Journal of Number Theory **80**, 273–290 (2000)
Article ID jnth.1999.2453, available online at http://www.idealibrary.com on IDEAL®

Engel Expansions and the Rogers–Ramanujan Identities

George E. Andrews[1]

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802
E-mail: andrews@math.psu.edu

Arnold Knopfmacher

Department of Computational and Applied Mathematics, University of the Witwatersrand, Johannesburg, Wits 2050, South Africa
E-mail: arnoldk@gauss.cam.wits.ac.za

and

John Knopfmacher[2]

Department of Mathematics, Centre for Applicable Analysis and Number Theory, University of Witwatersrand, and University of Melbourne, Parkville, Victoria, Australia 3052

Communicated by David Goss

Received December 21, 1998

We study the previously developed extension of the Engel expansion to the field of Formal Laurent series. We examine three separate aspects. First we consider the algorithm in relation to the work of Ramanujan. Second we show how the algorithm can be used to prove expansions such as those found by Euler, Rogers, and Ramanujan. Finally we remark briefly on its use in acceleration of convergence.

Key Words: Engel expansions; Rogers–Ramanujan identities; acceleration of convergence.

1. INTRODUCTION

The mysteries surrounding the methods used by Ramanujan to discover his amazing results have led to numerous speculations. In this paper we

The first author wishes to express his gratitude to the second who provided the hospitality and support which made his participation possible.

[1] Partially supported by National Science Foundation Grant DMS-9206993 and by The Centre for Applicable Analysis and Number Theory of the University of the Witwatersrand.

[2] It is with great sadness that we note that John Knopfmacher passed away on May 29, 1999 at age 62. He will live on through his research, in particular through his three books on Abstract Analytic Number Theory.

0022-314X/00 $35.00

shall probably throw no real light on what Ramanujan actually thought. However, we shall demonstrate that a series expansion algorithm introduced by two of us [10], [11], at least provides a plausible path to some of Ramanujan's hits [3, Chap. 7] and misses [5, p. 130].

We begin by recalling the Engel expansion [10] for the field $\mathcal{L} = \mathbb{C}((q))$ of formal Laurent series over the complex numbers, $\mathbb{C}$. If

$$A = \sum_{n=v}^{\infty} C_n q^n, \tag{1.1}$$

we call $v = v(A)$ the *order* of A and we define the *norm* of A to be

$$\|A\| = 2^{-v(A)}. \tag{1.2}$$

In addition, we define the *integral part* of A by

$$[A] = \sum_{v \leqq n \leqq 0} C_n Q^n. \tag{1.3}$$

Engel (c.f. [13, Sect. 34]) originally defined a series expansion for real numbers. In [10], this concept was extended to $\mathcal{L}$ in the following way:

Extended Engel Expansion Theorem ([10, Theorem 1.4]). *Every $A \in \mathcal{L}$ has a finite or convergent (relative to the above norm) series expansion of the form*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n}, \tag{1.4}$$

where $a_n \in \mathbb{C}[q^{-1}]$, $a_0 = [A]$,

$$v(a_n) \leqq -n, \qquad \textit{and} \qquad v(a_{n+1}) \leqq v(a_n) - 1. \tag{1.5}$$

The series (1.4) is unique for A (up to constants in $\mathbb{C}$), and it is finite if and only if $A \in \mathbb{C}(q)$. In addition, if

$$a_0 + \sum_{j=1}^{n} \frac{1}{a_1 \cdots a_j} = \frac{p_n}{q_n}, \qquad \textit{where} \quad q_n = a_1 a_2 \cdots a_n, \tag{1.6}$$

then

$$\left\| A - \frac{p_n}{q_n} \right\| \leqq \frac{1}{2^{n+1} \|q_n\|} \tag{1.7}$$

and

$$v\left(A-\frac{p_n}{q_n}\right)=-v(q_{n+1})\geqq\frac{(n+1)(n+2)}{2} \tag{1.8}$$

In fact, the a_n are given by

$$a_n=\left[\frac{1}{A_n}\right] \tag{1.9}$$

where $A_0=A$, $a_0=[A]$, and

$$A_{n+1}=a_n A_n-1. \tag{1.10}$$

At the time of preparation of [10] it was noticed that a number of famous expansions including those of Euler and the Rogers–Ramanujan identities are, in fact, special cases of the Engel expansion. In Section 2, we consider the empirical use that Ramanujan might have made of such an expansion. Section 3 is devoted to showing how the Engel algorithm can be used to prove such identities. We conclude with comments about possible further extensions.

