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General Section

On flagged framed deformation problems of local crystalline Galois representations ☆

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ABSTRACT

We prove that irreducible residual crystalline representations of the absolute Galois group of an unramified extension of $\mathbb{Q}_p$ have smooth representable crystalline framed deformation problems, provided that the Hodge–Tate weights lie in the Fontaine–Laffaille range. We then extend this result to the flagged lifting problem associated to any Fontaine–Laffaille upper triangular representation whose flag is of maximal length. We calculate the relative dimension of these various crystalline lifting functors in terms of the underlying Hodge–Tate weight structures, and also apply these results to give an alternative proof of the fact that every such residual representation admits a so-called "universally twistable lift". Finally we give some brief indications as to the various directions in which these results might be generalised.

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1. Introduction

1.1. Statement of results

Let K be a finite extension of $\mathbb{Q}_p$. Examples of crystalline representations of the Galois group G_K in characteristic zero are those arising from the étale cohomology of proper varieties over K with a smooth model over the ring of integers $\mathcal{O}_K$ due to the crystalline comparison theorem in cohomology ([17]; see also [10,8]), and can be classified in general via the theory of filtered ϕ-modules ([7]). To define crystalline representations over local artinian $W(k)$-algebras with residue field k, and thus approach the theory of crystalline framed deformation problems, we may in sufficiently simple situations[1] use the theory of Fontaine–Laffaille modules discussed in [12]. These objects are sufficiently explicit that we can actually parameterise all lifts in terms of filtration preserving endomorphisms of the underlying filtered module.

Consider residual crystalline representations as above together with a flag of maximal length. These take the form

$$\overline{\rho} = \begin{pmatrix} \overline{\rho_1} & * & \dots & * \\ 0 & \overline{\rho_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{\rho_r} \end{pmatrix}$$

for $\overline{\rho_i}$ (not necessarily distinct) irreducible crystalline representations of rank n_i ($i = 1, 2, \dots, r$). A framed deformation problem is specified by lifting the maximal flag, giving lifts ρ of this same form. By relating crystalline extensions of representations to filtration preserving homomorphisms between the corresponding Fontaine–Laffaille modules and adapting an argument from [15], we may prove the following theorem.

Theorem A. *The flagged crystalline framed deformation functor associated to $\overline{\rho}$ as above with labelled Hodge–Tate weights differing by at most $p-2$ is smoothly representable of relative dimension*

$$\sum_{i=1}^{r} (\sum_{j<i} n_j) n_i([K : \mathbb{Q}_p] + 1) - d_{\overline{\rho_{<i}}, \overline{\rho_i}}$$

where $d_{\overline{\rho_{<i}}, \overline{\rho_i}}$ is an explicit quantity depending only on the labelled Hodge–Tate weight structure of the representations $\overline{\rho_i}$ ($i = 1, 2, \dots, r$).

The above theorem gives a simple deduction of one of the main theorems of [13] on “universally twistable lifts”, a notion to be defined in Definition 3.10.

[1] Namely when K is unramified over $\mathbb{Q}_p$ and all labelled Hodge–Tate weights differ by at most $p-2$.

Theorem B. *Suppose K is unramified over $\mathbb{Q}_p$, and let $\overline{\rho}$ be a rank n Fontaine–Laffaille residual representation of G_K together with a maximal flag. Assume additionally that the labelled Hodge–Tate weights of $\overline{\rho}$ differ by at most $p-2$. Then $\overline{\rho}$ admits a universally twistable lift.*

These results rely heavily on the theory of Fontaine–Laffaille modules and so any significant generalisations would likely require additional theoretical input, for example the theory of Wach modules [18].

1.2. Notational conventions

Notational conventions will be introduced as needed throughout this article, but for the convenience of the reader we list some essential common notation here.

Throughout, p denotes a fixed prime number, and K is a finite unramified extension of $\mathbb{Q}_p$, with Frob (or sometimes ϕ) denoting arithmetic Frobenius on K. L is a finite extension of $\mathbb{Q}_p$, contained in $\overline{\mathbb{Q}_p}$ and containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, with residue field k_L. The p-adic valuation v_p will be normalised so that $v_p(p) = 1$, and we write $\mathcal{O}_K$ and $\mathcal{O}_L$ for the rings of integers of K and L, respectively. We let $\mathbb{C}_K$ denote the completion of the algebraic closure of K, with ring of integers $\mathcal{O}_{\mathbb{C}_K}$. We write G_K for the absolute Galois group of K, χ_p for the p-adic cyclotomic character of G_K, and adopt the sign convention that χ_p has all labelled Hodge–Tate weights equal to $+1$. Let $\mathrm{Rep}_L^{cris}(G_K)$ denote the category of crystalline representations of G_K valued in L, with associated category $MF_K^{\phi,w.a.} \bigotimes_{\mathbb{Q}_p} L$ of weakly admissible filtered ϕ-modules.

For a ring R and a finite module M over R, we write $\mathrm{lg}_R(M)$ for the length of M as an R-module.

We write $\mathcal{C}_{k_L}$ for the category of complete local artinian $\mathcal{O}_L$-algebras with residue field k_L, and $\hat{\mathcal{C}_{k_L}}$ for the category of complete local noetherian $\mathcal{O}_L$-algebras with residue field k_L; in both cases, morphisms are local $\mathcal{O}_L$-algebra homomorphisms reducing to the identity on residue fields. For $A \in \mathcal{C}_{k_L}$ (or $A \in \hat{\mathcal{C}_{k_L}}$), we write m_A for the maximal ideal of A.

2. Crystalline representations in characteristic p and Fontaine–Laffaille modules

Definition 2.1. Let $A \in \mathcal{C}_{k_L}$, and suppose $\rho : G_K \longrightarrow GL_n(A)$ is a representation. Fix an integer $r \geq 0$. We say that ρ is *crystalline* with Hodge–Tate weights at most r if there is $V \in \mathrm{Rep}_L^{cris}(G_K)$ with all labelled Hodge–Tate weights in the range $[0, r]$ containing G_K-stable $\mathcal{O}_L$-lattices $T' \subseteq T$, and an $\mathcal{O}_L$-algebra map $A \longrightarrow \mathrm{End}_{\mathcal{O}_L}(\frac{T}{T'})$ such that A^n with G_K-action given by ρ is isomorphic as an $A[G_K]$-module to $\frac{T}{T'}$. We denote by $\mathrm{Rep}_A^{cris,\leq r}(G_K)$ the category of such ρ.

Remark 2.2. In the above notation, if $r \leq p-2$ then we shall sometimes refer to ρ as simply being *Fontaine–Laffaille*, or say that ρ has weights in the *Fontaine–Laffaille range*.

As in the case of characteristic zero representations, we seek some semilinear algebra data which classify such representations. In certain cases the answer is given via the theory of Fontaine–Laffaille modules.

2.1. Fontaine–Laffaille modules

We now introduce various categories of algebraic objects that, in certain cases, allow us to classify crystalline representations in characteristic p in much the same way that one classifies crystalline representations in characteristic zero using $MF_K^{\phi,w.a.} \bigotimes_{\mathbb{Q}_p} L$. The main references for the results in this section are [12] and [14].

Definition 2.3.

(1) A Fontaine–Laffaille module is a finite length module over $\mathcal{O}_K$ together with a decreasing filtration by $\mathcal{O}_K$-module direct summands M^i such that $M^0 = M$, $M^p = 0$, and a collection of Frobenius-semilinear maps $\phi_M^i : M^i \longrightarrow M$ such that $\phi_M^i|_{M^{i+1}} = p\phi_M^{i+1}$ for all i, and $M = \sum_i \phi_M^i(M^i)$. The corresponding category is denoted $MF_{tor,\mathcal{O}_K}^{f,p-1}$; morphisms in this category are filtration-preserving $\mathcal{O}_K$-linear maps which are equivariant with respect to the corresponding ϕ_i for all i. When there is no risk of confusion we will write simply ϕ^i in place of ϕ_M^i.

