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PII: S0022-314X(16)30252-9
DOI: http://dx.doi.org/10.1016/j.jnt.2016.09.018
Reference: YJNTH 5584

To appear in: *Journal of Number Theory*

Received date: 4 March 2016
Revised date: 19 September 2016
Accepted date: 20 September 2016

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Yoshinori Mishiba*

Abstract

In this paper, we study multizeta values over function fields in characteristic p. For each $d \geq 2$, we show that when the constant field has cardinality > 2, the field generated by all multizeta values of depth d is of infinite transcendence degree over the field generated by all single zeta values.

1 Introduction

1.1 Classical case

The multiple zeta values (MZVs) in characteristic 0 was defined by Euler (depth two) and Hoffman (higher depth). For a d-tuple of positive integers $\underline{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ with $n_1 \geq 2$, it is defined by

$$\zeta_{\mathbb{Z}}(\underline{n}) = \zeta_{\mathbb{Z}}(n_1, \ldots, n_d) := \sum_{m_1 > \cdots > m_d \geq 1} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{R}^{\times}.$$

The sum $\mathrm{wt}(\underline{n}) := \sum_i n_i$ is called the weight and $\mathrm{dep}(\underline{n}) := d$ is called the depth of $\zeta_{\mathbb{Z}}(\underline{n})$. One of the goals of this topic is to determine all algebraic relations over $\overline{\mathbb{Q}}$ among the MZVs. Although many relations among MZVs are known, very few linear/algebraic independence results on MZVs are known. For example, Euler proved that when $d = 1$, the ratio $\zeta_{\mathbb{Z}}(n)/(2\pi\sqrt{-1})^n$ is a rational number if and only if $n \geq 2$ is an even integer. However, we do not know whether $\zeta_{\mathbb{Z}}(n)/\pi^n$ is a transcendental number for each odd integer $n \geq 3$. It is conjectured that $\pi, \zeta_{\mathbb{Z}}(3), \zeta_{\mathbb{Z}}(5), \zeta_{\mathbb{Z}}(7), \ldots$ are algebraically independent over $\overline{\mathbb{Q}}$. For the higher depth case, Goncharov ([G]) conjectured that MZVs of different weights are linearly independent over $\mathbb{Q}$. Moreover, it is considered that this conjecture is true even if we replace $\mathbb{Q}$ by $\overline{\mathbb{Q}}$. In characteristic 0, it is considered that to prove linear/algebraic independence of MZVs is very difficult. In this paper, we prove algebraic independence of certain multizeta values in the function field case.

1.2 Positive characteristic multizeta values

Let $K := \mathbb{F}_q(\theta)$ be the rational function field over the finite field of q elements with variable θ, p the characteristic of K, $K_\infty := \mathbb{F}_q(\!(\theta^{-1})\!)$ the ∞-adic completion of K, $\overline{K_\infty}$ a fixed

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This work was partially supported by the JSPS Research Fellowships for Young Scientists.

algebraic closure of K_∞, $\mathbb{C}_\infty$ the ∞-adic completion of $\overline{K_\infty}$, and $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. We fix a $(q-1)$-st root of $-\theta$ and let

$$\widetilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} \cdot K_\infty^\times$$

be the fundamental period of the Carlitz module. This is a generator of the kernel of the exponential map of the Carlitz module and a function field analogue of $2\pi\sqrt{-1}$ which is a generator of the kernel of the usual exponential map. As $\#\mathbb{F}_q[\theta]^\times = q-1$, we say that an integer n is "odd" if $q-1$ does not divide n, and "even" if $q-1$ divides n.

In this paper, an *index* means an element of $(\mathbb{Z}_{\geq 1})^d$ for some positive integer $d \geq 1$. Thakur ([T1]) defined the *positive characteristic multizeta values* (also denoted by MZVs) by

$$\zeta(\underline{n}) = \zeta(n_1, \ldots, n_d) := \sum \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in K_\infty^\times$$

for indices $\underline{n} = (n_1, \ldots, n_d)$, where the sum is over all monic polynomials a_i in $\mathbb{F}_q[\theta]$ such that $\deg a_1 > \cdots > \deg a_d \geq 0$. Thakur ([T2]) showed that MZVs are non-zero. Anderson and Thakur ([AT2]) showed that the MZVs have an interpretation as periods of certain t-motives, which are iterated extensions of the trivial t-motive by Carlitz motives. It is clear that $\zeta(p^e\underline{n}) = \zeta(\underline{n})^{p^e}$ for all $e \geq 0$, where we set $p^e\underline{n} := (p^e n_1, \ldots, p^e n_d)$. Carlitz ([Ca]) showed that $\zeta(n)/\widetilde{\pi}^n \in K^\times$ if and only if $n \geq 1$ is an "even" integer. This is an analogue of the relations of the special zeta values at even integers. Yu ([Y]) proved that $\zeta(n)/\widetilde{\pi}^n \notin \overline{K}$ for each "odd" integer $n \geq 1$. Moreover, Chang and Yu ([CY]) proved that the elements $\widetilde{\pi}, \zeta(n_1), \ldots, \zeta(n_d)$ are algebraically independent over $\overline{K}$ if n_i is "odd" for each i and n_i/n_j is not an integral power of p for each $i \neq j$. Chang, Papanikolas and Yu ([CPY]) also showed the algebraic independence of MZVs when the constant field $\mathbb{F}_q$ varies. Several results on the higher depth case were also proved. Chang ([Ch2]) showed that MZVs of different weights are linearly independent over $\overline{K}$. In [M1], we proved that $\widetilde{\pi}$, $\zeta(n)$ and $\zeta(n,n)$ are algebraically independent over $\overline{K}$ if $2n$ is "odd". In this paper, we prove the following theorem:

Theorem 1.1. *Let $d \geq 1$ be a positive integer, and let $n_1, \ldots, n_d \geq 1$ be d distinct positive integers. If n_i is not divisible by $q-1$ for each i and n_i/n_j is not an integral power of p for each $i \neq j$, then the the following $1 + \frac{d(d+1)}{2}$ elements*

$$\begin{gathered}
\widetilde{\pi}, \quad \zeta(n_1), \quad \zeta(n_2), \quad \zeta(n_3), \quad \zeta(n_4), \ \ldots\ldots\ldots, \ \zeta(n_d), \\
\zeta(n_1, n_2), \quad \zeta(n_2, n_3), \quad \zeta(n_3, n_4), \ \ldots\ldots, \ \zeta(n_{d-1}, n_d), \\
\zeta(n_1, n_2, n_3), \zeta(n_2, n_3, n_4), \ldots, \zeta(n_{d-2}, n_{d-1}, n_d), \\
\vdots \\
\zeta(n_1, n_2, \ldots, n_{d-1}), \zeta(n_2, n_3, \ldots, n_d), \\
\zeta(n_1, n_2, \ldots, n_d)
\end{gathered}$$

are algebraically independent over $\overline{K}$.

