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On an explicit zero-free region for the Dedekind zeta-function

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A R T I C L E I N F O

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A B S T R A C T

This article establishes new explicit zero-free regions for the Dedekind zeta-function. Two key elements of our proof are a non-negative, even, trigonometric polynomial and explicit upper bounds for the explicit formula of the so-called differenced logarithmic derivative of the Dedekind zeta-function. The improvements we establish over the last result of this kind come from two sources. First, our computations use a polynomial which has been optimised by simulated annealing for a similar problem. Second, we establish sharper upper bounds for the aforementioned explicit formula.

1. Introduction

Let K be an algebraic number field and L be a normal extension of K with Galois group $G = \mathrm{Gal}(L/K)$. Suppose d_L, d_K denote the absolute values of the respective discriminant, $n_L = [L : \mathbb{Q}]$ and $n_K = [K : \mathbb{Q}]$. The Dedekind zeta-function of L is denoted and defined for $\mathfrak{Re}(s) > 1$ by

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$$\zeta_L(s) = \sum_{\mathfrak{P}} \frac{1}{N(\mathfrak{P})^s},$$

where $\mathfrak{P}$ ranges over the non-zero ideals of $\mathcal{O}_L$. If $n_L = a + b$, then one can also consider the completed zeta-function

$$\xi_L(s) = s(s-1)d_L^{\frac{s}{2}}\gamma_L(s)\zeta_L(s) \text{ such that}$$
$$\gamma_L(s) = \pi^{-\frac{as}{2}}\Gamma\left(\frac{s}{2}\right)^a \pi^{-\frac{b(s+1)}{2}}\Gamma\left(\frac{s+1}{2}\right)^b.$$

Here, ξ_L is an entire function satisfying the functional equation $\xi_L(s) = \xi_L(1-s)$. It can be seen that ζ_L is meromorphic on the complex plane with exactly one simple pole at $s = 1$. Let $\mathcal{P}$ denote a prime ideal of K and P denote a prime ideal of L. If $\mathcal{P}$ is unramified in L, then the Artin symbol,

$$\left[\frac{L/K}{\mathcal{P}}\right],$$

denotes the conjugacy class of Frobenius automorphisms corresponding to prime ideals $P|\mathcal{P}$. For each conjugacy class $C \subset G$, the prime ideal counting function is

$$\pi_C(x, L/K) = \#\left\{\mathcal{P} : \mathcal{P} \text{ unramified in } L, \left[\frac{L/K}{\mathcal{P}}\right] = C, N_K(\mathcal{P}) \leq x\right\}.$$

In 1926, Chebotarëv [2] proved the Chebotarëv density theorem, which states that

$$\pi_C(x, L/K) \sim \frac{\#C}{\#G}\mathrm{Li}(x) = \frac{\#C}{\#G}\int_2^x \frac{dt}{\log t} \text{ as } x \to \infty.$$

For example, if $L = K = \mathbb{Q}$, then the Chebotarëv density theorem restates the prime number theorem. Moreover, if $\omega_\ell = e^{\frac{2\pi i}{\ell}}$ is the ℓth root of unity, $K = \mathbb{Q}$ and $L = \mathbb{Q}(\omega_\ell)$, then the Chebotarëv density theorem identifies with the Dirichlet theorem for primes in arithmetic progressions.

In 1977, Lagarias–Odlyzko [9] provided explicit estimates for the error term of the Chebotarëv density theorem. There are two results contained therein; one version assumes the generalised Riemann hypothesis (GRH) for ζ_L and the other does not. Their error term is effectively computable, dependent only on x, n_L, d_L and $\frac{\#C}{\#G}$.

Under the GRH for ζ_L, one can obtain the best possible effective results. Without assuming the GRH for ζ_L, the better the zero-free region for ζ_L one has, the better the effective result one can achieve. Therefore, the objective of this paper is to improve the best known, explicit zero-free region for ζ_L, given by Kadiri [6] in 2012. We recall two famous forms of zero-free regions for the Riemann zeta-function.

Classical zero-free region. In 1899, de la Vallée Poussin [3] famously proved that there exists a positive constant R such that ζ is non-zero in the region $s = \sigma + it$ such that $t \geq T$ and

$$\sigma \geq 1 - \frac{1}{R \log t}. \tag{1}$$

The best known zero-free region for ζ of this kind is attributed to Mossinghoff–Trudgian [12], who verified (1) for $R \approx 5.573$ and $T = 2$.

Koborov–Vinogradov zero-free region. In 1958, Koborov [8] and Vinogradov [15] independently demonstrated that there exists a positive constant R_1 such that ζ is non-zero in the region $s = \sigma + it$ such that $t \geq T$ and

$$\sigma \geq 1 - \frac{1}{R_1 (\log t)^{\frac{2}{3}} (\log\log t)^{\frac{1}{3}}}. \tag{2}$$

The best known zero-free region for ζ of this kind is attributed to Ford [4], who has verified (2) for $R_1 = 57.54$ and $T = 3$. Ford [4] also establishes the zero-free region (2) for large t with $R_1 = 49.13$.

Naturally, the closest form of the zero-free region for ζ_L will also depend on the extra variables d_L and n_L. However, the method we adopt is based on de la Vallée Poussin's method for determining the classical zero-free region for ζ. One complication is that a so-called *exceptional* zero could exist inside a zero-free region for ζ_L. If this exceptional zero exists, then it must be simple and real.

Kadiri [6, Theorem 1.1] was the last to re-purpose de la Vallée Poussin's proof (using Stečkin's [14] so-called differencing trick) to obtain a zero-free region for ζ_L. In this paper, we will establish Theorem 1, a new zero-free region for ζ_L which builds upon Kadiri's zero-free region for ζ_L. We will also establish Theorem 2, which will reveal more information pertaining to the exceptional zero.