(2) For $A \in C$, a *Fontaine–Laffaille module over* A consists of giving an object $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$ together with a map $\theta : A \longrightarrow \mathrm{End}_{MF_{tor,\mathcal{O}_K}^{f,p-1}}(M)$ that makes M into a free finitely generated module over $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$ in such a way that the filtered pieces above are $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$-direct summands of M. A morphism between two such objects is required to additionally preserve the A-structure. We will denote this category as $MF_{tor,\mathcal{O}_K}^{f,p-1} \bigotimes_{\mathbb{Z}_p} A$.

Definition 2.4. Let $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$. A *submodule* of M is an $\mathcal{O}_K$ submodule $N \subseteq M$ given the subspace filtration $N^i = N \cap M^i$ such that $\phi^i(N^i) \subseteq N$ for all i. When M has A-structure for some $A \in \mathcal{C}_{k_L}$ we additionally demand that N be a free $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$-module direct summand of M. If M has no submodules then we say M is *irreducible.*

We will have occasion to use various full subcategories of $MF_{tor,\mathcal{O}_K}^{f,p-1}$; in particular the following.

Definition 2.5.

(1) $MF_{tor,\mathcal{O}_K}^{f,p-1,'}$ consists of those objects $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$ which have no non-trivial quotient object N with $N^{p-1} = N$.

(2) $MF_{tor,\mathcal{O}_K}^{f,p-1,''}$ consists of those objects $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$ which have no non-trivial subobject N with $N^1 = 0$.

(3) Let $0 \leq r \leq p-1$. Then $MF^{f,r}_{tor,\mathcal{O}_K}$ consists of those objects $M \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ such that $M^{r+1} = 0$.

We also have the analog of these subcategories for $MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$, where the objects are required additionally to have A-structure in the sense of part 2 of Definition 2.3. The analogous full subcategories will be denoted (respectively) as $MF^{f,p-1,'}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$, $MF^{f,p-1,''}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$, and $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$.

Remark 2.6. $MF^{f,r}_{tor,\mathcal{O}_K} \subseteq MF^{f,p-1,'}_{tor,\mathcal{O}_K}$ for $0 \leq r \leq p-2$.

We now give some basic facts about Fontaine–Laffaille modules; see [12,14].

Proposition 2.7. *Let $M, N \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ and $f \in Hom_{MF^{f,p-1}_{tor,\mathcal{O}_K}}(M,N)$. Then f is strict with filtrations in the sense that $f(M^i) = f(M) \cap N^i$ for all i.*

We note also that $MF^{f,p-1}_{tor,\mathcal{O}_K}$ is an abelian category, as follows from [12], Proposition 1.8.

Definition 2.8. Let $A \in \mathcal{C}_{k_L}$. For every embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, we denote by e_σ the element of $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$ whose component at σ is 1 and all other components are 0, in the sense of Lemma 2.19. If M is a Fontaine–Laffaille module over A, put $M_\sigma = e_\sigma \cdot M$ and $M^i_\sigma = e_\sigma \cdot M^i$ for all i.

Note that for all σ, $\tau(e_\sigma) = e_{\sigma \cdot Frob^{-1}}$; thus if M is a Fontaine–Laffaille module over A of rank n, then $\phi^i(M^i_\sigma) \subseteq M_{\sigma\cdot Frob^{-1}}$ for all i, and $M_{\sigma\cdot Frob^{-1}} = \sum_i \phi^i(M^i_\sigma)$. Considering M_σ as an A-module in the natural way, M_σ is free over A of rank n, and the M^i_σ are free direct summands. Any i for which $\frac{M^i_\sigma}{M^{i+1}_\sigma} \neq 0$ is called a labelled Hodge–Tate weight for M (with label σ); the multiplicity of the label is $rk_A(\frac{M^i_\sigma}{M^{i+1}_\sigma})$. The multiset of labelled Hodge–Tate weights (counted with multiplicity) of a Fontaine–Laffaille module M over A for any embedding σ will be denoted as $HT_\sigma(M)$.

Remark 2.9. The definition of Hodge–Tate weights for Fontaine–Laffaille modules given above is consistent with the contravariant Fontaine–Laffaille functor U_S introduced in Theorem 2.10 and the convention adopted that the cyclotomic character has weight +1. That is, under this convention, we have that $HT_\sigma(M) = HT_\sigma(U_S(M))$ for any $M \in MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$.

The main reason for our interest in Fontaine–Laffaille modules is the following fundamental Theorem of [12] – see also [4].

Theorem 2.10.

(1) *There is a contravariant functor*

$$U_S : MF^{f,p-1}_{tor,\mathcal{O}_K} \longrightarrow Rep^f_{\mathbb{Z}_p}(G_K)$$

which is exact, additive, faithful, and length preserving in the sense that $lg_{\mathbb{Z}_p}(U_S(M)) = lg_{\mathcal{O}_K}(M)$ *for all* $M \in MF^{f,p-1}_{tor,\mathcal{O}_K}$. *Moreover, M has the same invariant factors over $\mathcal{O}_K$ as $U_S(M)$ does over $\mathbb{Z}_p$.*

(2) *U_S is full when restricted to either of the full subcategories $MF^{f,p-1,'}_{tor,\mathcal{O}_K}$ or $MF^{f,p-1,''}_{tor,\mathcal{O}_K}$ of $MF^{f,p-1}_{tor,\mathcal{O}_K}$.*

(3) *For $A \in \mathcal{C}_{k_L}$, every object in the essential image of U_S on $MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ is crystalline in the sense of Definition 2.1. For any $0 \leq r \leq p-2$, U_S gives an anti-equivalence of categories between $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ and its essential image $Rep^{cris,\leq r}_A(G_K)$.*

For convenience we will sometimes replace the functor U_S above with the covariant version $U(M) = \mathrm{Hom}_{\mathbb{Z}_p}(U_S(M), \frac{\mathbb{Q}_p}{\mathbb{Z}_p})$.

We now give some simple results on the Fontaine–Laffaille functor that will be needed later.

Proposition 2.11.

(1) *Let M be a rank n Fontaine–Laffaille module over A. Then $U_S(M)$ is free over A of rank n.*

(2) *Let $A \longrightarrow B$ be a morphism in $\mathcal{C}_{k_L}$ and M be a rank n Fontaine–Laffaille module over A. Then $U(M \bigotimes_A B) = U(M) \bigotimes_A B$.*

(3) *Let $0 \leq r \leq p-2$ and $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be a residual crystalline representation with Hodge–Tate weights in the range $[0, r]$. Then the subfunctor $\mathcal{D}^{\square,cris}_{\overline{\rho}} \subseteq \mathcal{D}^{\square}_{\overline{\rho}}$, which associates to $A \in \mathcal{C}_{k_L}$ the set of crystalline lifts of $\overline{\rho}$ to A with Hodge–Tate weights in the range $[0, r]$, is a deformation condition.*

Proof. See [16]. $\square$

Remark 2.12. This is a straightforward extension of section 2 of [15].

Given two Fontaine–Laffaille representations ρ_1, ρ_2 and an embedding σ, we write $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$ if there is an integer j such that all elements of $HT_\sigma(\rho_1)$ are greater than or equal to j, and all elements of $HT_\sigma(\rho_2)$ are strictly less than j. Note this same notation works in characteristic 0 and p.

Proposition 2.13. *Let ρ_i $(i = 1, 2)$ be irreducible characteristic* 0 *Fontaine–Laffaille representations. Assume that for each σ, we have $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$. Then unless $\rho_1 \cong \chi_p \bigotimes \rho_2$, every extension of ρ_2 by ρ_1 is Fontaine–Laffaille. In the exceptional case, the space of Fontaine–Laffaille extensions is a subspace of the space of all extensions which is of codimension* 1.

Proof. This follows from a straightforward calculation of Euler characteristics and an application of local Tate duality; see [16] for the details. □

2.2. Classification of low rank Fontaine–Laffaille modules

As an illustration of these ideas, we classify Fontaine–Laffaille modules of small rank. See [16] for more details.