2. RAMANUJAN'S FAILED CONJECTURE

In his first Notebook [5, p. 130], Ramanujan wrote down and then crossed out the following assertion:

$$\prod_{n=1}^{\infty}\frac{1}{1-q^{p_n}}=1+\sum_{j=1}^{\infty}\frac{q^{p_1+p_2+\cdots+p_j}}{(1-q)(1-q^2)\cdots(1-q^j)}, \tag{2.1}$$

where p_n is the nth prime. A few moments of reflection will suggest to you that this is an amazing and probably unbelievable assertion. Indeed the assertion is false as was shown in [4]; the power series expansions of both sides disagree at q^{21}. In fact, the study and extension of (2.1) in [4] has led to numerous further results (cf. [1], [6], [7], [9]).

So if this assertion is false (and, to be fair, Ramanujan crossed it out), how did he ever think it up in the first place? Suppose he had the extended Engel expansion. If we take A to be the left-hand side of (2.1), we may directly calculate (via Mathematica) that

$$a_1 = \frac{1}{q^2} - \frac{1}{q}$$

$$a_2 = \frac{1}{q^3} - \frac{1}{q}$$

$$a_3 = \frac{1}{q^5} - \frac{1}{q^2}$$

$$a_4 = -1 + \frac{1}{q^7} - \frac{1}{q^2}$$

$$a_5 = 1 - \frac{1}{q^8} - \frac{1}{q^7} + \frac{1}{q}$$

$$a_6 = 240 + \frac{1}{q^{10}} - \frac{1}{q^9} + \frac{2}{q^8} - \frac{4}{q^7} + \frac{7}{q^6} - \frac{13}{q^5} + \frac{23}{q^4} - \frac{41}{q^3} + \frac{74}{q^2} - \frac{134}{q}$$

$$\begin{aligned} a_7 = & -\frac{782107492743072987592134 33872}{434364932556686502987836 63146561} + \frac{1}{433q^{11}} \\ & + \frac{347}{187489q^{10}} + \frac{37273}{81182737q^9} \\ & - \frac{32413493}{351552125121q^8} - \frac{7244501857}{152208701773 93q^7} - \frac{6001513330790}{659063678611169q^6} \\ & - \frac{2710837873954506}{285374572868923 6177q^5} + \frac{1214546819910349722}{123567190052243926 4641q^4} \\ & + \frac{206849587495087652597}{535045932926216201589553:q^3} - \frac{389726942232901898611707}{23167488895705161528827 6449q^2} \\ & - \frac{30836482706562208443781823 37}{10031522691840334941982370 2417q} \end{aligned}$$

Note that even the first three calculations are tiresome by hand but that they confirm the series on the right of (2.1) is valid through $j = 3$. The term $j = 4$ is just barely wrong. Things start to fall apart at $j = 5$ and 6 and go completely to pieces at $j = 7$.

Consequently, while (2.1) is wrong, we must at least concede that it is a natural conjecture for anyone who might possess the extended Engel algorithm for application to a variety of functions such as the product in (2.1).

We are not asserting that Ramanujan possessed the extended Engel expansion. We only note that knowledge of it and lack of a computer might well lead to (2.1) as a conjecture.

3. PROOFS VIA THE EXTENDED ENGEL EXPANSION

The first author of [10] noted during its preparation that the expansion (1.4) is actually exhibited in a number of famous results. We shall now show that these results can actually be proved using the algorithm that constructs (1.4). The hard part of each proof will be the presentation of a suitably tractable form for the A_n given in (1.10).