Theorem 1. *Suppose* $(C_1, C_2, C_3, C_4) = (12.2411, 9.5347, 0.05017, 2.2692)$, *then* $\zeta_L(\sigma + it)$ *is non-zero for*

$$\sigma \geq 1 - \frac{1}{C_1 \log d_L + C_2 \cdot n_L \log|t| + C_3 \cdot n_L + C_4} \text{ and } |t| \geq 1. \tag{3}$$

Theorem 2. *For asymptotically large* d_L *and* $R = 12.43436$, $\zeta_L(\sigma + it)$ *has at most one zero in the region*

$$\sigma \geq 1 - \frac{1}{R \log d_L} \text{ and } |t| < 1. \tag{4}$$

If this exceptional zero exists, then it is simple and real.

Kadiri [6] established (3) with $(C_1, C_2, C_3, C_4) = (12.55, 9.69, 3.03, 58.63)$. To yield Theorem 1, we will follow a similar process to Kadiri, but observe some improvements. An important step in the proof of Theorem 1 is to choose a polynomial $p_n(\varphi)$ from the so-called the class of non-negative, trigonometric polynomials of degree n; denoted and defined by

$$P_n := \left\{ p_n(\varphi) = \sum_{k=0}^{n} a_k \cos(k\varphi) : p_n(\varphi) \geq 0 \text{ for all } \varphi,\ a_k \geq 0 \text{ and } a_0 < a_1 \right\}.$$

Whereas Kadiri worked with polynomials from P_4, we will use the same polynomial from P_{16} as Mossinghoff–Trudgian [12]. This polynomial has been optimised by simulated annealing for computations pertaining to their computations for the zero-free region for ζ. This amendment contributed *all* of the improvements that can be seen for C_1 and C_2. In fact, if one re-runs Kadiri's computations, only updating the polynomial, then this establishes (3) with $(C_1, C_2, C_3, C_4) = (12.2411, 9.5347, 3.3492, 57.7027)$.

Another improvement follows from improvements we have made to [11, Lemma 2] from McCurley. In particular, we improve explicit values for $\mathcal{S}(k)$, a computable constant dependent on $k \in \mathbb{N}$. These improvements will contribute almost all of the improvement one observes for C_3.

Kadiri [6] also established (4) with $R = 12.7305$. To yield Theorem 2, we will recycle bounds from [6, §3] and apply the same higher degree polynomial from P_{16}. A corollary of the method we use to establish Theorem 2 is an improvement to a well-known region by Stark [13]. However, because we only update the polynomial for this method, we cannot improve Stark's result further than [6, Corollary 1.2] already does.

Finally, if an exceptional zero β_1 exists, then one can enlarge the zero-free region in Theorem 2 using the Deuring-Heilbronn phenomenon [10]. This was one of the key ingredients in work by Ahn–Kwon [1], Zaman [16] and Kadiri–Ng–Wong [7], which pertains to the least prime ideal in the Chebotarëv density theorem.

Remark. The method of proof which we follow does not use Heath-Brown's version of Jensen's formula [5, Lemma 3.2], although this might yield better zero-free regions than those we can obtain using this method. This is partially because there does *not* exist a *general* sub-convexity bound for *general* number fields, so it is difficult to apply his approach in the number field setting — see Kadiri [6] for an excellent explanation of this.

2. Proof of Theorem 1

The set-up of our proof for Theorem 1 is the same as that which Kadiri uses in her proof of [6, Theorem 1.1], which has a similar shape to Stečkin's argument [14] for ζ. Suppose $t \geq 1$. We introduce some definitions, which will hold for the remainder of this paper:

- $\kappa = \frac{1}{\sqrt{5}}$;
- $s_k = \sigma + ikt$ such that $k \in \mathbb{N}$, $1 < \sigma < 1 + \varepsilon$ for some $0 < \varepsilon \leq 0.15$;
- $s'_k = \sigma_1 + ikt$ such that $\sigma_1 = \frac{1+\sqrt{1+4\sigma^2}}{2}$.

Note that σ_1 depends on σ, so for convenience we will write $\sigma_1(a)$ to denote the value of σ_1 at $\sigma = a$. To prove Theorem 1, we will isolate a non-trivial zero $\rho = \beta + it$ of ζ_L such that $\beta > 1 - \varepsilon \geq 0.85$, choose a polynomial $p_n(\varphi)$ from P_n, and consider the function

$$S(\sigma, t) = \sum_{k=0}^{n} a_k f_L(\sigma, kt),$$

such that

$$\begin{aligned} f_L(\sigma, kt) &= -\mathfrak{Re}\left(\frac{\zeta'_L}{\zeta_L}(s_k) - \kappa\frac{\zeta'_L}{\zeta_L}(s'_k)\right) \\ &= \sum_{0 \neq \mathfrak{P} \subset \mathcal{O}_L} \Lambda(\mathfrak{P})(N(\mathfrak{P})^{-\sigma} - \kappa N(\mathfrak{P})^{-\sigma_1}) \cos(kt\log(N(\mathfrak{P})). \end{aligned}$$

It follows that

$$S(\sigma, t) = \sum_{0 \neq \mathfrak{P} \subset \mathcal{O}_L} \Lambda(\mathfrak{P})(N(\mathfrak{P})^{-\sigma} - \kappa N(\mathfrak{P})^{-\sigma_1}) p_n(t\log(N(\mathfrak{P})) \geq 0.$$