Theorem 1.1 provides many MZVs which are algebraically independent over $\overline{K}$. The next theorem gives a positive answer to the function field analogue of a question in [Andr, p. 231].

Theorem 1.2. *For each positive integer $d \geq 1$, we set K_d to be the field generated by the MZVs of depth 1 or d over K. When $q \neq 2$, we have*

$$\mathrm{tr.deg}_{K_1} K_d = \infty$$

for each $d \geq 2$.

Proof. Since $q \neq 2$, the set $\mathbb{Z}_{\geq 1} \setminus ((q-1)\mathbb{Z}_{\geq 1} \cup p\mathbb{Z}_{\geq 1})$ is an infinite set. We denote the elements of this set by $n_1, n_2, n_3, \ldots$. Hence we have $K_1 = K(\widetilde{\pi}, \zeta(n_1), \zeta(n_2), \zeta(n_3), \ldots)$. By Theorem 1.1, the elements $\zeta(n_1, \ldots, n_d), \zeta(n_{d+1}, \ldots, n_{2d}), \zeta(n_{2d+1}, \ldots, n_{3d}), \ldots$ are algebraically independent over K_1. $\square$

Remark 1.3. (1) Similarly, we can prove that for any integers $d_1, d_2, d_3, \cdots \geq 2$, there exist indices $\underline{n}_1, \underline{n}_2, \underline{n}_3, \ldots$ such that $\mathrm{dep}(\underline{n}_j) = d_j$ for each j and $\zeta(\underline{n}_1), \zeta(\underline{n}_2), \zeta(\underline{n}_3), \ldots$ are algebraically independent over K_1.

(2) When $q = 2$, Chang ([Ch2]) showed that either $\zeta(1, 2)$ or $\zeta(2, 1)$ is transcendental over K_1. However we do not know whether there exist infinitely many MZVs which are algebraically independent over K_1 when $q = 2$.

(3) In [M2], we obtained a generalization of Theorem 1.1. However its proof is too complicated, and to prove Theorem 1.2, we do not need such generalization.

By Theorem 1.1, we may obtain some lower bounds of the dimension of the vector space over K (or $\overline{K}$) spanned by the MZVs of fixed weight. We do not pursue this problem in this paper and content ourselves with stating the following lower bound of the transcendental degree of the field generated by the MZVs of bounded weights, which is easily obtained from Theorem 1.1:

Corollary 1.4. *Let $w \geq 1$ be a positive integer. If there exist positive integers $d_1, \ldots, d_r \geq 1$ and an "odd" positive integer $n_{ij} \geq 1$ for each $1 \leq i \leq r$ and $1 \leq j \leq d_i$ such that $n_{ij}/n_{i'j'}$ is not an integral power of p for each $(i, j) \neq (i', j')$ and $\sum_j n_{ij} \leq w$ for each i, then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{n}) |\, \mathrm{wt}(\underline{n}) \leq w) \geq 1 + \sum_{i=1}^{r} \frac{d_i(d_i+1)}{2}.$$

1.3 Carlitz multiple polylogarithms

In [Ch2], Chang defined the *Carlitz multiple polylogarithms* (CMPLs) by

$$\mathrm{Li}_{\underline{n}}(z_1, \ldots, z_d) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{z_1^{q^{i_1}} \cdots z_d^{q^{i_d}}}{((\theta - \theta^q) \cdots (\theta - \theta^{q^{i_1}}))^{n_1} \cdots ((\theta - \theta^q) \cdots (\theta - \theta^{q^{i_d}}))^{n_d}}$$

for indices $\underline{n}$. It converges if $|z_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, where $|-|_\infty$ is an ∞-adic valuation on $\mathbb{C}_\infty$. The weight and depth of (values of) CMPLs are also defined to be $\mathrm{wt}(\underline{n})$ and $\mathrm{dep}(\underline{n})$. In [AT1], Anderson and Thakur showed that $\zeta(n)$ is described as a K-linear combination of the values of CMPLs of weight n and depth one at rational points for each $n \geq 1$. Moreover, in [Ch2], Chang showed that for each index $\underline{n}$ with $\mathrm{wt}(\underline{n}) = w$ and $\mathrm{dep}(\underline{n}) = d$, $\zeta(\underline{n})$ is described as a K-linear combination of the values of CMPLs of weight w and depth d at rational points. He also proved that CMPLs take non-zero values when $z_i \neq 0$ for each i. We are interested in the algebraic independence of their values at algebraic points over $\overline{K}$.

Let $n \geq 1$ be a positive integer, and let $\alpha_1, \ldots, \alpha_r \in \overline{K}^\times$ be algebraic points such that $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j. Papanikolas ([P]), Chang and Yu ([CY]) proved that if $\widetilde{\pi}^n, \mathrm{Li}_n(\alpha_1), \ldots, \mathrm{Li}_n(\alpha_r)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$. Let $n_1, \ldots, n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, let $\alpha_{i1}, \ldots, \alpha_{ir_i} \in \overline{K}^\times$ be algebraic points such that $|\alpha_{ij}|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each j. Chang and Yu ([CY]) also proved that if $\widetilde{\pi}^{n_i}, \mathrm{Li}_{n_i}(\alpha_{i1}), \ldots, \mathrm{Li}_{n_i}(\alpha_{ir_i})$ are linearly independent over K for each i, then the elements of $\{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_i}(\alpha_{ij}) | i, j\}$ are algebraically independent over $\overline{K}$. As in the case of the MZVs, several results on the higher depth case were also proved. Chang ([Ch2]) showed that values of Carlitz multiple polylogarithms at algebraic points of different weights are linearly independent over $\overline{K}$. In [M1], we proved that $\widetilde{\pi}$, $\mathrm{Li}_n(\alpha)$ and $\mathrm{Li}_{n,n}(\alpha, \alpha)$ are algebraically independent over $\overline{K}$ if $2n$ is "odd" and $\alpha \in K^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$. In this paper, we prove the following theorem:

Theorem 1.5. *Let $d \geq 1$ be a positive integer, and let $n_1, \ldots, n_d \geq 1$ be d distinct positive integers. For each i, we take a rational point $\alpha_i \in K^\times$ such that $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$. If n_i is not divisible by $q-1$ for each i and n_i/n_j is not an integral power of p for each $i \neq j$, then the cardinality of the set $\{\widetilde{\pi}\} \cup \{\mathrm{Li}_{n_\ell, n_{\ell+1}, \ldots, n_k}(\alpha_\ell, \alpha_{\ell+1}, \ldots, \alpha_k) | 1 \leq \ell \leq k \leq d\}$ is $1 + \frac{d(d+1)}{2}$ and all elements of this set are algebraically independent over $\overline{K}$.*

1.4 Outline of this paper

In Section 2, we define notations which are used in this paper. In Section 3, at first we review the (pre-)t-motives which were originally defined by Anderson ([Ande]). We explain the way how we obtain periods from pre-t-motives following the work of Anderson and Thakur ([AT1], [AT2]). Then we recall Papanikolas' theory ([P]) which states that the transcendental degree of the field generated by periods in question over a base field coincides with the dimension of the "motivic Galois group" of a pre-t-motive. As an example (see Example 3.4), we see that MZVs and CMPLs at algebraic points appear as periods of some pre-t-motives. The primary tools of proving the main results are to apply Papanikolas' theory. In Section 4, we give proofs of Theorems 1.1 and 1.5. These are simultaneously proved as corollaries of Theorem 4.3. This theorem is proved by using the arguments of Section 3. In Appendix A, we give a proof of Theorem 4.2 which states a criterion of the algebraic independence of "depth one elements". This gives a generalization of the main result of [CY].