THEOREM 1. *Suppose $|q|<1$ with z any fixed complex number and let*

$$F_1(q)=\prod_{n=1}^{\infty}(1+zq^n),$$

then the Extended Engel Expansion of $F_1(q)$ is

$$1+\sum_{n=1}^{\infty}\frac{z^n q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Remark. Of course, an immediate corollary of Theorem 1 is the identity of these two representations of $F_1(q)$. I.e.

$$\prod_{n=1}^{\infty}(1+zq^n)=1+\sum_{n=1}^{\infty}\frac{z^n q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{3.1}$$

Proof. It is immediate by mathematical induction that

$$1+\sum_{n=1}^{N}(1+zq)(1+zq^2)\cdots(1+zq^{n-1})\,zq^n=\prod_{j=1}^{N}(1+zq^j). \tag{3.2}$$

Hence with

$$A=A_0=\prod_{n=1}^{\infty}(1+zq^n), \tag{3.3}$$

we see that $a_0=1$ and

$$A_1=a_0A_0-1=\sum_{m=1}^{\infty}(1+zq)(1+zq^2)\cdots(1+zq^{m-1})\,zq^m. \tag{3.4}$$

Next we prove that

$$a_n=z^{-1}(q^{-n}-1)\qquad\text{for}\quad n>0 \tag{3.5}$$

and

$$A_n = \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, zq^{mn}, \tag{3.6}$$

by noting that (1.10) is fulfilled, namely

$$\begin{aligned}
&z^{-1}(q^{-n}-1) \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, zq^{mn} - 1 \\
&\quad= \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, q^{m(n-1)} \\
&\qquad - \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, q^{mn} - 1 \\
&\quad= \sum_{m=2}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, q^{n(m-1)} \\
&\qquad - \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, q^{mn} \\
&\quad= \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, q^{mn}((1+zq^m)-1) \\
&\quad= \sum_{m=1}^{\infty} (1+zq)(1+zq^2)\cdots(1+zq^{m-1})\, zq^{m(n+1)},
\end{aligned} \tag{3.7}$$

and that

$$\begin{aligned}
\left[\frac{1}{A_n}\right] &= \left[\frac{1}{zq^n + (1-q)\, zq^{2n} + O(q^{3n})}\right] \\
&= \left[z^{-1}q^{-n} \frac{1}{1+q^n+q^{n+1}+O(q^{2n})}\right] \\
&= [z^{-1}q^{-n}(1-q^n-q^{n+1}+O(q^{2n}))] \\
&= z^{-1}q^{-n} - z^{-1} = z^{-1}(q^{-n}-1) \qquad \text{as desired.}
\end{aligned}$$

So (3.3) and (3.7) guarantee that the a_n and A_n given by (3.5) and (3.6) are the sequences arising in the extended Engel expansion. Therefore by (1.4)

$$\prod_{n=1}^{\infty}(1+zq^n)=A_0$$
$$=1+\sum_{j=1}^{\infty}\frac{1}{a_1a_2\cdots a_j}$$
$$=1+\sum_{j=1}^{\infty}\frac{z^jq^{j(j+1)/2}}{(1-q)(1-q^2)\cdots(1-q^j)}. \quad \blacksquare \tag{3.8}$$

The Rogers–Ramanujan identities follow similarly although now the initial step is made possible by Schur's polynomials [14].

THEOREM 2. *Suppose* $|q|<1$, *and let*

$$F_2(q)=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$

then the Extended Engel Expansion of $F_2(q)$ *is*

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Remark. These two representations of $F_2(q)$ yield immediately the first Rogers–Ramanujan identity [3, p. 104, Eq. (7.1.6)]:

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-4})(1-q^{5n-1})}=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{3.9}$$

Proof. We take as our starting point the fact proved by I. Schur [14] that if

$$D_n=\begin{cases}0 & \text{if} \quad n<-1\\ 1 & \text{if} \quad n=-1\\ D_{n-1}+q^n D_{n-2} & \text{otherwise,}\end{cases} \tag{3.10}$$

then

$$D_{n-1}=\sum_{j=-\infty}^{\infty}(-1)^j q^{j(5j+1)/2}\begin{bmatrix} n \\ \left\lfloor \dfrac{n-5j}{2}\right\rfloor\end{bmatrix}_q, \tag{3.11}$$

where

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{cases} 0 & \text{if} \quad \beta < 0 \quad \text{or} \quad \beta > \alpha \\ \dfrac{(1-q^{\alpha})(1-q^{\alpha-1})\cdots(1-q^{\alpha-\beta+1})}{(1-q^{\beta})(1-q^{\beta-1})\cdots(1-q)}, & 0 \leqq \beta \leqq \alpha \end{cases} \tag{3.12}$$