On the other hand, we can utilise the explicit formula [9, (8.3)],

$$-\frac{\zeta'_L}{\zeta_L}(s_k) = \frac{\log d_L}{2} + \frac{1}{s_k} + \frac{1}{s_k - 1} + \frac{\gamma'_L}{\gamma_L}(s_k) - \frac{1}{2}\sum_{\varrho \in Z(\zeta_L)} \left(\frac{1}{s_k - \varrho} + \frac{1}{s_k - \overline{\varrho}}\right). \tag{5}$$

Here, $Z(\zeta_L)$ denotes the set of non-trivial zeros of ζ_L. One can use (5) to show

$$0 \leq S(\sigma, t) \leq S_1 + S_2 + S_3 + S_4, \tag{6}$$

where $F(s, z) = \mathfrak{Re}\left(\frac{1}{s-z} + \frac{1}{s-1+\bar{z}}\right)$ such that

$$\begin{aligned} S_1 &= -\sum_{k=0}^{n} a_k \sum_{\varrho \in Z(\zeta_L)} \mathfrak{Re}\left(\frac{1}{s_k - \varrho} - \frac{\kappa}{s'_k - \varrho}\right), \\ S_2 &= \frac{1-\kappa}{2}\left(\sum_{k=0}^{n} a_k\right)\log d_L, \\ S_3 &= \sum_{k=0}^{n} a_k \left(F(s_k, 1) - \kappa F(s'_k, 1)\right), \text{ and} \\ S_4 &= \sum_{k=0}^{n} a_k \mathfrak{Re}\left(\frac{\gamma'_L(s_k)}{\gamma_L(s_k)} - \kappa\frac{\gamma'_L(s'_k)}{\gamma_L(s'_k)}\right). \end{aligned}$$

We will choose $n = 16$, so we can apply Mossinghoff–Trudgian's polynomial $p_{16}(\varphi) \in P_{16}$ from [12]. Taking $n = 16$, S_2 is directly computable, and we find upper bounds for S_1, S_3, and S_4 in Sections 2.1, 2.2, and 2.3. The resulting upper bound for $S_1 + S_2 + S_3 + S_4$ will depend on β, σ, t, the coefficients of $p_{16}(\varphi)$ and ε, therefore we may use (6) and rearrange the inequality to obtain Theorem 1 in Section 2.4.

2.1. Upper bound for S_1

Lemma 3 *(Stečkin [14]). Suppose $s = \sigma + it$ with $1 < \sigma \leq 1.25$, $z \in \mathbb{C}$, and $0 < \mathfrak{Re}(z) < 1$, then*

$$F(s, z) - \kappa F(s_1', z) \geq 0. \tag{7}$$

Moreover, if $\mathfrak{Im}(z) = \mathfrak{Im}(s) = t$ and $\frac{1}{2} \leq \mathfrak{Re}(z) < 1$, then

$$\mathfrak{Re}\left(\frac{1}{s - 1 + \bar{z}}\right) - \kappa F(s_1', z) \geq 0.$$

Note that κ is the largest value such that (7) holds. This subsection is *not* an improvement on [6, Lemma 2.3], rather a repeat for the purpose of clarity. By the positivity condition (7) in Lemma 3, we have

$$\ell(s_k) := \sum_{\varrho \in Z(\zeta_L)} \mathfrak{Re}\left(\frac{1}{s_k - \varrho} - \frac{\kappa}{s_k' - \varrho}\right) \leq \kappa F(s_k', \rho) - F(s_k, \rho). \tag{8}$$

If $k = 1$, then (8) implies that

$$\ell(s_1) \leq -\frac{1}{\sigma - \beta} - \frac{1}{\sigma - 1 + \beta} + \frac{\kappa}{\sigma_1 - \beta} + \frac{\kappa}{\sigma_1 - 1 + \beta} = -\frac{1}{\sigma - \beta} + g(\sigma, \beta).$$

We see that $g(\sigma, \beta) < g(1, 1)$ and $g(1, 1)$ is small and negative, so $\ell(s_1) \leq -\frac{1}{\sigma - \beta}$. Moreover, if $k \neq 1$, then (8) implies that $\ell(s_k) \leq 0$ by (7). One can package the preceding observations into the following lemma.

Lemma 4. *Isolate a zero $\rho = \beta + it \in Z(\zeta_L)$ such that $\beta \geq 1 - \varepsilon \geq 0.85$, then*

$$\ell(\sigma + ikt) \leq \begin{cases} -\frac{1}{\sigma - \beta} & \text{if } k = 1, \\ 0 & \text{if } k \neq 1. \end{cases}$$

Therefore, $S_1 \leq -\frac{a_1}{\sigma - \beta}$.

Table 1
Admissible values for $\mathcal{B}_{0.15}(k)$.

k	$\mathcal{B}_{0.15}(k)$	k	$\mathcal{B}_{0.15}(k)$
1	0.23445352	9	0.00235718
2	0.06869804	10	0.00188669
3	0.02783858	11	0.00154513
4	0.01427867	12	0.00128917
5	0.0085573	13	0.0010924
6	0.00568194	14	0.00093759
7	0.00404715	15	0.00081374
8	0.00303134	16	0.00071303

2.2. Upper bound for S_3

Suppose that

$$\begin{aligned}\Sigma_k(\sigma,t) &:= F(\sigma+ikt,1)-\kappa F(\sigma_1+ikt,1)\\ &= \frac{\sigma}{\sigma^2+k^2t^2}+\frac{\sigma-1}{(\sigma-1)^2+k^2t^2}-\kappa\frac{\sigma_1}{{\sigma_1}^2+k^2t^2}-\kappa\frac{\sigma_1-1}{(\sigma_1-1)^2+k^2t^2}.\end{aligned}$$