Example 2.14. Let M be a rank 1 Fontaine–Laffaille module over A, with labelled Hodge–Tate weights $(i_\sigma)_\sigma$. Then M is specified by the collection $(i_\sigma)_\sigma$ of labelled Hodge–Tate weights together with a unit $a \in A^\times$. In particular we see that the rank 1 crystalline deformation functor over an unramified base is smooth of relative dimension 1. The reader should compare this with Remark 3.3.

Example 2.15. Suppose $K = \mathbb{Q}_p$. We classify the rank 2 Fontaine–Laffaille modules over A.

Let $i \le j$ be the Hodge–Tate weights of a rank 2 Fontaine–Laffaille module M. If $i = j$ then ϕ^i is specified up to a choice of basis by a matrix $\phi^i \in GL_2(A)$. The corresponding representation will be the ith cyclotomic twist of an unramified representation whose action on Frobenius is specified by ϕ^i.

So suppose $i < j$, and consider the A-line $M^j \subset M$. For convenience, we use the notation $N \bigoplus_f M$ for extensions of Fontaine–Laffaille modules, details of which can be found in the proof of Proposition 2.17. Also, for an integer i in the range $0 \le i \le p-2$ and a unit $a \in A^\times$, (i, a) will denote the rank 1 Fontaine–Laffaille module of weight i and parameter a as in the previous example. There are then 3 cases.

(1) $\phi^j(M^j) = M^j$. In this case, there exist units $a, d \in A^\times$ depending only on M such that M splits as $(i, a) \bigoplus (j, d)$ (in other words, $M \cong (i, a) \bigoplus_0 (j, d)$ in the notation of Proposition 2.17).
(2) $\phi^j(M^j) \bigoplus M^j = M$. In this case M is irreducible and is specified by 2 parameters $a \in \mathfrak{m}_A$, $c \in A^\times$ which depend only on M. A basis for M as a filtered module can be chosen such that the matrix of ϕ takes the form

$$\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}.$$

This follows from the observation that any basis $\{e\}$ for M^j gives a basis $\{\phi^j(e), e\}$ for M. We require that $a \in \mathfrak{m}_A$ since otherwise the module constructed here is isomorphic to the non-split extension $(i, a+y\cdot p^{j-i}) \bigoplus_{-y^{-1}}(j, -y)$ in the notation of Proposition 2.17, where $y \in A^\times$ is any root of the polynomial $p^{j-i}y^2 + ay - c$ (that this polynomial has a root in $A^\times$ follows from Hensel's lemma).

The corresponding representation is irreducible. Note that in particular, when $A = \mathbb{F}_p$, there is only one such representation up to unramified twist. One sees also that if $\overline{\rho}$ corresponds to such a Fontaine–Laffaille module, then the crystalline deformation problem for $\overline{\rho}$ is smooth of relative dimension 2 over $\mathbb{Z}_p$, and so (since $\overline{\rho}$ is irreducible of dimension 2), the crystalline framed deformation problem is smooth of relative dimension 5. The reader should compare this with Theorem 3.2.

(3) $\phi^j(M^j) \neq M^j$ but $\phi^j(M^j) \cap M^j \neq 0$. In this case, by Hensel's lemma, one shows that $\exists\, 0 \neq b \in \mathfrak{m}_A$ (determined by M up to unit), and $a, d \in A^\times$ (determined uniquely by M) such that M is isomorphic to the non-split extension $(i, a) \bigoplus_b (j, d)$ in the notation of Proposition 2.17. All such modules, and the corresponding representations, are residually split.

In particular in the case when $\overline{\rho}$ is upper triangular with strictly decreasing Hodge–Tate weights, the upper triangular crystalline framed deformation problem associated to $\overline{\rho}$ (as in Theorem 3.6) is smooth of relative dimension 3, since one must specify lifts of the quantities $\overline{a}$, $\overline{b}$, and $\overline{d}$. The reader should compare this with Corollary 3.7.

2.3. Extensions of Fontaine–Laffaille modules

Finally we establish an important result on the structure of the group of extension classes in the category of Fontaine–Laffaille modules. We will need the following lemma.

Lemma 2.16. *Let $A \in \mathcal{C}_{k_L}$. Let M and N be Fontaine–Laffaille modules over A, and $y \in Hom_{Fil, \mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N)$. Suppose we have elements $m^i \in M^i$ such that $\sum_i \phi^i_M(m^i) = 0$. Then $\sum_i \phi^i_N(y(m^i)) = 0$.*

Proof. Beginning with the smallest filtered piece of M, we may repeatedly extend bases from the direct summands M^i until we obtain a basis for M. Denoting by $\tilde{M}^j$ the span of only those basis elements introduced when extending to M^j, we can therefore write each m^i as a sum:

$$m^i = \sum_{j \geq i} m^{i,j}$$

where $m^{i,j} \in \tilde{M}^j$. Since y preserves the filtration, $y(m^{i,j}) \in N^j$. By assumption,

$$0 = \sum_i \phi_M^i(m^i) = \sum_j \sum_{i \le j} \phi_M^j(p^{j-i} m^{i,j})$$

from which conclude that $\sum_{i=1}^{j} p^{j-i} m^{i,j} = 0$ for all j, since ϕ_M^j is injective on $\tilde{M}^j$, and $M = \bigoplus_j \phi_M^j(\tilde{M}^j)$. Thus

$$\sum_i \phi_N^i(y(m^i)) = \sum_j \sum_{i \le j} p^{j-i} \phi_N^j(y(m^{i,j})) = 0$$

as required. $\square$

We can now state the main result of this section.

Proposition 2.17. *Suppose $A \in \mathcal{C}_{k_L}$. Let M and N be Fontaine–Laffaille modules over A. Then we have an exact sequence:*

$$0 \longrightarrow Hom_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow Hom_{Fil,\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow$$
$$\longrightarrow Hom_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow Ext^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow 0.$$

Proof. The construction of this sequence is similar to that performed in the proof of Proposition 2.16 in [9]. For convenience we give some details here.

Given $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$, define $\phi(y) \in \mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ as follows: if $m = \sum_i \phi_M^i(m^i)$ then $\phi(y)(m) = \sum_i \phi_N^i(y(m^i))$. This is well defined by Lemma 2.16. The equation $\mathrm{Ker}(\phi - 1) = \mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$ is then an immediate consequence.

For any $f \in \mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ we can construct an extension which we will denote $N \bigoplus_f M$ as follows: the filtration structure is $(N \bigoplus_f M)^i = N^i \bigoplus M^i$, and ϕ^i sends the pair (n^i, m^i) to $(\phi_N^i(n^i) + f(\phi_M^i(m^i)), \phi_M^i(M^i))$. It is trivial to check that the resulting module satisfies the definition. In this way we have obtained a map $\mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow \mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$. Unravelling, we see that a map f lies in the kernel precisely when there is a θ, necessarily of the form $\theta(n,m) = (n + y(m), m)$ with $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$, fitting in to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d] & N \bigoplus_f M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N \arrow[r] & N \bigoplus M \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

(where the leftmost and rightmost arrows are the identity) and such that $\phi_N^i(y(m^i)) = y(\phi_M^i(m^i)) + f(\phi_M^i(m^i))$ for all i and elements $m^i \in M^i$, by ϕ-compatibility of θ. In other words, the kernel consists precisely of those f of the form $\phi(y) - y$ for $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$.

It remains to show this map is surjective. Given $D \in \mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$, we have $D \cong N \bigoplus M$ as filtered modules. Pick filtration compatible linear sections $a : D \longrightarrow N$ and $b : M \longrightarrow D$. We may then define a map $f : M \longrightarrow N$ by sending $\phi^i_M(m^i)$ to $a(\phi^i_D b(m^i))$ and extending linearly. Again this is well-defined by Lemma 2.16, thus exhibiting D as lying in same class as $N \bigoplus_f M$. The result follows. □

Note that the formation of this exact sequence commutes with base extension in the sense that, if $A \longrightarrow B$ is a map in the category $\mathcal{C}_{k_L}$, the resulting exact sequence over B is obtained by tensoring each term with B over A. Also by Fontaine–Laffaille theory, $\mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N) \cong \mathrm{Hom}_{A[G_K]}(U_S(N), U_S(M))$ provided that M and N lie in $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ (by Theorem 2.10, part 3). In particular, the first term in the exact sequence is free over A. The third term is also free over $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$ and hence over A. Thus by a repeated application of Lemma 2.20 below, we see that all terms are in fact free over A.