During the past submission of the present paper to another journal, which takes almost 2 years, a generalization of Theorem 1.1 has been given in [M2]. However, the original ideas are rooted in the present paper when dealing with algebraic independence question of MZV's, and some techniques are applied to tackle the more complicated situation in [M2].

2 Notations

We continue to use the notations of the previous section. Let t be a variable independent of θ. Let $\mathbb{T} := \{f \in \mathbb{C}_\infty[\![t]\!] | f$ converges on $|t|_\infty \leq 1\}$ be the Tate algebra and $\mathbb{L}$ the fractional field of $\mathbb{T}$. We set

$$\mathbb{E} := \left\{ \sum a_i t^i \in \mathbb{C}_\infty[\![t]\!] \,\middle|\, \lim_{i\to\infty} \sqrt[i]{|a_i|_\infty} = 0,\ [K_\infty(a_0, a_1, \dots) : K_\infty] < \infty \right\}.$$

For any integer $n \in \mathbb{Z}$ and any formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty(\!(t)\!)$, let

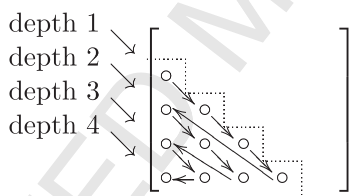
$$f^{(n)} := \sum_i a_i^{q^n} t^i$$

be the n-fold twist of f, and set $\sigma(f) := f^{(-1)}$. The fields $\mathbb{L}$ and $\overline{K}(t)$ are stable under the operation $f \mapsto f^{(n)}$ and we have $\mathbb{L}^{\sigma=1} = \mathbb{F}_q(t)$ where $(-)^{\sigma=1}$ is the σ-fixed part.

Definition 2.1. Let $d \geq 1$ be a positive integer. We set

$$I_d := \{(i, j) \in \mathbb{Z}^2 | 1 \leq j < i \leq d + 1\}.$$

We define a depth of $(i, j) \in I_d$ by $\mathrm{dep}(i, j) := i - j$ and a total order on I_d by setting $(i, j) \leq (k, \ell)$ if either $\mathrm{dep}(i, j) = \mathrm{dep}(k, \ell)$ and $j \leq \ell$ (hence $i \leq k$), or $\mathrm{dep}(i, j) < \mathrm{dep}(k, \ell)$. The order on I_d is illustrated as the following diagram:



For each $(i, j) \in I_d$ and a d-tuple of symbol $\underline{y} = (y_1, \dots, y_d)$, we set

$$\underline{y}_{ij} := (y_j, y_{j+1}, \dots, y_{i-1}).$$

Note that the MZV $\zeta(n_1, \dots, n_d)$ appears as a period of a t-motive of dimension $d+1$ (Example 3.4). Moreover, the MZV $\zeta(n_j, \dots, n_{i-1})$ for $(i, j) \in I_d$ appears as an (i, j)-th component of a matrix of periods of that t-motive.

We set

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{K_\infty}[\![t]\!]$$

which is an element of $\mathbb{E}$. Since Ω has a simple zero at θ^{q^i} for each $i = 1, 2, \dots,$ it is transcendental over $\overline{K}(t)$. It satisfies the equation

$$\Omega^{(-1)} = (t - \theta)\Omega$$

and we have

$$\Omega(\theta) = \frac{1}{\tilde{\pi}}.$$

We set $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$. For each integer $n \geq 0$ with the q-expansion $n = \sum_i n_i q^i$ $(0 \leq n_i < q)$, the Carlitz factorial is defined by

$$\Gamma_{n+1} := \prod_i D_i^{n_i}.$$

Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{u} = (u_1, \ldots, u_d) \in (\overline{K}[t])^d$ a d-tuple of polynomials. For a polynomial $u = \sum_j \alpha_j t^j \in \overline{K}[t]$, we set $\|u\|_\infty := \max_j |\alpha_j|_\infty$. When $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, we set

$$L_{\underline{u},\underline{n}}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{u_1^{(i_1)} \cdots u_d^{(i_d)}}{((t-\theta^q)\cdots(t-\theta)^{q^{i_1}})^{n_1} \cdots ((t-\theta^q)\cdots(t-\theta)^{q^{i_d}})^{n_d}} \in \overline{K_\infty}[\![t]\!],$$

which converges on $|t|_\infty < |\theta|_\infty^q$ and satisfies the equation

$$L_{\underline{u},\underline{n}}^{(-1)} = \frac{u_d^{(-1)}}{(t-\theta)^{n_1+\cdots+n_{d-1}}} L_{\underline{u}_{d1},\underline{n}_{d1}} + \frac{L_{\underline{u},\underline{n}}}{(t-\theta)^{n_1+\cdots+n_d}},$$

where we set $L_{\underline{u}_{11},\underline{n}_{11}} = L_{\emptyset,\emptyset} := 1$. When $\underline{u} = \underline{\alpha} \in \overline{K}^d$ with $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i, we have $L_{\underline{\alpha},\underline{n}}(\theta) = \mathrm{Li}_{\underline{n}}(\underline{\alpha})$. Anderson and Thakur ([AT1], [AT2]) showed that there exists a polynomial $H_{n-1} \in \mathbb{F}_q[\theta, t]$ for each $n \geq 1$ such that $\|H_{n-1}\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ and $L_{H(\underline{n}),\underline{n}}(\theta) = \Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n})$ where $H(\underline{n}) := (H_{n_1-1}, \ldots, H_{n_d-1})$.

