \Lambda_\pi^0(\mathfrak{n})}$  is an orthogonal basis of  $V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$ . Here  $V_\pi \subset L^2(\mathbb{Z}_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$  is equipped with the  $L^2$ -inner product (cf. [18, Proposition 17]).

We consider a character  $\eta$  of  $F^\times \backslash \mathbb{A}^\times$  satisfying

$$\begin{cases} \eta^2 = 1, \\ v \in \Sigma_\infty \Rightarrow \eta_v = 1_v, \\ \mathfrak{f}_\eta \text{ is relatively prime to } \mathfrak{n} \text{ and } \tilde{\eta}(\mathfrak{n}) = 1. \end{cases} \quad (\star)$$

For such a character  $\eta$  and  $\varphi \in V_\pi^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$ , we define the modified global zeta integral by

$$Z^*(s, \eta, \varphi) = \eta_{\text{fin}}(x_{\eta, \text{fin}}) Z\left(s, \eta, \pi \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix} \varphi\right), \quad s \in \mathbb{C}.$$

Here  $x_\eta = (x_{\eta, v})_{v \in \Sigma_F} \in \mathbb{A}$  is the adèle whose  $v$ -th component satisfies  $x_{\eta, v} = 0$  and  $x_{\eta, v} = \varpi_v^{-f(\eta_v)}$  for  $v \in \Sigma_\infty$  and  $v \in \Sigma_{\text{fin}}$ , respectively, and  $x_{\eta, \text{fin}}$  denotes the projection of  $x_\eta$  to  $\mathbb{A}_{\text{fin}}$ .

## 2.2. Regularized periods of cuspidal forms

We recall a definition of regularized periods of automorphic forms on  $G_{\mathbb{A}}$  defined in [20, §7]. Let  $\mathcal{B}$  be the space of all entire even functions  $\beta$  on  $\mathbb{C}$  such that for any  $l > 0$

and any interval  $[a, b] \subset \mathbb{R}$ , the estimate  $|\beta(\sigma + it)| \ll (1 + |t|)^{-l}$  holds uniformly in  $\sigma \in [a, b]$  and  $t \in \mathbb{R}$ . For  $\beta \in \mathcal{B}$  and  $\lambda \in \mathbb{C}$ , we define a function  $\hat{\beta}_\lambda$  on  $\mathbb{R}_{>0}$  by

$$\hat{\beta}_\lambda(t) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z + \lambda} t^z dz, \quad (\sigma > -\operatorname{Re}(\lambda)),$$

where  $L_\sigma = \{z \in \mathbb{C} \mid \operatorname{Re}(z) = \sigma\}$ . For  $\beta \in \mathcal{B}$ ,  $\lambda \in \mathbb{C}$ , a character  $\eta$  of  $F^\times \backslash \mathbb{A}^\times$  satisfying  $(\star)$  and a function  $\varphi : \mathfrak{A}_{GF} \backslash G_{\mathbb{A}} \rightarrow \mathbb{C}$ , we consider

$$P_{\beta, \lambda}^\eta(\varphi) = \int_{F^\times \backslash \mathbb{A}^\times} \{\hat{\beta}_\lambda(|t|_{\mathbb{A}}) + \hat{\beta}_\lambda(|t|_{\mathbb{A}}^{-1})\} \varphi \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix} \right) \eta(t) \eta_{\text{fin}}(x_{\eta, \text{fin}}) d^\times t.$$

Now we assume that for any  $\beta \in \mathcal{B}$ , there exists a constant  $C \in \mathbb{R}$  such that if  $\operatorname{Re}(\lambda) > C$  the integral  $P_{\beta, \lambda}^\eta(\varphi)$  converges and the function  $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > C\} \ni \lambda \mapsto P_{\beta, \lambda}^\eta(\varphi)$  is continued meromorphically to a neighborhood of  $\lambda = 0$ . Then a constant  $P_{\text{reg}}^\eta(\varphi)$  is called the *regularized  $\eta$ -period of  $\varphi$*  if  $\text{CT}_{\lambda=0} P_{\beta, \lambda}^\eta(\varphi) = P_{\text{reg}}^\eta(\varphi) \beta(0)$  for all  $\beta \in \mathcal{B}$ . Then the following was proved in [18].

**Proposition 7.** (See [18, Main Theorem A].) For any  $\rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\pi^0(\mathfrak{n})$  and  $\eta$  satisfying  $(\star)$ , the period  $P_{\text{reg}}^\eta(\varphi_{\pi, \rho})$  can be defined and we have

$$P_{\text{reg}}^\eta(\varphi_{\pi, \rho}) = Z^*(1/2, \eta, \varphi) = \mathcal{G}(\eta) \left\{ \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} Q_{\rho_k(v), v}^{\pi_v}(\eta_v, 1) \right\} L(1/2, \pi \otimes \eta),$$

where the constants  $Q_{\rho_k(v), v}^{\pi_v}(\eta_v, 1)$  are given as follows:

- If  $c(\pi_v) = 0$  and  $(\alpha_v, \alpha_v^{-1})$  is the Satake parameter of  $\pi_v$ , then

$$Q_{k, v}^{\pi_v}(\eta_v, 1) = \begin{cases} 1 & (\text{if } k = 0), \\ \eta_v(\varpi_v) - \frac{\alpha_v + \alpha_v^{-1}}{q_v^{1/2} + q_v^{-1/2}} & (\text{if } k = 1), \\ q_v^{-1} \eta_v(\varpi_v)^{k-2} (\alpha_v q_v^{1/2} \eta_v(\varpi_v) - 1) (\alpha_v^{-1} q_v^{1/2} \eta_v(\varpi_v) - 1) & (\text{if } k \geq 2). \end{cases}$$

- If  $c(\pi_v) = 1$ , then  $\pi_v$  is isomorphic to a special representation  $\sigma(\chi_v | \cdot |_v^{1/2}, \chi_v | \cdot |_v^{-1/2})$  for some unramified character  $\chi_v$  of  $F_v^\times$  and

$$Q_{k, v}^{\pi_v}(\eta_v, 1) = \begin{cases} 1 & (\text{if } k = 0), \\ \eta_v(\varpi_v)^{k-1} (\eta_v(\varpi_v) - q_v^{-1} \chi_v(\varpi_v)^{-1}) & (\text{if } k \geq 1). \end{cases}$$

- If  $c(\pi_v) \geq 2$ , then  $Q_{k, v}^{\pi_v}(\eta_v, 1) = \eta_v(\varpi_v)^k$  for any  $k \in \mathbb{N}_0$ .

### 2.3. Preliminaries for regularized periods of Eisenstein series

We fix a character  $\chi = \prod_{v \in \Sigma_F} \chi_v$  of  $F^\times \backslash \mathbb{A}^\times$ . For  $\nu \in \mathbb{C}$ , we denote by  $I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$  the space of all smooth  $\mathbb{C}$ -valued right  $\mathbf{K}$ -finite functions  $f$  on  $G_{\mathbb{A}}$  with the  $B_{\mathbb{A}}$ -equivariance

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right) = \chi(a/d) |a/d|_{\mathbb{A}}^{(\nu+1)/2} f(g)$$

for all  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B_{\mathbb{A}}$  and  $g \in G_{\mathbb{A}}$ . If  $\nu \in i\mathbb{R}$ , then the space  $I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$  is unitarizable and a  $G_{\mathbb{A}}$ -invariant hermitian inner product is given by

$$(f_1|f_2) = \int_{\mathbf{K}} f_1(k) \overline{f_2(k)} dk$$

for any  $f_1, f_2 \in I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$ .

For  $\nu \in \mathbb{C}$  and  $f^{(\nu)} \in I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$ , the family  $\{f^{(\nu)}\}_{\nu \in \mathbb{C}}$  is called a flat section if the restriction of  $f^{(\nu)}$  to  $\mathbf{K}$  is independent of  $\nu \in \mathbb{C}$ . We define the Eisenstein series for  $f^{(\nu)} \in I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$  by

$$E(f^{(\nu)}, g) = \sum_{\gamma \in B_F \backslash G_F} f^{(\nu)}(\gamma g), \quad g \in G_{\mathbb{A}}.$$

The defining series converges absolutely if  $\operatorname{Re}(\nu) > 1$ . If  $\{f^{(\nu)}\}_{\nu \in \mathbb{C}}$  is a flat section, then  $E(f^{(\nu)}, g)$  is continued meromorphically to  $\mathbb{C}$  as a function in  $\nu$ . We remark that the function  $E(f^{(\nu)}, g)$  is holomorphic on  $i\mathbb{R}$ . On the half plane  $\operatorname{Re}(\nu) > 0$ ,  $E(f^{(\nu)}, g)$  is holomorphic except for  $\nu = 1$ , and  $\nu = 1$  is a pole of  $E(f^{(\nu)}, g)$  if and only if  $\chi^2 = \mathbf{1}$ .

Let  $\mathfrak{n}$  be an ideal of  $\mathfrak{o}_F$ . Throughout Section 2, we assume that a character  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$  is contained in  $\Xi(\mathfrak{n})$ .

### 2.4. Zeta integrals of Eisenstein series

We consider Eisenstein series for  $f \in I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})^{\mathbf{K}_\infty \mathbf{K}_0(\mathfrak{n})}$ . Let  $n$  be the maximal non-negative integer  $m$  such that  $S_m(\mathfrak{n}\mathfrak{f}_\chi^{-2}) \neq \emptyset$ . For each  $v \in \Sigma_F$ , the space  $I(\chi_v| \cdot |_v^{\nu/2})$  is defined in the same way as the global case. For  $\rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\chi(\mathfrak{n}) = \prod_{k=1}^n \operatorname{Map}(S_k(\mathfrak{n}\mathfrak{f}_\chi^{-2}), \{0, \dots, k\})$ , let  $f_{\chi, \rho}^{(\nu)}$  denote the vector in  $I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})$  corresponding to

$$\bigotimes_{v \in \Sigma_\infty} f_{0, \chi_v}^{(\nu)} \otimes \bigotimes_{v \in S_1(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}_{\rho_1(v), \chi_v}^{(\nu)} \otimes \cdots \otimes \bigotimes_{v \in S_n(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}_{\rho_n(v), \chi_v}^{(\nu)} \otimes \bigotimes_{v \in \Sigma_{\text{fin}} - S(\mathfrak{n}\mathfrak{f}_\chi^{-2})} \tilde{f}_{0, \chi_v}^{(\nu)}$$

by the isomorphism  $I(\chi| \cdot |_{\mathbb{A}}^{\nu/2}) \cong \bigotimes_{v \in \Sigma_F} I(\chi_v| \cdot |_v^{\nu/2})$ , where  $f_{0, \chi_v}^{(\nu)}$  for  $v \in \Sigma_\infty$  is the spherical vector in  $I(\chi_v| \cdot |_v^{\nu/2})$  normalized so that  $f_{0, \chi_v}^{(\nu)}(e)$  equals one and  $\tilde{f}_{k, \chi_v}^{(\nu)}$  for  $v \in \Sigma_{\text{fin}}$  is the  $\mathbf{K}_0(\mathfrak{n}\mathfrak{o}_v)$ -invariant vector, which is constructed in [18, §7 and §8]. Then,

for any  $\rho = (\rho_k)_{1 \leq k \leq n} \in \Lambda_\chi(\mathbf{n})$ , the family  $\{f_{\chi, \rho}^{(\nu)}\}_{\nu \in \mathbb{C}}$  is a flat section. Moreover, if  $\nu \in i\mathbb{R}$ , the finite set  $\{f_{\chi, \rho}^{(\nu)}\}_{\rho \in \Lambda_\chi(\mathbf{n})}$  is an orthonormal basis of  $I(\chi| \cdot |_{\mathbb{A}}^{\nu/2})^{\mathbf{K}_\infty \mathbf{K}_0(\mathbf{n})}$  (cf. [18, Proposition 33]).

Let  $\rho \in \Lambda_\chi(\mathbf{n})$  and set  $E_{\chi, \rho}(\nu, g) = E(f_{\chi, \rho}^{(\nu)}, g)$ . The constant term of  $E(f_{\chi, \rho}^{(\nu)}, g)$  is defined by

$$E_{\chi, \rho}^\circ(\nu, g) = \int_{F \backslash \mathbb{A}} E_{\chi, \rho}(\nu, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g) dx.$$

For  $k \in \{1, \dots, n\}$ , the sets  $U_k(\rho)$ ,  $R_k(\rho)$  and  $R_0(\rho)$  are defined as follows:

$$\begin{aligned} U_k(\rho) &= \bigcup_{m=k}^n \rho_m^{-1}(k) - S(\mathfrak{f}_\chi), & R_k(\rho) &= \bigcup_{m=k}^n \rho_m^{-1}(k) \cap S(\mathfrak{f}_\chi), \\ R_0(\rho) &= \left( \bigcup_{m=0}^n \rho_m^{-1}(0) \cap S(\mathfrak{f}_\chi) \right) \bigcup (S(\mathfrak{f}_\chi) - S(\mathfrak{n}\mathfrak{f}_\chi^{-2})). \end{aligned}$$

Furthermore, for any  $k \in \mathbb{N}_0$ , set

$$S_k(\rho) = \begin{cases} R_0(\rho) & (\text{if } k = 0), \\ U_k(\rho) \cup R_k(\rho) & (\text{if } k \geq 1), \end{cases}$$

$R(\rho) = \bigcup_{k=0}^n R_k(\rho)$  and  $S(\rho) = \bigcup_{k=0}^n S_k(\rho)$ . Then, by [18, Proposition 34] we have

$$E_{\chi, \rho}^\circ(\nu, g) = f_{\chi, \rho}^{(\nu)}(g) + D_F^{-1/2} A_{\chi, \rho}(\nu) \frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)} f_{\chi^{-1}, \rho}^{(-\nu)}(g),$$

where

$$\begin{aligned} A_{\chi, \rho}(\nu) &= N(\mathfrak{f}_\chi)^{-\nu} \prod_{k=0}^n \prod_{v \in S_k(\rho)} \left\{ q_v^{d_v/2} q_v^{-k\nu} \frac{\epsilon(1 - \nu, \chi_v^{-2}, \psi_{F_v}) \epsilon(1 + \nu/2, \chi_v, \psi_{F_v})}{\epsilon(1 - \nu/2, \chi_v^{-1}, \psi_{F_v})} \right. \\ &\quad \left. \times \frac{L(1 + \nu, \chi_v^2)}{L(1 - \nu, \chi_v^{-2})} \right\}. \end{aligned}$$

We fix a character  $\eta$  of  $F^\times \backslash \mathbb{A}^\times$  satisfying  $(\star)$  in Section 2.1. For any  $v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\eta)$  and  $k \in \mathbb{N}_0$ , let  $Q_{k, \chi}^{(\nu)}(\eta_v, X)$  be the polynomial defined in [18, §9] as follows:

- For  $v \in \Sigma_{\text{fin}} - S(\mathfrak{f}_\chi)$ , set

$$Q_{k,\chi_v}^{(\nu)}(\eta_v, X) = \begin{cases} 1 & (\text{if } k = 0), \\ \eta_v(\varpi_v)X - \frac{\chi_v(\varpi_v)q_v^{-\nu/2} + \chi_v(\varpi_v)^{-1}q_v^{\nu/2}}{q_v^{1/2} + q_v^{-1/2}} & (\text{if } k = 1), \\ q_v^{-1}\eta_v(\varpi_v)^{k-2}X^{k-2}(\chi_v(\varpi_v)q_v^{(1-\nu)/2}\eta_v(\varpi_v)X - 1) \\ \quad \times (\chi_v(\varpi_v)^{-1}q_v^{(1+\nu)/2}\eta_v(\varpi_v)X - 1) & (\text{if } k \geq 2). \end{cases}$$

- For  $v \in S(\mathfrak{f}_\chi)$ , set

$$Q_{k,\chi_v}^{(\nu)}(\eta_v, X) := \eta_v(\varpi_v)^k X^k.$$

Then, we have the following.

**Proposition 8.** (See [18, Proposition 35].) We set  $E_{\chi,\rho}^{\natural}(\nu, g) = E_{\chi,\rho}(\nu, g) - E_{\chi,\rho}^{\circ}(\nu, g)$ . Then  $E_{\chi,\rho}^{\natural}(\nu, -)$  is left  $B_F$ -invariant and we have

$$Z^*(s, \eta, E_{\chi,\rho}^{\natural}(\nu, -)) = \mathcal{G}(\eta) D_F^{-\nu/2} \mathbf{N}(\mathfrak{f}_\chi)^{1/2-\nu} B_{\chi,\rho}^{\eta}(s, \nu) \frac{L(s + \nu/2, \chi\eta) L(s - \nu/2, \chi^{-1}\eta)}{L(1 + \nu, \chi^2)},$$

where

$$\begin{aligned} B_{\chi,\rho}^{\eta}(s, \nu) &= D_F^{s-1/2} \left\{ \prod_{k=0}^n \prod_{v \in S_k(\rho)} Q_{k,\chi_v}^{(\nu)}(\eta_v, q_v^{1/2-s}) L(1 + \nu, \chi_v^2) \right\} \\ &\times \prod_{v \in U_1(\rho)} (1 + q_v^{-1}) q_v^{-\nu/2} \prod_{k=2}^n \prod_{v \in U_k(\rho)} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k\nu/2} \\ &\times \left\{ \prod_{k=0}^n \prod_{v \in R_k(\rho)} q_v^{d_v/2 - k\nu/2} (1 - q_v^{-1})^{1/2} \overline{\mathcal{G}(\chi_v)} \right\} \prod_{v \in \Sigma_{\text{fin}} - R(\rho)} \chi_v(\varpi_v)^{d_v}. \end{aligned}$$

## 2.5. Regularized periods of Eisenstein series

For any characters  $\chi_1$  and  $\chi_2$  of  $F^\times \backslash \mathbb{A}^\times$ , we put  $\delta_{\chi_1, \chi_2} = \delta(\chi_1 = \chi_2)$ . The regularized period  $P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -))$  was computed as follows in [18].

**Proposition 9.** (See [18, Main Theorem B].) Assume  $\nu \in i\mathbb{R}$ . Then the integral  $P_{\beta,\lambda}^{\eta}(E_{\chi,\rho}(\nu, -))$  converges absolutely for any  $(\beta, \lambda) \in \mathcal{B} \times \mathbb{C}$  such that  $\text{Re}(\lambda) > 1$ . Moreover  $P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -))$  can be defined, and we have

$$P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -)) = \mathcal{G}(\eta) D_F^{-\nu/2} N(\mathfrak{f}_{\chi})^{1/2-\nu} \\ \times B_{\chi,\rho}^{\eta}(1/2, \nu) \frac{L((1+\nu)/2, \chi\eta) L((1-\nu)/2, \chi^{-1}\eta)}{L(1+\nu, \chi^2)}.$$

We define two functions  $\mathfrak{e}_{\chi,\rho,-1}$  and  $\mathfrak{e}_{\chi,\rho,0}$  on  $G_{\mathbb{A}}$  by the Laurent expansion

$$E_{\chi,\rho}(\nu, g) = \frac{\mathfrak{e}_{\chi,\rho,-1}(g)}{\nu-1} + \mathfrak{e}_{\chi,\rho,0}(g) + \mathcal{O}(\nu-1), \quad (\nu \rightarrow 1).$$

We explain the regularized  $\eta$ -periods of  $\mathfrak{e}_{\chi,\rho,-1}$  and that of  $\mathfrak{e}_{\chi,\rho,0}$ . Set  $R_F = \text{Res}_{s=1} \zeta_F(s) = \text{vol}(F^{\times} \backslash \mathbb{A}^{\times})$ , where  $\zeta_F(s)$  is the completed Dedekind zeta function of  $F$ . The regularized period  $P_{\text{reg}}^{\eta}(\mathfrak{e}_{\chi,\rho,-1})$  was computed as follows in [18].

**Proposition 10.** (See [18, Lemma 38 and Theorem 39].) *We have*

$$\mathfrak{e}_{\chi,\rho,-1}(g) = \delta(\chi^2 = \mathbf{1}, \mathfrak{f}_{\chi} = \mathfrak{o}_F, S(\rho) = \emptyset) \frac{D_F^{-1/2} R_F}{\zeta_F(2)} \chi(\det g)$$

for any  $g \in G_{\mathbb{A}}$ . Moreover, for  $\lambda \in \mathbb{C}$  such that  $\text{Re}(\lambda) > 0$ , we have

$$P_{\beta,\lambda}^{\eta}(\mathfrak{e}_{\chi,\rho,-1}) = \delta(\chi = \eta, \mathfrak{f}_{\chi} = \mathfrak{o}_F, S(\rho) = \emptyset) \frac{2D_F^{-1/2} R_F^2}{\zeta_F(2)} \frac{\beta(0)}{\lambda}$$

and  $P_{\text{reg}}^{\eta}(\mathfrak{e}_{\chi,\rho,-1}) = 0$ .

For any character  $\xi$  of  $F^{\times} \backslash \mathbb{A}^{\times}$ , we define  $R(\xi)$ ,  $C_0(\xi)$  and  $C_1(\xi)$  by the Laurent expansion

$$L(s, \xi) = \frac{R(\xi)}{s-1} + C_0(\xi) + C_1(\xi)(s-1) + \mathcal{O}((s-1)^2), \quad (s \rightarrow 1).$$

We note  $R(\xi) = \delta_{\xi,1} R_F$  for any character  $\xi$  of  $F^{\times} \backslash \mathbb{A}^{\times}$ . The regularized period  $P_{\text{reg}}^{\eta}(\mathfrak{e}_{\chi,\rho,0})$  is defined under some conditions, and  $P_{\beta,\lambda}^{\eta}(\mathfrak{e}_{\chi,\rho,0})$  was computed as follows in [18].