Definition 2.18. We denote the A-rank of $\mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ as $d_{M,N}$, or simply as d_M in the special case where $M = N$. Observer that this quantity depends only on the labelled Hodge–Tate weight structure of M and N.

There are two notable cases.

(1) Suppose for every $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$ that $HT_\sigma(M) > HT_\sigma(N)$. There is thus an integer i (depending on σ) such that $N^i_\sigma = 0$ and $M^i_\sigma = M_\sigma$. It follows that $\mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N) = 0$, so $d_{M,N} = 0$ and $\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$ has rank $[K : \mathbb{Q}_p]\mathrm{rk}(M_1)\mathrm{rk}(M_2)$ over A by Proposition 2.17. Note that this is the maximum possible value for $\mathrm{rk}_A(\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N))$. We call this case the *Hodge–Tate* case.

(2) Suppose conversely that for every σ, $HT_\sigma(N) > HT_\sigma(M)$. There is thus an integer i (depending on σ) such that $N^i_\sigma = N_\sigma$ and $M^i_\sigma = 0$. Pick any $f \in \mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$ and $m \in M_{\sigma \cdot Frob^{-1}}$. We may write $m = \sum_{j<i} \phi^j_M(m^j)$ for appropriate $m^j \in M^j_\sigma$. We then have

$$f(m) = \sum_{j<i} \phi^j_N(f(m^j)) = \sum_{j<i} p^{i-j} \phi^i_N(f(m^j)) \in p \cdot N_{\sigma \cdot Frob^{-1}}$$

and so $f(M) \subseteq p \cdot N$. By induction one then sees that in fact $f(M) \subseteq p^r \cdot N$ for every $r \in \mathbb{N}$, and so since $p^r \cdot N = 0$ for sufficiently large r we conclude that $f = 0$. On the other hand, since $N^i_\sigma = N_\sigma$ and $M^i_\sigma = 0$, it follows that every element of $\mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ preserves the filtration. We conclude $d_{M,N} = [K : \mathbb{Q}_p]\mathrm{rk}(M_1)\mathrm{rk}(M_2)$ and $\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N) = 0$. We call this case the *anti-Hodge–Tate* case.

2.4. Commutative algebra

We end this chapter with a few simple commutative algebra facts that were referred to in this chapter or will be needed in the following.

We record the following lemma, whose proof is straightforward.

Lemma 2.19. *Let $A \in \mathcal{C}_{k_L}$. Then the map $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A \longrightarrow \prod_\sigma A$ which sends $x \otimes y$ to $(\sigma(x) \cdot y)_\sigma$, where the product runs over all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, is an isomorphism.*

Given an element of $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$, we may thus refer to its σ-component for any embedding σ.

Lemma 2.20. *Let A be a local ring.*

(1) *Suppose N is a free finitely generated A-module and M is a free submodule of N such that the reduced map $\overline{M} \longrightarrow \overline{N}$ is an injection. Then M is a direct summand.*
(2) *Let*

$$0 \longrightarrow M \longrightarrow D \longrightarrow N \longrightarrow 0$$

be an exact sequence of finitely generated A-modules with M and N free over A. Then D is free.

Proof. See [16] for details. □

3. Smoothness of crystalline framed deformation functors and universally twistable lifts

In this section we prove the main results of this article on the representability and formal smoothness of framed deformation functors associated to various classes of Fontaine–Laffaille Galois representations. We also provide calculations on the dimensions of these functors, and apply the main theorems to the problem of the existence of universally twistable lifts of Fontaine–Laffaille representations.

3.1. Representability and formal smoothness

The first main result we demonstrate is the smooth representability of the framed deformation problem associated to an irreducible Fontaine–Laffaille representation. Note that in [6], Corollary 2.4.3, a similar result is demonstrated for (not necessarily irreducible) representations with the property that, for all $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, the labelled Hodge-weights for σ occur with multiplicity 1 (so in particular, $n \leq p + 1$). To remove this condition we are forced to take a different approach, motivated by a counting argument from [15] which there is applied to flat deformations.

Definition 3.1. For a Fontaine–Laffaille representation $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$, let $\mathcal{D}_{\overline{\rho}}^{\square,cris}$ denote the associated framed deformation functor that associates to $A \in \mathcal{C}_{k_L}$ the set of all continuous crystalline lifts $\rho : G_K \longrightarrow GL_n(A)$ with Hodge–Tate weights in the Fontaine–Laffaille range (and sends morphisms to the natural map).

Note that this is a relatively representable subfunctor of the (unframed) deformation functor which associates A with the strict equivalence class of continuous crystalline lifts of $\overline{\rho}$. For more discussion on deformation functors, see for example [16].

Theorem 3.2. *Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be an irreducible Fontaine–Laffaille representation, with associated crystalline framed deformation functor $\mathcal{D}_{\overline{\rho}}^{\square,cris}$. Let $\overline{M}$ be the rank n Fontaine–Laffaille module associated with $\overline{\rho}$ as in Theorem 2.10. Then $\mathcal{D}_{\overline{\rho}}^{\square,cris}$ is represented by a power series ring over $\mathcal{O}_L$ in $n^2([K : \mathbb{Q}_p] + 1) - d_{\overline{M}}$ variables.*

Proof. The argument is based on a generalisation of that found in [15].

By Proposition 2.17 we have (since $\overline{\rho}$ is irreducible) that $\mathrm{Ext}^1_{MF}(\overline{M}, \overline{M})$ is of rank $n^2[K : \mathbb{Q}_p] + 1 - d_{\overline{M}}$ over k_L. From the theory of representable functors, we see that the associated deformation problem is representable by a ring $R_{\overline{M}}$, and that there is a surjection

$$\mathcal{O}_L[[(T_i)_{i=1}^{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}]] \twoheadrightarrow R_{\overline{M}}$$

of $\mathcal{O}_L$-algebras. It suffices to prove that this is in fact an isomorphism, since in this case the associated framed deformation problem will be represented by a power series ring over $\mathcal{O}_L$ in $\mathrm{rk}_{\mathcal{O}_L}(R_{\overline{M}}) + n^2 - 1$ variables (as $\overline{\rho}$ is irreducible).

Supposing without loss of generality that L is unramified over $\mathbb{Q}_p$, we count for each $r \in \mathbb{N}$ the lifts M of $\overline{M}$ to $\mathcal{O}_L/p^r$. We will show that there are precisely $q^{(r-1)(n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}})}$, where q is the size of the residue field of L; since this is the number of $\mathcal{O}_L/p^r$-points of $\mathcal{O}_L[[(T_i)_{i=1}^{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}]]$ we deduce that the above map is an isomorphism (for any f in the kernel, $f(t_1, t_2, \ldots, t_{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}) = 0$ whenever $t_1, t_2, \ldots, t_{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}} \in p\mathcal{O}_L$, which implies $f = 0$).

We are thus seeking matrices $\phi \in GL_n(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} \mathcal{O}_L/p^r)$ with specified reduction $\overline{\phi} \in GL_n(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} k_L)$, determined up to τ-semilinear conjugation by a matrix R which preserves the filtration on M and reduces to the identity modulo p.

There are $q^{(r-1)n^2[K:\mathbb{Q}_p]}$ choices for ϕ, and $q^{(r-1)d_{\overline{M}}}$ choices for R. Of these, since M is irreducible, q^{r-1} commute with each ϕ in the sense that τ-semilinear conjugation by R preserves ϕ; or in other words, that

$$\phi_\sigma R_\sigma = R_{\sigma \cdot Frob^{-1}} \phi_\sigma$$

for every $\sigma : K \hookrightarrow L$. There are thus $q^{(r-1)(n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}})}$ lifts of $\overline{M}$ up to isomorphism, as required. $\square$

Remark 3.3. If $n = 1$ then $d_{\overline{M}} = [K : \mathbb{Q}_p]$ and so $\mathcal{D}_{\overline{\rho}}^{\square,cris}$ is smooth of rank 1, as expected.