As Schur [14] noted,

$$D_{\infty} = \frac{\sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2}}{\prod_{n=1}^{\infty} (1-q^n)}$$

$$= \frac{\prod_{n=0}^{\infty} (1-q^{5n+2})(1-q^{5n+3})(1-q^{5n+5})}{\prod_{n=1}^{\infty} (1-q^n)}$$

(by Jacobi's triple product [3, p. 21, Eq. (2.2.10)])

$$= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} := A.$$

Now note, by (3.10)

$$\sum_{j=0}^{N} q^j D_{j-2} = \sum_{j=0}^{N} (D_j - D_{j-1}) = D_N - 1. \tag{3.14}$$

So in the limit

$$A_1 = A_0 - 1 = A - 1 = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})} - 1 = \sum_{j=0}^{\infty} q^j D_{j-2} \tag{3.15}$$

It is now possible to prove that the relevant a_n and A_n for the extended Engel expansion are given by

$$a_n = \begin{cases} 1 & \text{if} \quad n = 0 \\ q^{-2n+1} - q^{-n+1} & \text{if} \quad n > 0, \end{cases} \tag{3.16}$$

and

$$A_n = \sum_{j=0}^{\infty} q^{nj+(n-1)} D_{j-2}. \tag{3.17}$$

These assertions follow immediately from the initial condition (3.15) and the following two facts. First, again by (3.10)

$$\begin{aligned}
&(q^{-2n+1}-q^{-n+1})\sum_{j=0}^{\infty} q^{nj+(n+1)} D_{j-2}-1 \\
&\quad=\sum_{j=1}^{\infty} q^{n(j-2)+n} D_{j-2}-\sum_{j=1}^{\infty} q^{n(j-1)+n} D_{j-2}-1 \\
&\quad=\sum_{j=0}^{\infty} q^{n(j-1)+n} D_{j-1}-\sum_{j=1}^{\infty} q^{n(j-1)+n} D_{j-2}-1 \\
&\quad=\sum_{j=1}^{\infty} q^{n(j-1)+n}(D_{j-1}-D_{j-2}) \\
&\quad=\sum_{j=1}^{\infty} q^{n(j-1)+n+j-1} D_{j-3} \\
&\quad=\sum_{j=2}^{\infty} q^{(n+1)(j-1)+n} D_{j-3} \\
&\quad=\sum_{j=0}^{\infty} q^{(n+1)j+n} D_{j-2}.
\end{aligned} \tag{3.18}$$

Finally, by (3.17)

$$\begin{aligned}
\left[\frac{1}{A_n}\right]&=\left[\frac{1}{q^{2n-1}+q^{3n-1}+O(q^{4n})}\right] \\
&=\left[q^{1-2n}\frac{1}{1+q^n+O(q^{2n+1})}\right] \\
&=[q^{1-2n}(1-q^n+O(q^{2n}))] \\
&=q^{1-2n}-q^{1-n},
\end{aligned} \tag{3.19}$$

as required.

Thus the extended Engel expansion for A of (3.13) is established. So

$$\begin{aligned}
&\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \\
&\quad=A \\
&\quad=1+\sum_{n=1}^{\infty}\frac{1}{a_1a_2\cdots a_n} \\
&\quad=1+\sum_{n=1}^{\infty}\frac{1}{(q^{-1}-1)(q^{-3}-q^{-1})\cdots(q^{1-2n}-q^{1-n})} \\
&\quad=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}. \quad \blacksquare
\end{aligned} \tag{3.20}$$

THEOREM 3. *Suppose $|q|<1$, and let*

$$F_3(q)=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-3})(1-q^{5n-2})},$$

then the Extended Engel Expansion of $F_3(q)$ is

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

Remark. This result giving two different representations of $F_3(q)$ immediately yields

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-3})(1-q^{5n-2})}=1+\sum_{n=1}^{\infty}\frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{3.21}$$

Proof. The proof here is completely parallel to that of Theorem 2; so we present the essentials and allow the reader to fill in the details.