**Proposition 11.** (See [18, Theorem 40 and Corollary 41].) *Let  $\eta$  be a character of  $F^{\times} \backslash \mathbb{A}^{\times}$  satisfying  $(\star)$  in Section 2.1. The integral  $P_{\beta,\lambda}^{\eta}(\mathfrak{e}_{\chi,\rho,0})$  converges absolutely for any  $(\beta, \lambda) \in \mathcal{B} \times \mathbb{C}$  such that  $\text{Re}(\lambda) > 1$ . There exists an entire function  $f(\lambda)$  on  $\mathbb{C}$  such that*

$$P_{\beta,\lambda}^{\eta}(\mathfrak{e}_{\chi,\rho,0}) = \delta_{\chi,\eta} R_F f_{\chi,\rho}^{(1)}(e) \left\{ \frac{1}{\lambda-1} + \frac{1}{\lambda+1} \right\} \beta(1) \\ + 2\delta_{\chi,\eta} R_F \frac{D_F^{-1/2} f_{\chi,\rho}^{(1)}(e)}{\zeta_F(2)} \left\{ R_F \left( -\frac{\zeta'_F(2)}{\zeta_F(2)} A_{\chi,\rho}(1) + A'_{\chi,\rho}(1) \right) \right. \\ \left. + C_0(\mathbf{1}) A_{\chi,\rho}(1) \right\} \frac{\beta(0)}{\lambda}$$

$$+ f(\lambda) - \mathcal{G}(\eta) D_F^{-1/2} R_F \delta_{\chi, \eta} \left\{ -\frac{\tilde{B}_{\chi, \rho}^{\eta}(1)}{\lambda + 1} + \frac{\tilde{B}_{\chi, \rho}^{\eta}(-1)}{\lambda - 1} \right\} \beta(1) \\ - \frac{\mathcal{G}(\eta) D_F^{-1/2}}{\zeta_F(2)} \delta_{\chi, \eta} \left\{ -(\tilde{B}_{\chi, \rho}^{\eta})'(0) R_F^2 \frac{\beta(0)}{\lambda} + \tilde{B}_{\chi, \rho}^{\eta}(0) R_F^2 \frac{\beta(0)}{\lambda^2} \right\},$$

where  $\tilde{B}_{\chi, \rho}^{\eta}(z) = \epsilon(-z, \chi^{-1}\eta) B_{\chi, \rho}^{\eta}(-z + 1/2, 1)$ . Moreover we have

$$\text{CT}_{\lambda=0} P_{\beta, \lambda}^{\eta}(\mathfrak{e}_{\chi, \rho, 0}) = \frac{\mathcal{G}(\eta) D_F^{-1/2} N(\mathfrak{f}_{\chi})^{-1/2}}{L(2, \chi^2)} \left\{ -\frac{1}{2} \delta_{\chi, \eta} \tilde{B}_{\chi, \rho}^{\eta}(0) R_F^2 \beta''(0) + a_{\chi, \rho}^{\eta}(0) \beta(0) \right\},$$

where

$$a_{\chi, \rho}^{\eta}(0) = -\frac{1}{2} \delta_{\chi, \eta} (\tilde{B}_{\chi, \rho}^{\eta})''(0) R_F^2 - 2 \delta_{\chi, \eta} \tilde{B}_{\chi, \rho}^{\eta}(0) R_F C_1(\mathbf{1}) + \tilde{B}_{\chi, \rho}^{\eta}(0) C_0(\chi \eta)^2.$$

## 2.6. An orthonormal basis of $V_{\pi}^{\mathbf{K}_{\infty} \mathbf{K}_0(\mathbf{n})}$

Let  $\pi$  be a cuspidal automorphic representation of  $G_{\mathbb{A}}$  such that  $\pi \in \Pi_{\text{cus}}(\mathbf{n})$ . We put

$$\mathbb{P}^{\eta}(\pi; \mathbf{K}_0(\mathbf{n})) = \sum_{\varphi \in B} \overline{Z^*(1/2, \mathbf{1}, \varphi)} Z^*(1/2, \eta, \varphi),$$

where  $B$  is an orthonormal basis of  $V_{\pi}^{\mathbf{K}_{\infty} \mathbf{K}_0(\mathbf{n})}$ . Set  $\varphi_{\pi}^{\text{new}} = \varphi_{\pi, \rho_{\pi}}$ , where  $\rho_{\pi}$  is a unique element of  $\Lambda_{\pi}^0(\mathfrak{f}_{\pi})$ .

**Lemma 12.** *The value  $\mathbb{P}^{\eta}(\pi; \mathbf{K}_0(\mathbf{n}))$  is independent of the choice of  $B$  and we have*

$$\mathbb{P}^{\eta}(\pi; \mathbf{K}_0(\mathbf{n})) = D_F^{-1/2} \mathcal{G}(\eta) w_{\mathbf{n}}^{\eta}(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{\|\varphi_{\pi}^{\text{new}}\|^2}.$$

Here  $w_{\mathbf{n}}^{\eta}(\pi)$  is an explicit non-negative constant defined as

$$w_{\mathbf{n}}^{\eta}(\pi) = \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{nf}_{\pi}^{-1})} r(\pi_v, \eta_v, k) = \prod_{v \in S(\mathfrak{nf}_{\pi}^{-1})} r(\pi_v, \eta_v, \text{ord}_v(\mathfrak{nf}_{\pi}^{-1})),$$

where  $r(\pi_v, \eta_v, k)$  is defined as follows:

- If  $c(\pi_v) \geq 2$ , then  $r(\pi_v, \eta_v, k) = \begin{cases} k+1 & (\text{if } \eta_v(\varpi_v) = 1), \\ 2^{-1}(1 + (-1)^k) & (\text{if } \eta_v(\varpi_v) = -1). \end{cases}$
- If  $c(\pi_v) = 1$ , then  $\pi_v$  is isomorphic to  $\sigma(\chi_v | \cdot |_v^{1/2}, \chi_v | \cdot |_v^{-1/2})$  for some unramified character  $\chi_v$  of  $F_v^{\times}$ . Then

$$r(\pi_v, \eta_v, k) = \begin{cases} 1 + \frac{1 - \chi_v(\varpi_v) q_v^{-1}}{1 + \chi_v(\varpi_v) q_v^{-1}} k & (\text{if } \eta_v(\varpi_v) = 1), \\ 2^{-1}(1 + (-1)^k) & (\text{if } \eta_v(\varpi_v) = -1). \end{cases}$$

- If  $c(\pi_v) = 0$  and  $(\alpha_v, \alpha_v^{-1})$  is the Satake parameter of  $\pi_v$ , then

$$r(\pi_v, \eta_v, k) = \begin{cases} \frac{2}{1+Q(\pi_v)} + \frac{1-Q(\pi_v)}{1+Q(\pi_v)} \frac{q_v+1}{q_v-1} (k-1) & (\text{if } \eta_v(\varpi_v) = 1), \\ \frac{q_v+1}{q_v-1} \frac{1+(-1)^k}{2} & (\text{if } \eta_v(\varpi_v) = -1), \end{cases}$$

where  $Q(\pi_v) = (\alpha_v + \alpha_v^{-1})(q_v^{1/2} + q_v^{-1/2})^{-1}$ .

Moreover,  $\mathcal{G}(\eta)^{-1} \mathbb{P}^\eta(\pi; \mathbf{K}_0(\mathfrak{n}))$  is non-negative.

**Proof.** The first assertion is obvious. Thus we may take  $\{||\varphi_{\pi, \rho}||^{-1} \varphi_{\pi, \rho}\}_{\rho \in \Lambda_\pi^0(\mathfrak{n})}$  as  $B$ . By virtue of Proposition 7, we have

$$\begin{aligned} \mathbb{P}^\eta(\pi; \mathbf{K}_0(\mathfrak{n})) &= \sum_{\rho \in \Lambda_\pi^0(\mathfrak{n})} \frac{1}{||\varphi_{\pi, \rho}||^2} Z^*(1/2, \mathbf{1}, \varphi) Z^*(1/2, \eta, \varphi) \\ &= \sum_{\rho \in \Lambda_\pi^0(\mathfrak{n})} \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \left\{ \frac{\overline{Q_{\rho_k(v), v}^{\pi_v}(\mathbf{1}_v, 1)} Q_{\rho_k(v), v}^{\pi_v}(\eta_v, 1)}{\tau_{\pi_v}(\rho_k(v), \rho_k(v))} \right\} \\ &\quad \times \frac{\mathcal{G}(\mathbf{1}) \mathcal{G}(\eta) L(1/2, \pi) L(1/2, \pi \otimes \eta)}{||\varphi_\pi^{\text{new}}||^2}. \end{aligned}$$

Then, we obtain the second assertion by setting

$$w_{\mathfrak{n}}^\eta(\pi) = \sum_{\rho \in \Lambda_\pi^0(\mathfrak{n})} \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \left\{ \frac{\overline{Q_{\rho_k(v), v}^{\pi_v}(\mathbf{1}_v, 1)} Q_{\rho_k(v), v}^{\pi_v}(\eta_v, 1)}{\tau_{\pi_v}(\rho_k(v), \rho_k(v))} \right\}.$$

Here  $\tau_{\pi_v}(j, j) = ||\phi_{j, v}||_v^2$  for  $k \in \mathbb{N}$  and  $||\cdot||_v$  is the norm on  $V_{\pi_v}$  defined by the  $G_v$ -invariant inner product normalized so that  $||\phi_{0, v}||_v = 1$ . We remark that an explicit formula of  $\tau_{\pi_v}(j, j)$  was given in [18, Corollaries 12, 16 and Lemma 3]. By definition and a direct computation, we have

$$w_{\mathfrak{n}}^\eta(\pi) = \prod_{k=1}^n \left\{ \sum_{(j_v)_v \in \{0, \dots, k\} \cdot S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \prod_{v \in S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} r_{v, j_v} \right\} = \prod_{k=1}^n \prod_{v \in S_k(\mathfrak{n} \mathfrak{f}_\pi^{-1})} \sum_{j=0}^k r_{v, j}$$

and  $\sum_{j=0}^k r_{v, j} = r(\pi_v, \eta_v, k)$ , where  $r_{v, j} = \overline{Q_{j, v}^{\pi_v}(\mathbf{1}_v, 1)} Q_{j, v}^{\pi_v}(\eta_v, 1) \tau_{\pi_v}(j, j)^{-1}$ .

Then, one can check  $w_{\mathfrak{n}}^\eta(\pi) \in \mathbb{R}_{\geq 0}$  easily by noting  $|Q(\pi_v)| < 1$  when  $c(\pi_v) = 0$ . The estimate  $L(1/2, \pi) L(1/2, \pi \otimes \eta) \geq 0$  by [5] gives us  $\mathcal{G}(\eta)^{-1} \mathbb{P}^\eta(\pi; \mathbf{K}_0(\mathfrak{n})) \geq 0$ .  $\square$

Since  $\eta^2 = \mathbf{1}$ , we have  $\tilde{\eta}(\mathfrak{n}) = \pm 1$ . We consider only the case of  $\tilde{\eta}(\mathfrak{n}) = 1$  because of the following reason.



**Lemma 13.** *Let  $\pi$  be a  $\mathbf{K}_\infty$ -spherical irreducible cuspidal automorphic representation of  $G_\mathbb{A}$  with trivial central character. Let  $\eta$  be a character of  $F^\times \backslash \mathbb{A}^\times$  such that  $\eta^2 = 1$  and  $\mathfrak{f}_\eta$  is relatively prime to  $\mathfrak{f}_\pi$ . Suppose that  $\eta_v(-1) = 1$  for all  $v \in \Sigma_\infty$ . Then,  $L(1/2, \pi)L(1/2, \pi \otimes \eta) = 0$  unless  $\tilde{\eta}(\mathfrak{f}_\pi) = 1$ .*

**Proof.** By the argument in the proof of [20, Lemma 2.3], it is enough to show  $\epsilon(1/2, \pi_v, \psi_{F_v})\epsilon(1/2, \pi_v \otimes \eta_v, \psi_{F_v}) = \eta_v(\varpi_v^{c(\pi_v)})$  for any  $v \in \bigcup_{k \geq 2} S_k(\mathfrak{f}_\pi)$ . It follows immediately from fundamental properties of  $\epsilon$ -factors (cf. [16, 1.1]). We note that  $\eta_v$  is unramified if  $v \in S(\mathfrak{f}_\pi)$ .  $\square$

## 2.7. Adjoint $L$ -functions

Let  $\pi$  be a cuspidal automorphic representation of  $G_\mathbb{A}$  contained in  $\Pi_{\text{cus}}(\mathfrak{n})$ . To examine an explicit description of  $\|\varphi_\pi^{\text{new}}\|^2$  in terms of the adjoint  $L$ -function of  $\pi$ , we compute the Rankin–Selberg convolution of  $\varphi_\pi^{\text{new}}$  and  $E_{1, \rho_0}(\nu, g)$ , where  $\rho_0$  denotes a unique element of  $\Lambda_1(\mathfrak{o})$ . For any  $v \in \Sigma_F$ , we denote by  $Z_v(s)$  the local Rankin–Selberg integral

$$\int_{\mathbf{K}_v} \int_{F_v^\times} \phi_{0,v} \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} k_v \right) \overline{\phi_{0,v} \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} k_v \right)} |t_v|_v^{s-1} d^\times t_v dk_v.$$

**Lemma 14.** *Set  $S_\pi = \{v \in \Sigma_{\text{fin}} \mid \text{ord}_v(\mathfrak{f}_\pi) \geq 2\}$ . We have*

$$\begin{aligned} & \int_{Z_\mathbb{A} G_F \backslash G_\mathbb{A}} \varphi_\pi^{\text{new}}(g) \overline{\varphi_\pi^{\text{new}}(g)} E_{1, \rho_0}(2s-1, g) dg \\ &= [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1} N(\mathfrak{f}_\pi)^s D_F^{s-3/2} \zeta_F(2s)^{-1} \zeta_F(s) L(s, \pi; \text{Ad}) \\ & \quad \times \prod_{v \in S_\pi} \frac{q_v^{d_v(3/2-s)} q_v^{c(\pi_v)(1-s)} Z_v(s)}{L(s, \pi_v; \text{Ad})} \frac{1 + q_v^{-1}}{1 + q_v^{-s}} \end{aligned}$$

for  $\text{Re}(s) \gg 0$ . Moreover, we have  $\|\varphi_\pi^{\text{new}}\|^2 = 2N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]^{-1} L^{S_\pi}(1, \pi; \text{Ad})$ .

**Proof.** If  $v \in \Sigma_F - S_\pi$ , then  $Z_v(s)$  is computed in [20, Lemma 2.14]. Hence, it suffices to examine  $Z_v(s)$  when  $v \in S_\pi$ . By  $[\mathbf{K}_v : \mathbf{K}_0(\mathfrak{p}_v^{c(\pi_v)})] = q_v^{c(\pi_v)}(1 + q_v^{-1})$ , we obtain the first assertion.

We note  $Z_v(1) = q_v^{-d_v/2}$  for  $v \in S_\pi$ . Then we obtain the second assertion by taking the residue at  $s = 1$  with the aid of  $\text{Res}_{s=1} E_{1, \rho_0}(s, g) = D_F^{-1/2} R_F \zeta_F(2)^{-1}$  by Proposition 10.  $\square$

### 3. Adelic Green functions

We define the adelic Green function on  $G_{\mathbb{A}}$  associated to an ideal  $\mathfrak{n}$  of  $\mathfrak{o}_F$ . This was introduced in [20] in the case where  $\mathfrak{n}$  is square-free. We define the function in the case where  $\mathfrak{n}$  is an arbitrary ideal of  $\mathfrak{o}_F$ .

Let  $v \in \Sigma_{\infty}$ . For  $s, z \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 2|\operatorname{Re}(z)|$ , set

$$\begin{aligned} \Psi_v^{(z)}(s; g) &= |\det g|^{(s+1)/2} \frac{-1}{8\sqrt{\pi}} \frac{\Gamma((s+2z+1)/4) \Gamma((s-2z+1)/4)}{\Gamma(s/2+1)} (a^2 + b^2)^{-(s-2z+1)/4} \\ &\quad \times (c^2 + d^2)^{-(s+2z+1)/4} \\ &\quad \times {}_2F_1 \left( \frac{s+2z+1}{4}, \frac{s-2z+1}{4}; \frac{s}{2} + 1; \frac{(\det g)^2}{(a^2 + b^2)(c^2 + d^2)} \right) \end{aligned}$$

for any  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_v \cong GL(2, \mathbb{R})$ . Then,  $\Psi_v^{(z)}(s; -)$  has nice properties on the Casimir operator  $\Omega_v$  of  $G_v$  as in [20, §2.5.2] (cf. [20, §4]). We call this the Green function on  $G_v$ .

Let  $v \in \Sigma_{\text{fin}}$ . For  $s, z \in \mathbb{C}$  such that  $\operatorname{Re}(s) > 2|\operatorname{Re}(z)|$ , we define  $\Psi_v^{(z)}(s; -)$  as a function satisfying the  $(H_v, \mathbf{K}_v)$ -equivariance

$$\Psi_v^{(z)} \left( s; \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} g k \right) = |t_1/t_2|_v^z \Psi_v^{(z)}(s; g), \quad \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in H_v, k \in \mathbf{K}_v$$

and

$$\begin{aligned} \Psi_v^{(z)} \left( s; \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) &= -q_v^{-(s+1)/2} (1 - q_v^{-(s-2z+1)/2})^{-1} (1 - q_v^{-(s+2z+1)/2})^{-1} \\ &\quad \times \sup(1, |x|_v)^{-(s-2z+1)/2}, \quad x \in F_v. \end{aligned}$$

Then  $\Psi_v^{(z)}(s_v; -)$  is uniquely determined and has nice properties on the  $v$ -th Hecke operator  $\mathbb{T}_v$ , where  $\mathbb{T}_v$  denotes the characteristic function of  $\mathbf{K}_v \begin{pmatrix} \varpi_v & 0 \\ 0 & 1 \end{pmatrix} \mathbf{K}_v$  on  $G_v$  divided by  $\operatorname{vol}(\mathbf{K}_v, dg_v)$  (cf. [20, §5]). We call  $\Psi_v^{(z)}(s_v; -)$  the Green function on  $G_v$ .

For any  $v \in S(\mathfrak{n})$ , we set

$$\Phi_{\mathfrak{n},v}^{(z)} \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} k \right) = |t_1/t_2|_v^z \delta(x \in \mathfrak{o}_v) \delta(k \in \mathbf{K}_0(\mathfrak{no}_v))$$

for any  $t_1, t_2 \in F_v^\times$ ,  $x \in F_v$  and  $k \in \mathbf{K}_v$  and put  $\Phi_{0,v}^{(z)} = \Phi_{\mathfrak{n},v}^{(z)}$  for  $v \in \Sigma_{\text{fin}} - S(\mathfrak{n})$ .

Fix a finite subset  $S$  of  $\Sigma_F$  such that  $\Sigma_{\infty} \subset S$ , and set  $\mathfrak{X}_S = \prod_{v \in \Sigma_{\infty}} \mathbb{C} \times \prod_{v \in S_{\text{fin}}} (\mathbb{C}/4\pi i(\log q_v)^{-1} \mathbb{Z})$  and  $q(\mathfrak{s}) = \inf_{v \in S} (\operatorname{Re}(s_v) + 1)/4$  for any  $\mathfrak{s} \in \mathfrak{X}_S$ . Then, for any  $\mathfrak{s} \in \mathfrak{X}_S$  and  $z \in \mathbb{C}$  such that  $q(\mathfrak{s}) > |\operatorname{Re}(z)| + 1$ , the adelic Green function is defined by

$$\Psi^{(z)}(\mathfrak{n}|\mathfrak{s}, g) = \prod_{v \in \Sigma_{\infty}} \Psi_v^{(z)}(s_v, g_v) \prod_{v \in S_{\text{fin}}} \Psi_v^{(z)}(s_v, g_v) \prod_{v \in S(\mathfrak{n})} \Phi_{\mathfrak{n},v}^{(z)}(g_v) \prod_{v \notin S \cup S(\mathfrak{n})} \Phi_{0,v}^{(z)}(g_v)$$

for any  $g = (g_v)_{v \in \Sigma_F} \in G_{\mathbb{A}}$ . Note that the function  $\Psi^{(z)}(\mathfrak{n}|\mathbf{s}; -)$  on  $G_{\mathbb{A}}$  is right  $\mathbf{K}_{\infty}\mathbf{K}_0(\mathfrak{n})$ -invariant and continuous on  $G_{\mathbb{A}}$ . Moreover, we have  $\Psi^{(z)}\left(\mathfrak{n}|\mathbf{s}; \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} g\right) = |t_1/t_2|_{\mathbb{A}}^z \Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g)$  for any  $\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \in H_{\mathbb{A}}$  and  $g \in G_{\mathbb{A}}$ .

To state a main property of adelic Green functions, for any  $\varphi \in C_c^{\infty}(\mathfrak{A}G_F \backslash G_{\mathbb{A}})$  we consider the integral

$$\varphi^{H, (z)}(g) = \int_{\mathfrak{A}H_F \backslash H_{\mathbb{A}}} \varphi(hg) \chi_z(h) dh,$$

where  $\chi_z : H_F \backslash H_{\mathbb{A}} \rightarrow \mathbb{C}^{\times}$  is defined by

$$\chi_z \left( \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \right) = |t_1/t_2|_{\mathbb{A}}^z$$

for any  $t_1, t_2 \in \mathbb{A}^{\times}$ . The integral  $\varphi^{H, (z)}(g)$  converges absolutely and  $\varphi^{H, (z)}(hg) = \chi_z(h)^{-1} \varphi^{H, (z)}(g)$  holds for any  $h \in H_{\mathbb{A}}$  (cf. [20, §6.2]).

Let  $\mathfrak{Z}(\mathfrak{g}_{\infty})$  be the center of the universal enveloping algebra of the complexification of the Lie algebra of  $GL(2, F \otimes_{\mathbb{Q}} \mathbb{R})$ . For  $v \in \Sigma_{\text{fin}}$ , the spherical Hecke algebra of  $G_v$  is denoted by  $\mathcal{H}(G_v, \mathbf{K}_v)$ . For  $\mathbf{s} \in \mathfrak{X}_S$ , the element  $\Omega_S(\mathbf{s})$  of the algebra  $\mathfrak{Z}(\mathfrak{g}_{\infty}) \otimes \{\otimes_{v \in \Sigma_{\text{fin}}} \mathcal{H}(G_v, \mathbf{K}_v)\}$  is defined as

$$\Omega_S(\mathbf{s}) = \bigotimes_{v \in \Sigma_{\infty}} \left( \Omega_v - \frac{s_v^2 - 1}{2} \right) \bigotimes_{v \in \Sigma_{\text{fin}}} \left( \mathbb{T}_v - (q_v^{(1-s_v)/2} + q_v^{(1+s_v)/2}) \mathbf{1}_{\mathbf{K}_v} \right),$$

where  $\mathbf{1}_{\mathbf{K}_v}$  is the unit element of  $\mathcal{H}(G_v, \mathbf{K}_v)$ . The following proposition is proved in the same way as [20, Lemma 6.3].