Case I. If $k=0$, then Σ_k is only dependent on σ, with a singularity occurring at $\sigma=1$. In fact,

$$\Sigma_0(\sigma,t)=\frac{1}{\sigma}+\frac{1}{\sigma-1}-\frac{\kappa}{\sigma_1}-\frac{\kappa}{\sigma_1-1}:=\frac{1}{\sigma-1}+h(\sigma).$$

We observe that $h(\sigma)$ *increases* as σ increases, so for $\alpha_\varepsilon=h(1+\varepsilon)<0.021467$, we have

$$\Sigma_0(\sigma,t)\le\frac{1}{\sigma-1}+\alpha_\varepsilon.$$

Case II. Suppose $1\le k\le 16$, then $\Sigma_k(\sigma,t)$ depends on σ and t. For each σ, $\Sigma_k(\sigma,t)$ *decreases* as t increases, because the derivative of $\Sigma_k(\sigma,t)$ with respect to t is negative for all $t\ge1$. Therefore, $\Sigma_k(\sigma,t)\le\Sigma_k(\sigma,1)$, which *increases* as σ increases, because the derivative of $\Sigma_k(\sigma,1)$ with respect to σ is positive for all $1\le\sigma\le1.15$. It follows that

$$\Sigma_k(\sigma,t)\le\Sigma_k(1+\varepsilon,1)<\mathcal{B}_\varepsilon(k),$$

where admissible values for $\mathcal{B}_\varepsilon(k)$ are easily computed using a computer. To further verify this bound, the Maximize command in Maple confirms that the maximum of $\Sigma_k(\sigma,t)$ occurs at $\sigma=1+\varepsilon$ and $t=1$. For example, if $\varepsilon=0.15$ or $\varepsilon=0.01$, then admissible values of $\mathcal{B}_{0.15}(k)$ and $\mathcal{B}_{0.01}(k)$ are given in Table 1 and Table 2 respectively. Note that we round up at 8 decimal places, to account for any possible rounding errors.

Now, we can collect the preceding observations to yield Lemma 5.

Table 2
Admissible values for $\mathcal{B}_{0.01}(k)$.

k	$\mathcal{B}_{0.01}(k)$	k	$\mathcal{B}_{0.01}(k)$
1	0.10919579	9	0.00029396
2	0.03040152	10	0.00021655
3	0.00958566	11	0.00016557
4	0.00384196	12	0.00013046
5	0.00185609	13	0.00010535
6	0.00102853	14	0.00008684
7	0.00063099	15	0.00007282
8	0.00041809	16	0.00006196

Lemma 5. *For $0 \leq k \leq 16$, we have that*

$$\Sigma_k(\sigma, t) \leq \begin{cases} \frac{1}{\sigma - 1} + \alpha_\varepsilon & \text{if } k = 0, \\ \mathcal{B}_\varepsilon(k) & \text{if } k \neq 0. \end{cases}$$

Under a choice of polynomial from P_{16}, it follows that

$$S_3 \leq a_0 \left(\frac{1}{\sigma - 1} + \alpha_\varepsilon \right) + \sum_{k=1}^{16} a_k \mathcal{B}_\varepsilon(k).$$

Remark. The benefits of Lemma 5 over [6, Lemma 2.4] lie in the computed constants $\mathcal{B}_\varepsilon(k)$. That is, Kadiri established $\Sigma_k(\sigma, t) \leq 1.6666$ for $1 \leq k \leq 4$.

2.3. Upper bound for S_4

We bring forward an observation from Kadiri [6, §2.4],

$$\begin{aligned} \mathfrak{Re} \left(\frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \kappa \frac{\gamma_L'(s_k')}{\gamma_L(s_k')} \right) &\leq -\frac{1-\kappa}{2} \cdot \log \pi \cdot n_L \\ &+ \frac{n_L}{2} \max_{\delta \in \{0,1\}} \left\{ \mathfrak{Re} \left(\frac{\Gamma'}{\Gamma} \left(\frac{s_k + \delta}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left(\frac{s_k' + \delta}{2} \right) \right) \right\}. \end{aligned}$$

Case I. If $k = 0$, then we directly compute that

$$\frac{1}{2} \max_{\delta \in \{0,1\}} \left\{ \mathfrak{Re} \left(\frac{\Gamma'}{\Gamma} \left(\frac{\sigma + \delta}{2} \right) - \kappa \frac{\Gamma'}{\Gamma} \left(\frac{\sigma_1 + \delta}{2} \right) \right) \right\} \leq d_\varepsilon(0), \tag{9}$$

where $d_\varepsilon(0)$ is the maximum of the functions such that $\sigma = 1 + \varepsilon$. To see this, one can observe that the left-hand side of (9) is maximised at $\sigma = 1 + \varepsilon$ visually *or* use the Maximize command in Maple. For example, if $\varepsilon = 0.01$, then

$$d_{0.01}(0) = -0.2500763736.$$

Case II. Suppose $1 \leq k \leq 16$. McCurley [11, Lemma 2] establishes that

$$\frac{1}{2}\mathfrak{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s_k+\delta}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{s_k'+\delta}{2}\right)\right) = \frac{1-\kappa}{2}\log\frac{kt}{2} + \Xi(\sigma,k,t,\delta)$$
$$+ \frac{\theta_1}{2k}\left(\frac{\pi}{2} - \arctan\left(\frac{1+\delta}{k}\right)\right) + \kappa\frac{\theta_2}{2k}\left(\frac{\pi}{2} - \arctan\left(\frac{\sigma_1(1)+\delta}{k}\right)\right), \tag{10}$$

where $|\theta_i| \leq 1$ and

$$\Xi(\sigma,k,t,\delta) = \frac{1}{4}\log\left[1 + \left(\frac{\sigma+\delta}{kt}\right)^2\right] - \frac{\kappa}{4}\log\left[1 + \left(\frac{\sigma_1+\delta}{kt}\right)^2\right]$$
$$- \frac{\sigma+\delta}{2((\sigma+\delta)^2+k^2t^2)} + \kappa\frac{\sigma_1+\delta}{2((\sigma_1+\delta)^2+k^2t^2)}.$$

Next, we will bound $\Xi(\sigma,k,t,\delta)$ using two different methods, then choose the best bound for each k.