This time Schur [14] tells us that if

$$E_n=\begin{cases}0 & \text{if} \quad n<0\\ 1 & \text{if} \quad n=0\\ E_{n-1}+q^nE_{n-2} & \text{for} \quad n>0,\end{cases} \tag{3.22}$$

then

$$E_{n-1}=\sum_{j=-\infty}^{\infty}(-1)^j\,q^{j(5j-3)/2}\begin{bmatrix} n \\ \left\lfloor \dfrac{n-5j+2}{2}\right\rfloor\end{bmatrix}_q. \tag{3.23}$$

Consequently by Jacobi's triple product identity [3, p. 21, Eq. (2.2.10)]:

$$E_\infty=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+3})}=A=A_0. \tag{3.24}$$

As before

$$A_1=A_0-1=A-1=\prod_{n=0}^{\infty}\frac{1}{(1-q^{5n+2})(1-q^{5n+2})}-1=\sum_{j=0}^{\infty}q^jE_{j-2}-1 \tag{3.25}$$

In this case, we prove that

$$a_n = \begin{cases} 1 & \text{if} \quad n=0 \\ q^{-2n} - q^{-n} & \text{for} \quad n>0, \end{cases} \tag{3.26}$$

and

$$A_n = \sum_{j=0}^{\infty} q^{nj} E_{j-2}. \tag{3.27}$$

This follows from a perfect analog of (3.18) and the fact that for $n \geqq 1$

$$\begin{aligned} \left[\frac{1}{A_n}\right] &= \left[\frac{1}{q^{2n} + q^{3n} + q^{4n} + O(q^{4n+2})}\right] \\ &= \left[q^{-2n} \frac{1}{1 + q^n + q^{2n} + O(q^{2n+2})}\right] \\ &= [q^{-2n}(1 - (q^n + q^{2n} + O(q^{2n+2})) + (q^n + q^{2n} + O(q^{2n+2}))^2 + O(q^{3n}))] \\ &= [q^{-2n}(1 - q^n - q^{2n} + q^{2n} + O(q^{2n+2}))] \\ &= q^{-2n} - q^{-n}. \end{aligned} \tag{3.28}$$

Thus the extended Engel expansion for the A of (3.24) is established. So

$$\begin{aligned} \prod_{n=0}^{\infty} & \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \\ &= A \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)}. \quad \blacksquare \end{aligned} \tag{3.29}$$

We now turn to what ought to be the most straight forward example of all, the generating function for $p(n)$, the total number of partitions of n [3, p. 4]. Now

$$A := \sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}. \tag{3.30}$$

Direct computation of the relevant a_n reveals these first few values:

$$a_1 = -2 + \frac{1}{q}$$

$$a_2 = 2 - \frac{1}{q^2} + \frac{1}{q}$$

$$a_3 = -\frac{1}{2q^3} + \frac{1}{2q^2} + \frac{3}{4q}$$

$$a_4 = -\frac{51388}{16807} - \frac{4}{7q^4} + \frac{48}{49q^3} - \frac{184}{343q^2} + \frac{4560}{2401q}$$

$$\begin{aligned} a_5 = {} & \frac{369186144138061664381 1569037}{324140365758786997492 9000000} \\ & + \frac{16807}{76940q^5} + \frac{14470827}{1479940900q^4} \\ & - \frac{44936558936787}{113866652846000q^3} - \frac{338838765638461333}{547556266873202500q^2} \\ & + \frac{842236168527139770 0067}{168515916692896801400000q} \end{aligned}$$

However, if we instead apply the variant of the Engel algorithm given in [11; p. 251] with $s_n = a_n$ and $r_n = q^{-1}$, we may then prove the familiar [3; p. 21, Eq. (2.2.9)].