**Proposition 15.** *Suppose  $q(\mathbf{s}) > 2|\operatorname{Re}(z)| + 1$ . Then, for any  $\varphi \in C_c^{\infty}(\mathfrak{A}G_F \backslash G_{\mathbb{A}})^{\mathbf{K}_{\infty}\mathbf{K}_0(\mathfrak{n})}$ , the function  $g \mapsto \Psi^{(z)}(\mathfrak{n}|\mathbf{s}; g) \varphi^{H, (z)}(g)$  is integrable on  $H_{\mathbb{A}} \backslash G_{\mathbb{A}}$  and we have*

$$\int_{H_{\mathbb{A}} \backslash G_{\mathbb{A}}} \Psi^{(z)}(\mathfrak{n}|\mathbf{s}, g) [R(\Omega_S(\mathbf{s})) \varphi^{H, (z)}](g) dg = \operatorname{vol}(H_{\text{fin}} \backslash H_{\text{fin}} \mathbf{K}_0(\mathfrak{n})) \varphi^{H, (z)}(e).$$

#### 4. Spectral expansions of renormalized Green functions

In this section, we introduce  $\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; g)$  for arbitrary ideal  $\mathfrak{n}$  by using the  $L^2$ -expansion of  $\hat{\Psi}_{\beta, \lambda}(\mathfrak{n}|\alpha; g)$  for a parameter  $(\beta, \lambda) \in \mathcal{B} \times \mathbb{C}$ . The cuspidal and residual spectra of  $\hat{\Psi}_{\beta, \lambda}(\mathfrak{n}|\alpha; g)$  are analyzed in the same way as [20, §9]. Contrary to this, we need harder calculation for the continuous spectrum.

The set  $\mathfrak{X}_S = \prod_{v \in \Sigma_{\infty}} \mathbb{C} \times \prod_{v \in \Sigma_{\text{fin}}} (\mathbb{C}/4\pi i(\log q_v)^{-1} \mathbb{Z})$  is considered as a complex manifold with respect to a usual complex structure. Let  $\mathcal{A}_S$  be the space of holomorphic

functions  $\alpha(\mathbf{s})$  on  $\mathfrak{X}_S$  such that for any  $v \in S$  and  $\mathbf{s}' \in \mathfrak{X}_{S-\{v\}}$ , the function  $s_v \mapsto \alpha(\mathbf{s}', s_v)$  is contained in  $\mathcal{B}$ .

For  $\mathbf{c} \in \mathbb{R}^S$ , we put  $\mathbb{L}_S(\mathbf{c}) = \{\mathbf{s} \in \mathfrak{X}_S \mid \operatorname{Re}(\mathbf{s}) = \mathbf{c}\}$ . A multidimensional contour integral of a holomorphic function  $f(\mathbf{s})$  on  $\mathfrak{X}_S$  along  $\mathbb{L}_S(\mathbf{c})$  is defined inductively as

$$\int_{\mathbb{L}_S(\mathbf{c})} f(\mathbf{s}) d\mu_S(\mathbf{s}) = \int_{L_v(c_v)} \left\{ \int_{\mathbb{L}_{S-\{v\}}(\mathbf{c}')} f(\mathbf{s}', s_v) d\mu_{S-\{v\}}(\mathbf{s}') \right\} d\mu_v(s_v)$$

for  $\mathbf{c} = (\mathbf{c}', c_v) \in \mathbb{R}^S$ , where

$$d\mu_v(s) = \begin{cases} s ds & (\text{if } v \in \Sigma_\infty), \\ \frac{1}{2}(\log q_v)(q_v^{(1+s)/2} - q_v^{(1-s)/2}) ds & (\text{if } v \in \Sigma_{\text{fin}}) \end{cases}$$

and  $L(c_v)$  stands for  $c_v + i\mathbb{R}$  and  $c_v + i\mathbb{R}/4\pi i(\log q_v)^{-1}\mathbb{Z}$  for  $v \in \Sigma_\infty$  and  $v \in \Sigma_{\text{fin}}$ , respectively. Then, for  $\mathbf{c} \in \mathbb{R}^S$  and  $z \in \mathbb{C}$  such that  $q(\mathbf{c}) > |\operatorname{Re}(z)| + 1$ , the integral

$$\hat{\Psi}^{(z)}(\mathbf{n}|\alpha; g) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \Psi^{(z)}(\mathbf{n}|\mathbf{s}, g) \alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

is absolutely convergent and is independent of the choice of  $\mathbf{c}$ , and the function  $z \mapsto \hat{\Psi}^{(z)}(\mathbf{n}|\alpha; g)$  is entire. Furthermore, for  $\beta \in \mathcal{B}$ ,  $\lambda \in \mathbb{C}$  and  $g \in G_{\mathbb{A}}$ , we consider the integral

$$\hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g) = \frac{1}{2\pi i} \int_{L_\sigma} \frac{\beta(z)}{z + \lambda} \{\hat{\Psi}^{(z)}(\mathbf{n}|\alpha; g) + \hat{\Psi}^{(-z)}(\mathbf{n}|\alpha; g)\} dz$$

for  $\sigma \in \mathbb{R}$  such that  $-\inf(q(\mathbf{s}) - 1, \operatorname{Re}(\lambda)) < \sigma < q(\mathbf{s}) - 1$ . The integral of the right-hand side is absolutely convergent and is independent of the choice of  $\sigma$ . Moreover, for  $\alpha \in \mathcal{A}_S$ ,  $\beta \in \mathcal{B}$  and  $\lambda \in \mathbb{C}$  with  $\operatorname{Re}(\lambda) > 0$ , the Poincaré series of  $\hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g)$  is defined to be

$$\hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g) = \sum_{\gamma \in H_F \backslash G_F} \hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; \gamma g)$$

for  $g \in G_{\mathbb{A}}$ . We have the following in the same way as [20, Proposition 9.1]:

1. The series  $\hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g)$  is absolutely convergent locally uniformly in  $\{\operatorname{Re}(\lambda) > 0\} \times G_{\mathbb{A}}$ . Moreover, the function  $\lambda \mapsto \hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g)$  on  $\operatorname{Re}(\lambda) > 0$  is holomorphic and the function  $g \mapsto \hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; g)$  on  $G_{\mathbb{A}}$  is continuous, left  $G_F$ -invariant and right  $\mathbf{K}_\infty \mathbf{K}_0(\mathbf{n})$ -invariant.
2. For  $\operatorname{Re}(\lambda) > 0$ , we have  $\hat{\Psi}_{\beta, \lambda}(\mathbf{n}|\alpha; -) \in L^l(\mathfrak{A}G_F \backslash G_{\mathbb{A}})$  for any  $l > 0$  such that  $l(1 - \operatorname{Re}(\lambda)) < 1$ .

Let us compute the spectral expansion of  $\hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; -)$  explicitly. Recall spectral parameters at  $S$  of automorphic forms (cf. [20, 9.1.3]). For a given automorphic form  $\varphi$  on  $G_{\mathbb{A}}$ , if there exists  $\nu_{\varphi,S} = (\nu_{\varphi,v})_{v \in S} \in \mathfrak{X}_S$  such that

$$R(\Omega_v)\varphi = \frac{\nu_{\varphi,v}^2 - 1}{2}\varphi$$

and

$$R(\mathbb{T}_v)\varphi = (q_v^{(1-\nu_{\varphi,v})/2} + q_v^{(1+\nu_{\varphi,v})/2})\varphi$$

hold for all  $v \in \Sigma_{\infty}$  and all  $v \in S_{\text{fin}}$ , respectively, then we call  $\nu_{\varphi,S}$  the spectral parameter at  $S$  of  $\varphi$ . Set

$$C(\mathbf{n}, S) = (-1)^{\#S} \text{vol}(H_{\text{fin}} \backslash H_{\text{fin}} \mathbf{K}_0(\mathbf{n})) = (-1)^{\#S} D_F^{-1/2} [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})]^{-1}.$$

By using Proposition 15 and the argument similar to [20, Lemma 9.4], we have the following.

**Lemma 16.** Assume  $\text{Re}(\lambda) > 1$ . Then, for any automorphic form  $\varphi$  on  $G_{\mathbb{A}}$  with spectral parameter  $\nu_{\varphi,S}$ , we have

$$\langle \hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; -) | \varphi \rangle_{L^2} = C(\mathbf{n}, S) \alpha(\nu_{\varphi,S}) P_{\beta,\lambda}^1(\bar{\varphi}),$$

where  $\langle \cdot | \cdot \rangle_{L^2}$  is the  $L^2$ -inner product on  $L^2(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})$ .

For any character  $\chi$  of  $F^{\times} \backslash \mathbb{A}^{\times}$  and  $\alpha \in \mathcal{A}_S$ , we define the function  $\tilde{\alpha}_{\chi}$  on  $\mathbb{C}$  by

$$\tilde{\alpha}_{\chi}(\nu) = \alpha((\nu + 2ib(\chi_v))_{v \in S})$$

and write  $\tilde{\alpha}(\nu)$  for  $\tilde{\alpha}_1(\nu)$ .

Fix an orthonormal basis  $\mathcal{B}_{\text{cus}}(\mathbf{n})$  of  $\sum_{\pi \in \Pi_{\text{cus}}(\mathbf{n})} V_{\pi}^{\mathbf{K}_{\infty} \mathbf{K}_0(\mathbf{n})}$  and let  $\mathcal{B}_{\text{res}}(\mathbf{n})$  be the orthonormal system consisting of all functions  $\varphi_{\chi} = \text{vol}(Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}})^{-1/2} \chi \circ \det$  on  $G_{\mathbb{A}}$  for any  $\chi \in \Xi_0(\mathfrak{o}_F)$  such that  $\chi^2 = \mathbf{1}$ . We write  $\Lambda(\mathbf{n})$  for  $\Lambda_1(\mathbf{n})$ . By Lemma 16, the following spectral expansion of  $\hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; g)$  is given in the same way as [20, Lemma 9.6].

**Lemma 17.** Assume  $\text{Re}(\lambda) > 1$ . Then we have the expression

$$\begin{aligned} \hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; g) = C(\mathbf{n}, S) & \left\{ \sum_{\varphi \in \mathcal{B}_{\text{cus}}(\mathbf{n})} \alpha(\nu_{\varphi,S}) P_{\beta,\lambda}^1(\bar{\varphi}) \varphi(g) + \sum_{\varphi \in \mathcal{B}_{\text{res}}(\mathbf{n})} \alpha(\nu_{\varphi,S}) P_{\beta,\lambda}^1(\bar{\varphi}) \varphi(g) \right. \\ & \left. + \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_{\chi}(\mathbf{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) P_{\beta,\lambda}^1(\overline{E_{\chi,\rho}(\nu, -)}) E_{\chi,\rho}(\nu, g) d\nu \right\}. \end{aligned}$$

The series and integrals in the right-hand side converge absolutely and locally uniformly on  $Z_{\mathbb{A}}G_F \backslash G_{\mathbb{A}}$ .

**Lemma 18.** For any  $g \in G_{\mathbb{A}}$ , the function  $\lambda \mapsto \hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$  on  $\operatorname{Re}(\lambda) > 1$  is continued to a meromorphic function on  $\operatorname{Re}(\lambda) > -1/2$ .

**Proof.** We give a proof in the same way as [20, Lemma 9.8]. Let  $\Psi_{\text{cus}}(\lambda) = \Psi_{\text{cus}}(\lambda, \alpha, g)$ ,  $\Psi_{\text{res}}(\lambda) = \Psi_{\text{res}}(\lambda, \alpha, g)$  and  $\Psi_{\text{ct}}(\lambda) = \Psi_{\text{ct}}(\lambda, \alpha, g)$  be the cuspidal part, the residual part and the Eisenstein part divided by  $C(\mathfrak{n}, S)$  in the spectral expansion of  $\hat{\Psi}_{\beta,\lambda}(\mathfrak{n}|\alpha; g)$  given in Lemma 17, respectively.

First we examine  $\Psi_{\text{res}}(\lambda)$ . For  $\operatorname{Re}(\lambda) > 0$ , by applying Proposition 10, the function  $\Psi_{\text{res}}(\lambda)$  is written as

$$\Psi_{\text{res}}(\lambda) = \sum_{\chi \in \Xi_0(\mathfrak{o}_F), \chi^2=1} \alpha(\nu_{\varphi_{\chi}, S}) P_{\beta,\lambda}^1(\overline{\varphi_{\chi}}) \varphi_{\chi}(g) = 2\tilde{\alpha}(1) \frac{R_F}{\operatorname{vol}(Z_{\mathbb{A}}G_F \backslash G_{\mathbb{A}})} \frac{\beta(0)}{\lambda}$$

and has a meromorphic continuation to  $\mathbb{C}$ . From this,  $\operatorname{CT}_{\lambda=0} \Psi_{\text{res}}(\lambda) = 0$  holds.

Next we examine  $\Psi_{\text{cus}}(\lambda)$ . By the same computation as in the proof of [20, Lemma 9.8], the series  $\Psi_{\text{cus}}(\lambda)$  converges absolutely and the estimate

$$|\Psi_{\text{cus}}(\lambda, \alpha, g)| \ll y(g)^{-m}, \quad g \in \mathfrak{S} \cap G_{\mathbb{A}}^1$$

holds, where  $\mathfrak{S}$  denotes a Siegel set of  $G_{\mathbb{A}}$  such that  $G_{\mathbb{A}} = G_F \mathfrak{S}$  and  $y$  denotes the height function on  $G_{\mathbb{A}}$ . Moreover,  $\Psi_{\text{cus}}(\lambda)$  is analytically continued to an entire function and we have

$$\operatorname{CT}_{\lambda=0} \Psi_{\text{cus}}(\lambda) = \sum_{\varphi \in \mathcal{B}_{\text{cus}}(\mathfrak{n})} \alpha(\nu_{\varphi, S}) \overline{P_{\text{reg}}^1(\varphi)} \varphi(g).$$

Therefore, it is enough to examine  $\Psi_{\text{ct}}(\lambda)$ . Assume  $\operatorname{Re}(\lambda) > 1$  and  $\nu \in i\mathbb{R}$ . By the proof of [18, Theorem 37], the integral  $P_{\lambda,\beta}^1(E_{\chi^{-1},\rho}(-\nu, -))$  can be expressed as

$$\begin{aligned} P_{\lambda,\beta}^1(E_{\chi^{-1},\rho}(-\nu, -)) &= P_{\chi^{-1}}(\mathbf{1}, \lambda, -\nu) + D_F^{-1/2} A_{\chi^{-1},\rho}(-\nu) \frac{L(-\nu, \chi^{-2})}{L(1-\nu, \chi^{-2})} P_{\chi}(\mathbf{1}, \lambda, \nu) \\ &\quad + Q_{\chi^{-1},\rho}^+(\mathbf{1}, \lambda, -\nu) + Q_{\chi^{-1},\rho}^-(\mathbf{1}, \lambda, -\nu), \end{aligned}$$

where

$$P_{\chi^{\pm 1}}(\eta, \lambda, \pm \nu) = f_{\chi^{\pm 1},\rho}^{(\pm \nu)}(e) \delta_{\chi,\eta} R_F \left\{ \frac{\beta((\mp \nu - 1)/2)}{\lambda - (\pm \nu + 1)/2} + \frac{\beta((\pm \nu + 1)/2)}{\lambda + (\pm \nu + 1)/2} \right\}$$

and

$$Q_{\chi^{-1},\rho}^{\pm}(\eta, \lambda, -\nu) = \frac{1}{2\pi i} \int_{L_{\pm\sigma}} Z^*(\pm z + 1/2, \eta, E_{\chi^{-1},\rho}^{\natural}(-\nu, -)) \frac{\beta(z)}{\lambda + z} dz.$$

We remark  $\overline{E_{\chi,\rho}(\nu, -)} = E_{\chi^{-1},\rho}(-\nu, -)$ . Furthermore, by the residue theorem, we have

$$\begin{aligned} & P_{\lambda,\beta}^1(E_{\chi^{-1},\rho}(-\nu, -)) \\ &= P_{\chi^{-1}}(\mathbf{1}, \lambda, -\nu) + D_F^{-1/2} A_{\chi^{-1},\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} P_{\chi}(\mathbf{1}, \lambda, \nu) + Q_{\chi^{-1},\rho}^0(\mathbf{1}, \lambda, -\nu) \\ &\quad - \left\{ \frac{\beta((- \nu + 1)/2)}{\lambda + (-\nu + 1)/2} \operatorname{Res}_{z=(-\nu+1)/2} + \frac{\beta((\nu + 1)/2)}{\lambda + (\nu + 1)/2} \operatorname{Res}_{z=(\nu+1)/2} \right. \\ &\quad \left. + \frac{\beta((- \nu - 1)/2)}{\lambda + (-\nu - 1)/2} \operatorname{Res}_{z=(-\nu-1)/2} + \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} \operatorname{Res}_{z=(\nu-1)/2} \right\} f_{\chi^{-1},\rho}^1(-z, -\nu), \end{aligned}$$

where we put

$$f_{\chi,\rho}^{\eta}(z, \nu) = Z^*(z + 1/2, \eta, E_{\chi,\rho}^{\natural}(\nu, -))$$

and

$$Q_{\chi,\rho}^0(\eta, \lambda, \nu) = \frac{1}{2\pi i} \int_{L_{\sigma}} \{f_{\chi,\rho}^{\eta}(z, \nu) + f_{\chi,\rho}^{\eta}(-z, \nu)\} \frac{\beta(z)}{\lambda + z} dz$$

for  $\operatorname{Re}(\lambda) > -\sigma$ . Thus we express  $\Psi_{\text{ct}}(\lambda)$  as the sum of the following four terms:

$$\begin{aligned} \Phi_1(\lambda) &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{\mathbf{1},\rho}^{(-\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) \beta((\nu - 1)/2) \left\{ \frac{1}{\lambda - (-\nu + 1)/2} + \frac{1}{\lambda + (-\nu + 1)/2} \right\} \\ &\quad \times E_{\mathbf{1},\rho}(\nu; g) d\nu, \\ \Phi_2(\lambda) &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{\mathbf{1},\rho}^{(\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) D_F^{-1/2} A_{\mathbf{1},\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} \beta((\nu + 1)/2) \\ &\quad \times \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} E_{\mathbf{1},\rho}(\nu, g) d\nu, \\ \Phi_3(\lambda) &= \frac{1}{8\pi i} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) Q_{\chi^{-1},\rho}^0(\mathbf{1}, \lambda, -\nu) E_{\chi,\rho}(\nu, g) d\nu, \\ \Phi_4(\lambda) &= - \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \left\{ \frac{\beta((\nu + 1)/2)}{\lambda + (\nu + 1)/2} \operatorname{Res}_{z=(\nu+1)/2} \right. \\ &\quad + \frac{\beta((- \nu + 1)/2)}{\lambda + (-\nu + 1)/2} \operatorname{Res}_{z=(-\nu+1)/2} + \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} \operatorname{Res}_{z=(\nu-1)/2} \\ &\quad \left. + \frac{\beta((- \nu - 1)/2)}{\lambda + (-\nu - 1)/2} \operatorname{Res}_{z=(-\nu-1)/2} \right\} \{f_{\chi^{-1},\rho}^1(-z, -\nu)\} \tilde{\alpha}_{\chi}(\nu) E_{\chi,\rho}(\nu, g) d\nu. \end{aligned}$$

By the functional equation

$$D_F^{-1/2} A_{1,\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} E_{1,\rho}(\nu, g) = E_{1,\rho}(-\nu, g)$$

of the Eisenstein series, the following equalities hold:

$$\begin{aligned} \Phi_2(\lambda) &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) D_F^{-1/2} A_{1,\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} E_{1,\rho}(\nu, g) \beta((\nu+1)/2) \\ &\quad \times \left\{ \frac{1}{\lambda - (\nu+1)/2} + \frac{1}{\lambda + (\nu+1)/2} \right\} d\nu \\ &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) \beta((-\nu+1)/2) \\ &\quad \times \left\{ \frac{1}{\lambda - (-\nu+1)/2} + \frac{1}{\lambda + (-\nu+1)/2} \right\} d\nu \\ &= \Phi_1(\lambda). \end{aligned}$$

Thus we have to consider only  $\Phi_1(\lambda)$ ,  $\Phi_3(\lambda)$  and  $\Phi_4(\lambda)$ .