We will now extend this result to the situation where $\overline{\rho}$ is block upper triangular with n diagonal blocks. To this end, we make the following definition.

Definition 3.4. Suppose

$$\rho = \begin{pmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r \end{pmatrix}$$

is a block upper triangular representation of a group G valued in some ring A. Given a positive integer $i \le r$, the ith *truncation* of ρ is the representation

$$\rho_{\le i} = \begin{pmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_i \end{pmatrix}.$$

Remark 3.5. Assuming that ρ and $\rho_1, \rho_2, \dots, \rho_r$ as above are Fontaine–Laffaille representations of G_K valued in some ring $A \in \mathcal{C}_{k_L}$, all truncations $\rho_{\le i}$ of ρ are also Fontaine–Laffaille. This follows from Theorem 2.10 together with the fact that $MF_{tor,\mathcal{O}_K}^{f,p-1}$ is an abelian category ([12], Proposition 1.8).

In the situation of the above remark, we will denote the Fontaine–Laffaille module associated to $\rho_{\le i}$ as $M_{\le i}$, and the Fontaine–Laffaille module associated to $\rho_{\le i-1}$ as $M_{<i}$, for $i \le r$. Note that $M_{\le i} \in \mathrm{Ext}^1_{MF_{tor,\mathcal{O}_K}^{f,p-1} \otimes_{\mathbb{Z}_p} A}(M_{<i}, M_i)$, where M_i is the Fontaine–Laffaille module associated to ρ_i.

We are now in a position to prove the following theorem.

Theorem 3.6. *For $i = 1, 2, \dots, r$, let $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ be irreducible Fontaine–Laffaille representations, with framed deformation functors $\mathcal{D}_i^{\square,cris}$, and associated Fontaine–Laffaille modules $\overline{M_i}$, of rank n_i. Fix a Fontaine–Laffaille representation $\overline{\rho}$ which is block upper triangular with $\overline{\rho_1}, \overline{\rho_2}, \dots, \overline{\rho_r}$ (not necessarily distinct) on the diagonal, and define a functor $\mathcal{F}_{\overline{\rho}} : \mathcal{C}_{k_L} \longrightarrow Set$ which sends a ring A to the set of crystalline lifts ρ of $\overline{\rho}$ to A of the form*

$$\rho = \begin{pmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r \end{pmatrix}$$

for $\rho_i \in \mathcal{D}_i^{\square,cris}(A)$ *($i = 1, 2, \ldots, r$). Then* $\mathcal{F}_{\overline{\rho}}$ *is representable. Moreover, the natural map* $\mathcal{F}_{\overline{\rho}} \longrightarrow \prod_{i=1}^r \mathcal{D}_i^{\square,cris}$ *is smooth of relative dimension* $\sum_{i=1}^r d_i$*, where*

$$d_i = (\sum_{j<i} n_j)n_i([K : \mathbb{Q}_p] + 1) - d_{\overline{M_{<i}},\overline{M_i}}.$$

Proof. Suppose first that $r = 2$. It suffices to prove that the collection $Z^1_{crys,A}(\rho_2, \rho_1)$ of crystalline cocyles is a free A-module of rank $n_1n_2([K : \mathbb{Q}_p]+1)-d_{\overline{M_1},\overline{M_2}}$, for any $A \in \mathcal{C}_{k_L}$ and pair of lifts $(\rho_1, \rho_2) \in (\mathcal{D}_1^{\square,cris} \times \mathcal{D}_2^{\square,cris})(A)$. Note that *crystalline cocyle* refers to any cocyle map whose image lies in $H^1_{crys,A}(\rho_2, \rho_1)$. Letting $B^1_{crys,A}(\rho_2, \rho_1)$ denote the kernel of this map (the collection of *crystalline coboundaries*), we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{G_K}(\rho_2, \rho_1) \longrightarrow Mat_{n_1\times n_2}(A) \longrightarrow B^1_{crys,A}(\rho_2, \rho_1) \longrightarrow 0$$

and so $B^1_{crys,A}(\rho_2, \rho_1)$ is free over A of rank $n_1n_2 - \dim(\mathrm{Hom}_{G_K}(\rho_2, \rho_1))$ by part 1 of Lemma 2.20.

By Proposition 2.17 and the discussion following it, $H^1_{crys,A}(\rho_2, \rho_1)$ is free over A of rank $n_1n_2[K : \mathbb{Q}_p] + \dim(\mathrm{Hom}_{G_K}(\rho_2, \rho_1)) - d_{\overline{M_1},\overline{M_2}}$.[2] From the exact sequence

$$0 \longrightarrow B^1_{crys,A}(\rho_2, \rho_1) \longrightarrow Z^1_{crys,A}(\rho_2, \rho_1) \longrightarrow H^1_{crys,A}(\rho_2, \rho_1) \longrightarrow 0$$

and Lemma 2.20 (part 2), we deduce the result.

Now let r be arbitrary. For $i = 1, 2, \ldots, r$, let $\mathcal{F}_{\overline{\rho},i}$ be the crystalline lift functor corresponding to the ith truncation of $\overline{\rho}$. We have just seen that $\mathcal{F}_{\overline{\rho},2} \longrightarrow \mathcal{D}_1^{\square,cris}\times\mathcal{D}_2^{\square,cris}$ is relatively representable and smooth of relative dimension d_2. Suppose that $\mathcal{F}_{\overline{\rho},r-1} \longrightarrow \mathcal{F}_{\overline{\rho},r-2} \times \mathcal{D}_{r-1}^{\square,cris}$ is relatively representable and smooth of relative dimension d_{r-1}. Then by the same reasoning as for the case $r = 2$, $\mathcal{F}_{\overline{\rho},r} \longrightarrow \mathcal{F}_{\overline{\rho},r-1} \times \mathcal{D}_r^{\square,cris}$ is relatively representable and smooth of relative dimension d_r. Thus the result follows by induction on r. □

Corollary 3.7. *Let notation be as in Theorem 3.6. Then* $\mathcal{F}_{\overline{\rho}}$ *is represented by a power series ring over* $\mathcal{O}_L$ *in* $([K : \mathbb{Q}_p] + 1)(\sum_{i,j:i\leq j} n_in_j) - \sum_{i=1}^r d_{\overline{M_{\leq i}},\overline{M_i}}$ *variables.*

Proof. By Theorem 3.2, each $\mathcal{D}_i^{\square,cris}$ for $i = 1, 2, \ldots, r$ is smooth of relative dimension c_i, for $c_i = n_i^2([K : \mathbb{Q}_p] + 1) - d_{\overline{M_i}}$. The result then follows from Theorem 3.6 after observing that

$$c_i + d_i = (\sum_{j\leq i} n_j)n_i([K : \mathbb{Q}_p] + 1) - d_{\overline{M_{\leq i}},\overline{M_i}},$$

using the fact that $\overline{M_{\leq i}} = \overline{M_{<i}} \bigoplus \overline{M_i}$ as filtered modules. □

[2] Note that $H^1_{crys,A}(\rho_2, \rho_1)$ is identified with $\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K}\otimes_{\mathbb{Z}_p} A}(\overline{M_1}, \overline{M_2})$; the rank then follows by dimension counting along the exact sequence in Proposition 2.17.

Remark 3.8. In Corollary 2.4.3 of [6] it is shown that if $\overline{\rho}$ is an n-dimensional crystalline representation of G_K with labelled Hodge–Tate weights all of multiplicity 1, then the crystalline framed deformation problem for $\overline{\rho}$ is representable by a power series ring over $\mathcal{O}_L$ in $n^2 + [K : \mathbb{Q}_p]\frac{n(n-1)}{2}$ variables. Note that if $\overline{\rho}$ is irreducible and corresponds to the Fontaine–Laffaille module $\overline{M}$ then $d_{\overline{M}} = [K : \mathbb{Q}_p]\frac{n(n+1)}{2}$ and so this result is a special case of Theorem 3.2.