3 Review of pre-t-motives

In this section, we review the notions of pre-t-motives and Papanikolas' theory for pre-t-motives. For more details, see [P]. A *pre-t-motive* is an étale φ-module over $(\overline{K}(t), \sigma)$; this means a finite-dimensional $\overline{K}(t)$-vector space M equipped with a σ-semilinear bijective map $\varphi\colon M \to M$. A morphism of pre-t-motives is a $\overline{K}(t)$-linear map which is compatible with the φ's. A tensor product of two pre-t-motives are defined naturally. For any pre-t-motive M, the *Betti realization* of M is defined by

$$M^B := \left(\mathbb{L} \otimes_{\overline{K}(t)} M\right)^{\sigma\otimes\varphi=1},$$

where $(-)^{\sigma\otimes\varphi=1}$ is the $\sigma \otimes \varphi$-fixed part. A pre-t-motive M is called *rigid analytically trivial* if the natural injection $\mathbb{L} \otimes_{\mathbb{F}_q(t)} M^B \hookrightarrow \mathbb{L} \otimes_{\overline{K}(t)} M$ is an isomorphism. The category of rigid analytically trivial pre-t-motives forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor $M \mapsto M^B$. For any such M, we denote by G_M the fundamental group of the Tannakian subcategory generated by M with respect to the Betti realization. By definition, G_M is an $\mathbb{F}_q(t)$-subgroup scheme of $\mathrm{GL}(M^B)$. Thus for each $\mathbb{F}_q(t)$-algebra R, $G_M(R)$ is a subgroup of $GL(R \otimes_{\mathbb{F}_q(t)} M^B)$.

Let $\Phi \in \mathrm{GL}_r(\overline{K}(t))$ be a matrix. We consider the system of Frobenius difference equations

$$\Psi^{(-1)} = \Phi\Psi \tag{1}$$

with solution entries of $\Psi = (\Psi_{ij})$ in $\mathbb{L}$. The matrix Φ defines the pre-t-motive $M_\Phi := \overline{K}(t)^r$ with

$$\varphi(x_1, \ldots, x_r) = (x_1^{(-1)}, \ldots, x_r^{(-1)})\Phi.$$

The pre-t-motive M_Φ is rigid analytically trivial if and only if the system of Frobenius difference equations (1) has a solution matrix Ψ in $\mathrm{GL}_r(\mathbb{L})$, and in this case $\Psi^{-1}\mathbf{m}$ forms an $\mathbb{F}_q(t)$-basis of $(M_\Phi)^B$, where $\mathbf{m} \in \mathrm{Mat}_{r\times 1}(M_\Phi)$ is the standard basis of $M_\Phi = \overline{K}(t)^r$ on which the action of φ is represented by Φ. Such matrix Ψ is called a *rigid analytic trivialization* of Φ, and the values $\Psi_{ij}(\theta)$ of its components at $t = \theta$ (if they converge) are called *periods* of M_Φ. For such Ψ, we set $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$, where Ψ_1 (resp. Ψ_2) is the matrix in $\mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$ such that $(\Psi_1)_{ij} = \Psi_{ij} \otimes 1$ (resp. $(\Psi_2)_{ij} = 1 \otimes \Psi_{ij}$). Let $X = (X_{ij})$ be the $r \times r$ matrix of independent variables X_{ij} and $\mathbb{F}_q(t)[X, 1/\det X]$ the localization of the polynomial ring over $\mathbb{F}_q(t)$ with r^2 variables $X_{11}, X_{12}, \ldots, X_{rr}$ with respect to the polynomial $\det X$. Thus it is the coordinate ring of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$. We define an $\mathbb{F}_q(t)$-algebra homomorphism ν_Ψ by

$$\nu_\Psi\colon \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L};\ X_{ij} \mapsto \widetilde{\Psi}_{ij}$$

and set

$$G_\Psi := \mathrm{Spec}(\mathbb{F}_q(t)[X, 1/\det X]/\ker \nu_\Psi) \subset \mathrm{GL}_{r/\mathbb{F}_q(t)}\,.$$

Let R be an $\mathbb{F}_q(t)$-algebra. Since $\Psi^{-1}\mathbf{m}$ forms an R-basis of $R \otimes_{\mathbb{F}_q(t)} (M_\Phi)^B$, each element of this space is written as $\mathbf{f} \cdot \Psi^{-1}\mathbf{m}$ for $\mathbf{f} \in R^r$. Then we have a well-defined map given by

$$G_\Psi(R) \to G_{M_\Phi}(R);\ g \mapsto (\mathbf{f} \cdot \Psi^{-1}\mathbf{m} \mapsto \mathbf{f}g^{-1} \cdot \Psi^{-1}\mathbf{m}). \tag{2}$$

Theorem 3.1 ([P, Theorems 4.3.1, 4.5.10, 5.2.2])**.** *Let Φ and Ψ be matrices satisfying (1), and let G_{M_Φ} and G_Ψ be as above.*

(1) *The scheme G_Ψ is a smooth subgroup scheme of $\mathrm{GL}_{r/\mathbb{F}_q(t)}$ and the above map $G_\Psi \to G_{M_\Phi}$ is an isomorphism of group schemes over $\mathbb{F}_q(t)$.*

(2) *Let $\overline{K}(t)(\Psi)$ be the field generated by the entries of Ψ over $\overline{K}(t)$. Then we have*

$$\dim G_\Psi = \mathrm{tr.deg}_{\overline{K}(t)}\, \overline{K}(t)(\Psi).$$

(3) *Assume that $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$, $\Psi \in \mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$, and $\det \Phi = c(t-\theta)^d$ for some $c \in \overline{K}^\times$ and $d \geq 0$. Let $\overline{K}(\Psi(\theta))$ be the field generated by the entries of $\Psi(\theta)$ over $\overline{K}$. Then we have*

$$\mathrm{tr.deg}_{\overline{K}(t)}\, \overline{K}(t)(\Psi) = \mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\Psi(\theta)).$$

Remark 3.2. The result (3) in Theorem 3.1 is rooted in the deep result in [ABP], which is addressed as ABP-criterion. However, the restriction of the condition on $\det \Phi$ originated from Anderson t-motives but such restriction indeed can be relaxed (see [Ch1]). But for our purpose, the above is sufficient and so we do not state the refined version given in [Ch1].

Example 3.3. The Carlitz pre-t-motive C is the pre-t-motive defined by the 1×1-matrix $\left[t-\theta\right]$. Since $\Omega^{(-1)} = (t-\theta)\Omega$, the Carlitz pre-$t$-motive is rigid analytically trivial. Since Ω is transcendental over $\overline{K}(t)$, we have $\dim G_{[\Omega]} = 1$, and thus $G_C = G_{[\Omega]} = \mathbb{G}_m$.