We take  $c > 1$ . Then  $\Phi_1(\lambda)$  is expressed as

$$\begin{aligned} \Phi_1(\lambda) &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(-\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) \beta((\nu-1)/2) \left\{ \frac{1}{\lambda - (-\nu+1)/2} + \frac{1}{\lambda + (-\nu+1)/2} \right\} \\ &\quad \times E_{1,\rho}(\nu, g) d\nu \\ &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(-\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) \beta((\nu-1)/2) \frac{1}{\lambda + (-\nu+1)/2} E_{1,\rho}(\nu, g) d\nu \\ &\quad + \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(-\nu)}(e) \left\{ \int_{L_c} \tilde{\alpha}(\nu) \beta((\nu-1)/2) \frac{1}{\lambda - (-\nu+1)/2} E_{1,\rho}(\nu, g) d\nu \right. \\ &\quad \left. - 2\pi i \tilde{\alpha}(1) \beta(0) \frac{2}{\lambda} \mathfrak{e}_{1,\rho,-1}(g) \right\}. \end{aligned}$$

The first term is holomorphic on  $\operatorname{Re}(\lambda) > -1/2$ , the second term is holomorphic on  $\operatorname{Re}(\lambda) > (-c+1)/2$  and the third term is holomorphic on  $\mathbb{C} - \{0\}$ . Hence  $\Phi_1(\lambda) = \Phi_2(\lambda)$  has a meromorphic continuation to  $\operatorname{Re}(\lambda) > -1/2$ . Since  $\Phi_3(\lambda)$  is described as an absolutely convergent double integral,  $\Phi_3(\lambda)$  has an analytic continuation to  $\mathbb{C}$ . We note that the integral  $Q_{\chi^{-1},\rho}^0(\mathbf{1}, \lambda, -\nu)$  is absolutely convergent and is entire as a function in  $\lambda$ . In order to examine  $\Phi_4(\lambda)$ , we consider the following residues:

$$\begin{aligned} \operatorname{Res}_{z=(\nu+1)/2} f_{\chi^{-1},\rho}^{\mathbf{1}}(-z, -\nu) &= D_F^{-1/2+\nu/2} N(\mathfrak{f}_\chi)^{1/2+\nu} B_{\chi^{-1},\rho}^{\mathbf{1}}(-\nu/2, -\nu) \\ &\quad \times \frac{L(-\nu, \chi^{-1})}{L(1-\nu, \chi^{-2})} \delta_{\chi,1} D_F^{1/2} R_F, \end{aligned}$$



$$\begin{aligned}
\operatorname{Res}_{z=(-\nu+1)/2} f_{\chi^{-1},\rho}^1(-z, -\nu) &= D_F^{-1/2+\nu/2} N(\mathfrak{f}_\chi)^{1/2+\nu} B_{\chi^{-1},\rho}^1(\nu/2, -\nu) \\
&\quad \times \frac{L(\nu, \chi^{-1})}{L(1-\nu, \chi^{-2})} \delta_{\chi,1} D_F^{1/2} R_F, \\
\operatorname{Res}_{z=(\nu-1)/2} f_{\chi^{-1},\rho}^1(-z, -\nu) &= D_F^{-1/2+\nu/2} N(\mathfrak{f}_\chi)^{1/2+\nu} B_{\chi^{-1},\rho}^1(1-\nu/2, -\nu) \\
&\quad \times \frac{L(1-\nu, \chi^{-1})}{L(1-\nu, \chi^{-2})} (-\delta_{\chi,1} R_F), \\
\operatorname{Res}_{z=(-\nu-1)/2} f_{\chi^{-1},\rho}^1(-z, -\nu) &= D_F^{-1/2+\nu/2} N(\mathfrak{f}_\chi)^{1/2+\nu} B_{\chi^{-1},\rho}^1(1+\nu/2, -\nu) \\
&\quad \times \frac{L(1+\nu, \chi)}{L(1-\nu, \chi^{-2})} (-\delta_{\chi,1} R_F).
\end{aligned}$$

The functions  $\operatorname{Res}_{z=(\pm\nu\pm 1)/2} f_{\chi^{-1},\rho}^1(-z, -\nu)$  are holomorphic on  $i\mathbb{R}$  as functions in  $\nu$  and vanish unless  $\chi = \mathbf{1}$ . Therefore, the integral

$$\begin{aligned}
&\int_{i\mathbb{R}} \left\{ \frac{\beta((\nu+1)/2)}{\lambda + (\nu+1)/2} \operatorname{Res}_{z=(\nu+1)/2} + \frac{\beta((-\nu+1)/2)}{\lambda + (-\nu+1)/2} \operatorname{Res}_{z=(-\nu+1)/2} \right\} \\
&\quad \times f_{\chi^{-1},\rho}^1(-z, -\nu) \tilde{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu
\end{aligned}$$

is holomorphic on  $\operatorname{Re}(\lambda) > -1/2$ .

Consider the integral

$$\begin{aligned}
&\int_{i\mathbb{R}} \left\{ \frac{\beta((\nu-1)/2)}{\lambda + (\nu-1)/2} \operatorname{Res}_{z=(\nu-1)/2} + \frac{\beta((-\nu-1)/2)}{\lambda + (-\nu-1)/2} \operatorname{Res}_{z=(-\nu-1)/2} \right\} \\
&\quad \times f_{\chi^{-1},\rho}^1(-z, -\nu) \tilde{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu.
\end{aligned}$$

Set  $F_\rho^+(\nu) = \operatorname{Res}_{z=(\nu-1)/2} f_{\mathbf{1},\rho}^1(-z, -\nu)$ . We note that  $F_\rho^+(\nu)$  is entire. By taking  $c > 1$ , we obtain

$$\begin{aligned}
&\int_{i\mathbb{R}} \frac{\beta((\nu-1)/2)}{\lambda + (\nu-1)/2} \operatorname{Res}_{z=(\nu-1)/2} f_{\mathbf{1},\rho}^1(-z, -\nu) \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) d\nu \\
&= \int_{L_c} \frac{\beta((\nu-1)/2)}{\lambda + (\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) d\nu - 2\pi i \frac{\beta(0)}{\lambda} F_\rho^+(1) \tilde{\alpha}(1) \mathfrak{e}_{\mathbf{1},\rho,-1}(g).
\end{aligned}$$

The first term of the right-hand side is holomorphic on  $\operatorname{Re}(\lambda) > (-c+1)/2$  and the second term is meromorphic on  $\mathbb{C}$ . Set  $F_\rho^-(\nu) = \operatorname{Res}_{z=(-\nu-1)/2} f_{\mathbf{1},\rho}^1(-z, -\nu)$ . By the relation  $B_{\mathbf{1},\rho}^1(1-\nu/2, -\nu) = B_{\mathbf{1},\rho}^1(1-\nu/2, \nu) A_{\mathbf{1},\rho}(-\nu)$ , we have

$$\begin{aligned}
F_\rho^-(\nu) D_F^{-1/2} A_{\mathbf{1},\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} &= B_{\mathbf{1},\rho}^1(1-\nu/2, \nu) D_F^{\nu/2} (-R_F) D_F^{-1/2} A_{\mathbf{1},\rho}(-\nu) \\
&= F_\rho^+(\nu),
\end{aligned}$$

and hence, we obtain

$$\begin{aligned}
 & \int_{i\mathbb{R}} \frac{\beta((- \nu - 1)/2)}{\lambda + (-\nu - 1)/2} F_{\rho}^{-}(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= \int_{i\mathbb{R}} \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} F_{\rho}^{-}(-\nu) \tilde{\alpha}(\nu) E_{1,\rho}(-\nu, g) d\nu \\
 &= \int_{i\mathbb{R}} \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} F_{\rho}^{-}(-\nu) D_F^{-1/2} A_{1,\rho}(-\nu) \frac{\zeta_F(-\nu)}{\zeta_F(1-\nu)} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= \int_{i\mathbb{R}} \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} F_{\rho}^{+}(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= \int_{L_c} \frac{\beta((\nu - 1)/2)}{\lambda + (\nu - 1)/2} F_{\rho}^{+}(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu - 2\pi i \frac{\beta(0)}{\lambda} F_{\rho}^{+}(1) \tilde{\alpha}(1) \mathbf{e}_{1,\rho,-1}(g).
 \end{aligned}$$

Then, in the last line of the equalities above, the first term is holomorphic on  $\operatorname{Re}(\lambda) > (-c+1)/2$  and the second term is meromorphic on  $\mathbb{C}$ . Hence we can prove that  $\Phi_4(\lambda)$  has a meromorphic continuation to  $\operatorname{Re}(\lambda) > -1/2$ . This gives us a meromorphic continuation of  $\Psi_{\text{ct}}(\lambda)$  to  $\operatorname{Re}(\lambda) > -1/2$ .  $\square$

**Lemma 19.** *We have*

$$\begin{aligned}
 \text{CT}_{\lambda=0} \hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; g) &= C(\mathbf{n}, S) \left\{ \sum_{\varphi \in \mathcal{B}_{\text{cus}}(\mathbf{n})} \alpha(\nu_{\varphi,S}) \overline{P_{\text{reg}}^{\mathbf{1}}(\varphi)} \varphi(g) \right. \\
 &\quad + \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_{\chi}(\mathbf{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) P_{\text{reg}}^{\mathbf{1}}(E_{\chi^{-1},\rho}(-\nu, -)) E_{\chi,\rho}(\nu, g) d\nu \\
 &\quad \left. + \sum_{\rho \in \Lambda(\mathbf{n})} \{f_{1,\rho}^{(0)}(e) + D_1(\rho)\} \{\tilde{\alpha}'(1) \mathbf{e}_{1,\rho,-1}(g) + \tilde{\alpha}(1) \mathbf{e}_{1,\rho,0}(g)\} \right\} \beta(0),
 \end{aligned}$$

where  $D_{\eta}(\rho) = \delta(\cup_{k=2}^n S_k(\rho) = \emptyset) \prod_{v \in S_1(\rho)} (-\eta_v(\varpi_v) q_v^{-1/2})$  for  $\eta \in \Xi_0(\mathfrak{o}_F)$ .

**Proof.** In the proof of Lemma 18, we gave the constant terms of the cuspidal and residual parts at  $\lambda = 0$ . Therefore, it is enough to evaluate the constant term of the Eisenstein part  $\Psi_{\text{ct}}(\lambda) = 2\Phi_1(\lambda) + \Phi_3(\lambda) + \Phi_4(\lambda)$ . By the residue theorem, we have

$$\begin{aligned}
 \text{CT}_{\lambda=0} \Phi_1(\lambda) &= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathbf{n})} f_{1,\rho}^{(-\nu)}(e) \int_{i\mathbb{R}} \tilde{\alpha}(\nu) \beta((\nu - 1)/2) \frac{-1}{(\nu - 1)/2} E_{1,\rho}(\nu, g) d\nu \\
 &\quad + \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathbf{n})} f_{1,\rho}^{(-\nu)}(e) \int_{L_c} \tilde{\alpha}(\nu) \beta((\nu - 1)/2) \frac{1}{(\nu - 1)/2} E_{1,\rho}(\nu, g) d\nu
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(-\nu)}(e) 2\pi i \operatorname{Res}_{\nu=1} \left\{ \tilde{\alpha}(\nu) \beta((\nu-1)/2) \frac{1}{(\nu-1)/2} E_{1,\rho}(\nu, g) \right\} \\
&= \frac{1}{2} \sum_{\rho \in \Lambda(\mathfrak{n})} f_{1,\rho}^{(-\nu)}(e) \{ \tilde{\alpha}'(1) \mathbf{e}_{1,\rho,-1}(g) + \tilde{\alpha}(1) \mathbf{e}_{1,\rho,0}(g) \} \beta(0)
\end{aligned}$$

and the integral  $Q_{\chi^{-1},\rho}^0(\mathbf{1}, 0, -\nu)$  is written as

$$\begin{aligned}
Q_{\chi^{-1},\rho}^0(\mathbf{1}, 0, -\nu) &= \frac{1}{2\pi i} \int_{L_\sigma} \{ f_{\chi^{-1},\rho}^{\mathbf{1}}(z, -\nu) + f_{\chi^{-1},\rho}^{\mathbf{1}}(-z, -\nu) \} \frac{\beta(z)}{z} dz \\
&= f_{\chi^{-1},\rho}^{\mathbf{1}}(0, -\nu) \beta(0) + \{ \operatorname{Res}_{z=(1+\nu)/2} + \operatorname{Res}_{z=(1-\nu)/2} + \operatorname{Res}_{z=(-1+\nu)/2} \\
&\quad + \operatorname{Res}_{z=(-1-\nu)/2} \} \left\{ f_{\chi^{-1},\rho}^{\mathbf{1}}(z, -\nu) \frac{\beta(z)}{z} \right\}.
\end{aligned}$$

Thus the constant term of  $\Phi_3(\lambda)$  is evaluated as

$$\begin{aligned}
&\operatorname{CT}_{\lambda=0} \Phi_3(\lambda) \\
&= \frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) Q_{\chi^{-1},\rho}^0(\mathbf{1}, 0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \\
&= \frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \left\{ \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) f_{\chi^{-1},\rho}^{\mathbf{1}}(0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \beta(0) + \int_{i\mathbb{R}} \{ \operatorname{Res}_{z=(1+\nu)/2} \right. \\
&\quad + \operatorname{Res}_{z=(1-\nu)/2} + \operatorname{Res}_{z=(-1+\nu)/2} + \operatorname{Res}_{z=(-1-\nu)/2} \} \left\{ f_{\chi^{-1},\rho}^{\mathbf{1}}(z, -\nu) \frac{\beta(z)}{z} \right\} \\
&\quad \times \tilde{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu \Big\}.
\end{aligned}$$

We examine the constant term of  $\Phi_4(\lambda)$ . By the expression of  $\Phi_4(\lambda)$  given in the proof of [Lemma 18](#), we have

$$\begin{aligned}
&\operatorname{CT}_{\lambda=0} \Phi_4(\lambda) \\
&= -\frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \int_{i\mathbb{R}} \left\{ \frac{\beta((\nu+1)/2)}{(\nu+1)/2} \operatorname{Res}_{z=(\nu+1)/2} \right. \\
&\quad + \frac{\beta((- \nu+1)/2)}{(-\nu+1)/2} \operatorname{Res}_{z=(-\nu+1)/2} \Big\} f_{\chi^{-1},\rho}^{\mathbf{1}}(-z, -\nu) \\
&\quad \times \tilde{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu \\
&\quad - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{L_c} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \Big\}.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
 & \text{CT}_{\lambda=0}\{\Phi_3(\lambda) + \Phi_4(\lambda)\} \\
 &= \frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \left\{ \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) f_{\chi^{-1},\rho}^1(0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \beta(0) \right. \\
 & \quad \left. + \int_{i\mathbb{R}} \{\text{Res}_{z=(-1+\nu)/2} + \text{Res}_{z=(-1-\nu)/2}\} \left\{ f_{\chi^{-1},\rho}^1(z, -\nu) \frac{\beta(z)}{z} \right\} \tilde{\alpha}_\chi(\nu) E_{\chi,\rho}(\nu, g) d\nu \right\} \\
 & \quad - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \left\{ \int_{L_c} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \right\}.
 \end{aligned}$$

By noting the relation

$$\int_{i\mathbb{R}} F_\rho^+(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu = \int_{i\mathbb{R}} F_\rho^-(\nu) \frac{\beta((- \nu - 1)/2)}{(-\nu-1)/2} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu,$$

we have

$$\begin{aligned}
 & \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{i\mathbb{R}} \{\text{Res}_{z=(-1+\nu)/2} + \text{Res}_{z=(-1-\nu)/2}\} \left\{ f_{1,\rho}^1(z, -\nu) \frac{\beta(z)}{z} \right\} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 & \quad - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{L_c} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{i\mathbb{R}} \left\{ F_\rho^+(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} + F_\rho^-(\nu) \frac{\beta((- \nu - 1)/2)}{(-\nu-1)/2} \right\} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 & \quad - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{L_c} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{i\mathbb{R}} F_\rho^+(\nu) \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 & \quad - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} \int_{L_c} \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) d\nu \\
 &= \frac{R_F^{-1}}{4\pi i} \sum_{\rho \in \Lambda(\mathfrak{n})} (-2\pi i) \text{Res}_{\nu=1} \left\{ \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_\rho^+(\nu) \tilde{\alpha}(\nu) E_{1,\rho}(\nu, g) \right\}.
 \end{aligned}$$

Here the residue is expressed as

$$\begin{aligned} \text{Res}_{\nu=1} & \left\{ \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_{\rho}^{+}(\nu) \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) \right\} \\ &= \text{Res}_{\nu=1} \left\{ \frac{\beta((\nu-1)/2)}{(\nu-1)/2} \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) D_F^{-1/2+\nu/2} B_{\mathbf{1},\rho}^{\mathbf{1}}(1-\nu/2, -\nu)(-R_F) \right\} \\ &= \{2\tilde{\alpha}'(1)\mathbf{e}_{\mathbf{1},\rho,-1}(g) + 2\tilde{\alpha}(1)\mathbf{e}_{\mathbf{1},\rho,0}(g)\} D_{\mathbf{1}}(\rho)(-R_F)\beta(0). \end{aligned}$$

We note  $D_F^{(\nu-1)/2} B_{\eta,\rho}^{\eta}(1-\nu/2, -\nu) = \tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}}) D_{\eta}(\rho)$  for any  $\eta \in \Xi_0(\mathfrak{o}_F)$  satisfying  $\eta^2 = \mathbf{1}$ . Therefore we obtain

$$\begin{aligned} & \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathbf{n})} \int_{i\mathbb{R}} \{ \text{Res}_{z=(-1+\nu)/2} + \text{Res}_{z=(-1-\nu)/2} \} \left\{ f_{\mathbf{1},\rho}^{\mathbf{1}}(z, -\nu) \frac{\beta(z)}{z} \right\} \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) d\nu \\ & - 2 \times \frac{R_F^{-1}}{8\pi i} \sum_{\rho \in \Lambda(\mathbf{n})_{L_c}} \int \frac{\beta((\nu-1)/2)}{(\nu-1)/2} F_{\rho}^{+}(\nu) \tilde{\alpha}(\nu) E_{\mathbf{1},\rho}(\nu, g) d\nu \\ &= \frac{R_F^{-1}}{4\pi i} \sum_{\rho \in \Lambda(\mathbf{n})} 2\pi i \{ 2\tilde{\alpha}'(1)\mathbf{e}_{\mathbf{1},\rho,-1}(g) + 2\tilde{\alpha}(1)\mathbf{e}_{\mathbf{1},\rho,0}(g) \} D_{\mathbf{1}}(\rho) R_F \beta(0) \\ &= \sum_{\rho \in \Lambda(\mathbf{n})} D_{\mathbf{1}}(\rho) \{ \tilde{\alpha}'(1)\mathbf{e}_{\mathbf{1},\rho,-1}(g) + \tilde{\alpha}(1)\mathbf{e}_{\mathbf{1},\rho,0}(g) \} \beta(0), \end{aligned}$$

and hence

$$\begin{aligned} \text{CT}_{\lambda=0} \Psi_{\text{ct}}(\lambda) &= \frac{R_F^{-1}}{8\pi i} \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_{\chi}(\mathbf{n})} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) f_{\chi^{-1},\rho}^{\mathbf{1}}(0, -\nu) E_{\chi,\rho}(\nu, g) d\nu \beta(0) \\ &+ \sum_{\rho \in \Lambda(\mathbf{n})} \{ f_{\mathbf{1},\rho}^{(0)}(e) + D_{\mathbf{1}}(\rho) \} \{ \tilde{\alpha}'(1)\mathbf{e}_{\mathbf{1},\rho,-1}(g) + \tilde{\alpha}(1)\mathbf{e}_{\mathbf{1},\rho,0}(g) \} \beta(0). \end{aligned}$$

This gives the expression of  $\text{CT}_{\lambda=0} \hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; g)$ .  $\square$

Then, we define the regularized automorphic smoothed kernel  $\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha; g)$  by the relation

$$\text{CT}_{\lambda=0} \hat{\Psi}_{\beta,\lambda}(\mathbf{n}|\alpha; g) = \hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha; g) \beta(0), \quad \beta \in \mathcal{B}.$$

**Lemma 20.** *Let  $\mathfrak{S}$  be a Siegel set of  $G_{\mathbb{A}}$  such that  $G_{\mathbb{A}} = G_F \mathfrak{S}$ . We have the following.*

(1) *For  $m \in \mathbb{N}$ , there exists  $C_{\alpha,\mathbf{n},m} > 0$  such that*

$$\sum_{\varphi \in B_{\text{cus}}(\mathbf{n})} |\alpha(\nu_{\varphi,S}) \overline{P_{\text{reg}}^{\mathbf{1}}(\varphi)} \varphi(g)| \leq C_{\alpha,\mathbf{n},m} y(g)^{-m}, \quad g \in G_{\mathbb{A}}^1 \cap \mathfrak{S}.$$

(2) There exist  $N \in \mathbb{N}$  and  $C'_{\alpha, \mathfrak{n}, N} > 0$  such that

$$\sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} |\tilde{\alpha}_\chi(\nu) P_{\text{reg}}^1(E_{\chi^{-1}, \rho}(-\nu, -)) E_{\chi, \rho}(\nu, g)| d\nu \leq C'_{\alpha, \mathfrak{n}, N} y(g)^N,$$

$$g \in G_{\mathbb{A}}^1 \cap \mathfrak{S}.$$

(3) We have  $|\mathfrak{e}_{1, \rho, 0}(g)| + |\mathfrak{e}_{1, \rho, -1}(g)| \ll y(g)$ ,  $g \in G_{\mathbb{A}}^1 \cap \mathfrak{S}$ .

(4) For  $N$  as in (2), there exists  $C''_{\alpha, \mathfrak{n}, N} > 0$  such that  $|\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; g)| \leq C''_{\alpha, \mathfrak{n}, N} y(g)^N$ ,  $g \in G_{\mathbb{A}}^1 \cap \mathfrak{S}$ .

**Proof.** By virtue of [20, Propositions 15.1 and 15.2], which are valid for any non-square-free ideal  $\mathfrak{n}$ , we have the first, third and fourth inequalities in the same way as in the proof of [20, Lemma 9.9]. The constant  $N \geq 1$  is given by using [20, Proposition 15.1].

We show the proof of the second inequality. In [20, Lemma 9.9], we considered  $\Xi_0(\mathfrak{o}_F)$  since  $\Xi(\mathfrak{n}) = \Xi_0(\mathfrak{o}_F)$  holds for any square-free ideal  $\mathfrak{n}$ . Contrary to this, now we must consider contribution of values  $L(1 - \nu, \chi^{-2})^{-1}$  for all  $\chi \in \Xi(\mathfrak{n})$ . We give a lower uniform bound of  $L_{\text{fin}}(1, \chi)$  for all characters  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$  in the following way. There exists a positive constant  $C$  such that  $L_{\text{fin}}(s, \chi)$  does not vanish for any non-quadratic character  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$  if  $\text{Re}(s) \geq 1 - C / \log\{\mathfrak{q}(\chi)(3 + |\text{Im}(s)|)\}$  (cf. [6, Theorem 5.10]). Hence, by virtue of the proof of [19, Theorem 3.11], the estimate

$$\frac{1}{|L_{\text{fin}}(1, \chi)|} \ll \log \mathfrak{q}(\chi)$$

holds uniformly for non-quadratic characters  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$ . Next we give a generalized Siegel's theorem for quadratic characters of  $F^\times \backslash \mathbb{A}^\times$ . By [11, Theorem 2.3.1], for any  $\epsilon > 0$ , the estimate

$$|L_{\text{fin}}(1, \chi)| \gg \mathfrak{q}(\chi)^{-\epsilon}$$

holds uniformly for quadratic characters  $\chi$  of  $F^\times \backslash \mathbb{A}^\times$ . Indeed, [11, Theorem 2.3.1] works for general  $L$ -functions over  $F$  in the sense of [2].

As a consequence, we have the estimate

$$\frac{1}{|L_{\text{fin}}(1 + \nu, \chi^2)|} \ll \mathfrak{q}(\chi^2) \cdot |\mathbb{A}^\nu|^\epsilon, \quad \nu \in i\mathbb{R}$$

with the implied constant independent of  $\chi \in \Xi(\mathfrak{n})$  and  $\mathfrak{n}$ . Combining this with the argument of the proof of [20, Lemma 9.9], we have the assertion (2).  $\square$

## 5. Periods of regularized automorphic smoothed kernels: spectral side

By (4) in Lemma 20, the integral  $P_{\beta,\lambda}^\eta(\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha;-))$  converges absolutely for  $\text{Re}(\lambda) > N$  and is holomorphic on  $\text{Re}(\lambda) > N$ . We have the following expression in the same way as [20, Lemma 10.1].