Method I. For any $t > 0$, we have

$$\Xi_1(\sigma,k,t,\delta) := -\frac{\sigma+\delta}{2((\sigma+\delta)^2+k^2t^2)} + \kappa\frac{\sigma_1+\delta}{2((\sigma_1+\delta)^2+k^2t^2)}$$
$$\leq \frac{\kappa(\sigma_1+\delta)-\sigma-\delta}{2((\sigma_1+\delta)^2+k^2t^2)} \leq 0,$$

because $\sigma < \sigma_1$ and $\kappa(\sigma_1+\delta)-\sigma-\delta \leq 0$. Moreover, for fixed σ, observe that

$$\Xi_2(\sigma,k,t,\delta) := \frac{1}{4}\log\left[1 + \left(\frac{\sigma+\delta}{kt}\right)^2\right] - \frac{\kappa}{4}\log\left[1 + \left(\frac{\sigma_1+\delta}{kt}\right)^2\right]$$

is positive for $t \geq 1$ and decreases as t increases, because the derivative of $\Xi_2(\sigma,k,t,\delta)$ with respect to t is negative for all $t \geq 1$. Therefore,

$$\Xi_2(\sigma,k,t,\delta) \leq \Xi_2(\sigma,k,1,\delta)$$

for $t \geq 1$, which increases as σ increases in the range $1 \leq \sigma \leq 1.15$, because the derivative of $\Xi_2(\sigma,k,1,\delta)$ with respect to σ is positive for $1 \leq \sigma \leq 1.15$. Hence, for each k,

$$\Xi_2(\sigma,k,t,\delta) \leq \Xi_2(1+\varepsilon,k,1,\delta).$$

To verify the preceding bound, the Maximize command in Maple confirms that the maximum of $\Xi_2(\sigma,k,t,\delta)$ occurs at $\sigma = 1+\varepsilon$ and $t = 1$. It follows that $\Xi(\sigma,k,t,\delta) \leq \Xi_2(1+\varepsilon,k,1,\delta)$ for each k and

$$\frac{1}{2}\max_{\delta\in\{0,1\}}\left\{\mathfrak{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s_k+\delta}{2}\right)-\kappa\frac{\Gamma'}{\Gamma}\left(\frac{s_k'+\delta}{2}\right)\right)\right\}\leq\frac{1-\kappa}{2}\log t+\mathcal{S}_1(k,\varepsilon),$$

where $\mathcal{S}_1(k,\varepsilon)=\max_{\delta\in\{0,1\}}\{\mathcal{C}_1(k,\delta,\varepsilon)\}$ such that

$$\begin{aligned}\mathcal{C}_1(k,\delta,\varepsilon):=&\frac{1-\kappa}{2}\log\frac{k}{2}+\Xi_2(1+\varepsilon,k,1,\delta)\\&+\frac{1}{2k}\left(\frac{\pi}{2}-\arctan\left(\frac{1+\delta}{k}\right)\right)+\frac{\kappa}{2k}\left(\frac{\pi}{2}-\arctan\left(\frac{\sigma_1(1)+\delta}{k}\right)\right).\end{aligned}$$

Method II. We will verify that for $0<\varepsilon\leq 0.15$,

$$\Xi(\sigma,k,t,\delta)\leq\mathcal{A}(k,\delta,\varepsilon):=\begin{cases}0 & \text{if } \delta=0,\\ 0 & \text{if } \delta=1 \text{ and } k\notin\{1,2\},\\ \Xi(1+\varepsilon,k,1,1) & \text{if } \delta=1 \text{ and } k=1.\\ \Xi(1.15,k,1,1) & \text{if } \delta=1 \text{ and } k=2.\end{cases} \tag{11}$$

First, for fixed σ and $\delta=0$, the derivative of $\Xi(\sigma,k,t,\delta)$ with respect to t is positive for $t\geq 1$, so $\Xi(\sigma,k,t,0)$ is increasing as $t\to\infty$. Therefore, for each $\sigma\in[1,1.15]$,

$$\Xi(\sigma,k,t,0)\leq\lim_{t\to\infty}\Xi(\sigma,k,t,0)=0.$$

Next, for fixed σ and $\delta=1$, the derivative of $\Xi(\sigma,k,t,\delta)$ with respect to t is positive for $t\geq 1$ whenever $k\notin\{1,2,3\}$, so $\Xi(\sigma,k,t,1)$ is increasing as $t\to\infty$ for $k\notin\{1,2,3\}$. Therefore, for each $k\notin\{1,2,3\}$ and $1\leq\sigma\leq 1.15$,

$$\Xi(\sigma,k,t,1)\leq\lim_{t\to\infty}\Xi(\sigma,k,t,1)=0.$$

To completely verify (11), we now establish bounds for the special cases $\delta=1$ and $k\in\{1,2,3\}$. Observe that for each $t\geq 1$, the derivative of $\Xi(\sigma,k,t,1)$ with respect to σ is positive for $1\leq\sigma\leq 1.15$ whenever $k\in\{1,2,3\}$, so