THEOREM 4. *Suppose* $|q| < 1$, *and let*

$$F_4(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

then the variant of the Extended Engel Expansion [11; p. 251; *with* $s_n = a_n$, $r_n = q^{-1}$] *of* $F_4(q)$ *is*

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2 (1-q^2)^2 \cdots (1-q^n)^2}$$

Remark. Theorem 4 has the following familiar identity for the partition generating function as a corollary

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}. \tag{3.31}$$

Proof. It is immediate by mathematical induction that

$$1+\sum_{j=1}^{N}\frac{q^j}{(1-q)(1-q^2)\cdots(1-q^j)}=\frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{3.32}$$

Hence with

$$A=A_0=\prod_{n=1}^{\infty}\frac{1}{1-q^n}=1+\sum_{j=1}^{\infty}\frac{q^j}{(1-q)(1-q^2)\cdots(1-q^j)} \tag{3.33}$$

we see that $a_0=[A]=1$ and (following the variant of the Engel expansion mentioned just before the statement of this theorem)

$$\begin{aligned} A_1&=q(a_0A_0-1)\\ &=\sum_{j=1}^{\infty}\frac{q^{j+1}}{(1-q)(1-q^2)\cdots(1-q^j)} \end{aligned} \tag{3.34}$$

Next we prove that

$$a_n=q^{-2n}(1-q^n)^2 \qquad \text{for} \quad n>0 \tag{3.35}$$

and

$$A_n=q^{2n}\sum_{j=0}^{\infty}\frac{q^{nj}}{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+j})} \tag{3.36}$$

(noting that (3.36) coincides with (3.34) when $n=1$).

The proof resembles what has gone before; namely

$$\begin{aligned} &q\left(q^{-2n}(1-q^n)^2\sum_{j=0}^{\infty}\frac{q^{nj+2n}}{(1-q^n)(1-q^{n+1})\cdots(1-q^{n+j})}-1\right)\\ &=q\left(\sum_{j=0}^{\infty}\frac{q^{nj}(1-q^n)}{(1-q^{n+1})\cdots(1-q^{n+j})}-1\right)\\ &=q\left(\sum_{j=1}^{\infty}\frac{q^{nj}}{(1-q^{n+1})\cdots(1-q^{n+j})}-\sum_{j=0}^{\infty}\frac{q^{n(j+1)}}{(1-q^{n+1})\cdots(1-q^{n+j})}\right)\\ &=q\left(\sum_{j=1}^{\infty}\frac{q^{nj}}{(1-q^{n+1})\cdots(1-q^{n+j})}-\sum_{j=1}^{\infty}\frac{q^{nj}(1-q^{n+j})}{(1-q^{n+1})\cdots(1-q^{n+j})}\right)\\ &=q\left(\sum_{j=1}^{\infty}\frac{q^{j(n+1)+n}}{(1-q^{n+1})\cdots(1-q^{n+j})}\right)\\ &=q^{2n+2}\sum_{j=0}^{\infty}\frac{q^{j(n+1)}}{(1-q^{n+1})\cdots(1-q^{n+1+j})}, \end{aligned} \tag{3.37}$$

thus confirming $A_{n+1} = q(a_n A_n - 1)$, and

$$\begin{aligned}\left[\frac{1}{A_n}\right] &= \left[q^{-2n} \frac{1}{1+q^n+q^{2n}+q^n+q^{2n+1}+2q^{2n}+O(q^{3n})}\right] \\ &= [q^{-2n}(1-(2q^n+3q^{2n}+q^{2n+1}+O(q^{3n})) \\ &\quad +(2q^n+\cdots)^2+O(q^{3n}))] \\ &= [q^{-2n}(1-2q^n+q^{2n}+O(q^{2n+1}))] \\ &= q^{-2n}(1-q^n)^2, \qquad \text{as desired.} \end{aligned} \tag{3.38}$$

Hence, we see that by Prop. 2 of [11; p. 251] with $s_n = a_n$ and $r_n = q^{-1}$

$$\begin{aligned}\prod_{n=1}^{\infty} \frac{1}{1-q^n} &= A = 1 + q^{-1} A_1 \\ &= 1 + q^{-1} \sum_{n=1}^{\infty} \frac{q^{1-n}}{a_1 a_2 \cdots a_n} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{1+3+\cdots+(2n-1)}}{(1-q)^2 (1-q^2)^2 \cdots (1-q^n)^2} \\ &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2 \ (1-q^2)^2 \cdots (1-q^n)^2}. \quad \blacksquare \end{aligned} \tag{3.39}$$

It should be pointed out that the Engel algorithm need not be restricted to identities relating infinite series to infinite products. To this end we close this section by proving a formula of N. J. Fine [8, p. 55, Eq. (26.22)] for one of Ramanujan's third order mock theta functions.