**Lemma 21.** *For  $\text{Re}(\lambda) > N$ , we have the expression*

$$P_{\beta,\lambda}^\eta(\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha;-)) = C(\mathbf{n}, S) \{ \mathbb{P}_{\text{cus}}^\eta(\beta, \lambda, \alpha) + \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) + \mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha) \},$$

where

$$\begin{aligned} \mathbb{P}_{\text{cus}}^\eta(\beta, \lambda, \alpha) &= \sum_{\varphi \in \mathcal{B}_{\text{cus}}(\mathbf{n})} \alpha(\nu_{\varphi, S}) \overline{P_{\text{reg}}^{\mathbf{1}}(\varphi)} P_{\beta,\lambda}^\eta(\varphi), \\ \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) &= \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_\chi(\mathbf{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) P_{\text{reg}}^{\mathbf{1}}(E_{\chi^{-1}, \rho}(-\nu, -)) P_{\beta,\lambda}^\eta(E_{\chi, \rho}(\nu, -)) d\nu \end{aligned}$$

and

$$\mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha) = \sum_{\rho \in \Lambda(\mathbf{n})} \{ f_{\mathbf{1}, \rho}^{(0)}(e) + D_{\mathbf{1}}(\rho) \} ( \tilde{\alpha}'(1) P_{\beta,\lambda}^\eta(\mathbf{e}_{\mathbf{1}, \rho, -1}) + \tilde{\alpha}(1) P_{\beta,\lambda}^\eta(\mathbf{e}_{\mathbf{1}, \rho, 0}) ).$$

Here the series converge absolutely and locally uniformly on  $\text{Re}(\lambda) > N$ .

The value  $\mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha)$  is described by Propositions 10 and 11 as follows.

**Lemma 22.** *The function  $\lambda \mapsto \mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha)$  on  $\text{Re}(\lambda) > N$  is analytically continued to a meromorphic function on  $\mathbb{C}$ . Its constant term at  $\lambda = 0$  is given by*

$$\begin{aligned} \text{CT}_{\lambda=0} \mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha) &= \sum_{\rho \in \Lambda(\mathbf{n})} \{ f_{\mathbf{1}, \rho}^{(0)}(e) + D_{\mathbf{1}}(\rho) \} \tilde{\alpha}(1) \frac{\mathcal{G}(\eta) D_F^{-1/2}}{\zeta_F(2)} \\ &\quad \times \left\{ -\frac{1}{2} \delta_{\eta, \mathbf{1}} \tilde{B}_{\mathbf{1}, \rho}^{\mathbf{1}}(0) R_F^2 \beta''(0) + a_{\mathbf{1}, \rho}^\eta(0) \beta(0) \right\}. \end{aligned}$$

Here  $\tilde{B}_{\chi, \rho}^\eta(z) = \epsilon(-z, \chi^{-1}\eta) B_{\chi, \rho}^\eta(-z + 1/2, 1)$  and

$$a_{\mathbf{1}, \rho}^\eta(0) = -\frac{1}{2} (\tilde{B}_{\mathbf{1}, \rho}^{\mathbf{1}})''(0) \delta_{\eta, \mathbf{1}} R_F^2 - 2 \tilde{B}_{\mathbf{1}, \rho}^{\mathbf{1}}(0) R_F C_1(\mathbf{1}) \delta_{\eta, \mathbf{1}} + \tilde{B}_{\mathbf{1}, \rho}^\eta(0) C_0(\eta)^2.$$

**Lemma 23.** *The function  $\lambda \mapsto \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha)$  on  $\text{Re}(\lambda) > N$  is analytically continued to a meromorphic function on  $\text{Re}(\lambda) > -1/2$ .*

**Proof.** By Proposition 8, we have

$$Z^*(s, \eta, E_{\chi, \rho}^{\natural}(\nu, -)) = \mathcal{G}(\eta) D_F^{-\nu/2} N(\mathfrak{f}_{\chi})^{1/2-\nu} B_{\chi, \rho}^{\eta}(s, \nu) \frac{L(s + \nu/2, \chi\eta) L(s - \nu/2, \chi^{-1}\eta)}{L(1 + \nu, \chi^2)}.$$

Set

$$\mathfrak{L}_{\chi, \rho}^{\eta}(\nu) = D_F^{\nu/2} N(\mathfrak{f}_{\chi})^{1/2+\nu} B_{\chi^{-1}, \rho}^{\eta}(1/2, -\nu) \frac{L((1 + \nu)/2, \chi\eta) L((1 - \nu)/2, \chi^{-1}\eta)}{L(1 - \nu, \chi^{-2})}$$

and recall the expression

$$\begin{aligned} P_{\beta, \lambda}^{\eta}(E_{\chi, \rho}(\nu, -)) &= P_{\chi}(\eta, \lambda, \nu) + D_F^{-1/2} A_{\chi, \rho}(\nu) \frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)} P_{\chi^{-1}}(\eta, \lambda, -\nu) \\ &\quad + Q_{\chi, \rho}^{+}(\eta, \lambda, \nu) + Q_{\chi, \rho}^{-}(\eta, \lambda, \nu). \end{aligned}$$

We remark

$$\begin{aligned} \mathbb{P}_{\text{eis}}^{\eta}(\beta, \lambda, \alpha) &= \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) \mathcal{G}(\mathbf{1}) \mathfrak{L}_{\chi, \rho}^{\mathbf{1}}(\nu) \left\{ P_{\chi}(\eta, \lambda, \nu) \right. \\ &\quad \left. + D_F^{-1/2} A_{\chi, \rho}(\nu) \frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)} P_{\chi^{-1}}(\eta, \lambda, -\nu) + Q_{\chi, \rho}^0(\eta, \lambda, \nu) \right. \\ &\quad \left. - \sum_{a=(\pm\nu \pm 1)/2} \frac{\beta(a)}{\lambda + a} \text{Res}_{z=a} \{f_{\chi, \rho}^{\eta}(-z, \nu)\} \right\} d\nu. \end{aligned}$$

In order to examine  $\mathbb{P}_{\text{eis}}^{\eta}(\beta, \lambda, \alpha)$ , we decompose this into the following four terms:

$$\begin{aligned} \Phi_1^{+}(\lambda) &= \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) D_F^{-1/2} \mathfrak{L}_{\chi, \rho}^{\mathbf{1}}(\nu) f_{\chi, \rho}^{(0)}(e) \\ &\quad \times \delta_{\chi, \eta} R_F \left\{ \frac{1}{\lambda - (\nu + 1)/2} + \frac{1}{\lambda + (\nu + 1)/2} \right\} \beta((\nu + 1)/2) d\nu, \\ \Phi_1^{-}(\lambda) &= \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) D_F^{-1/2} \mathfrak{L}_{\chi, \rho}^{\mathbf{1}}(\nu) D_F^{-1/2} A_{\chi, \rho}(\nu) \frac{L(\nu, \chi^2)}{L(1 + \nu, \chi^2)} f_{\chi^{-1}, \rho}^{(0)}(e) \\ &\quad \times \delta_{\chi, \eta} R_F \left\{ \frac{1}{\lambda - (-\nu + 1)/2} + \frac{1}{\lambda + (-\nu + 1)/2} \right\} \beta((-\nu + 1)/2) d\nu, \\ \Phi_2(\lambda) &= \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_{\chi}(\nu) D_F^{-1/2} \mathfrak{L}_{\chi, \rho}^{\mathbf{1}}(\nu) Q_{\chi, \rho}^0(\eta, \lambda, \nu) d\nu, \end{aligned}$$



$$\begin{aligned}\Phi_3(\lambda) = & - \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \\ & \times \sum_{a=(\pm\nu\pm 1)/2} \frac{\beta(a)}{\lambda+a} \operatorname{Res}_{z=a} \{f_{\chi,\rho}^\eta(-z, \nu)\} d\nu.\end{aligned}$$

When  $\chi = \eta$ , by using the functional equations

$$\mathfrak{L}_{\eta,\rho}^1(\nu) D_F^{-1/2} A_{\eta,\rho}(\nu) \frac{\zeta_F(\nu)}{\zeta_F(1+\nu)} = \mathfrak{L}_{\eta,\rho}^1(-\nu)$$

and  $B_{\eta,\rho}^1(1/2, \nu) = B_{\eta,\rho}^1(1/2, -\nu) A_{\eta,\rho}(\nu)$ , we obtain  $\Phi_1^+(\lambda) = \Phi_1^-(\lambda)$ . The term  $\Phi_1^+(\lambda)$  is expressed as

$$\begin{aligned}\Phi_1^+(\lambda) = & \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \left\{ \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) f_{\chi,\rho}^{(0)}(e) \delta_{\chi,\eta} \right. \\ & \times R_F \frac{1}{\lambda + (\nu+1)/2} \beta((\nu+1)/2) d\nu \\ & \left. + \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) f_{\chi,\rho}^{(0)}(e) \delta_{\chi,\eta} R_F \frac{1}{\lambda - (\nu+1)/2} \beta((\nu+1)/2) d\nu \right\}.\end{aligned}$$

Then the first term in the summation is holomorphic on  $\operatorname{Re}(\lambda) > -1/2$ . For any fixed  $\sigma > 1$ , the second term in the summation is transformed into

$$\begin{aligned}& \frac{R_F^{-1}}{8\pi i} D_F^{-1/2} \left\{ \int_{L_{-\sigma}} \tilde{\alpha}_\chi(\nu) \mathfrak{L}_{\chi,\rho}^1(\nu) f_{\chi,\rho}^{(0)}(e) \delta_{\chi,\eta} R_F \frac{1}{\lambda - (\nu+1)/2} \beta((\nu+1)/2) d\nu \right. \\ & \left. + \delta_{\chi,\eta} 2\pi i \operatorname{Res}_{\nu=-1} \left( \frac{\beta((\nu+1)/2)}{\lambda - (\nu+1)/2} \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \right) f_{\chi,\rho}^{(0)}(e) R_F \right\}.\end{aligned}$$

The first term in the expression above is meromorphic on  $\operatorname{Re}(\lambda) > (-\sigma+1)/2$ . In order to prove the meromorphicity of the second term in the expression above, we put

$$\begin{aligned}D_F^{\nu/2} \frac{L((1+\nu)/2, \eta) L((1-\nu)/2, \eta)}{\zeta_F(1-\nu)} &= \frac{D_{-2}^\eta}{(\nu+1)^2} + \frac{D_{-1}^\eta}{\nu+1} + D_0^\eta + \mathcal{O}((\nu+1)), \quad (\nu \rightarrow -1), \\ B_{\eta,\rho}^1(1/2, -\nu) &= p_0^\eta(\rho) + p_1^\eta(\rho)(\nu+1) + p_2^\eta(\rho)(\nu+1)^2 + \mathcal{O}((\nu+1)^3), \quad (\nu \rightarrow -1)\end{aligned}$$

and

$$\frac{\beta((\nu+1)/2)}{\lambda - (\nu+1)/2} \tilde{\alpha}_\eta(\nu) = q_0^\eta(\lambda) + q_1^\eta(\lambda)(\nu+1) + \mathcal{O}((\nu+1)^2), \quad (\nu \rightarrow -1).$$

Then these give the following expressions:

$$\begin{aligned} \operatorname{Res}_{\nu=-1} \left\{ \frac{\beta((\nu+1)/2)}{\lambda - (\nu+1)/2} \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \right\} &= p_0^\eta(\rho) q_1^\eta(\lambda) D_{-2}^\eta + p_0^\eta(\rho) q_0^\eta(\lambda) D_{-1}^\eta \\ &\quad + p_1^\eta(\rho) q_0^\eta(\lambda) D_{-2}^\eta, \\ q_0^\eta(\lambda) &= \frac{\tilde{\alpha}_\eta(1) \beta(0)}{\lambda}, \quad q_1^\eta(\lambda) = \left( \frac{\tilde{\alpha}'_\eta(1)}{\lambda} + \frac{\tilde{\alpha}_\eta(1)}{2\lambda^2} \right) \beta(0). \end{aligned}$$

Therefore  $\Phi_1^+(\lambda) = \Phi_1^-(\lambda)$  has a meromorphic continuation to  $\operatorname{Re}(\lambda) > -1/2$ . Since  $\Phi_2(\lambda)$  is described as an absolutely convergent double integral,  $\Phi_2(\lambda)$  is entire.

We examine  $\Phi_3(\lambda)$ . This is written as

$$\begin{aligned} \Phi_3(\lambda) &= - \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \left\{ \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \right. \\ &\quad \times \sum_{a=(\pm\nu+1)/2} \frac{\beta(a)}{\lambda+a} \operatorname{Res}_{z=a} f_{\chi,\rho}^\eta(-z, \nu) d\nu \\ &\quad \left. + \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \sum_{a=(\pm\nu-1)/2} \frac{\beta(a)}{\lambda+a} \operatorname{Res}_{z=a} f_{\chi,\rho}^\eta(-z, \nu) d\nu \right\}. \end{aligned}$$

In the bracket of the right-hand side, the first term is holomorphic on  $\operatorname{Re}(\lambda) > -1/2$  and the part of  $a = (-\nu-1)/2$  in the second term is transposed into

$$\begin{aligned} &\int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{\lambda + (-\nu - 1)/2} \operatorname{Res}_{z=(-\nu-1)/2} f_{\chi,\rho}^\eta(-z, \nu) d\nu \\ &= \int_{L-\sigma} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{\lambda + (-\nu - 1)/2} \operatorname{Res}_{z=(-\nu-1)/2} f_{\chi,\rho}^\eta(-z, \nu) d\nu \\ &\quad + 2\pi i \delta_{\chi,\eta} \operatorname{Res}_{\nu=-1} \mathfrak{L}_{\eta,\rho}^1(\nu) \times \tilde{\alpha}_\eta(1) D_F^{-1/2} \frac{\beta(0)}{\lambda} \mathcal{G}(\eta) D_F^{1/2} B_{\eta,\rho}^\eta(1/2, -1) (-R_F) \end{aligned}$$

for any fixed  $\sigma > 1$ . We note  $\operatorname{Res}_{\nu=-1} \mathfrak{L}_{\eta,\rho}^1(\nu) = p_0^\eta(\rho) D_{-1}^\eta + p_1^\eta(\rho) D_{-2}^\eta$ . Thus the part of  $a = (-\nu-1)/2$  is meromorphic on  $\operatorname{Re}(\lambda) > -1/2$ . Noting that the part of  $a = (\nu-1)/2$  equals that of  $a = (-\nu-1)/2$ , the function  $\Phi_3(\lambda)$  has a meromorphic continuation to  $\operatorname{Re}(\lambda) > -1/2$ . This completes the proof.  $\square$

**Lemma 24.** *We have*

$$\begin{aligned} &\operatorname{CT}_{\lambda=0} \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) + \operatorname{CT}_{\lambda=0} \mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha) \\ &= \left\{ \mathcal{G}(\eta) D_F^{-1/2} R_F^{-1} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{1}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) \mathfrak{L}_{\chi,\rho}^1(\nu) \mathfrak{L}_{\chi^{-1},\rho}^\eta(-\nu) d\nu \right. \\ &\quad \left. + \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \{ Y_2^\eta(\mathfrak{n}) \tilde{\alpha}_\eta''(1) + Y_1^\eta(\mathfrak{n}) \tilde{\alpha}_\eta'(1) + Y_0^\eta(\mathfrak{n}) \tilde{\alpha}_\eta(1) \} + Y_{-1}^\eta(\mathfrak{n}) \tilde{\alpha}(1) \right\} \beta(0), \end{aligned}$$

where we put

$$\begin{aligned} Y_2^\eta(\mathbf{n}) &= \sum_{\rho \in \Lambda(\mathbf{n})} D_F^{-1/2} \{f_{\eta,\rho}^{(0)}(e) + D_\eta(\rho)\} \frac{1}{2} p_0^\eta(\rho) D_{-2}^\eta, \\ Y_1^\eta(\mathbf{n}) &= \sum_{\rho \in \Lambda(\mathbf{n})} D_F^{-1/2} \{f_{\eta,\rho}^{(0)}(e) + D_\eta(\rho)\} \{D_{-1}^\eta p_0^\eta(\rho) + D_{-2}^\eta p_1^\eta(\rho)\}, \\ Y_0^\eta(\mathbf{n}) &= \sum_{\rho \in \Lambda(\mathbf{n})} D_F^{-1/2} \{f_{\eta,\rho}^{(0)}(e) + D_\eta(\rho)\} \{D_{-2}^\eta p_2^\eta(\rho) + D_{-1}^\eta p_1^\eta(\rho) + D_0^\eta p_0^\eta(\rho)\} \end{aligned}$$

and

$$Y_{-1}^\eta(\mathbf{n}) = \sum_{\rho \in \Lambda(\mathbf{n})} \frac{\mathcal{G}(\eta) D_F^{-1/2}}{\zeta_F(2)} \{f_{1,\rho}^{(0)}(e) + D_1(\rho)\} a_{1,\rho}^\eta(0).$$

**Proof.** Let  $\Phi_1^+$ ,  $\Phi_2$  and  $\Phi_3$  be the functions defined in the proof of Lemma 23. Then, we obtain  $\text{CT}_{\lambda=0} \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) = \text{CT}_{\lambda=0} (2\Phi_1^+(\lambda) + \Phi_2(\lambda) + \Phi_3(\lambda))$ . A direct computation gives us

$$\begin{aligned} \text{CT}_{\lambda=0} \Phi_1^+(\lambda) &= \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathbf{n})} \frac{1}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\eta(\nu) D_F^{-1/2} \mathfrak{L}_{\eta,\rho}^1(\nu) f_{\eta,\rho}^{(0)}(e) \frac{\beta(0)}{(\nu+1)/2} d\nu \\ &\quad + \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathbf{n})} \frac{1}{8\pi i} \int_{L-\sigma} \tilde{\alpha}_\eta(\nu) D_F^{-1/2} \mathfrak{L}_{\eta,\rho}^1(\nu) f_{\eta,\rho}^{(0)}(e) \frac{\beta(0)}{-(\nu+1)/2} d\nu \end{aligned}$$

and

$$\text{CT}_{\lambda=0} \Phi_2(\lambda) = \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_\chi(\mathbf{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) Q_{\chi,\rho}^0(\eta, 0, \nu) d\nu,$$

where

$$Q_{\chi,\rho}^0(\eta, 0, \nu) = f_{\chi,\rho}^\eta(0, \nu) \beta(0) + \sum_{a=(\pm\nu\pm 1)/2} \text{Res}_{z=a} \left\{ f_{\chi,\rho}^\eta(-z, \nu) \frac{\beta(z)}{z} \right\}.$$

The constant term of  $\Phi_3(\lambda)$  at  $\lambda = 0$  is evaluated as

$$\begin{aligned} \text{CT}_{\lambda=0} \Phi_3(\lambda) &= - \sum_{\chi \in \Xi(\mathbf{n})} \sum_{\rho \in \Lambda_\chi(\mathbf{n})} \frac{R_F^{-1}}{8\pi i} \left\{ \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \right. \\ &\quad \times \left. \sum_{a=(\pm\nu+1)/2} \frac{\beta(a)}{a} \text{Res}_{z=a} f_{\chi,\rho}^\eta(-z, \nu) d\nu \right\} \end{aligned}$$

$$\begin{aligned}
& -2\delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \left\{ \int_{L-\sigma} \tilde{\alpha}_\eta(\nu) D_F^{-1/2} \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{(- \nu - 1)/2} \right. \\
& \quad \left. \times \operatorname{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-z, \nu) d\nu \right\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \operatorname{CT}_{\lambda=0} \mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) \\
& = 2\delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathfrak{n})} \frac{1}{8\pi i} D_F^{-1/2} f_{\eta,\rho}^{(0)}(e) \left( \int_{i\mathbb{R}} - \int_{L-\sigma} \right) \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((\nu+1)/2)}{(\nu+1)/2} d\nu \\
& \quad + \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) f_{\chi,\rho}^\eta(0, \nu) \beta(0) d\nu \\
& \quad + 2\delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \left\{ \left( \int_{i\mathbb{R}} - \int_{L-\sigma} \right) \tilde{\alpha}_\eta(\nu) D_F^{-1/2} \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{(- \nu - 1)/2} \right. \\
& \quad \left. \times \operatorname{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-z, \nu) d\nu \right\} \\
& = \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathfrak{n})} \frac{1}{2} D_F^{-1/2} f_{\eta,\rho}^{(0)}(e) \operatorname{Res}_{\nu=-1} \left\{ \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((\nu+1)/2)}{(\nu+1)/2} \right\} \\
& \quad + \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) D_F^{-1/2} \mathfrak{L}_{\chi,\rho}^1(\nu) \mathcal{G}(\eta) \mathfrak{L}_{\chi^{-1},\rho}^\eta(-\nu) \beta(0) d\nu \\
& \quad + \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \sum_{\rho \in \Lambda(\mathfrak{n})} \frac{R_F^{-1}}{2} \operatorname{Res}_{\nu=-1} \left\{ \tilde{\alpha}_\eta(\nu) D_F^{-1/2} \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{(- \nu - 1)/2} \right. \\
& \quad \left. \times \operatorname{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-z, \nu) \right\}.
\end{aligned}$$

We remark

$$\operatorname{Res}_{z=(-\nu-1)/2} f_{\eta,\rho}^\eta(-z, \nu) = \mathcal{G}(\eta)(-R_F) D_F^{-\nu/2} B_{\eta,\rho}^\eta(\nu/2 + 1, \nu) = -R_F D_\eta(\rho)$$

and compute the residues as follows:

$$\begin{aligned}
& \operatorname{Res}_{\nu=-1} \left\{ \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((\nu+1)/2)}{(\nu+1)/2} \right\} \\
& = -\operatorname{Res}_{\nu=-1} \left\{ \tilde{\alpha}_\eta(\nu) \mathfrak{L}_{\eta,\rho}^1(\nu) \frac{\beta((- \nu - 1)/2)}{(- \nu - 1)/2} \right\}
\end{aligned}$$

$$= \tilde{\alpha}_\eta''(1)p_0^\eta(\rho)D_{-2}^\eta\beta(0) + 2\tilde{\alpha}_\eta'(1)\{p_0^\eta(\rho)D_{-1}^\eta + p_1^\eta(\rho)D_{-2}^\eta\}\beta(0) \\ + \tilde{\alpha}_\eta(1)\left\{D_{-2}^\eta\left(2p_2^\eta(\rho)\beta(0) + \frac{1}{4}p_0^\eta(\rho)\beta''(0)\right) + 2D_{-1}^\eta p_1^\eta(\rho)\beta(0) + 2D_0^\eta p_0^\eta(\rho)\beta(0)\right\}.$$

One can check that the sum of all terms containing  $\beta''(0)$  in  $\text{CT}_{\lambda=0}\mathbb{P}_{\text{eis}}^\eta(\beta, \lambda, \alpha) + \text{CT}_{\lambda=0}\mathbb{P}_{\text{res}}^\eta(\beta, \lambda, \alpha)$  vanishes with the aid of [Lemma 22](#). As a consequence, we obtain the assertion.  $\square$

**Lemma 25.** *For any  $\epsilon > 0$ , we have the following estimate*

$$|Y_j^\eta(\mathbf{n})| \ll N(\mathbf{n})^\epsilon, \quad j \in \{-1, 0, 1, 2\},$$

where the implied constant is independent of  $\mathbf{n}$ .