$$\Xi(\sigma,k,t,1)\leq\Xi(1+\varepsilon,k,t,1). \tag{12}$$

Suppose that $k\in\{1,2,3\}$ and observe that in the range $t\geq 1$, $\Xi(1+\varepsilon,k,t,1)$ *either* has one minimum point at $t=t_k(\varepsilon)$ *or* increases as $t\to\infty$. Here, $t_k(\varepsilon)$ equals the only root of the derivative of $\Xi(1+\varepsilon,k,t,1)$ with respect to t in the range $t\geq 1$. If this root does not exist, then set $t_k(\varepsilon)=1$ for convenience. For example, $t_1(0.15)=3.2308\ldots$, $t_2(0.15)=1.6154\ldots$, $t_3(0.15)=1.0769\ldots$ and $t_3(0.01)=1$. It follows that $\Xi(1+\varepsilon,k,t,1)$ decreases for $1\leq t\leq t_k(\varepsilon)$ and $\Xi(1+\varepsilon,k,t,1)$ increases for $t>t_k(\varepsilon)$, so

$$\Xi(1+\varepsilon,k,t,1)\leq\begin{cases}\Xi(1+\varepsilon,k,1,1) & \text{if } 1\leq t\leq t_k(\varepsilon),\\ \lim_{t\to\infty}\Xi(1+\varepsilon,k,t,1) & \text{if } t>t_k(\varepsilon),\end{cases}$$

in which $\lim_{t\to\infty} \Xi(1+\varepsilon, k, t, 1) = 0$ for each k. If $k = 1$, then for $t \geq 1$, we have

$$\Xi(1+\varepsilon, 1, t, 1) \leq \max\{\Xi(1+\varepsilon, 1, 1, 1), 0\} = \Xi(1+\varepsilon, 1, 1, 1). \tag{13}$$

Observe that $\Xi(1+\varepsilon, 2, 1, 1)$ increases as $0 < \varepsilon \leq 0.15$ increases. So, if $k = 2$, then for $t \geq 1$, we have

$$\begin{aligned}\Xi(1+\varepsilon, 2, t, 1) \leq \max\{\Xi(1+\varepsilon, 2, 1, 1), 0\} &\leq \max\{\Xi(1.15, 2, 1, 1), 0\} \\ &\leq \Xi(1.15, 2, 1, 1).\end{aligned} \tag{14}$$

In this case, the final bound is convenient and not too wasteful, because $\Xi(1.15, 2, 1, 1)$ is small. Finally, if $k = 3$, then for $t \geq 1$, we have

$$\Xi(1+\varepsilon, 3, t, 1) \leq \max\{\Xi(1+\varepsilon, 3, 1, 1), 0\} = 0. \tag{15}$$

Combining the observation (12) with (13), (14), and (15) implies (11). For each k, it follows from (10) and (11) that

$$\frac{1}{2}\max_{\delta\in\{0,1\}}\left\{\mathfrak{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s_k+\delta}{2}\right) - \kappa\frac{\Gamma'}{\Gamma}\left(\frac{s_k'+\delta}{2}\right)\right)\right\} \leq \frac{1-\kappa}{2}\log t + \mathcal{S}_2(k,\varepsilon),$$

where $\mathcal{S}_2(k,\varepsilon) = \max_{\delta\in\{0,1\}}\{\mathcal{C}_2(k,\delta,\varepsilon)\}$ such that

$$\begin{aligned}\mathcal{C}_2(k,\delta,\varepsilon) :=& \frac{1-\kappa}{2}\log\frac{k}{2} + \mathcal{A}(k,\delta,\varepsilon) \\ &+ \frac{1}{2k}\left(\frac{\pi}{2} - \arctan\left(\frac{1+\delta}{k}\right)\right) + \frac{\kappa}{2k}\left(\frac{\pi}{2} - \arctan\left(\frac{\sigma_1(1)+\delta}{k}\right)\right).\end{aligned}$$

Combination. We say that $\mathcal{S}(k,\varepsilon) = \min(\mathcal{S}_1(k,\varepsilon), \mathcal{S}_2(k,\varepsilon))$ and (for $1 \leq k \leq 16$) present the quantities $\mathcal{S}_1(k, 0.15)$, $\mathcal{S}_2(k, 0.15)$ and $\mathcal{S}(k, 0.15)$ alongside each other in Table 3. It turns out that $\mathcal{S}_2(k, 0.15)$ yields a better bound for cases $k = 1, 2, 3, 4$ and $\mathcal{S}_1(k, 0.15)$ yields the better bound otherwise. Finally, we package our observations into a useful lemma (Lemma 6).

Lemma 6. *For $0 \leq k \leq 16$, we have shown that*

$$\mathfrak{Re}\left(\frac{\gamma_L'(s_k)}{\gamma_L(s_k)} - \kappa\frac{\gamma_L'(s_k')}{\gamma_L(s_k')}\right) \leq \begin{cases} n_L\left(d_\varepsilon(0) - \frac{1-\kappa}{2}\cdot\log\pi\right) & \textit{if } k = 0, \\ n_L\left(\frac{1-\kappa}{2}\left(\log t + \log\left(\frac{k}{\pi}\right)\right) + \mathcal{S}(k,\varepsilon)\right) & \textit{if } k \neq 0.\end{cases}$$

Under a choice of polynomial from P_{16}, it follows that

$$S_4 \leq a_0 n_L\left(d_\varepsilon(0) - \frac{1-\kappa}{2}\cdot\log\pi\right) + \sum_{k=1}^{16} a_k n_L\left(\frac{1-\kappa}{2}\left(\log t + \log\left(\frac{k}{\pi}\right)\right) + \mathcal{S}(k,\varepsilon)\right).$$

Table 3
Computed values for $\mathcal{S}_1(k, 0.15)$, $\mathcal{S}_2(k, 0.15)$ and $\mathcal{S}(k, 0.15)$.