THEOREM 5. *Suppose* $|q| < 1$, *and let*

$$F_5(q) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^n}{(1+q)(1+q^2) \cdots (1+q^n)},$$

then the variant of the Extended Engel Expansion [11; p. 251, *with* $s_n = a_n$, $r_n = q^{-1}$] *of* $F_5(q)$ *is*

$$1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 (1+q^2)^2 \cdots (1+q^n)^2}.$$

Remark. As before, we deduce directly here that

$$1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n}{(1+q)(1+q^2)\cdots(1+q^n)}$$
$$=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}. \tag{3.40}$$

Proof. This proof is very similar to that of Theorem 4. Indeed it is possible to prove a common generalization of Theorems 4 and 5; however for simplicity we restrict ourselves to (3.40).

In contrast with (3.33) we now have

$$A=A_0=1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n}{(1+q)(1+q^2)\cdots(1+q^n)}, \tag{3.41}$$

$$A_1=q(A_0-1)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n+1}}{(1+q)(1+q^2)\cdots(1+q^n)}, \tag{3.42}$$

and in general

$$a_n=q^{-2n}(1+q^n)^2, \tag{3.43}$$

$$A_n=\sum_{j=1}^{\infty}\frac{(-1)^{j-1}q^{n(j+1)}}{(1+q^n)(1+q^{n+1})\cdots(1+q^{n+j-1})}. \tag{3.44}$$

In exactly the way (3.37) was proved, one may show that

$$A_{n+1}=q(a_nA_n-1), \tag{3.45}$$

and

$$\left[\frac{1}{A_n}\right]=\left[q^{-2n}\frac{1}{1-2q^n+3q^{2n}+O(q^{2n+1})}\right]$$
$$=[q^{-2n}\{1+(2q^n-3q^{2n}+O(q^{2n+1}))$$
$$+(2q^n+O(q^{2n}))^2+O(q^{2n+1})\}]$$
$$=q^{-2n}(1+2q^n+q^{2n})$$
$$=q^{-2n}(1+q^n)^2. \tag{3.46}$$

Finally exactly parallel to (3.39) we see that

$$\begin{aligned}
1+\sum_{n=1}^{\infty} & \frac{(-1)^{n-1} q^n}{(1+q)(1+q^2)\cdots(1+q^n)} \\
&= 1+q^{-1}A_1 \\
&= 1+q^{-1}\sum_{n=1}^{\infty}\frac{q^{1-n}}{a_1 a_2 \cdots a_n} \\
&= 1+\sum_{n=1}^{\infty}\frac{q^{1+3+\cdots+(2n-1)}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2} \\
&= 1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots s(1+q^n)^2}. \quad \blacksquare \qquad (3.47)
\end{aligned}$$

Our five theorems in this section are just a sampling of applications of Engel expansions. A variety of other identities are Engel expansions such as two more theorems of L. J. Rogers:

$$\begin{aligned}
1+\sum_{n=1}^{\infty} & \frac{q^{n(3n-1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)(1-q)(1-q^3)\cdots(1-q^{2n-1})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{10n-6})(1-q^{10n-4})(1-q^{10n})}{(1-q^n)}, \qquad (3.48)
\end{aligned}$$

and [15, p. 156, Eq. (46)]

$$\begin{aligned}
\sum_{n=1}^{\infty} & \frac{q^{3n(n-1)/2}}{(1-q)(1-q^2)\cdots(1-q^{n-1})(1-q)(1-q^3)\cdots(1-q^{2n-1})} \\
&= \prod_{n=1}^{\infty}\frac{(1-q^{10n-8})(1-q^{10n-2})(1-q^{10n})}{(1-q^n)} \qquad (3.49)
\end{aligned}$$

[15, p. 156, Eq. (44)] as well as Gauss's familiar theorem

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{n=1}^{\infty}\frac{(1-q^{2n})}{(1-q^{2n-1})} \qquad (3.50)$$

[3, p. 23, Eq. (2.2.13)]

In addition to these results from q-series, we note that there are other applications of the Engel expansion that are in quite a different direction; indeed [12] considers Engel expansion related to certain algebraic functions. The a_n in this case turn out to be instances of Lucas polynomials.