Example 3.9. Suppose $\overline{\rho}$ is an r-dimensional upper triangular Fontaine–Laffaille representation with Fontaine–Laffaille characters $\overline{\chi_1}, \overline{\chi_2}, \dots, \overline{\chi_r}$ on the diagonal, and that $\overline{\chi_j}$ has labelled Hodge–Tate weights $i_{j,\sigma}$ for $1 \leq j \leq r$. Then $\mathcal{F}_{\overline{\rho}}$ is represented by a power series ring over $\mathcal{O}_L$ in

$$\frac{r(r+1)}{2} + \sum_{a \leq b} \#\{\sigma \mid i_{b,\sigma} < i_{a,\sigma}\}$$

variables. In particular, if $\overline{\rho}$ is ordinary in the sense that $i_{b,\sigma} < i_{a,\sigma}$ for all σ when $a < b$ then the dimension is $r + \frac{r(r-1)}{2}([K : \mathbb{Q}_p] + 1)$.

3.2. Dimension bounds

Suppose $\overline{\rho}$ is irreducible Fontaine–Laffaille of dimension n, and corresponds to the Fontaine–Laffaille module $\overline{M}$. It is straightforward to show that

$$[K : \mathbb{Q}_p]\frac{n(n+1)}{2} \leq d_{\overline{M}} \leq [K : \mathbb{Q}_p]n^2.$$

Hence the crystalline framed deformation problem for $\overline{\rho}$ is smooth of relative dimension $c_{\overline{\rho}}$, where

$$n^2 \leq c_{\overline{\rho}} \leq n^2 + [K : \mathbb{Q}_p]\frac{n(n-1)}{2}.$$

The lower bound here comes from being able to twist by any representation which is unramified and residually trivial, and is attained only if all labelled Hodge–Tate weights occur with multiplicity n; in other words, when $\overline{\rho}$ is a twist by some character of an unramified representation.

Now suppose $\overline{\rho_1}, \overline{\rho_2}, \dots, \overline{\rho_r}$ are irreducible Fontaine–Laffaille of dimension n, and correspond to the Fontaine–Laffaille modules $\overline{M_1}, \overline{M_2}, \dots, \overline{M_r}$. Let $\overline{\rho}$ be a Fontaine–Laffaille representation of G_K which is upper triangular with $\overline{\rho_1}, \overline{\rho_2}, \dots, \overline{\rho_r}$ on the diagonal. Using the notation of Theorem 3.6,

$$(i-1)n^2 \leq d_i \leq ([K : \mathbb{Q}_p] + 1)(i-1)n^2$$

for each i, which implies by taking the sum over i that

$$\frac{r(r+1)}{2}n^2 \leq \dim_{\mathcal{O}_L}\mathcal{F}_{\overline{\rho}} \leq \frac{r(r+1)}{2}n^2 + \frac{rn(rn-1)}{2}[K:\mathbb{Q}_p].$$

3.3. Application to universal twistable lifts

We now discuss how the results of this chapter can be applied to the work on so-called "universally twistable lifts" carried out in [13]. We first review some of the definitions in that paper. We temporarily remove all restrictions on notation that have been present up until now, and assume only that K is a finite extension of $\mathbb{Q}_p$, and L is a subfield of $\overline{\mathbb{Q}_p}$ containing the images of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$ with residue field k_L.

Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be any representation, and denote by $\overline{V}$ the underlying $k_L[G_K]$-module. Suppose $0 = \overline{U_0} \subsetneq \overline{U_1} \subsetneq \ldots \subsetneq \overline{U_r} = \overline{V}$ is an increasing filtration of $\overline{V}$ by $k_L[G_K]$-submodules, and put $\overline{V_i} = \overline{U_i}/\overline{U_{i-1}}$ for $i = 1, 2, \ldots, r$.

Definition 3.10. $\overline{\rho}$ *admits a universally twistable lift with respect to the filtration* $(\overline{U_i})_{i=0}^r$ if there exists lifts V_i of $\overline{V_i}$ to $\mathcal{O}_L$ for $i = 1, 2, \ldots, r$, together with, for every r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K, a lift $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$, satisfying the following additional properties for $i = 1, 2, \ldots, r$:

(1) $V(\psi_1, \psi_2, \ldots, \psi_r)$ has an increasing filtration by $\mathcal{O}_L[G_K]$-submodules $0 = U(\psi_1, \psi_2, \ldots, \psi_r)_0 \subsetneq U(\psi_1, \psi_2, \ldots, \psi_r)_1 \subsetneq \ldots \subsetneq U(\psi_1, \psi_2, \ldots, \psi_r)_r = V(\psi_1, \psi_2, \ldots, \psi_r)$ which are free $\mathcal{O}_L$-direct summands, such that $U(\psi_1, \psi_2, \ldots, \psi_r)_i/U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1} \cong V_i \bigotimes_{\mathcal{O}_L} \psi_i$.
(2) The reduction $V(\psi_1, \psi_2, \ldots, \psi_r) \bigotimes_{\mathcal{O}_L} k_L \cong \overline{V}$ induces reductions $U(\psi_1, \psi_2, \ldots, \psi_r)_i \bigotimes_{\mathcal{O}_L} k_L \cong \overline{U_i}$.
(3) The submodule $U(\psi_1, \psi_2, \ldots, \psi_r)_i$ depends up to isomorphism only on $(\psi_1, \psi_2, \ldots, \psi_i)$, and not on $(\psi_{i+1}, \psi_{i+2}, \ldots, \psi_r)$.

Definition 3.11. $\overline{\rho}$ *admits a universally twistable lift* if it does so with respect to a saturated filtration $(\overline{U_i})_{i=0}^r$; here we say that a filtration $(\overline{U_i})_{i=0}^r$ is *saturated* if the graded pieces $\overline{V_i} = \overline{U_i}/\overline{U_{i-1}}$ are absolutely irreducible, for all $i = 1, 2, \ldots, r$.

Example 3.12. If $\overline{\rho}$ is semisimple then it admits a universally twistable lift. Indeed, choosing the saturated filtration $(\overline{U_i})_{i=0}^r$ displaying $\overline{\rho}$ as block diagonal with representations $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ on the diagonal, we know from [16] that each $\overline{\rho_i}$ lifts to $\mathcal{O}_L$ (for $i = 1, 2, \ldots, r$). We can therefore choose any lifts V_i of $\overline{V_i}$ and a family of block diagonal lifts $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$.

We now show that, in addition to semisimple representations as discussed above, the class of Fontaine–Laffaille representations also admit universally twistable lifts.

Theorem 3.13. *Suppose K is unramified over $\mathbb{Q}_p$, and let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be a representation with underlying space $\overline{V}$ and an increasing saturated filtration $(\overline{U_i})_{i=0}^r$ by $k_L[G_K]$-submodules. Assume additionally that $\overline{\rho}$ is Fontaine–Laffaille (in the sense of Definition 2.1) with Hodge–Tate weights in the range $[a, a+p-2]$ for some integer a. Then $\overline{\rho}$ admits a universally twistable lift. Moreover, the lifts $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$ are Fontaine–Laffaille for every r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K.*

Remark 3.14. This is essentially the statement of Proposition 2.2.1 of [13]. We now give a proof using the methods of this article.

Proof. For $i = 1, 2, \ldots, r$, let $\overline{\rho_i}$ be the representation of G_K whose underlying space is $\overline{V_i}$. Then $\overline{\rho_i}$ is irreducible and Fontaine–Laffaille for all i. Put $n_i = \dim(\overline{\rho_i})$. By Theorem 3.2, each $\overline{V_i}$ lifts to a crystalline V_i since $\mathcal{D}_i^{\square,cris}$ is smooth of dimension at least $n_i^2 \geq 1$ from the discussion of the previous section.