k	$\mathcal{S}_1(k, 0.15)$	$\mathcal{S}_2(k, 0.15)$	$\mathcal{S}(k, 0.15)$
1	0.3784516540	0.3249009026	0.3249009026
2	0.3839873212	0.3763572015	0.3763572015
3	0.4018562060	0.4004551145	0.4004551145
4	0.4238223974	0.4236306767	0.4236306767
5	0.4467597648	0.4468482525	0.4467597648
6	0.4693610537	0.4695098183	0.4693610537
7	0.4910902618	0.4912403488	0.4910902618
8	0.5117562107	0.5118920810	0.5117562107
9	0.5313238925	0.5314428586	0.5313238925
10	0.5498280118	0.5499312088	0.5498280118
11	0.5673323540	0.5674218683	0.5673323540
12	0.5839104248	0.5839883668	0.5839104248
13	0.5996362678	0.5997044990	0.5996362678
14	0.6145802698	0.6146403531	0.6145802698
15	0.6288074426	0.6288606647	0.6288074426
16	0.6423769295	0.6424243440	0.6423769295

Table 4
Table of coefficients for Mossinghoff–Trudgian's polynomial $p_{16}(\varphi) \in P_{16}$.

a_0	1
a_1	1.74126664022806
a_2	1.128282822804652
a_3	0.5065272432186642
a_4	0.1253566902628852
a_5	$2.372710620 \cdot 10^{-26}$
a_6	$2.818732841 \cdot 10^{-22}$
a_7	0.01201214561729989
a_8	0.006875849760911001
a_9	$2.064157910 \cdot 10^{-23}$
a_{10}	$6.601587090 \cdot 10^{-11}$
a_{11}	0.001608306592372963
a_{12}	0.001017994683287104
a_{13}	$6.728831293 \cdot 10^{-11}$
a_{14}	$3.682448595 \cdot 10^{-11}$
a_{15}	$2.949853019 \cdot 10^{-6}$
a_{16}	0.00003713656497

Remark. The benefits of Lemma 6 over [6, Lemma 2.5] lie in the computed constants $d_\varepsilon(0)$ and $\mathcal{S}(k, \varepsilon)$. Kadiri imports results from McCurley [11, Lemma 2] for her bound, so the improvements we see follow from our observations pertaining to McCurley's work.

2.4. Computations

As declared in the introduction, we will choose the polynomial $p_{16}(\varphi) \in P_{16}$ from [12], whose coefficients are given in Table 4. Suppose $r > 0$ and σ is chosen such that $\sigma - 1 = r(1 - \beta)$ where $\rho = \beta + it \in Z(\zeta_L)$ is an isolated zero such that $\beta \geq 1 - \varepsilon \geq 0.85$. Applying the upper bounds for each S_i, which can be found in Lemmas 4, 5 and 6, then rearranging inequality (6) will yield

Table 5
Constants for the explicit zero-free region in Theorem 1 given $\varepsilon = 0.15$ or $\varepsilon = 0.01$.

	$\varepsilon = 0.15$	$\varepsilon = 0.01$
M	0.1021253857	0.1021253857
$\frac{c_1}{M}$	12.24106100	12.24106100
$\frac{c_2}{M}$	9.534650638	9.534650638
$\frac{c_3}{M}$	0.444485082	0.050168175
$\frac{c_4}{M}$	5.123026304	2.269182727

$$\beta \leq 1 - \frac{\frac{a_1}{1+r} - \frac{a_0}{r}}{c_1 \log d_L + c_2 n_L \log t + c_3 n_L + c_4}, \tag{16}$$

where

$$c_1 = \frac{1-\kappa}{2} \sum_{k=0}^{16} a_k,$$

$$c_2 = \frac{1-\kappa}{2} \sum_{k=1}^{16} a_k,$$

$$c_3 = a_0 \left(d_\varepsilon(0) - \frac{1-\kappa}{2} \log \pi \right) + \sum_{k=1}^{16} a_k \left(\frac{1-\kappa}{2} \log \left(\frac{k}{\pi} \right) + \mathcal{S}(k, \varepsilon) \right) \text{ and}$$

$$c_4 = \alpha_\varepsilon a_0 + \sum_{k=1}^{16} a_k \mathcal{B}_\varepsilon(k).$$

For the remainder of this proof, we replicate the process which Kadiri [6] followed. The maximum value of $\frac{a_1}{1+r} - \frac{a_0}{r}$ occurs at $r = \frac{\sqrt{a_0}}{\sqrt{a_1} - \sqrt{a_0}}$. Therefore, dividing the numerator and denominator of (16) by

$$M = \frac{a_1}{1 + \frac{\sqrt{a_0}}{\sqrt{a_1} - \sqrt{a_0}}} - \frac{a_0}{\frac{\sqrt{a_0}}{\sqrt{a_1} - \sqrt{a_0}}},$$

we see that

$$\beta \leq 1 - \frac{1}{\frac{c_1}{M} \log d_L + \frac{c_2}{M} n_L \log t + \frac{c_3}{M} n_L + \frac{c_4}{M}}. \tag{17}$$

In Table 5, we present the constants for two choices of ε. Observing the values for $\varepsilon = 0.01$, inequality (17) will yield the explicit zero-free region (3) for $t \geq 1$, which completes the proof of Theorem 1.