**Proof.** First let us examine  $Y_j^\eta(\mathbf{n})$  for  $j \in \{0, 1, 2\}$ . We have

$$f_{\eta, \rho}^{(0)}(e) = \prod_{v \in S_1(\rho)} \eta_v(\varpi_v) q_v^{1/2} \prod_{k=2}^n \prod_{v \in S_k(\rho)} (1 - q_v^{-1}) \eta_v(\varpi_v)^k \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{k/2}$$

and

$$p_0^\eta(\rho) = \tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}}) \prod_{v \in S_1(\rho)} (1 - \eta_v(\varpi_v)) \frac{q_v}{q_v - 1} q_v^{-1/2} \\ \times \prod_{k=2}^n \prod_{v \in S_k(\rho)} \left\{ \frac{(\eta_v(\varpi_v) - 1)(\eta_v(\varpi_v) q_v - 1)}{q_v - q_v^{-1}} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k/2} \right\}.$$

Moreover, we obtain expressions

$$p_1^\eta(\rho) = \tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}}) \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) - \{w\}} Y_v^\eta(-1) \right\} (Y_w^\eta)'(-1)$$

and

$$p_2^\eta(\rho) = \frac{\tilde{\eta}(\mathfrak{D}_{F/\mathbb{Q}})}{2} \sum_{w \in S(\rho)} \left[ \sum_{x \in S(\rho) - \{w\}} \left\{ \prod_{v \in S(\rho) - \{w, x\}} Y_v^\eta(-1) \right\} (Y_w^\eta)'(-1) (Y_x^\eta)'(-1) \right. \\ \left. + \left\{ \prod_{v \in S(\rho) - \{w\}} Y_v^\eta(-1) \right\} (Y_w^\eta)''(-1) \right],$$

where we set

$$C_v = \delta(v \in S_1(\rho)) + \delta\left(v \in \prod_{k=2}^n S_k(\rho)\right) \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2}$$

and

$$Y_v^\eta(\nu) = C_v \{q_v + 1 + \eta_v(\varpi_v)(q_v^{(1+\nu)/2} + q_v^{(1-\nu)/2})\} \frac{q_v^{k\nu/2}}{q_v - q_v^\nu}.$$

Furthermore we have

$$\begin{aligned} (Y_v^\eta)'(-1) &= C_v (\log q_v^k) q_v^{-k/2} \\ &\quad \times \frac{-\eta_v(\varpi_v) q_v (q_v - 1)^2 + k(1 + \eta_v(\varpi_v)) q_v (q_v^2 - 1) + 2(1 + \eta_v(\varpi_v)) q_v}{2k(q_v^2 - 1)(q_v - 1)} \end{aligned}$$

and

$$\begin{aligned} (Y_v^\eta)''(-1) &= C_v \left[ \eta_v(\varpi_v) (\log q_v^k)^2 q_v^{-k/2} \frac{(1 + q_v)(q_v - q_v^{-1}) + (1 - q_v)\{k(q_v - q_v^{-1}) + 2q_v^{-1}\}}{4k^2(q_v - q_v^{-1})^2} \right. \\ &\quad + (\log q_v^k)^2 q_v^{-k/2} \frac{\eta_v(\varpi_v)\{k(q_v^3 - q_v) + 2q_v\}}{4k^2(1 + q_v)(1 - q_v^2)} \\ &\quad + (\log q_v^k)^2 q_v^{-k/2} \frac{(1 + \eta_v(\varpi_v)) q_v}{k^2(q_v^2 - 1)^3(q_v - 1)} \left\{ \left( \frac{k^2}{4}(q_v^2 - 1) + 1 \right) (q_v^2 - 1)^2 \right. \\ &\quad \left. \left. + (k(q_v^2 - 1) + 2)(q_v^2 - 1) \right\} \right]. \end{aligned}$$

Thus, by noting  $\Lambda(\mathfrak{n}) \ll N(\mathfrak{n})^\epsilon$ , we obtain the estimates of  $Y_j^\eta(\mathfrak{n})$  for  $j \in \{0, 1, 2\}$ .

Next let us examine  $Y_{-1}^\eta(\mathfrak{n})$ . We have the following expressions:

$$\tilde{B}_{1,\rho}^\eta(0) = \epsilon(0, \eta) B_{1,\rho}^\eta(1/2, 1),$$

$$\begin{aligned} B_{1,\rho}^\eta(1/2, 1) &= \prod_{v \in S_1(\rho)} \frac{(\eta_v(\varpi_v) - 1) q_v^{-1/2}}{(1 - q_v^{-1})} \\ &\quad \times \prod_{k=2}^n \prod_{v \in S_k(\rho)} \left\{ \frac{\eta_v(\varpi_v)^k (\eta_v(\varpi_v) - 1) (\eta_v(\varpi_v) - q_v^{-1})}{1 - q_v^{-2}} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-k/2} \right\}, \\ (\tilde{B}_{1,\rho}^1)''(0) &= \epsilon''(0, 1) B_\rho(0) + 2\epsilon'(0, 1) B'_\rho(0) + \epsilon(0, 1) B''_\rho(0). \end{aligned}$$

Here we set  $B_\rho(z) = B_{1,\rho}^1(-z + 1/2, 1) = D_F^{-z} \prod_{v \in S(\rho)} B_v(z)$  and

$$\begin{aligned} B_v(z) &= \delta(v \in S_1(\rho)) (q_v^z - 1) \frac{q_v^{-1/2}}{1 - q_v^{-1}} \\ &\quad + \sum_{k=2}^n \delta(v \in S_k(\rho)) (q_v^{kz} - q_v^{(k-1)z-1} - q_v^{(k-1)z} + q_v^{(k-2)z-1}) \\ &\quad \times \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} \frac{q_v^{-k/2}}{1 - q_v^{-2}}. \end{aligned}$$

A direct computation gives us

$$\begin{aligned}
 B'_\rho(0) &= (\log D_F^{-1}) \prod_{v \in S(\rho)} B_v(0) + \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) - \{w\}} B_v(0) \right\} B'_w(0), \\
 B''_\rho(0) &= (\log D_F^{-1})^2 \prod_{v \in S(\rho)} B_v(0) + 2(\log D_F^{-1}) \sum_{w \in S(\rho)} \left\{ \prod_{v \in S(\rho) - \{w\}} B_v(0) \right\} B'_w(0) \\
 &\quad + \sum_{w \in S(\rho)} \left\{ \sum_{x \in S(\rho) - \{w\}} \left\{ \prod_{v \in S(\rho) - \{w, x\}} B_v(0) \right\} B'_w(0) B'_x(0) \right. \\
 &\quad \left. + \left\{ \prod_{v \in S(\rho) - \{w\}} B_v(0) \right\} B''_w(0) \right\}, \\
 B'_v(0) &= \delta(v \in S_1(\rho)) (\log q_v) \frac{q_v^{-1/2}}{1 - q_v^{-1}} + \sum_{k=2}^n \delta(v \in S_k(\rho)) (\log q_v) \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} \frac{q_v^{-k/2}}{1 + q_v^{-1}}
 \end{aligned}$$

and

$$\begin{aligned}
 B''_v(0) &= \delta(v \in S_1(\rho)) \frac{(\log q_v)^2 q_v^{-1/2}}{1 - q_v^{-1}} \\
 &\quad + \sum_{k=2}^n \delta(v \in S_k(\rho)) \frac{2k - 1 - (2k - 3)q_v^{-1}}{k^2} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} \frac{(\log q_v)^2 q_v^{-k/2}}{1 - q_v^{-2}}.
 \end{aligned}$$

This completes the proof of the estimate of  $Y_{-1}^\eta(\mathfrak{n})$ .  $\square$

With the aid of [Lemmas 21 and 24](#), we obtain the expression of the spectral side of  $P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -))$ .

**Theorem 26.** *The value  $P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -))$  can be defined and we have*

$$P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -)) = C(\mathfrak{n}, S) \{ \mathbb{I}_{\text{cus}}^\eta(\mathfrak{n}|\alpha) + \mathbb{I}_{\text{eis}}^\eta(\mathfrak{n}|\alpha) + \mathbb{D}^\eta(\mathfrak{n}|\alpha) \}.$$

Here we put

$$\begin{aligned}
 \mathbb{I}_{\text{cus}}^\eta(\mathfrak{n}|\alpha) &= \sum_{\varphi \in \mathcal{B}_{\text{cus}}(\mathfrak{n})} \alpha(\nu_{\varphi, S}) \overline{P_{\text{reg}}^{\mathbf{1}}(\varphi)} P_{\text{reg}}^\eta(\varphi), \\
 \mathbb{I}_{\text{eis}}^\eta(\mathfrak{n}|\alpha) &= \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_\chi(\mathfrak{n})} \frac{R_F^{-1}}{8\pi i} \int_{i\mathbb{R}} \tilde{\alpha}_\chi(\nu) P_{\text{reg}}^{\mathbf{1}}(E_{\chi^{-1}, \rho}(-\nu, -)) P_{\text{reg}}^\eta(E_{\chi, \rho}(\nu, -)) d\nu
 \end{aligned}$$

and

$$\mathbb{D}^\eta(\mathfrak{n}|\alpha) = \delta(\mathfrak{f}_\eta = \mathfrak{o}_F) \{ Y_2^\eta(\mathfrak{n}) \tilde{\alpha}_\eta''(1) + Y_1^\eta(\mathfrak{n}) \tilde{\alpha}_\eta'(1) + Y_0^\eta(\mathfrak{n}) \tilde{\alpha}_\eta(1) \} + Y_{-1}^\eta(\mathfrak{n}) \tilde{\alpha}(1).$$

## 6. Periods of regularized automorphic smoothed kernels: geometric side

In this section, we describe the geometric expression of  $\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -)$  and its regularized  $\eta$ -period  $P_{\text{reg}}^{\eta}(\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -))$ . For  $\delta \in G_F$ , we put  $\text{St}(\delta) = H_F \cap \delta^{-1}H_F\delta$ . By [20, Lemma 11.1], the following elements of  $G_F$  form a complete system of representatives of the double coset space  $H_F \backslash G_F / H_F$ :

$$\begin{aligned} e &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ u &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad uw_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \bar{u}w_0 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \\ \delta_b &= \begin{pmatrix} 1+b^{-1} & 1 \\ 1 & 1 \end{pmatrix}, \quad b \in F^{\times} - \{-1\}. \end{aligned}$$

Moreover, we have  $\text{St}(e) = \text{St}(w_0) = H_F$  and  $\text{St}(\delta) = Z_F$  for any  $\delta \in \{u, \bar{u}, uw_0, \bar{u}w_0\} \cup \{\delta_b | b \in F^{\times} - \{-1\}\}$ . We note

$$H_F \backslash G_F = \coprod_{\delta \in H_F \backslash G_F / H_F} H_F \backslash (H_F \delta H_F) \cong \coprod_{\delta \in H_F \backslash G_F / H_F} \text{St}(\delta) \backslash H_F.$$

Thus we obtain the following expression for  $\text{Re}(\lambda) > 0$ :

$$\hat{\Psi}_{\beta, \lambda}(\mathfrak{n}|\alpha; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{\eta} \\ 0 & 1 \end{pmatrix}) = \sum_{\delta \in H_F \backslash G_F / H_F} \sum_{\gamma \in \text{St}(\delta) \backslash H_F} \hat{\Psi}_{\beta, \lambda}(\mathfrak{n}|\alpha; \delta \gamma \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{\eta} \\ 0 & 1 \end{pmatrix}).$$

Set

$$J_{\delta}(\beta, \lambda, \alpha; t) = \sum_{\gamma \in \text{St}(\delta) \backslash H_F} \hat{\Psi}_{\beta, \lambda}(\mathfrak{n}|\alpha; \delta \gamma \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_{\eta} \\ 0 & 1 \end{pmatrix})$$

for any  $\delta \in H_F \backslash G_F / H_F$ . We obtain Lemmas 27 and 28 by the same computation as [20, Lemma 11.2] and [20, Lemma 11.3], respectively.

**Lemma 27.** *Both functions  $\lambda \mapsto J_{\epsilon}(\beta, \lambda, \alpha; t)$  and  $\lambda \mapsto J_{w_0}(\beta, \lambda, \alpha; t)$  are analytically continued to entire functions. The values of these functions at  $\lambda = 0$  are equal to  $J_{\text{id}}(\alpha; t)\beta(0)$  and  $\delta(\mathfrak{n} = \mathfrak{o}_F)J_{\text{id}}(\alpha; t)\beta(0)$ , respectively, where*

$$J_{\text{id}}(\alpha; t) = \delta(\mathfrak{f}_{\eta} = \mathfrak{o}_F) \left( \frac{1}{2\pi i} \right)^{\#S} \int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^1(\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s})$$

with

$$\Upsilon_S^1(\mathbf{s}) = \left\{ \prod_{v \in \Sigma_{\infty}} \frac{-1}{8} \frac{\Gamma((s_v + 1)/4)^2}{\Gamma((s_v + 3)/4)^2} \right\} \left\{ \prod_{v \in S_{\text{fin}}} (1 - q_v^{-(s_v + 1)/2})^{-1} (1 - q_v^{(s_v + 1)/2})^{-1} \right\}.$$



We put

$$J_u(\beta, \lambda, \alpha; t) = J_u(\beta, \lambda, \alpha, t) + J_{\bar{u}w_0}(\beta, \lambda, \alpha, t)$$

and

$$J_{\bar{u}}(\beta, \lambda, \alpha; t) = J_{uw_0}(\beta, \lambda, \alpha, t) + J_{\bar{u}}(\beta, \lambda, \alpha, t).$$

**Lemma 28.** *For any  $* \in \{u, \bar{u}\}$ , the function  $\lambda \mapsto J_*(\beta, \lambda, \alpha, t)$  is analytically continued to an entire function and the value at  $\lambda = 0$  is equal to  $J_*(\alpha; t)\beta(0)$ , where*

$$\begin{aligned} J_u(\alpha; t) = & \left(\frac{1}{2\pi i}\right)^{\#S} \sum_{a \in F^\times \mathbb{L}_S(\mathbf{c})} \int \left\{ \hat{\Psi}^{(0)} \left( \mathbf{n} | \mathbf{s}; \begin{pmatrix} 1 & at^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix} \right) \right. \\ & \left. + \hat{\Psi}^{(0)} \left( \mathbf{n} | \mathbf{s}; \begin{pmatrix} 1 & 0 \\ at^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_\eta & 1 \end{pmatrix} w_0 \right) \right\} \alpha(\mathbf{s}) d\mu_S(\mathbf{s}) \end{aligned}$$

and

$$\begin{aligned} J_{\bar{u}}(\alpha; t) = & \left(\frac{1}{2\pi i}\right)^{\#S} \sum_{a \in F^\times \mathbb{L}_S(\mathbf{c})} \int \left\{ \hat{\Psi}^{(0)} \left( \mathbf{n} | \mathbf{s}; \begin{pmatrix} 1 & 0 \\ at & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix} \right) \right. \\ & \left. + \hat{\Psi}^{(0)} \left( \mathbf{n} | \mathbf{s}; \begin{pmatrix} 1 & at \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x_\eta & 1 \end{pmatrix} w_0 \right) \right\} \alpha(\mathbf{s}) d\mu_S(\mathbf{s}). \end{aligned}$$

These series-integrals are absolutely convergent.

We put

$$\begin{aligned} J_{\text{hyp}}(\beta, \lambda, \alpha; t) &= \sum_{b \in F^\times - \{-1\}} J_{\delta_b}(\beta, \lambda, \alpha; t) \\ &= \sum_{b \in F^\times - \{-1\}} \sum_{a \in F^\times} \hat{\Psi}_{\beta, \lambda}(\mathbf{n} | \alpha; \delta_b \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}). \end{aligned}$$

**Lemma 29.** *The function  $J_{\text{hyp}}(\beta, \lambda, \alpha; t)$  on  $\text{Re}(\lambda) > 1$  is analytically continued to an entire function and the value at  $\lambda = 0$  is  $J_{\text{hyp}}(\alpha; t)\beta(0)$ , where*

$$J_{\text{hyp}}(\alpha; t) = \sum_{b \in F^\times - \{-1\}} \sum_{a \in F^\times} \hat{\Psi}^{(0)}(\mathbf{n} | \alpha; \delta_b \begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}).$$

The series converges absolutely and locally uniformly in  $t \in \mathbb{A}^\times$ .

**Proof.** We obtain the assertion by the same proof as in [20, Lemma 11.21]. Let  $\mathbf{n}$  be an ideal of  $\mathfrak{o}_F$  such that  $S(\mathbf{n}) \cap S = \emptyset$  and  $S(\mathbf{n}) \cap S(\mathfrak{f}_\eta) = \emptyset$ . In [20, Lemma 11.21], it is

assumed that  $\mathfrak{n}$  is square-free. However, the argument in [20, §11.4] is generalized to the case of arbitrary ideal  $\mathfrak{n}$  as follows: The last sentence in [20, Lemma 11.4] is replaced with “If the relation (11.9) holds with  $k \in \mathbf{K}_0(\mathfrak{no}_v)$ , then  $t \in \mathfrak{no}_v$ .” In [20, Corollary 11.7],  $\delta(t \in \mathfrak{p}_v)$  is replaced with  $\delta(t \in \mathfrak{no}_v)$ . In the definition of  $\mathbf{N}(\mathfrak{n}|\sigma, \mathbf{c}; t, b)$  in [20, §11.4.4],  $\delta(t_v \in \mathfrak{p}_v)$  is replaced with  $\delta(t_v \in \mathfrak{no}_v)$ . Then, [20, Lemmas 11.15, 11.16, 11.19, 11.20 and Corollary 11.17] hold for our  $\mathfrak{n}$ . This completes the proof.  $\square$

Lemmas 27, 28 and 29 give us the geometric expression of  $\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix})$ .

**Proposition 30.** *Let  $\mathfrak{n}$  be an ideal of  $\mathfrak{o}_F$  and  $S$  a finite subset of  $\Sigma_F$  satisfying  $\Sigma_\infty \subset S$  and  $S \cap S(\mathfrak{n}) = \emptyset$ . Let  $\eta$  be a character satisfying  $(\star)$  in §2.1. Then, for any  $\alpha \in \mathcal{A}_S$ , we have*

$$\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}) = (1 + \delta(\mathfrak{n} = \mathfrak{o}_F))J_{\text{id}}(\alpha; t) + J_u(\alpha; t) + J_{\bar{u}}(\alpha; t) + J_{\text{hyp}}(\alpha; t),$$

$$t \in \mathbb{A}^\times.$$

Next let us compute  $P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathfrak{n}|\alpha; -))$  explicitly. Set

$$\mathbb{J}_*(\beta, \lambda; \alpha) = \int_{F^\times \setminus \mathbb{A}^\times} J_*(\alpha; t) \{ \hat{\beta}_\lambda(|t|_{\mathbb{A}}) + \hat{\beta}_\lambda(|t|_{\mathbb{A}}^{-1}) \} \eta(t) \eta_{\text{fin}}(x_{\eta, \text{fin}}) d^\times t$$

for  $* \in \{\text{id}, u, \bar{u}, \text{hyp}\}$  and

$$\Upsilon_S^\eta(\mathbf{s}) = \left\{ \prod_{v \in \Sigma_\infty} \frac{-1}{8} \frac{\Gamma((s_v + 1)/4)^2}{\Gamma((s_v + 3)/4)^2} \right\}$$

$$\times \left\{ \prod_{v \in S_{\text{fin}}} (1 - q_v^{(s_v + 1)/2})^{-1} (1 - \eta_v(\varpi_v) q_v^{-(s_v + 1)/2})^{-1} \right\}.$$

For any ideal  $\mathfrak{a}$  of  $\mathfrak{o}_F$ , we set

$$\mathfrak{C}_{S, \mathfrak{a}}^\eta(\mathbf{s}) = C_0(\eta) + R(\eta) \left\{ \log(D_F \mathbf{N}(\mathfrak{a})) + \frac{d_F}{2} (C_{\text{Euler}} + 2 \log 2 - \log \pi) \right.$$

$$\left. + \sum_{v \in S_{\text{fin}}} \frac{\log q_v}{1 - q_v^{(s_v + 1)/2}} + \frac{1}{2} \sum_{v \in \Sigma_\infty} \left( \psi\left(\frac{s_v + 1}{4}\right) + \psi\left(\frac{s_v + 3}{4}\right) \right) \right\},$$

where  $\psi(z) = \Gamma'(z)/\Gamma(z)$  is the digamma function and  $C_{\text{Euler}}$  is the Euler constant. We note that if  $\eta \neq \mathbf{1}$ , then  $\mathfrak{C}_{S, \mathfrak{a}}^\eta(\mathbf{s})$  is independent of the choice of  $\mathfrak{a}$ , and  $\mathfrak{C}_{S, \mathfrak{a}}^\eta(\mathbf{s}) = C_0(\eta) = L(1, \eta)$ . Put

$$\mathfrak{K}_\eta(\mathfrak{n}|\mathbf{s}) = \sum_{b \in F^\times - \{-1\}_{\mathbb{A}^\times}} \int \Psi^{(0)}(\mathfrak{n}|\mathbf{s}; \delta_b \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x_\eta \\ 0 & 1 \end{pmatrix}) \eta(t) \eta_{\text{fin}}(x_{\eta, \text{fin}}) d^\times t.$$

The defining series-integral converges absolutely if we take  $c \in \mathbb{R}$  such that  $\operatorname{Re}(\mathbf{s}) = \underline{c} = (c)_{v \in S}$  and  $(c+1)/4 > 1$ . By the expression of  $\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha; -)$  in Proposition 30 and the same computation as in the proof of [20, Theorem 12.1], we can express the geometric side of  $P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha; -))$  as follows.