Example 3.4. Let $\underline{n} = (n_1, \ldots, n_d)$ be an index and $\underline{u} = (u_1, \ldots, u_d) \in (\overline{K}[t])^d$ be a d-tuple of polynomials such that $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. We consider $(d+1) \times (d+1)$-matrices

$$\Phi := \begin{bmatrix} (t-\theta)^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ u_1^{(-1)}(t-\theta)^{n_1+\cdots+n_d} & (t-\theta)^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ 0 & u_2^{(-1)}(t-\theta)^{n_2+\cdots+n_d} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{n_d} & 0 \\ 0 & \cdots & 0 & u_d^{(-1)}(t-\theta)^{n_d} & 1 \end{bmatrix}$$

and

$$\Psi := \begin{bmatrix} \Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{21},\underline{n}_{21}} & \Omega^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{31},\underline{n}_{31}} & \Omega^{n_2+\cdots+n_d} L_{\underline{u}_{32},\underline{n}_{32}} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \Omega^{n_d} & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{u}_{d+1,1},\underline{n}_{d+1,1}} & \Omega^{n_2+\cdots+n_d} L_{\underline{u}_{d+1,2},\underline{n}_{d+1,2}} & \cdots & \Omega^{n_d} L_{\underline{u}_{d+1,d},\underline{n}_{d+1,d}} & 1 \end{bmatrix},$$

where the notations $\underline{n}_{ij}$ and $\underline{u}_{ij}$ are defined in Definition 2.1. These satisfy the Frobenius difference equations (1). Hence Ψ is a rigid analytic trivialization of Φ. Let M be the pre-t-motive defined by Φ. By Theorem 3.1, we have an isomorphism $G_\Psi \to G_M$ and

$$\operatorname{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, L_{\underline{u}_{ij},\underline{n}_{ij}}(\theta) | (i,j) \in I_d) = \dim G_\Psi.$$

Thus when $\underline{u} = H(\underline{n})$ (resp. $\underline{u} = \underline{\alpha} \in \overline{K}^d$ with $|\alpha_i|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i), the multizeta values $\zeta(\underline{n}_{ij})$ (resp. the Carlitz multiple polylogarithms $\mathrm{Li}_{\underline{n}_{ij}}(\underline{\alpha}_{ij})$) appear as periods of the pre-t-motive M. By the definition of G_Ψ, we also have the inclusion

$$G_\Psi \subset \left\{ \begin{bmatrix} a^{n_1+\cdots+n_d} & & & \\ x_{21} & a^{n_2+\cdots+n_d} & & \\ \vdots & \ddots & \ddots & \\ x_{d+1,1} & \cdots & x_{d+1,d} & 1 \end{bmatrix} \right\}.$$

We can calculate $\widetilde{\Psi}$ explicitly as

$$\widetilde{\Psi}_{ij} = (\Omega^{-1} \otimes \Omega)^{n_i+\cdots+n_d} \sum_{s=j}^{i} \sum_{r=0}^{i-s} (-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=i}} L_{i_1 i_0} \cdots L_{i_r i_{r-1}} \otimes \Omega^{n_j+\cdots+n_{i-1}} L_{sj}$$

for each $(i,j) \in I_d$, where we write $L_{k\ell} := L_{\underline{u}_{k\ell},\underline{n}_{k\ell}}$.

4 Algebraic independence

In this section, we prove Theorems 1.1 and 1.5. For square matrices A and B, we denote by $A \oplus B$ the diagonal block matrix made of A and B. We use the letters a and x_{ij}'s as coordinate variables of algebraic groups. In our proofs, our purpose is to show that the dimension of the algebraic group in question is as maximal as possible, and so we always work on the $\overline{\mathbb{F}_q(t)}$-valued points without studying the reduced/non-reduced structures, where $\overline{\mathbb{F}_q(t)}$ is a fixed algebraic closure of $\mathbb{F}_q(t)$. So for an algebraic group G over $\mathbb{F}_q(t)$, when it is clear from the contents, without confusion we still denote by G the $\overline{\mathbb{F}_q(t)}$-valued points of G.

Before starting the proof of Theorems 1.1 and 1.5, we state several algebraic independence results concerning the case of depth one. Papanikolas, Chang and Yu proved the following theorem which states a criterion of the algebraic independence of MZVs and CMPLs at algebraic points of depth one. Note that they discussed only the case where $u_j = \alpha_j \in \overline{K}$ with $|\alpha_j|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, but their arguments work also for any $u_j \in \overline{K}[t]$ with $\|u_j\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$.

Theorem 4.1 ([P, Theorem 6.3.2], [CY, Theorem 3.1])**.** *Let $n \geq 1$ be a positive integer and $u_1, \dots, u_r \in \overline{K}[t]$ polynomials with $\|u_j\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for each j. If $\widetilde{\pi}^n, L_{u_1,n}(\theta), \dots, L_{u_r,n}(\theta)$ are linearly independent over K, then they are algebraically independent over $\overline{K}$.*

Thus $\widetilde{\pi}$ and $\zeta(n)$ (or $\mathrm{Li}_n(\alpha)$) are algebraically independent over $\overline{K}$ for each "odd" integer $n \geq 1$ and $\alpha \in K^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$, because $\widetilde{\pi}^n \notin K_\infty$ and $\zeta(n), \mathrm{Li}_n(\alpha) \in K_\infty^\times$ for such n and α.

Theorem 4.2. *Let $n_1, \dots, n_d \geq 1$ be positive integers such that n_i/n_j is not an integral power of p for each $i \neq j$. For each i, we take polynomials $u_{i1}, \dots, u_{ir_i} \in \overline{K}[t]$ with $\|u_{ij}\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for $j = 1, \dots, r_i$. If $\widetilde{\pi}^{n_i}, L_{u_{i1},n_i}(\theta), \dots, L_{u_{ir_i},n_i}(\theta)$ are linearly independent over K for each i, then the $1 + \sum_{i=1}^d r_i$ elements $\{\widetilde{\pi}, L_{u_{ij},n_i}(\theta) | 1 \leq i \leq d,\ 1 \leq j \leq r_i\}$ are algebraically independent over $\overline{K}$.*

This is almost proved in [CY]. For the sake of completeness, we give a proof of it in Appendix A.

The next theorem is the main result in this paper. Clearly, Theorems 1.1 and 1.5 follow from Theorems 4.1, 4.2 and 4.3. Recall that I_d is the set defined in Definition 2.1. The notations $\underline{n}_{ij}$ and $\underline{u}_{ij}$ are also defined there.

Theorem 4.3. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index and $\underline{u} = (u_1, \dots, u_d) \in (\overline{K}[t])^d$ a d-tuple of polynomials such that $\|u_i\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$ for each i. If $\widetilde{\pi}, L_{u_1,n_1}(\theta), \dots, L_{u_d,n_d}(\theta)$ are algebraically independent over $\overline{K}$, then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, L_{\underline{u}_{ij},\underline{n}_{ij}}(\theta) | (i,j) \in I_d) = 1 + \#I_d = 1 + \frac{d(d+1)}{2}.$$

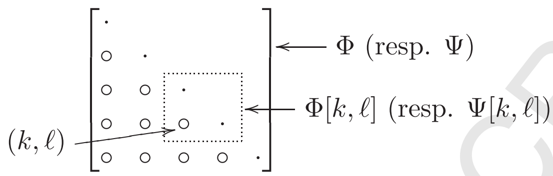
To prove Theorems 4.2 and 4.3, we use the following lemma. This lemma is clear, but very useful.