Now, given any r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K, we may inductively lift $\overline{U_i}$ to an $\mathcal{O}_L[G_K]$-submodule $U(\psi_1, \psi_2, \ldots, \psi_r)_i$ containing $U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1}$ which is a free $\mathcal{O}_L$-direct summand depending only on $(\psi_1, \psi_2, \ldots, \psi_i)$ with $U(\psi_1, \psi_2, \ldots, \psi_r)_i / U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1} \cong V_i \bigotimes_{\mathcal{O}_L} \psi_i$, since, in the notation of Theorem 3.6, $\mathcal{F}_{\overline{\rho},i} \longrightarrow \mathcal{F}_{\overline{\rho},i-1} \times \mathcal{D}_i^{\square,cris}$ is smooth of relative dimension at least $n_i \sum_{j<i} n_j \geq 1$, again from the discussion of the previous section.

In this way we have constructed the lift $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$. $\square$

We note in passing that the proof of this theorem gives an explicit understanding of the freedom we have in constructing the universally twistable lift of $\overline{V}$ as above. In particular, we may lift each $\overline{V_i}$ arbitrarily (and the set of allowable lifts is of rank $n^2([K : \mathbb{Q}_p] + 1) - d_{\overline{V_i}}$ over $\mathcal{O}_L$ for $i = 1, 2, \ldots, r$), and given any lifts, the collection of allowable lifts of $\overline{U_i}$ at each stage i is parametrised by a free $\mathcal{O}_L$-module of explicitly calculable (non-zero) rank.

4. Generalisations

We can hope to generalise the theorems of Section 3.1 in several ways. Three particular avenues for generalisation are:

(1) Relax the unramified condition on the extension K of $\mathbb{Q}_p$.
(2) Consider other classes of representations, such as semistable representations.
(3) Relax the condition on the Hodge–Tate weights being inside the Fontaine–Laffaille range.

It seems that all of these questions will need techniques outside those discussed in this article to be answered fully; however, in this section we will make a few brief remarks about the second question, and also explore the third question in the situation where the departure from the Fontaine–Laffaille range is relatively "mild". In full generality however, note that we do not expect formal smoothness, as in for example Theorem 3.2, to continue to hold.

4.1. Semistable representations

A natural approach to the second question would be to establish a category of semilinear algebra data analogous to Fontaine–Laffaille modules which, in a similar way to Theorem 2.10, correspond with subcategories of semistable representations. There is some hope here as, in the characteristic zero situation, semistable representations (with coefficients) correspond to so-called "weakly admissible (ϕ, N)-modules" (see Theorem A of [7]).

However, there are issues with simply adding a "monodromy operator" N to the definition of a Fontaine–Laffaille module, since demanding that $N\phi^{i+1} = \phi^i N$ requires that $N(M^{i+1}) \subseteq M^i$ for all i, but the analogous property is not true of all weakly admissible (ϕ, N)-modules. See [2] and [3] for an alternative approach, which may give a sufficiently concrete correspondence to allow the methods of this article to extend to certain subcategories of semistable representations.

4.2. Hodge–Tate weights outside the Fontaine–Laffaille range

We now discuss two possible approaches to the third question.

4.2.1. (ϕ, Γ)*-modules, Wach modules, and crystalline representations*

A possible approach to this question would be to seek semilinear algebraic data to categorise crystalline representations where no restrictions on the labelled Hodge–Tate weights are imposed. This is done in [18] and [1]; a good summary, particularly of the generalisation needed to allow coefficients, can be found in [5], upon which this subsection is based. See also [16] and [14], particularly sections 3.2 and 3.3.

Assume K is unramified over $\mathbb{Q}_p$ and L is a finite extension of $\mathbb{Q}_p$ containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. Recall the construction (see for example [16]) of the rings $\mathbf{A}^+_{K,L} \subseteq \mathbf{A}_{K,L}$ (respectively $\mathbf{B}^+_{K,L} \subseteq \mathbf{B}_{K,L}$) equipped with an $\mathcal{O}_L$-linear (respectively L-linear) action of ϕ and $\Gamma_K = G_K/\mathrm{Ker}(\chi_p)$. We write π for the distinguished element on which ϕ and Γ_K act as $\phi(\pi) = (1+\pi)^p - 1$ and $g(\pi) = (1+\pi)^{\chi_p(g)} - 1$ for $g \in \Gamma_K$. We then make the following definition.

Definition 4.1. A (ϕ, Γ)-*module* over $\mathbf{A}_{K,L}$ (respectively $\mathbf{B}_{K,L}$) is a finitely generated $\mathbf{A}_{K,L}$-module (respectively $\mathbf{B}_{K,L}$-module) M with continuous semilinear commuting actions of a Frobenius endomorphism ϕ_M and of Γ_K. M is *étale* if the image of ϕ_M spans M over $\mathbf{A}_{K,L}$ (respectively if M contains a ϕ_M-stable $\mathbf{A}_{K,L}$-lattice which is étale over $\mathbf{A}_{K,L}$). We denote the category of *étale* (ϕ, Γ)-modules over $\mathbf{A}_{K,L}$ (respectively $\mathbf{B}_{K,L}$) as $M^{\phi,\Gamma,et}_{\mathbf{A}_{K,L}}$ (respectively $M^{\phi,\Gamma,et}_{\mathbf{B}_{K,L}}$).

Our main interest in (ϕ, Γ)-modules is the following result.

Proposition 4.2. *There is a tensor-equivalence of categories*

$$\mathbf{D} : Rep_{\mathcal{O}_L}(G_K) \longrightarrow M^{\phi,\Gamma,et}_{\mathbf{A}_{K,L}}$$

which preserves rank in the sense that $V \in Rep_{\mathcal{O}_L}(G_K)$ is free over $\mathcal{O}_L$ of rank d if and only if $\mathbf{D}(V)$ is free over $\mathbf{A}_{K,L}$ of rank d. Moreover, inverting p leads to an equivalence

$$\mathbf{D}[\frac{1}{p}] : Rep_L(G_K) \longrightarrow M^{\phi,\Gamma,et}_{\mathbf{B}_{K,L}}$$

Proof. This is essentially Theorem 3.4.3 of [11], augmented to coefficients in $\mathcal{O}_L$ (or L). For more details, see [5], especially Corollary 2.13. □

It remains to determine which objects in $M^{\phi,\Gamma,et}_{\mathbf{A}_{K,L}}$ correspond to crystalline representations.

Definition 4.3. Let $k \in \mathbb{N}$. A *Wach module* over $\mathbf{A}^+_{K,L}$ (respectively $\mathbf{B}^+_{K,L}$) with weights at most k is a free finite rank $\mathbf{A}^+_{K,L}$-module (respectively $\mathbf{B}^+_{K,L}$-module) N with an action of Γ_K that is trivial modulo π, together with a commuting action ϕ_N of Frobenius on $N[\frac{1}{\pi}]$ such that N is stable by ϕ_N and $N/\phi_N(N)$ is killed by $(\phi(\pi)/\pi)^k$.

Proposition 4.4.

(1) *$V \in Rep_L(G_K)$ is crystalline with labelled Hodge–Tate weights between 0 and k if and only if $\mathbf{D}[\frac{1}{p}](V)$ contains a Wach module of rank $dim_L(V)$ with weights at most k. This Wach module is unique if it exists, and is denoted $N(V)$.*

(2) *For $V \in Rep^{cris}_L(G_K)$, there is a bijection*

$$T \mapsto N(V) \cap D(T)$$

between G_K-stable $\mathcal{O}_L$-lattices $T \subseteq V$ and $\mathbf{A}^+_{K,L}$-lattices in $N(V)$ which are Wach modules over $\mathbf{A}^+_{K,L}$.

Proof. See [1], Theorem 2 and [5], especially Corollary 2.19. □

The above result gives some hope that Definition 2.1 may be used to define and study crystalline representations with Hodge–Tate weights outside the Fontaine–Laffaille range by understanding Wach modules with A-structure for $A \in \mathcal{C}_{k_L}$.

4.2.2. An outlook on smooth representability of crystalline framed deformation functors outside the Fontaine–Laffaille range

Provided that the departure from the Fontaine–Laffaille range is relatively elementary, there are some more concrete results we can give. To approach this, we first establish a more general result on smooth representability of framed deformation functors in the situation where there is no restriction on the extension classes we allow. We then apply this to establish a generalisation of the main results of this chapter in the situation where the Hodge–Tate weights may fall outside the Fontaine–Laffaille range, but where we nevertheless have an understanding of crystalline extensions by mimicking the "Hodge–Tate" case.