4. CONCLUSION

We should note that in fact the extended Engel expansion may be viewed as an algorithm for the acceleration of convergence. In each of our five theorems, $A_0 = A_1 - 1$ was given as a series whose nth term went to 0 like q^n. The expansion

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \tag{4.1}$$

in each of our five theorems possesses an nth term going to 0 like q^{cn^2}.

The possibilities for deriving further such algorithms are numerous. Indeed in earlier papers [10], [11], a number of such other algorithms are considered. Furthermore our near miss and retrieval of Theorem 4 suggests that only slight variations in the algorithm greatly alter the outcome.

Finally the existence of multiple series generalizations of the Rogers–Ramanujan identities [2] such as

$$\prod_{\substack{n=1 \\ n \not\equiv 0,\ \pm 3 (\mathrm{mod} 7)}}^{\infty} \frac{1}{1-q^n} = \sum_{n,m=0}^{\infty} \frac{q^{(n+m)^2+m^2}}{(1-q)(1-q^2)\cdots(1-q^n)(1-q)(1-q^2)\cdots(1-q^m)}$$

suggests the possibility of multidimensional analogs of the extended Engel expansion.

Note added in proof. We have subsequently provided complete accounts of the extended Engel expansions for (3.48)–(3.50). These will appear in An Algorithmic Approach to Discovering and Proving q-Series Identities, *Algorithmica* (to appear).

REFERENCES

1. D. Acreman and J. Loxton, Asymptotic analysis of Ramanujan pairs, *Aequationes Math.* **29** (1985), 319.
2. G. E. Andrews, An analytic generalization of the Rogers–Ramanujan identities for odd moduli, *Proc. Nat. Acad. Sci. U.S.A.* **71** (1974), 4082–4085.
3. G. E. Andrews, "The Theory of Partitions," Encyclopedia of Math. and Its Applications, Vol. 2, Addison–Wesley, Reading, 1976. (Reissued; Cambridge University Press, Cambridge, 1984 and 1998).
4. G. E. Andrews, An incredible formula of Ramanujan, *Austral. Math. Soc. Gaz.* **6** (1979), 80–89.
5. B. C. Berndt, "Ramanujan's Notebooks, Part I," Springer, New York, 1985.
6. R. Blecksmith, "The Determination of Ramanujan Pairs," Ph.D. thesis, Univ. Arizona, Tucson, 1983.

7. R. Blecksmith, J. Brillhart, and I. Gerst, A computer assisted investigation of Ramanujan pairs, *Math. Comp.* **46** (1986), 731–749.
8. N. J. Fine, "Basic Hypergeometric Series and Applications," Math. Surveys and Monographs, No. 27, Amer. Math. Soc., Providence, 1988.
9. M. D. Hirschhorn, Two further Ramanujan pairs, *J. Austral. Math. Soc. Ser. A* **30** (1980), 1–4.
10. A. Knopfmacher and J. Knopfmacher, Inverse polynomial expansions of Laurent series, *Constructive Approx.* **4** (1988), 379–389.
11. A. Knopfmacher and J. Knopfmacher, Inverse polynomial expansions of Laurent series, II, *J. Comp. and Appl. Math.* **28** (1989), 249–257.
12. A. Knopfmacher and M. Mays, Pierce expansions of ratios of Fibonacci and Lucas numbers and polynomials, *Fibo. Quart.* **33** (1995), 153–163.
13. O. Perron, "Irrationalzahlen," reprinted, Chelsea, New York, 1951.
14. I. Schur, Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche, *in* "I. Schur," Ges. Abh., Vol. 2, Springer, Berlin, 1973, pp. 117–136.
15. L. J. Slater, Further identities of the Rogers–Ramanujan type, *Proc. London Math. Soc. Ser. 2* **54** (1952), 147–167.