Lemma 4.4. *Let $V \subset {\mathbb{G}_a}^r$ be an algebraic subgroup of dimension zero. Let $m_1, \dots, m_r \in \mathbb{Z}$ be non-zero integers. Assume that V is stable under the $\mathbb{G}_m$-action on ${\mathbb{G}_a}^r$ defined by*

$$a.(x_1, \dots, x_r) = (a^{m_1} x_1, \dots, a^{m_r} x_r) \quad (a \in \mathbb{G}_m, (x_i) \in {\mathbb{G}_a}^r).$$

Then $V(\overline{\mathbb{F}_q(t)})$ is trivial.

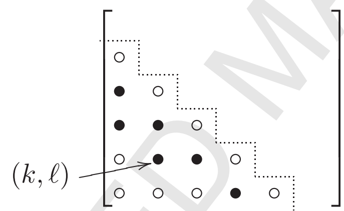
Proof of Theorem 4.3. In this proof, (i, j) and (k, ℓ) are always assumed to be elements of the totally ordered set I_d. Let Φ and Ψ be the $(d+1)\times(d+1)$-matrices defined in Example 3.4. These satisfy the Frobenius difference equations (1). For $(k, \ell) \in I_d$, we define $(\mathrm{dep}(k, \ell)+1)\times(\mathrm{dep}(k, \ell)+1)$-matrices $\Phi[k, \ell] = (\Phi[k, \ell]_{ij})$ and $\Psi[k, \ell] = (\Psi[k, \ell]_{ij})$ which are sub-matrices of Φ and Ψ, where $\Phi[k, \ell]_{ij} = \Phi_{i+\ell-1,j+\ell-1}$ and $\Psi[k, \ell]_{ij} = \Psi_{i+\ell-1,j+\ell-1}$. In particular, the lower left corner of $\Phi[k, \ell]$ (resp. $\Psi[k, \ell]$) is the (k, ℓ)-th entry of Φ (resp. Ψ). The following is the illustration of the relative positions of the matrices:



Let $M[k, \ell]$ be the pre-t-motive defined by $\Phi[k, \ell]$ and $G(k, \ell)$ the fundamental group of the pre-t-motive

$$M(k, \ell) := C \oplus \bigoplus_{\substack{\mathrm{dep}(i,j)\geq \mathrm{dep}(k,\ell)-1 \\ (i,j)\leq(k,\ell)}} M[i, j],$$

where C is the Carlitz pre-t-motive (see Example 3.3). The closed circles in the following illustration is the range in which (i, j) runs over in the above direct sum:



We identify $G(k, \ell)$ with the algebraic group defined by $\left[\Omega\right] \oplus \bigoplus_{(i,j)} \Psi[i, j]$ as in Theorem 3.1. Then we have the inclusion

$$G(k, \ell) \subset \left\{ \left[a\right] \oplus \bigoplus_{\substack{\mathrm{dep}(i,j)\geq \mathrm{dep}(k,\ell)-1 \\ (i,j)\leq(k,\ell)}} \begin{bmatrix} a^{n_j+\cdots+n_d} & & & \\ x_{j+1,j} & a^{n_{j+1}+\cdots+n_d} & & \\ \vdots & \ddots & \ddots & \\ x_{ij} & \cdots & x_{i,i-1} & a^{n_i+\cdots+n_d} \end{bmatrix} \right\}$$

for each (k, ℓ). Note that some different entries/coordinates of different block matrices may be the same and denoted by same letters; this means that for $(i, j), (i', j')$ and r, r', s, s' with $1 \leq s < r \leq \mathrm{dep}(i, j) + 1$ and $1 \leq s' < r' \leq \mathrm{dep}(i', j') + 1$, if $(r + j - 1, s + j - 1) = (r' + j' - 1, s' + j' - 1)$, then the (r, s)-th entry of the (i, j)-th component matrix and the (r', s')-th entry of the (i', j')-th component matrix are the same and they are denoted by $x_{r+j-1,s+j-1}$. In fact, since Ψ is a lower triangular matrix, the (r, s)-th entry of $\widetilde{\Psi[i, j]}$ is equal to the $(r + j - 1, s + j - 1)$-th entry of $\widetilde{\Psi}$ (for the explicit description of $\widetilde{\Psi}$, see Example 3.4). Thus if $(r + j - 1, s + j - 1) = (r' + j' - 1, s' + j' - 1)$, then the (r, s)-th entry of $\widetilde{\Psi[i, j]}$ and the (r', s')-th entry of $\widetilde{\Psi[i', j']}$ coincide. Therefore the corresponding positions in the algebraic group $G(k, \ell)$ are the same.

By Theorem 3.1, it suffices to show that the above inclusion is actually an equality for each (k,ℓ). We prove this by induction on $(k,\ell) \in I_d$ via the total order "$\leq$".

By the assumption, this is true for $(2,1) \leq (k,\ell) \leq (d+1,d)$, these are the depth one cases. Let $(k,\ell) \geq (3,1)$ (this means $\mathrm{dep}(k,\ell) \geq 2$) and assume that the inclusion is an equality for (k',ℓ') the greatest element of $\{(i,j) \in I_d | (i,j) < (k,\ell)\}$, which means that $(k',\ell') = (k-1,\ell-1)$ if $\ell \neq 1$ and $(k',\ell') = (d+1,d+3-k)$ if $\ell = 1$. By definition, $M(k',\ell')$ is a subobject of $M(k,\ell)$ and C is a subobject of $M(k,\ell)$ and $M(k',\ell')$. By Tannakian duality, we have surjections $\psi\colon G(k,\ell) \to G(k',\ell')$, $\pi\colon G(k,\ell) \to \mathbb{G}_m$ and $\pi'\colon G(k',\ell') \to \mathbb{G}_m$, where we identify G_C with $\mathbb{G}_m$. These are projection maps. More precisely, π and π' map the matrices of the above forms to a and ψ maps to the same matrices with the (k,ℓ)-th component matrices removed. These follow from the description of the map (2). The arguments are as same as in [P, §6.2.2], [CY, §4.3] and [CPY, Remark 2.3.2]. We set $V := \mathrm{Ker}\,\pi$ and $V' := \mathrm{Ker}\,\pi'$ to be the unipotent radicals of $G(k,\ell)$ and $G(k',\ell')$. Then we have the following diagram

$$\begin{tikzcd}
1 \arrow[r] & V \arrow[r] \arrow[d, "\psi|_V"] & G(k,\ell) \arrow[r, "\pi"] \arrow[d, "\psi"] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & V' \arrow[r] & G(k',\ell') \arrow[r, "\pi'"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact.