Fix representations $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ $(i = 1, 2)$ and assume that both pairs $(\overline{\rho_1}, \overline{\rho_2})$ and $(\overline{\rho_1}, \chi_p \otimes \overline{\rho_2})$ have no Jordan–Hölder factors in common. Fix any deformation problems $\mathcal{D}^{\square, X_i}_{\overline{\rho_i}}$ $(i = 1, 2)$, together with an extension $\overline{\rho} \in \mathrm{Ext}^1(\overline{\rho_2}, \overline{\rho_1})$, and define a functor $\mathcal{F}^X_{\overline{\rho}} : \mathcal{C}_{k_L} \longrightarrow \mathrm{Set}$ sending A to the set of lifts ρ of $\overline{\rho}$ to A with the property that

$$\rho = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}$$

for $\rho_i \in \mathcal{D}^{\square, X_i}_{\overline{\rho_i}}(A)$ $(i = 1, 2)$. We then have the following result.

Proposition 4.5. *The natural map $\mathcal{F}^X_{\overline{\rho}} \longrightarrow \mathcal{D}^{\square, X_1}_{\overline{\rho_1}} \times \mathcal{D}^{\square, X_2}_{\overline{\rho_2}}$ is smooth of relative dimension $n_1 n_2 (1 + [K : \mathbb{Q}_p])$.*

Proof. For $A \in \mathcal{C}_{k_L}$ and $\rho_i \in \mathcal{D}^{\square, X_i}_{\overline{\rho_i}}(A)$ $(i = 1, 2)$, put $M_A = \mathrm{Hom}_A(\rho_2, \rho_1)$. As in the proof of Theorem 3.6, it suffices to prove that $H^1(G_K, M_A)$ is free over A of rank $d = \dim_{k_L}(H^1(G_K, M_{k_L}))$, and that $d = n_1 n_2 [K : \mathbb{Q}_p]$ (since $\mathrm{Hom}_{G_K}(\rho_2, \rho_1) = 0$ by assumption).

Supposing $x_1, x_2, \ldots, x_r$ generate the maximal ideal m_A, we get (for an appropriate A-module B) an exact sequence

$$0 \longrightarrow B \longrightarrow A^r \longrightarrow m_A \longrightarrow 0$$

of A-modules, where the second map is $(a_1, a_2, \ldots, a_r) \mapsto \sum_j x_j a_j$. Since A is artinian, B is finitely generated as an A-module, and so there is a surjection $A^e \twoheadrightarrow B$ of A-modules, for appropriate integer e. Putting $N_A = M_A \bigotimes_A B$, we get an exact sequence

$$0 \longrightarrow N_A \longrightarrow M_A^r \longrightarrow m_A M_A \longrightarrow 0$$

of $A[G_K]$-modules, as well as a G_K-equivariant surjection $M_A^e \twoheadrightarrow N_A$, since M_A is free, and thus flat, over A. Since G_K has cohomological dimension 2, the H^2 functor is right exact and so we see that $H^2(G_K, N_A) = 0$ (using $H^2(G_K, M_A)^e = 0$, which follows from considering the Tate dual of M_A and recalling that $\overline{\rho_1}$ and $chi_p \otimes \overline{\rho_2}$ have no Jordan–Hölder factors in common).

Having established that $H^2(G_K, N_A) = 0$ it follows by taking cohomology of the exact sequence above that $H^1(G_K, m_A M_A) = m_A H^1(G_K, M_A)$. Observe further that $m_A M_A$ has Tate dual $\mathrm{Hom}_A(m_A\rho_1, \chi_p \otimes \rho_2)$, which by assumption has trivial H^0, and so $H^2(G_K, m_A M_A) = 0$ by local Tate duality. We conclude by taking cohomology of the exact sequence

$$0 \longrightarrow m_A M_A \longrightarrow M_A \longrightarrow M_{k_L} \longrightarrow 0$$

that $H^1(G_K, M_{k_L}) = H^1(G_K, M_A)/m_A$.

By Nakayama's lemma, we conclude that $H^1(G_K, M_A)$ is generated by d elements. On the other hand, from the local Euler characteristic formula together with the fact that $H^0(G_K, M_A)$ and $H^2(G_K, M_A)$ both vanish (by local Tate duality and the assumptions on ρ_1 and ρ_2), we see that $H^1(G_K, M_A)$ has size

$$\#H^1(G_K, M_A) = \#\left(\frac{\mathcal{O}_K}{\#M_A}\right) = (\#A)^{n_1 n_2 [K:\mathbb{Q}_p]}$$

from which we conclude that $d = n_1 n_2 [K : \mathbb{Q}_p]$ and that $H^1(G_K, M_A)$ is free over A of rank d, as required. $\square$

Remark 4.6. The author wishes to thank the reviewers for pointing out that, in the general setting, it is not clear what one could hope to prove beyond this.

We now aim to apply the above result to certain kinds of crystalline representations. We first make the following definition.

Definition 4.7. Let a be any integer. A representation $\rho : G_K \longrightarrow GL_n(A)$ is *crystalline with Hodge–Tate weights in the range* $[a, a+p-2]$ if $\rho \bigotimes \chi_p^{-a} \in \mathrm{Rep}_A^{cris,\leq p-2}(G_K)$, where χ_p denotes the cyclotomic character. The *labelled Hodge–Tate weights* of ρ are defined to be those of $\rho \bigotimes \chi_p^{-a}$ with each weight increased by a (and counted with multiplicity). With some abuse of notation we denote the multiset of labelled Hodge–Tate weights for a given label $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$ as either $HT_\sigma(\rho)$ or as $HT_\sigma(U_S^{-1}(\rho))$ where convenient.

The situation where the difference between the largest and smallest Hodge–Tate weights exceeds $p-2$ falls outside of the scope of Fontaine–Laffaille theory. However, we can at least make the following definition, inspired by Proposition 2.13.

Definition 4.8. Let $A \in \mathcal{C}_{k_L}$, and suppose $\rho_i : G_K \longrightarrow GL_{n_i}(A)$ $(i = 1, 2)$ are crystalline representations with Hodge–Tate weights in the range $[a_i, a_i + p - 2]$ in the sense of the preceding definition. Suppose further that $\overline{\rho_1} \ncong \chi_p \bigotimes \overline{\rho_2}$ and that for all σ, $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$. Then we put $\mathrm{Ext}^1_{cris,A}(\rho_2, \rho_1) = \mathrm{Ext}^1_A(\rho_2, \rho_1)$.

We then have the following result.

Theorem 4.9. *For $i = 1, 2, \ldots, r$, let $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ be pairwise distinct irreducible crystalline representations with Hodge–Tate weights in the range $[a_i, a_i + p - 2]$. Suppose further that for all pairs of integers $j < k$, $\overline{\rho_j} \ncong \chi_p \bigotimes \overline{\rho_k}$ and that for all σ, $HT_\sigma(\rho_j) > HT_\sigma(\rho_k)$. Let $\overline{M_i}$ be the rank n_i Fontaine–Laffaille module associated with $\overline{\rho_i} \bigotimes \chi_p^{-a_i}$. Fix a representation $\overline{\rho}$ which is block upper triangular with $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ on the diagonal. Then the functor $\mathcal{F}_{\overline{\rho}}$ as defined in Theorem 3.6 is represented by a power series ring over $\mathcal{O}_L$ in $([K : \mathbb{Q}_p] + 1)(\sum_{i,j:i \leq j} n_i n_j) - \sum_{i=1}^{r} d_{\overline{M_i}}$ variables.*

Proof. This follows from Proposition 4.5 and Theorem 3.2 analogously to the reasoning in the proof of Corollary 3.7. □

Remark 4.10. Note that since every extension is crystalline and we are in the Hodge–Tate case, we should expect an analog of the statement that $d_{\overline{M_{\leq i}}, \overline{M_i}} = d_{\overline{M_i}}$ for all i in the above Theorem, so this result is unsurprising.

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