3. Proof of Theorem 2

Theorem 2 is an improvement of part of [6, Theorem 1.2]. We will recycle Kadiri's proof, except we use the polynomial $p_{16}(\varphi)$ in place of a polynomial from P_4. Suppose $\log d_L$ is asymptotically large and consider three regions,

$$\mathbb{I}_A = \left(0, \frac{d_1}{\log d_L}\right], \mathbb{I}_B = \left(\frac{d_1}{\log d_L}, \frac{d_2}{\log d_L}\right], \mathbb{I}_C = \left(\frac{d_2}{\log d_L}, 1\right),$$

where d_1, d_2 are constants to be chosen. Suppose further, that

$$\sigma - 1 = \frac{r}{\log d_L} \text{ and } 1 - \beta = \frac{c}{\log d_L}.$$

In the regions $\mathbb{I}_B$ and $\mathbb{I}_C$, we impose further restrictions. Suppose $0 < c, r < 1$ such that

$$\frac{a_0}{a_1 - a_0} c < r \text{ and } d_2 > \frac{\sqrt{r(r+c)}}{2}.$$

Combining analogous arguments to those results in [6, §3.2, §3.3, §3.4], one can easily establish that

$$0 \leq \frac{1}{r} - 2\frac{r+c}{(r+c)^2 + {d_1}^2} + \frac{1-\kappa}{2} \tag{18}$$

in the region $\mathbb{I}_A$,

$$\begin{aligned} 0 \leq &\mathcal{E}_B(d_1, d_2, r, c) \\ &:= \frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2 + {d_1}^2} - \frac{a_0(r+c)}{(r+c)^2 + {d_1}^2} \\ &- \frac{a_0(r+c)}{(r+c)^2 + {d_2}^2} - \frac{a_1(r+c)}{(r+c)^2 + 4{d_2}^2} + \frac{1-\kappa}{2}\sum_{k=0}^{16} a_k \\ &+ \sum_{k=2}^{16} a_k \left(\frac{r}{r^2 + k^2{d_1}^2} - \frac{r+c}{(r+c)^2 + (k-1)^2{d_2}^2} - \frac{r+c}{(r+c)^2 + (k+1)^2{d_2}^2}\right) \end{aligned} \tag{19}$$

in the region $\mathbb{I}_B$ and

$$\begin{aligned} 0 \leq &\mathcal{E}_C(d_2, r, c) \\ &:= \frac{a_0}{r} - \frac{a_1}{r+c} + \frac{a_1 r}{r^2 + {d_2}^2} - \frac{a_0(r+c)}{(r+c)^2 + {d_2}^2} + \frac{1-\kappa}{2}\sum_{k=0}^{16} a_k \end{aligned} \tag{20}$$

in the region $\mathbb{I}_C$. Suppose d_1 and r are fixed. The admissible values of c which one can input into (18) are those c such that

$$c \geq \frac{\sqrt{r^2 - {d_1}^2 \left(1 + \frac{1-\kappa}{2} r\right)^2} - \frac{1-\kappa}{2} r^2}{1 + \frac{1-\kappa}{2} r}. \tag{21}$$

Denote the smallest value for c in (21) by c_A. Next, let c_B denote the root of $\mathcal{E}_B(d_1, d_2, r, c)$, where r is chosen such that the root c_B is as small as possible. Similarly, let c_C denote the smallest root of $\mathcal{E}_C(d_2, r, c)$ for some optimally chosen r. It follows that ζ_L has at most one zero in the region $s = \sigma + it$ such that $t < 1$ and

$$\sigma \geq 1 - \frac{1}{R \log d_L}$$

such that $R = \max(1/c_A, 1/c_B, 1/c_C)$. Moreover, if an exceptional zero exists then it is real and simple by [6, §3.5]. To complete our proof of Theorem 2, it will suffice to show that $R = 12.43436$ is an admissible value.

First, suppose that we choose the same values that Kadiri chose; $d_1 = 1.021$ and $d_2 = 2.374$. One can establish that $1/c_A = 12.5494$ when $r = 2.1426$. Moreover, using our higher degree polynomial, we can compute the roots of $\mathcal{E}_B(1.021, 2.374, r, c)$ and $\mathcal{E}_C(2.374, r, c)$ over a selection of r. The results of these computations are presented below.

Root of	r	$1/c$
$\mathcal{E}_B(1.021, 2.374, r, c)$	0.2366	12.43922
$\mathcal{E}_C(2.374, r, c)$	0.2477	12.42548

Therefore, these choices of d_1 and d_2 would yield Theorem 2 with

$$R = \max(12.5494, 12.43922, 12.42548) = 12.5494.$$

Above, the limiting factor appears to be the value for $1/c_A$. We can reduce the value of $1/c_A$ by decreasing the value of d_1, however, we are also limited by the sizes of $1/c_B$ and $1/c_C$ which we can obtain. Therefore, we only need to choose d_1 such that $1/c_A$ is small enough. The cost of choosing d_1 too small is a larger interval $\mathbb{I}_B$, which might not be ideal.

Given d_1, to find a good enough choice for d_2, we have tested many values for d_2 and computed the optimal outcomes in each case. If one chooses $d_1 = 1.0015$, then we found (to 3 decimal places) that $d_2 = 2.318$ yielded the best results. For this d_1, one can determine that $1/c_A = 9.7946$ when $r = 2.1163$. The results of the remaining computations for $1/c_B$ and $1/c_C$ are presented below.

Root of	r	$\frac{1}{c}$
$\mathcal{E}_B(1.0015, 2.318, r, c)$	0.2363	12.43355
$\mathcal{E}_C(2.318, r, c)$	0.2473	12.43436

Therefore — as required — these choices of d_1 and d_2 will yield Theorem 2 with

$$R = \max(9.7946, 12.43355, 12.43436) = 12.43436.$$

CRediT authorship contribution statement

Ethan S. Lee: Conceptualization, Data curation, Formal analysis, Investigation, Methodology, Project administration, Software, Validation, Visualization, Writing – original draft, Writing – review & editing.

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