It is clear that $\psi|_V$ is surjective. Since V is non-commutative, the $G(k,\ell)$-action $A.X := A^{-1}XA$ on V $(X \in V,\ A \in G(k,\ell))$ depends not only on $\pi(A)$ but also on the other entries of A. Note that the coordinate variable $x_{k\ell}$ of $G(k,\ell)$ is the only coordinate variable which does not appear as a coordinate variable of $G(k',\ell')$. Thus we know that $\dim G(k',\ell') \leq \dim G(k,\ell) \leq \dim G(k',\ell') + 1$. This also follows from Theorem 3.1 (2). It suffices to show that the second inequality is an equality.

Now, assume that $\dim G(k,\ell) = \dim G(k',\ell')$. Then $\dim \mathrm{Ker}(\psi|_V) = 0$. It is clear that $\mathrm{Ker}(\psi|_V)$ is a normal subgroup of $G(k,\ell)$ and $A.x_{k\ell} = \pi(A)^{n_\ell + \cdots + n_{k-1}} x_{k\ell}$ for each $x_{k\ell} \in \mathrm{Ker}(\psi|_V)$ and $A \in G(k,\ell)$, where we identify $\mathrm{Ker}(\psi|_V) \subset \mathbb{G}_a$ by means of the coordinate $x_{k\ell}$. By Lemma 4.4 we have that $\mathrm{Ker}(\psi|_V)$ is trivial. We take any elements

$$X = [1] \oplus \bigoplus_{\substack{\mathrm{dep}(i,j) \geq \mathrm{dep}(k,\ell)-1 \\ (i,j) \leq (k,\ell)}} \begin{bmatrix} 1 & & & \\ x_{j+1,j} & 1 & & \\ \vdots & \ddots & \ddots & \\ x_{ij} & \cdots & x_{i,i-1} & 1 \end{bmatrix} \in V$$

and

$$\begin{aligned} A = [1] \quad &\oplus \bigoplus_{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell)-1} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 0 & \ddots & \\ a_{ij} & & & 1 \end{bmatrix} \\ &\oplus \bigoplus_{\substack{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell) \\ (i,j) \leq (k,\ell)}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ a_{i-1,j} & 0 & & \ddots & \\ a_{ij} & a_{i,j+1} & & & 1 \end{bmatrix} \in V, \end{aligned}$$

where we can take any $x_{ij} \in \overline{\mathbb{F}_q(t)}$ (resp. any $a_{ij} \in \overline{\mathbb{F}_q(t)}$) for each $(i,j) \in I_d$ such that $(i,j) \neq (k,\ell)$ (resp. $\mathrm{dep}(i,j) \geq \mathrm{dep}(k,\ell) - 1$ and $(i,j) < (k,\ell)$) by the assumption on (k',ℓ') and the surjectivity of $\psi|_V$. Then $X^{-1}(A^{-1}XA)$ is equal to

$$
\begin{aligned}
[1] \quad &\oplus \bigoplus_{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell)-1} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \\
&\oplus \bigoplus_{\substack{\mathrm{dep}(i,j)=\mathrm{dep}(k,\ell) \\ (i,j)\leq(k,\ell)}} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ a_{i-1,j}x_{i,i-1} - a_{i,j+1}x_{j+1,j} & & & & 1 \end{bmatrix}.
\end{aligned}
$$

Now we take $a_{ij} = 0$ for $(k-\ell, 1) \leq (i,j) < (k, \ell+1)$ and $a_{k,\ell+1} = 1$. Then we see that $X^{-1}(A^{-1}XA) \in \mathrm{Ker}(\psi|_V) = \{0\}$ and so we have $x_{\ell+1,\ell} = 0$. Since $(\ell+1,\ell) \neq (k,\ell)$, this is a contradiction. Therefore we have $\dim G(k,\ell) = \dim G(k',\ell') + 1$. $\square$

Appendix A

In this appendix, we prove Theorem 4.2. In [CY], they treated some special case, but their proof works also for any n_i not divisible by p and any $u_{ij} \in \overline{K}[t]$ with $\|u_{ij}\|_\infty < |\theta|_\infty^{\frac{n_i q}{q-1}}$. By a slight modification of their proof, we can weaken the condition on n_i's as in our statement. Note that when $u_{ij} \in \overline{K}$ or $u_{ij} = H_{n_i - 1}$, we can reduce Theorem 4.2 to the case where n_i is not divisible by p, and we do not need the following proof in such cases. As in Section 4, for an algebraic group G over $\mathbb{F}_q(t)$, when it is clear from the contents, without confusion we still denote by G the $\overline{\mathbb{F}_q(t)}$-valued points of G.

Proof of Theorem 4.2. We set $I := \{(i,j) \in \mathbb{Z}^2 | 1 \leq i \leq d,\ 1 \leq j \leq r_i\}$. In this proof, (i,j) and (k,ℓ) are always assumed to be elements of I. We define an order on I by the lexicographic order; this means $(i,j) \leq (k,\ell)$ if and only if $i = k$ and $j \leq \ell$, or $i < k$. For $(k,\ell) \in I$, we define 2×2-matrices

$$
\Phi[k,\ell] := \begin{bmatrix} (t-\theta)^{n_k} & 0 \\ u_{k\ell}^{(-1)}(t-\theta)^{n_k} & 1 \end{bmatrix} \quad \text{and} \quad \Psi[k,\ell] := \begin{bmatrix} \Omega^{n_k} & 0 \\ \Omega^{n_k} L_{u_{k\ell}, n_k} & 1 \end{bmatrix}.
$$

Then they satisfy the Frobenius difference equations $\Psi[k,\ell]^{(-1)} = \Phi[k,\ell]\Psi[k,\ell]$. Let $M[k,\ell]$ be the pre-t-motive defined by $\Phi[k,\ell]$ and $G(k,\ell)$ (resp. $G_k(\ell)$) the fundamental group of the pre-t-motive

$$
M(k,\ell) := C \oplus \bigoplus_{(i,j)\leq(k,\ell)} M[i,j] \quad \left(\text{resp. } M_k(\ell) := C \oplus \bigoplus_{j\leq\ell} M[k,j] \right).
$$

We identify $G(k,\ell)$ (resp. $G_k(\ell)$) with the algebraic group defined by $\left[\Omega\right] \oplus \bigoplus_{(i,j)} \Psi[i,j]$ (resp. $\left[\Omega\right] \oplus \bigoplus_j \Psi[k,j]$) as in Theorem 3.1. Then we have the inclusion (resp. equality)

$$
G(k,\ell) \subset \left\{ [a] \oplus \bigoplus_{(i,j)\leq(k,\ell)} \begin{bmatrix} a^{n_i} & 0 \\ x_{ij} & 1 \end{bmatrix} \right\} \quad \left(\text{resp. } G_k(\ell) = \left\{ [a] \oplus \bigoplus_{j\leq\ell} \begin{bmatrix} a^{n_k} & 0 \\ x_{kj} & 1 \end{bmatrix} \right\} \right)
$$

for each (k, ℓ). By Theorem 3.1, it suffices to show that the above inclusion is actually an equality for each (k, ℓ). We prove this by induction on $(k, \ell) \in I$ via the total order "$\leq$".