**Theorem 31.** *For any  $*$   $\in \{\text{id}, \text{u}, \bar{\text{u}}, \text{hyp}\}$ , the integral  $\mathbb{J}_*^\eta(\beta, \lambda; \alpha)$  converges absolutely and locally uniformly in  $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 1\}$ . The function  $\lambda \mapsto \mathbb{J}_*^\eta(\beta, \lambda; \alpha)$  is analytically continued to a meromorphic function on  $\{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > -1\}$ . Moreover, the constant term  $\text{CT}_{\lambda=0} \mathbb{J}_*^\eta(\beta, \lambda; \alpha)$  is equal to  $\mathbb{J}_*^\eta(\mathbf{n}|\alpha)\beta(0)$ , where*

$$\begin{aligned}\mathbb{J}_{\text{id}}^\eta(\mathbf{n}|\alpha) &= 0, \\ \mathbb{J}_{\text{u}}^\eta(\mathbf{n}|\alpha) &= (1 + \delta(\mathbf{n} = \mathbf{o}_F)) D_F^{1/2} \mathcal{G}(\eta) \int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^\eta(\mathbf{s}) \mathfrak{C}_{S, \mathbf{o}_F}^\eta(\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s}), \\ \mathbb{J}_{\bar{\text{u}}}^\eta(\mathbf{n}|\alpha) &= (1 + \delta(\mathbf{n} = \mathbf{o}_F)) D_F^{1/2} \mathcal{G}(\eta) \int_{\mathbb{L}_S(\mathbf{c})} \Upsilon_S^\eta(\mathbf{s}) \mathfrak{C}_{S, \mathbf{n}}^\eta(\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s})\end{aligned}$$

and

$$\mathbb{J}_{\text{hyp}}^\eta(\mathbf{n}|\alpha) = \left(\frac{1}{2\pi i}\right)^{\#S} \int_{\mathbb{L}_S(\underline{c})} \mathfrak{K}_\eta(\mathbf{n}|\mathbf{s}) \alpha(\mathbf{s}) d\mu_S(\mathbf{s}).$$

In particular, we have

$$P_{\text{reg}}^\eta(\hat{\Psi}_{\text{reg}}(\mathbf{n}|\alpha; -)) = \mathbb{J}_{\text{u}}^\eta(\mathbf{n}|\alpha) + \mathbb{J}_{\bar{\text{u}}}^\eta(\mathbf{n}|\alpha) + \mathbb{J}_{\text{hyp}}^\eta(\mathbf{n}|\alpha).$$

## 7. Proofs of main theorems

Fix a character  $\eta$  of  $F^\times \backslash \mathbb{A}^\times$  so that  $\eta^2 = \mathbf{1}$  and  $\eta_v(-1) = 1$  for all  $v \in \Sigma_\infty$ . Let  $S$  be a finite subset of  $\Sigma_F$  such that  $S \supset \Sigma_\infty$  and  $S_{\text{fin}} \cap S(f_\eta) = \emptyset$ . Let  $J'_{S, \eta}$  be the set of all ideals  $\mathbf{n}$  of  $\mathbf{o}_F$  such that  $S(\mathbf{n}) \cap (S \cup S(f_\eta)) = \emptyset$  and  $\tilde{\eta}(\mathbf{n}) = 1$ . By Theorems 26 and 31, we obtain the relative trace formula

$$C(\mathbf{n}, S) \{ \mathbb{I}_{\text{cus}}^\eta(\mathbf{n}|\alpha) + \mathbb{I}_{\text{eis}}^\eta(\mathbf{n}|\alpha) + \mathbb{D}^\eta(\mathbf{n}|\alpha) \} = \mathbb{J}_{\text{u}}^\eta(\mathbf{n}|\alpha) + \mathbb{J}_{\bar{\text{u}}}^\eta(\mathbf{n}|\alpha) + \mathbb{J}_{\text{hyp}}^\eta(\mathbf{n}|\alpha)$$

for any  $\alpha \in \mathcal{A}_S$  and  $\mathbf{n} \in J'_{S, \eta}$ . The following estimate of  $\mathbb{J}_{\text{hyp}}^\eta(\mathbf{n}|\alpha)$  is given by the same argument in the proof of [20, Lemma 12.9].

**Lemma 32.** *For any  $\alpha \in \mathcal{A}_S$  and  $q > 0$ , we have  $|\mathbb{J}_{\text{hyp}}^\eta(\mathbf{n}|\alpha)| \ll N(\mathbf{n})^{-q}$  with the implied constant independent of  $\mathbf{n} \in J'_{S, \eta}$ .*

**Lemma 33.** For any  $\epsilon > 0$ , we have

$$|B_{\chi,\rho}^{\eta}(1/2, \nu)| \ll N(\mathfrak{f}_{\chi})^{-1/2-\epsilon} N(\mathfrak{n})^{\epsilon}, \quad \nu \in i\mathbb{R}, \quad \rho \in \Lambda_{\chi}(\mathfrak{n}), \quad \chi \in \Xi(\mathfrak{n})$$

with the implied constant independent of  $\mathfrak{n} \in J'_{S,\eta}$ .

**Proof.** Assume  $\nu \in i\mathbb{R}$ . Then, the following estimate holds for any  $\epsilon > 0$ :

$$\begin{aligned} & |B_{\chi,\rho}^{\eta}(1/2, \nu)| \\ &= \prod_{k=0}^n \prod_{v \in S_k(\rho)} |Q_{k,\chi_v}^{(\nu)}(\eta_v, 1)| |L(1 + \nu, \chi_v^2)| \prod_{v \in U_1(\rho)} (1 + q_v^{-1}) \\ &\quad \times \prod_{k=2}^n \prod_{v \in U_k(\rho)} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} \prod_{k=0}^n \prod_{v \in R_k(\rho)} q_v^{d_v/2} (1 - q_v^{-1})^{1/2} |\overline{\mathcal{G}(\chi_v)}| \\ &\ll \prod_{v \in U_1(\rho)} (1 + q_v^{-1}) \left( 1 + \frac{2}{q_v^{1/2} + q_v^{-1/2}} \right) \frac{1}{1 - q_v^{-1}} \\ &\quad \times \prod_{k=2}^n \prod_{v \in U_k(\rho)} \left( \frac{q_v + 1}{q_v - 1} \right)^{1/2} q_v^{-1} (q_v^{1/2} + 1)^2 \frac{1}{1 - q_v^{-1}} \\ &\quad \times \prod_{k=0}^n \prod_{v \in R_k(\rho)} q_v^{d_v/2} (1 - q_v^{-1})^{1/2} \frac{q_v^{-f(\chi_v)/2} q_v^{-d_v/2}}{1 - q_v^{-1}} \frac{1}{1 - q_v^{-1}} \\ &\ll N(\mathfrak{f}_{\chi})^{-1/2+\epsilon} N(\mathfrak{n}\mathfrak{f}_{\chi}^{-2})^{\epsilon}. \end{aligned}$$

This completes the proof.  $\square$

Note that  $[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})] = N(\mathfrak{n}) \prod_{v \in S(\mathfrak{n})} (1 + q_v^{-1})$  holds by an easy computation.

**Lemma 34.** For any  $\alpha \in \mathcal{A}_S$ , there exists  $\delta > 0$  such that  $|C(\mathfrak{n}, S) \mathbb{I}_{\text{eis}}^{\eta}(\mathfrak{n}|\alpha)| \ll N(\mathfrak{n})^{-\delta}$  with the implied constant independent of  $\mathfrak{n} \in J'_{S,\eta}$ .

**Proof.** We recall that for any  $\epsilon > 0$ , the estimate  $|L_{\text{fin}}(1 + \nu, \chi^2)|^{-1} \ll \mathfrak{q}(\chi^2 \cdot |\cdot|_{\mathbb{A}}^{\nu})^{\epsilon}$ ,  $\nu \in i\mathbb{R}$  holds with the implied constant independent of  $\chi \in \Xi(\mathfrak{n})$  and  $\mathfrak{n}$ . This was given in the proof of Lemma 20. Let  $\theta$  be a real number such that  $|L_{\text{fin}}(1/2 + it, \chi)| \ll \mathfrak{q}(\chi \cdot |\cdot|_{\mathbb{A}}^{it})^{1/4+\theta}$ ,  $t \in \mathbb{R}$  uniformly for any  $\chi \in \Xi(\mathfrak{n})$  and  $\mathfrak{n}$ . We can take such  $\theta$  so that  $-1/4 < \theta < 0$  by [10, Theorem 1.1]. Thus, with the aid of Lemma 33 and

$$\prod_{v \in \Sigma_{\infty}} \left| \frac{L((1 + \nu)/2, \chi_v) L((1 - \nu)/2, \chi_v^{-1})}{L(1 - \nu, \chi_v^{-2})} \right| \asymp \prod_{v \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_v)|)^{-1/2},$$

which follows from Stirling's formula, the explicit description of  $P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -))$  in Proposition 9 gives us the estimate

$$\begin{aligned}
|P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -))| &\ll N(\mathfrak{f}_{\chi})^{1/2} N(\mathfrak{f}_{\chi})^{-1/2-\epsilon} N(\mathfrak{n})^{\epsilon} (N(\mathfrak{f}_{\chi})^{1/4+\theta})^2 N(\mathfrak{f}_{\chi})^{\epsilon} \\
&\quad \times \prod_{v \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_v)|)^{2\theta+\epsilon} \\
&= N(\mathfrak{f}_{\chi})^{1/2+2\theta} N(\mathfrak{n})^{\epsilon} \prod_{v \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_v)|)^{2\theta+\epsilon} \\
&\ll N(\mathfrak{n})^{1/4+\theta+\epsilon} \prod_{v \in \Sigma_{\infty}} (1 + |\nu + 2ib(\chi_v)|)^{2\theta+\epsilon}
\end{aligned}$$

for any  $\epsilon > 0$ . Here the implied constant is independent of  $\nu \in i\mathbb{R}$ ,  $\chi \in \Xi(\mathfrak{n})$  and  $\mathfrak{n} \in J'_{S,\eta}$ .

Take a point  $(\nu_{0,\infty}, \nu_{0,\text{fin}}) \in \mathfrak{X}_S$  which gives the maximal value of  $|\alpha(\nu)|$  on  $\mathfrak{X}_S$ . We set  $\alpha_0(\nu) = \alpha(\nu, \nu_{0,\text{fin}}) \in \mathcal{A}_{\Sigma_{\infty}}$ . With the aid of [Lemma 6](#), we have

$$\begin{aligned}
|C(\mathfrak{n}, S) \mathbb{I}_{\text{eis}}^{\eta}(\mathfrak{n}|\alpha)| &\ll [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1} \sum_{\chi \in \Xi(\mathfrak{n})} \sum_{\rho \in \Lambda_{\chi}(\mathfrak{n})} \int_{i\mathbb{R}} |P_{\text{reg}}^1(E_{\chi^{-1},\rho}(-\nu, -))| \\
&\quad \times |P_{\text{reg}}^{\eta}(E_{\chi,\rho}(\nu, -))| |\tilde{\alpha}_{\chi}(\nu)| |d\nu| \\
&\ll N(\mathfrak{n})^{-1} \sum_{\chi \in \Xi(\mathfrak{n})} \left( \sum_{\mathfrak{a}|\mathfrak{n}} 1 \right) \int_{y \in \mathbb{R}} N(\mathfrak{n})^{1/2+2\theta+2\epsilon} \\
&\quad \times \left\{ \prod_{v \in \Sigma_{\infty}} (1 + |y + 2b(\chi_v)|)^{4\theta+2\epsilon} \right\} |\tilde{\alpha}_{\chi}(iy)| dy \\
&\ll N(\mathfrak{n})^{-1/2+2\theta+3\epsilon} \sum_{\chi \in \Xi_{\ker}(\mathfrak{n})} \sum_{b \in L_0 y \in \mathbb{R}} \int \left\{ \prod_{v \in \Sigma_{\infty}} (1 + |y + 2b_v|)^{4\theta+2\epsilon} \right\} \\
&\quad \times |\alpha_0((iy + 2ib_v)_{v \in \Sigma_{\infty}})| dy \\
&\ll N(\mathfrak{n})^{2\theta+4\epsilon} \int_{y \in \mathbb{R}^{d_F}} (1 + \|y\|^2)^{4\theta+2\epsilon} |\alpha_0(iy)| dy.
\end{aligned}$$

Note  $\sum_{\mathfrak{a}|\mathfrak{n}} 1 \ll N(\mathfrak{n})^{\epsilon}$ . Since we can take  $\epsilon > 0$  so that  $2\theta + 4\epsilon < 0$ , we obtain the assertion.  $\square$

**Lemma 35.** *For any  $\epsilon > 0$  and  $\alpha \in \mathcal{A}_S$ , we have  $|C(\mathfrak{n}, S) \mathbb{D}^{\eta}(\mathfrak{n}|\alpha)| \ll N(\mathfrak{n})^{-1+\epsilon}$  with the implied constant independent of  $\mathfrak{n} \in J'_{S,\eta}$ .*

**Proof.** This follows immediately from [Lemma 25](#).  $\square$

For a fixed  $\mathfrak{n} \in J'_{S,\eta}$ , we set  $\langle \lambda_S^{\eta}(\mathfrak{n}), f \rangle = 2D_F^{1/2} \mathcal{G}(\eta)^{-1} [\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]^{-1} \times \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} \mathbb{P}^{\eta}(\pi, \mathbf{K}_0(\mathfrak{n})) f(\nu_{\pi,S})$  for any  $f \in C_c(\mathfrak{X}_S^{0+})$ . Since the argument in [\[20, §13.1\]](#) is generalized to the case of arbitrary levels,  $\langle \lambda_S^{\eta}(\mathfrak{n}), f \rangle$  is convergent and  $\lambda_S^{\eta}(\mathfrak{n})$  is extended to a linear functional on the Schwartz space  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$  by [\[20, Lemma 13.16\]](#). For the definition of  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ , see [Appendix A](#).

Combining Lemmas 12, 14, 32, 34 and 35 with the argument in [20, Lemma 13.18], we obtain the following.

**Theorem 36.** *For a fixed  $\alpha \in \mathcal{A}_S$ , there exists  $\delta > 0$  such that for any infinite subset  $\Lambda \subset J'_{S,\eta}$ , we have*

$$\begin{aligned} \langle \lambda_S^\eta(\mathfrak{n}), \alpha \rangle &= \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]} w_\mathfrak{n}^\eta(\pi) \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \text{Ad})} \alpha(\nu_{\pi, S}) \\ &= \langle \lambda_S^\eta, \alpha \rangle + \mathcal{O}(N(\mathfrak{n})^{-\delta}) \end{aligned}$$

as  $N(\mathfrak{n}) \rightarrow \infty$  in  $\mathfrak{n} \in \Lambda$ .

### 7.1. Proof of Theorem 2

For  $\mathfrak{n} \in J_{S,\eta}$ , let  $\mathfrak{n} = \prod_{k=1}^s \mathfrak{p}_k^{a_k} \prod_{k=s+1}^{s+l} \mathfrak{p}_k$  with  $a_k \geq 2$  be a prime ideal decomposition of  $\mathfrak{n}$ . For  $\pi \in \Pi_{\text{cus}}(\mathfrak{n})$  with  $\mathfrak{f}_\pi = \prod_{k=1}^s \mathfrak{p}_k^{b_k} \prod_{k=s+1}^{s+l} \mathfrak{p}_k^{\epsilon_k}$ , Lemma 12 gives us

$$w_\mathfrak{n}^\eta(\pi) = \delta((\epsilon_{s+k})_k \in \{1\}^l, (a_k - b_k)_k \in (2\mathbb{N}_0)^s) \prod_{k=1}^s \left( \frac{N(\mathfrak{p}_k) + 1}{N(\mathfrak{p}_k) - 1} \right)^{\delta(b_k=0)}.$$

Hence, by setting  $L(\pi) = \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \text{Ad})} \alpha(\nu_{\pi, S})$  for a fixed  $\alpha \in \mathcal{A}_S$ , we obtain

$$\begin{aligned} \text{AL}(\mathfrak{n}; \alpha) &= \frac{1}{N(\mathfrak{n})} \sum_{\pi \in \Pi_{\text{cus}}^*(\mathfrak{n})} \frac{L(1/2, \pi) L(1/2, \pi \otimes \eta)}{L^{S_\pi}(1, \pi; \text{Ad})} \alpha(\nu_{\pi, S}) \\ &= \left( \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} + \sum_{j=1}^{s+l} (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq s+l} \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-1})} \right) \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]} \\ &\quad \times w_\mathfrak{n}^\eta(\pi) L(\pi) \\ &= \left( \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} + \sum_{j=1}^s (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq s} \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})} \right) \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]} \\ &\quad \times w_\mathfrak{n}^\eta(\pi) L(\pi) \\ &= \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n})} \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]} w_\mathfrak{n}^\eta(\pi) L(\pi) \\ &\quad + \sum_{j=1}^s \sum_{1 \leq i_1 < \dots < i_j \leq s} (-1)^j \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})]}{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n})]} \\ &\quad \times \sum_{\pi \in \Pi_{\text{cus}}(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})} \frac{[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{f}_\pi)]}{N(\mathfrak{f}_\pi)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})]} \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \prod_{v \in S_2(\mathfrak{n}) \cap S(\prod_{k=1}^j \mathfrak{p}_{i_k})} \frac{q_v + 1}{q_v - 1} \right\} w_{\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2}}^\eta(\pi) L(\pi) \\
& = \langle \lambda_S^\eta, \alpha \rangle + \mathcal{O}(N(\mathfrak{n})^{-\delta}) \\
& \quad + \sum_{j=1}^s (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq s} \frac{N(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2}) \prod_{v \in S(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})} (1 + q_v^{-1})}{N(\mathfrak{n}) \prod_{v \in S(\mathfrak{n})} (1 + q_v^{-1})} \\
& \quad \times \left\{ \prod_{v \in S_2(\mathfrak{n}) \cap S(\prod_{k=1}^j \mathfrak{p}_{i_k})} \frac{q_v + 1}{q_v - 1} \right\} \left\{ \langle \lambda_S^\eta, \alpha \rangle + \mathcal{O}(N(\mathfrak{n} \prod_{k=1}^j \mathfrak{p}_{i_k}^{-2})^{-\delta}) \right\} \\
& = \left( 1 + \sum_{j=1}^s (-1)^j \sum_{1 \leq i_1 < \dots < i_j \leq s} \frac{\prod_{v \in S_2(\mathfrak{n}) \cap S(\prod_{k=1}^j \mathfrak{p}_{i_k})} (1 - q_v^{-1})^{-1}}{\prod_{k=1}^j N(\mathfrak{p}_{i_k})^2} \right) \langle \lambda_S^\eta, \alpha \rangle \\
& \quad + \mathcal{O} \left( N(\mathfrak{n})^{-\delta} \left( 1 + \sum_{j=1}^s \sum_{1 \leq i_1 < \dots < i_j \leq s} \frac{\prod_{v \in S_2(\mathfrak{n}) \cap S(\prod_{k=1}^j \mathfrak{p}_{i_k})} (1 - q_v^{-1})^{-1}}{\prod_{k=1}^j N(\mathfrak{p}_{i_k})^{2-2\delta}} \right) \right) \\
& = \left( \prod_{v \in S_2(\mathfrak{n})} \{1 - (1 - q_v^{-1})^{-1} q_v^{-2}\} \prod_{v \in S(\mathfrak{n}) - (S_1(\mathfrak{n}) \cup S_2(\mathfrak{n}))} (1 - q_v^{-2}) \right) \langle \lambda_S^\eta, \alpha \rangle \\
& \quad + \mathcal{O}(N(\mathfrak{n})^{-\delta}).
\end{aligned}$$

Here we use [Theorem 36](#) and an explicit formula of  $w_{\mathfrak{n}}^\eta(\pi)$ .

With the aid of the proof of [\[20, Theorem 13.17\]](#) and the result as above, for any  $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$  we have

$$C(\mathfrak{n})^{-1} \text{AL}(\mathfrak{n}; f) \rightarrow \langle \lambda_S^\eta, f \rangle$$

as  $N(\mathfrak{n}) \rightarrow \infty$  in  $\mathfrak{n} \in J_{S, \eta}$ . Indeed, by [Proposition 38](#), for any  $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ , any  $\epsilon > 0$  and any  $m \in \mathbb{N}$ , there exists a function  $\alpha \in A_\infty \otimes A_{\text{fin}}$  such that

$$\sup_{\mathfrak{s} \in \overline{\mathfrak{X}_S^{0+}}} |f(\mathfrak{s}) - \alpha(\mathfrak{s})| (1 + \|\mathfrak{s}\|^2)^m < \epsilon.$$

We regard naturally  $\alpha$  as a function on  $\mathfrak{X}_S$ ; then  $\alpha$  is an element of  $\mathcal{A}_S$ . From this, the argument in the proof of [\[20, Lemma 13.17\]](#) is valid by using  $\text{AL}(\mathfrak{n}; -)$  in place of  $\lambda_S^\eta(\mathfrak{n})$ .

The assertion for  $f \in \mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$  deduces to that for  $f \in C_c(\overline{\mathfrak{X}_S^{0+}})$  in the following way. It suffices to prove the assertion for  $\Lambda = J_{S, \eta}$  and  $f \in C_c(\overline{\mathfrak{X}_S^{0+}})$ .

Take any  $f \in C_c(\overline{\mathfrak{X}_S^{0+}})$  and any  $\epsilon > 0$ . Fix a locally compact bounded open subset  $U$  of  $\overline{\mathfrak{X}_S^{0+}}$  such that  $\text{supp}(f) \subset U$ . We may suppose  $U = U_\infty \times U_{\text{fin}}$  for some  $U_\infty \subset \overline{\mathfrak{X}_\infty^{0+}}$  and  $U_{\text{fin}} \subset \overline{\mathfrak{X}_{\text{fin}}^{0+}}$ , where both  $U_\infty$  and  $U_{\text{fin}}$  are locally compact bounded open subsets. Then, we have  $f|_U \in C_c(U)$ . Let  $C_c^\infty(U_\infty)$  be the space of all compactly supported functions  $h$  on  $U_\infty$  such that  $h(s) = \varphi((\frac{1-s^2}{4})_{v \in \Sigma_\infty})$  for some  $C^\infty$ -function  $\varphi$  on the

set  $\{x = (x_v)_{v \in \Sigma_\infty} \in (\mathbb{R}_{\geq 0})^{\Sigma_\infty} \mid (\sqrt{1 - 4x_v})_{v \in \Sigma_\infty} \in U_\infty\}$ . By the Stone–Weierstrass theorem,  $C_c^\infty(U_\infty) \otimes C_c(U_{\text{fin}})$  is dense in  $C_c(U)$  with respect to the topology by supremum norm. Thus there exists  $g_\epsilon \in C_c^\infty(U_\infty) \otimes C_c(U_{\text{fin}})$  satisfying

$$\sup_{\mathbf{s} \in U} |f(\mathbf{s}) - g_\epsilon(\mathbf{s})| < \epsilon.$$

By the extension by zero, the function  $g_\epsilon$  is naturally extended as an element of  $C_c^\infty(\mathfrak{X}_{\Sigma_\infty}^{0+}) \otimes C(\mathfrak{X}_{S_{\text{fin}}}^{0+})$ , which is also denoted by  $g_\epsilon$ . Then, we have  $g_\epsilon \in \mathcal{S}(\mathfrak{X}_S^{0+})$ .

Hence, there exists  $M > 0$  such that for any  $\mathbf{n} \in J_{S,\eta}$  with  $N(\mathbf{n}) > M$ , we have

$$|C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; g_\epsilon) - \langle \lambda_S^\eta, g_\epsilon \rangle| < \epsilon.$$

In the same way as [20, Lemmas 13.14 and 13.16], we have the estimates

$$|\nu(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; f - g_\epsilon)| < C \sup_{\mathbf{s} \in U} (1 + \|\mathbf{s}\|^2)^m \epsilon \quad \text{and}$$

$$|\langle \lambda_S^\eta, f - g_\epsilon \rangle| < C \sup_{\mathbf{s} \in U} (1 + \|\mathbf{s}\|^2)^m \epsilon,$$

where  $C > 0$  and  $m \in \mathbb{N}$  are independent of  $\mathbf{n} \in J_{S,\eta}$ , the function  $f$  and  $\epsilon > 0$ . As a consequence, for any  $\mathbf{n} \in J_{S,\eta}$  with  $N(\mathbf{n}) > M$ , we obtain

$$\begin{aligned} |C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; f) - \langle \lambda_S^\eta, f \rangle| &\leq |C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; f - g_\epsilon)| + |C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; g_\epsilon) - \langle \lambda_S^\eta, g_\epsilon \rangle| \\ &\quad + |\langle \lambda_S^\eta, g_\epsilon - f \rangle| \\ &< \{1 + 2C \sup_{\mathbf{s} \in U} (1 + \|\mathbf{s}\|^2)^m\} \epsilon. \end{aligned}$$

This completes the proof of Theorem 2.  $\square$

## 7.2. Proof of Theorem 3

The proof of Theorem 3 is the same as in [20, Corollary 1.2]. Let  $\mathbf{J}$  be the set of all  $(\nu_v)_{v \in S} \in \mathfrak{X}_S^0$  such that  $(1 - \nu_v^2)/4 \in J_v$  for all  $v \in \Sigma_\infty$  and  $q_v^{-\nu_v/2} + q_v^{\nu_v/2} \in J_v$  for all  $v \in S_{\text{fin}}$ . Then,  $\mathbf{J}$  is a bounded Borel set of  $\mathfrak{X}_S^{0+}$  such that  $\text{vol}(\mathbf{J}, \lambda_S^\eta) > 0$ . Since [20, Corollary 13.19] is generalized to the case of  $\mathbf{n} \in J_{S,\eta}$  with the aid of Theorem 2 and Proposition 37, for any  $M > 0$  there exists  $\mathbf{n} \in \Lambda$  with  $N(\mathbf{n}) > M$  such that  $|C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; \text{ch}_{\mathbf{J}}) - \text{vol}(\mathbf{J}, \lambda_S^\eta)| < 2^{-1} \text{vol}(\mathbf{J}, \lambda_S^\eta)$ , where  $\text{ch}_{\mathbf{J}}$  is the characteristic function of  $\mathbf{J}$  on  $\mathfrak{X}_S^{0+}$ . Therefore,  $C(\mathbf{n})^{-1} \text{AL}(\mathbf{n}; \text{ch}_{\mathbf{J}}) > 2^{-1} \text{vol}(\mathbf{J}, \lambda_S^\eta) > 0$  holds. This completes the proof of Theorem 3.  $\square$



### 7.3. Remarks on Theorems 4 and 5

We remark that Theorems 4 and 5 are proved in the same way as [20, Theorem 1.3, Corollary 1.4] since we can generalize [20, Theorem 14.1] to the case of arbitrary levels by using the relative trace formula explained in Section 7. The key idea is as follows. Set  $S = \Sigma_\infty$  and we consider  $\alpha(\mathbf{s}) \in \mathcal{A}_{\Sigma_\infty}$  of the form

$$\alpha_{\mathbf{K}, \Delta}^{(m)}(\mathbf{s}) = (1 - \langle \mathbf{s} \rangle^2)^m \sum_{\epsilon \in \{\pm 1\}^{\Sigma_\infty}} \exp\left(\frac{\langle \epsilon \mathbf{s} - i\mathbf{K} \rangle^2}{\Delta^2}\right)$$

for parameters  $m \in \mathbb{N}_0$ ,  $\mathbf{K} \in \mathbb{R}^{\Sigma_\infty}$  and  $\Delta > 1$ , where  $\langle \mathbf{s} \rangle = \sum_{v \in \Sigma_\infty} s_v^2$ . By substituting  $\alpha = \alpha_{\mathbf{K}, \Delta}^{(m)}$  in the relative trace formula and following [20, §14], the tempered part of  $\mathbb{I}_{\text{cus}}^\eta(\mathbf{n}|\alpha)$ , which gives the left-hand side of [20, (14.1)] is evaluated. The sum  $\mathbb{J}_{\mathbf{u}}^\eta(\mathbf{n}|\alpha) + \mathbb{J}_{\mathbf{u}}^\eta(\mathbf{n}|\alpha)$  gives us the main term of [20, (14.1)]. The error term results from all other terms in the relative trace formula. As a consequence, Theorem 4 follows from [20, Theorem 14.1] for arbitrary  $\mathbf{n}$ .

Here we note that the main term of the formula in [20, Theorem 1.3] should be corrected as

$$\frac{4(1 + \delta_{\mathbf{n}, \mathbf{o}_F}) D_F^{3/2} \text{vol}(J)}{(2\pi)^{d_F}} t^{d_F} \{R(\eta)(d_F \log(t/4) + V(J)) + \mathbf{C}^\eta(F, \mathbf{n})\},$$

where  $R(\eta) = \text{Res}_{s=1} L(s, \eta)$  and  $V(J) = \text{vol}(J)^{-1} \int_J (\sum_{v \in \Sigma_\infty} \log |x_v|) dx$ . To correct the proof of [20, Theorem 1.3], it is sufficient to note the following two points. First, the main term of the asymptotic formula in [20, Lemma 14.8] should be replaced with

$$\left(\frac{1}{4\pi}\right)^{d_F} D_F \mathcal{G}(\eta)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})] (1 + \delta_{\mathbf{n}, \mathbf{o}_F}) \{ \mathbf{C}^\eta(F, \mathbf{n}) + R(\eta) \sum_{v \in \Sigma_\infty} \log(|\mathbf{K}_v|/4) \}$$

when  $\min_{v \in \Sigma_\infty} |\mathbf{K}_v| \asymp \|\mathbf{K}\| \rightarrow \infty$ . Here we use

$$\begin{aligned} & \psi((1 + i\mathbf{K}_v)/4) + \psi((1 - i\mathbf{K}_v)/4) + \psi((3 + i\mathbf{K}_v)/4) + \psi((3 - i\mathbf{K}_v)/4) \\ &= 4 \log(|\mathbf{K}_v|/4) + \mathcal{O}((1 + |\mathbf{K}|)^{-1} \log(2 + \|\mathbf{K}\|)) \end{aligned}$$

as  $|\mathbf{K}_v| \rightarrow \infty$ , where  $\psi$  is the digamma function. We remark that  $\sum_{v \in \Sigma_\infty} \log(|\mathbf{K}_v|/4)$  cannot be evaluated as  $d_F \log \|\mathbf{K}\| + \mathcal{O}((1 + |\mathbf{K}|)^{-1} \log(2 + \|\mathbf{K}\|))$ .

Second, the main term in the fourth line of [20, Proof of Theorem 1.3] should be corrected as

$$\begin{aligned} & \left(\frac{1}{4\pi}\right)^{d_F} D_F \mathcal{G}(\eta)[\mathbf{K}_{\text{fin}} : \mathbf{K}_0(\mathbf{n})] (1 + \delta_{\mathbf{n}, \mathbf{o}_F}) \text{vol}(D) t^{d_F} \\ & \times \{R(\eta)(d_F \log(t/4) + V(D)) + \mathbf{C}^\eta(F, \mathbf{n})\}. \end{aligned}$$

Here we use

$$\int_D \sum_{v \in \Sigma_\infty} \log(|K_v|/4) dK = \text{vol}(D) t^{d_F} d_F \log(t/4) + \int_D \sum_{v \in \Sigma_\infty} \log |K_v| dK.$$

**Theorem 5** also follows from [20, Theorem 14.1] for arbitrary  $\mathbf{n}$ . Here  $|L_{\text{fin}}^{S_\pi}(1, \pi, \text{Ad})| \asymp |L_{\text{fin}}(1, \pi, \text{Ad})| \ll (1 + \|\nu_{\Sigma_\infty}(\pi)\|)^\epsilon$  is due to [9, Theorem 2]. Although [12, Corollary] is referred to in the proof of [20, Corollary 1.4], the Rankin–Selberg condition (A5) in [12] is valid only for general  $L$ -functions over  $\mathbb{Q}$ .

## Acknowledgments

The author would like to thank Professor Takao Watanabe for useful comments. He would also like to thank Professor Masao Tsuzuki for a lot of fruitful discussions and constructive comments, and for informing him of the thesis [11]. Furthermore, he would like to thank the referee for giving him a lot of comments for his draft which was hard to read. The author was supported by Grant-in-Aid for JSPS Fellows (25-668).

## Appendix A

**Proposition 37** is needed to prove [20, Corollary 13.19]. By **Proposition 37**, any Borel set  $\mathbf{J}$  in [20, Corollary 13.19] should be bounded. The following is regarded as a generalization of [17, Proposition 1].

**Proposition 37.** *Let  $X$  be a locally compact Hausdorff space and  $\mu$  a positive Radon measure on  $X$ . Let  $\{\mu_\lambda\}_{\lambda \in \Lambda}$  be a directed sequence of positive Radon measures on  $X$  such that  $\mu_\lambda$  converges weakly to  $\mu$  on  $C_c(X)$ . Suppose that  $A$  is a  $\mu$ -measurable set of  $X$  satisfying (1) its boundary  $\partial A$  is a  $\mu$ -null set, (2) there exists a compact subset  $K$  of  $X$  containing  $A$  such that its boundary  $\partial K$  is a  $\mu$ -null set.*

*Then, we have  $\lim_{\lambda \in \Lambda} \mu_\lambda(A) = \mu(A)$ .*

**Proof.** The restriction of  $\mu_\lambda$  and  $\mu$  to  $K$  gives Radon measures  $\mu_\lambda|_K$  and  $\mu|_K$  on  $K$ , respectively. By [1, Chap. IV, §5, n°12, Proposition 22], it is sufficient to prove  $\lim_{\lambda \in \Lambda} \mu_\lambda|_K(K) = \mu|_K(K)$  for any relatively compact subset  $K$  of  $X$  such that  $\partial K$  is a  $\mu$ -null set. The proof is given in the following way, which was suggested by Tsuzuki.

Let  $K^\circ$  and  $\bar{K}$  be the interior and closure of  $K$ , respectively. By  $\mu(\bar{K}) - \mu(K^\circ) = \mu(\bar{K} - K^\circ) = \mu(\partial K) = 0$ , we have  $\mu(K^\circ) = \mu(K) = \mu(\bar{K})$ . Take any  $\epsilon > 0$ . By inner regularity of  $\mu$  (and Urysohn's lemma), there exists  $f_\epsilon \in C_c(X)$  such that  $0 \leq f_\epsilon \leq \text{ch}_{K^\circ}$  and  $\mu(K^\circ) - \epsilon/2 < \mu(f_\epsilon)$ . In a similar way, by outer regularity of  $\mu$ , there exists  $g_\epsilon \in C_c(X)$  such that  $\text{ch}_{\bar{K}} \leq g_\epsilon$  and  $\mu(g_\epsilon) < \mu(\bar{K}) + \epsilon/2$ .

For  $\epsilon$ ,  $f_\epsilon$  and  $g_\epsilon$ , there exists  $\lambda_\epsilon \in \Lambda$  such that we have

$$|\mu_\lambda(f_\epsilon) - \mu(f_\epsilon)| < \epsilon/2 \quad \text{and} \quad |\mu_\lambda(g_\epsilon) - \mu(g_\epsilon)| < \epsilon/2$$

for any  $\lambda \geq \lambda_\epsilon$ . Then, we obtain  $\mu(K) = \mu(K^\circ) < \mu(f_\epsilon) + \epsilon/2 < \mu_\lambda(f_\epsilon) + \epsilon \leq \mu_\lambda(K) + \epsilon$  and  $\mu_\lambda(K) \leq \mu_\lambda(g_\epsilon) < \mu(g_\epsilon) + \epsilon/2 < \mu(\bar{K}) + \epsilon = \mu(K) + \epsilon$  for any  $\lambda \geq \lambda_\epsilon$ . This completes the proof.  $\square$

Next we introduce  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ . The Schwartz space  $\mathcal{S}(\mathfrak{X}_S^{0+})$  was introduced in [20, §13.2]. However, the definition is inaccurate; indeed, the Weierstrass approximation theorem does not work in the proof of [20, Lemma 13.17]. In the appendix, we introduce another Schwartz space  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ . Set  $\overline{\mathfrak{X}_v^{0+}} = i\mathbb{R}_{\geq 0} \cup (0, 1]$  for  $v \in \Sigma_\infty$ ,  $\overline{\mathfrak{X}_v^{0+}} = i[0, 2\pi(\log q_v)^{-1}] \cup (0, 1] \cup \{(0, 1] + 2\pi i(\log q_v)^{-1}\}$  for  $v \in S_{\text{fin}}$  and  $\overline{\mathfrak{X}_S^{0+}} = \prod_{v \in S} \overline{\mathfrak{X}_v^{0+}}$ . We note  $\overline{\mathfrak{X}_v^{0+}} \cong \mathbb{R}_{\geq 0}$  by the homeomorphism  $s \mapsto (1-s^2)/4$  if  $v \in \Sigma_\infty$ , and  $\overline{\mathfrak{X}_v^{0+}} \cong [-(q_v^{1/2} + q_v^{-1/2}), q_v^{1/2} + q_v^{-1/2}]$  by the homeomorphism  $s \mapsto q_v^{-s/2} + q_v^{s/2}$  if  $v \in S_{\text{fin}}$ .

**Definition.** We define  $\mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}})$  as the space of all functions  $f$  on  $\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}$  such that  $f$  is of the form  $\varphi((\frac{1-s_v^2}{4})_{v \in \Sigma_\infty})$  for some  $\varphi \in \mathcal{S}((\mathbb{R}_{\geq 0})^{\Sigma_\infty})$ . Here  $\mathcal{S}((\mathbb{R}_{\geq 0})^{\Sigma_\infty})$  is the Schwartz space in the usual sense.

We define the Schwartz space on  $\overline{\mathfrak{X}_S^{0+}} = \prod_{v \in S} \overline{\mathfrak{X}_v^{0+}}$ , which is denoted by  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$ , as

$$\mathcal{S}(\overline{\mathfrak{X}_S^{0+}}) = \mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}) \otimes C(\overline{\mathfrak{X}_{S_{\text{fin}}}^{0+}}) \quad (\text{algebraic tensor}).$$

Both measures  $\lambda_S^\eta(\mathbf{n})$  and  $\lambda_S^\eta$  on  $\mathfrak{X}_S^{0+}$  are naturally extended as linear functionals on  $\mathcal{S}(\overline{\mathfrak{X}_S^{0+}})$  (cf. [20, Lemmas 13.14 and 13.16]).

**Proposition 38.**

- (1) Let  $A_{\text{fin}}$  denote the  $\mathbb{C}$ -vector space of all functions on  $\overline{\mathfrak{X}_{S_{\text{fin}}}^{0+}}$  generated by  $\prod_{v \in S} Q_v(q_v^{-s_v/2} + q_v^{s_v/2})$  for any polynomials  $Q_v[X] \in \mathbb{C}[X]$ , ( $v \in S_{\text{fin}}$ ). Then,  $A_{\text{fin}}$  is dense in  $C(\overline{\mathfrak{X}_{S_{\text{fin}}}^{0+}})$  with respect to the topology by supremum norm.
- (2) Let  $A_\infty$  denote the  $\mathbb{C}$ -vector space of all functions on  $\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}$  generated by the functions

$$\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}} \ni \mathbf{s} = (s_v)_{v \in \Sigma_\infty} \mapsto \prod_{v \in \Sigma_\infty} Q_v(s_v^2) \exp((s_v^2 - 1)/4)$$

for any polynomials  $Q_v(X) \in \mathbb{C}[X]$ , ( $v \in \Sigma_\infty$ ). Then,  $A_\infty$  is dense in  $\mathcal{S}(\overline{\mathfrak{X}_{\Sigma_\infty}^{0+}})$  with respect to the Fréchet topology determined by the semi-norms

$$p_{m,\mathbf{n}}(f) = \sup_{\mathbf{s} \in \overline{\mathfrak{X}_{\Sigma_\infty}^{0+}}} |\partial^\mathbf{n} f(\mathbf{s})| (1 + \|\mathbf{s}\|^2)^m$$

for all  $m \in \mathbb{N}_0$  and all  $\mathbf{n} \in (\mathbb{N}_0)^{\Sigma_\infty}$ . Here  $\partial^\mathbf{n}$  denotes the higher order partial derivative  $\prod_{v \in \Sigma_\infty} \partial^{n_v} / \partial s_v^{n_v}$  for any multi-index  $\mathbf{n} = (n_v)_{v \in \Sigma_\infty} \in (\mathbb{N}_0)^{\Sigma_\infty}$ , and we put  $\|\mathbf{s}\| = (\sum_{v \in \Sigma_\infty} |s_v|^2)^{1/2}$ .

**Proof.** To prove (1), we only have to use the Stone–Weierstrass theorem for the compact Hausdorff space  $\mathfrak{X}_{S_{\text{fin}}}^{0+}$ . The assertion (2) follows from [15, Theorem V.13 (p. 143)] and [3, Lemma 9.3].  $\square$

## References

- [1] N. Bourbaki, *Intégration Chaps. I–IV*, 3rd edition, Springer, 2007.
- [2] E. Carletti, G. Monti Bragadin, A. Perelli, On general  $L$ -functions, *Acta Arith.* LXVI (1994) 147–179.
- [3] D.L. DeGeorge, N. Wallach, Limit formulas for multiplicities in  $L^2(\Gamma \backslash G)$ , *Ann. of Math.* (2) 107 (1978) 133–150.
- [4] B. Feigon, D. Whitehouse, Averages of central  $L$ -values of Hilbert modular forms with an application to subconvexity, *Duke Math. J.* 149 (2009) 347–410.
- [5] J. Guo, On the positivity of the central critical values of automorphic  $L$ -functions, *Duke Math. J.* 83 (1) (1996) 157–190.
- [6] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ., vol. 53, Amer. Math. Soc., Providence, RI, 2004.
- [7] M. Jutila, Y. Motohashi, Uniform bound for Hecke  $L$ -functions, *Acta Math.* 195 (2005) 61–115.
- [8] A. Knightly, C. Li, Kuznetsov’s trace formula and the Hecke eigenvalues of Maass forms, *Mem. Amer. Math. Soc.* 224 (1055) (2013).
- [9] X. Li, Upper bounds on  $L$ -functions at the edge of the critical strip, *Int. Math. Res. Not. IMRN* (2010) 727–755.
- [10] P. Michel, A. Venkatesh, The subconvexity problem for  $GL_2$ , *Publ. Math. Inst. Hautes Études Sci.* 111 (2010) 171–271.
- [11] G. Molteni,  $L$ -functions: Siegel-type theorems and structure theorems, Ph.D. thesis, University of Milan, Milan, 1999.
- [12] G. Molteni, Upper and lower bounds at  $s = 1$  for certain Dirichlet series with Euler product, *Duke Math. J.* 111 (1) (2002) 133–158.
- [13] Y. Motohashi, Spectral mean values of Maass waveform  $L$ -functions, *J. Number Theory* 42 (1992) 258–284.
- [14] D. Ramakrishnan, J. Rogawski, Average values of modular  $L$ -series via the relative trace formula, *Pure Appl. Math. Q.* 1 (4) (2005) 701–735.
- [15] M. Reed, B. Simon, *Methods of Mathematical Physics I, Functional Analysis*, Academic Press, N.Y., 1972.
- [16] R. Schmidt, Some remarks on local newforms for  $GL(2)$ , *J. Ramanujan Math. Soc.* 17 (2002) 115–147.
- [17] J.P. Serre, Répartition asymptotique des valeurs propres de l’opérateur de Hecke  $T_p$ , *J. Amer. Math. Soc.* 10 (1) (1997) 75–102.
- [18] S. Sugiyama, Regularized periods of automorphic forms on  $GL(2)$ , *Tohoku Math. J.* 65 (3) (2013) 373–409.
- [19] E.C. Titchmarsh, *The Theory of Riemann Zeta-Function*, revised by H. Brown, second edition, Oxford Science Publications, Clarendon Press, Oxford, 1986.
- [20] M. Tsuzuki, Spectral means of central values of automorphic  $L$ -functions for  $GL(2)$ , *Mem. Amer. Math. Soc.* 235 (1110) (2014).