By the assumption, this is true for $(1, 1) \leq (k, \ell) \leq (1, r_1)$. Let $(k, \ell) \geq (2, 1)$ and assume that the inclusion is an equality for (k', ℓ') the greatest element of $\{(i, j) \in I | (i, j) < (k, \ell)\}$. Thus $(k', \ell') = (k, \ell - 1)$ if $\ell \neq 1$ and $(k', \ell') = (k - 1, r_{k-1})$ if $\ell = 1$. By definition, $M(k', \ell')$ and $M_k(\ell)$ are subobjects of $M(k, \ell)$ and C is a subobject of $M(k, \ell)$, $M(k', \ell')$ and $M_k(\ell)$. By the Tannakian duality, we have surjections $\psi\colon G(k, \ell) \to G(k', \ell')$, $\psi_k\colon G(k, \ell) \to G_k(\ell)$, $\pi\colon G(k, \ell) \to \mathbb{G}_m$, $\pi'\colon G(k', \ell') \to \mathbb{G}_m$ and $\pi''\colon G_k(\ell) \to \mathbb{G}_m$, where we identify G_C with $\mathbb{G}_m$. The projections π, π' and π'' map the matrices of the above forms to a and ψ (resp. ψ_k) maps to the same matrices with the (k, ℓ)-th component matrices (resp. all (i, j)-th component matrices $(i \neq k)$) removed. We set $V := \operatorname{Ker} \pi$, $V' := \operatorname{Ker} \pi'$ and $V'' := \operatorname{Ker} \pi''$ to be the unipotent radicals of $G(k, \ell)$, $G(k', \ell')$ and $G_k(\ell)$. Then we have the following diagram

$$\begin{tikzcd}
1 \arrow[r] & V'' \arrow[r] & G_k(\ell) \arrow[r, "\pi''"] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & V \arrow[r] \arrow[u, "\psi_k|_V"] \arrow[d, "\psi|_V"] & G(k, \ell) \arrow[r, "\pi"] \arrow[u, two heads, "\psi_k"] \arrow[d, two heads, "\psi"] & \mathbb{G}_m \arrow[r] \arrow[u, equal] \arrow[d, equal] & 1 \\
1 \arrow[r] & V' \arrow[r] & G(k', \ell') \arrow[r, "\pi'"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

which is commutative and whose rows are exact.

It is clear that $\psi|_V$ is surjective. Note that the coordinate variable $x_{k\ell}$ of $G(k, \ell)$ is the only coordinate variable which does not appear as a coordinate variable of $G(k', \ell')$. Thus we know that $\dim G(k', \ell') \leq \dim G(k, \ell) \leq \dim G(k', \ell') + 1$. This also follows from Theorem 3.1 (2). It suffices to show that the second inequality is an equality.

Now, assume that $\dim G(k, \ell) = \dim G(k', \ell')$. Then $\dim \operatorname{Ker}(\psi|_V) = 0$. We identify $V \subset \prod_{(i,j) \leq (k,\ell)} \mathbb{G}_a$, $V' = \prod_{(i,j)<(k,\ell)} \mathbb{G}_a$ and $V'' = \prod_{j \leq \ell} \mathbb{G}_a$ by means of the coordinates x_{ij}. The $\mathbb{G}_m$-action on V (resp. V', resp. V'') defined by $a.X := \widetilde{a}^{-1} X \widetilde{a}$, where $\widetilde{a} \in G(k, \ell)$ (resp. $G(k', \ell')$, resp. $G_k(\ell)$) is a lift of $a \in \mathbb{G}_m$, is described as $x_{ij} \mapsto a^{n_i} x_{ij}$ on each coordinate. By Lemma 4.4 we have $\operatorname{Ker}(\psi|_V) = 1$. Thus the morphism $\psi|_V$ is bijective (but not necessary an isomorphism of varieties) and we have the surjective map

$$\psi_k|_V \circ \psi|_V^{-1}\colon V' \xleftarrow{\sim} V \twoheadrightarrow V''.$$

For each $(i, j) \neq (k, \ell)$, we set V_{ij} (resp. V'_{ij}) to be the subvariety of V (resp. V') defined by $x_{i'j'} = 0$ for each $(i', j') \neq (i, j), (k, \ell)$. Then $\psi|_{V_{ij}}\colon V_{ij} \to V'_{ij} = \mathbb{G}_a$ is a bijective $\mathbb{G}_m$-homomorphism. Thus we have $\dim V_{ij} = 1$. Hence the algebraic set[1] V_{ij} is defined by a separable polynomial of the form $x_{k\ell}^{p^e} - \sum_{n=0}^{m} b_n x_{ij}^{p^n}$ for some $e, m \geq 0$ and $b_n \in \overline{\mathbb{F}_q(t)}$ (See [Co, Corollary 1.8]). Now we take $i \neq k$ and assume that the $\mathbb{G}_m$-homomorphism $\psi_k|_{V_{ij}} \circ \psi|_{V_{ij}}^{-1}$ is non-zero. Then we can take $b_m \neq 0$ and we have $(\sum_n b_n (a^{n_i} x_{ij})^{p^n})^{p^{-e}} = a^{n_k} (\sum_n b_n x_{ij}^{p^n})^{p^{-e}}$ for each $a \in \mathbb{G}_m$. By comparing the coefficients of $x_{ij}^{p^{m-e}}$, we have $n_i p^{m-e} = n_k$, which is a contradiction. Thus we conclude that $\psi_k|_{V_{ij}} \circ \psi|_{V_{ij}}^{-1} = 0$. Therefore we have $\psi_k|_V (\psi|_V^{-1}(\mathbb{G}_a{}^{\ell-1})) = V''$, whence a contradiction since $\dim V'' = \ell$. $\square$

[1] More precisely, the smooth algebraic group $(V_{ij} \otimes \overline{\mathbb{F}_q(t)})_{\mathrm{red}}$ is defined by such polynomial.

Acknowledgments. The author would like to thank Yuichiro Taguchi for reading this paper carefully and giving him many useful comments. He also would like to thank Chieh-Yu Chang for his careful reading and helpful discussions about Theorem 1.2. Most part of this paper was written when the author was a graduate student at Kyushu University. This work was partially supported by the JSPS Research Fellowships for Young Scientists.

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