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Overpartitions and real quadratic fields

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Abstract

It is shown that counting certain differences of overpartition functions is equivalent to counting elements of a given norm in appropriate real quadratic fields.

Keywords: Overpartitions; Real quadratic fields; Bailey pairs

1. Introduction

In 1988, Andrews et al. [2] studied the coefficient of q^m in

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)}, \tag{1.1}$$

a q-series that had appeared in Ramanujan's "lost" notebook [1]. They found that these coefficients have multiplicative properties determined by a certain Hecke character associated to the real quadratic field $\mathbb{Q}(\sqrt{6})$ (cf. [4]). Although predicted to exist by Dyson [7], no other examples of q-series determined by such characters had been observed until recent work of Corson et al. [5], who showed that the coefficient of q^m in the series

$$\sum_{n\geqslant 0} \frac{(1-q)(1-q^2)\cdots(1-q^n)(-1)^n q^{n(n+1)/2}}{(1+q)(1+q^2)\cdots(1+q^n)} \tag{1.2}$$

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is equal to $(-1)^m$ times the number of ideals of norm $8m+1$ in $\mathbb{Z}(\sqrt{2})$.

One pleasant surprise in their work is that (1.2) is also the generating function for certain overpartitions. Recall that an overpartition of n is a partition of n in which the first occurrence of a number may be overlined. Specifically, let $f(m)$ denote the number of overpartitions of m into distinct parts such that $\bar{1}$ does not occur and consecutive parts differ by at least two if the larger is overlined. If $f^{\pm}(m)$ denotes the number of overpartitions counted by $f(m)$ with largest part even (odd), then $f^+(m) - f^-(m)$ is the coefficient of q^m in (1.2). Using some basic arithmetic in $\mathbb{Z}(\sqrt{2})$, it is then possible to give an exact formula for $f^+(m) - f^-(m)$ in terms of the factorization of m [5, Section 6].

As observed in [5], series (1.2) is closely related to the series in the left-hand side of the Rogers–Ramanujan type identity,

$$\sum_{n\geqslant 0} \frac{(1+q)(1+q^2)\cdots(1+q^n)q^{n(n+1)/2}}{(1-q)(1-q^2)\cdots(1-q^n)} = \prod_{n=1}^{\infty} (1+q^n)(1-q^{2n}). \tag{1.3}$$

In fact, this series is the generating function for $f(m)$ [10,12]. Recent works have embedded this identity in a family of overpartition theorems [10, Theorem 1.2], and uncovered three other families of overpartition theorems which are analogues of some classical and celebrated results in the theory of partitions [10,11]. We shall demonstrate here that the objects in the base cases of these three other families are also connected in a precise way to real quadratic fields. In view of the rarity of such connections, this is both surprising and satisfying. It will allow us to give exact formulas for the relevant overpartition functions, and further demonstrates that the combinatorics of overpartitions is an excellent guide to finding q-series with interesting behavior. The three main theorems follow.

Theorem 1.1. *Let m have the prime factorization $m = 2^a p_1^{e_1}\cdots p_j^{e_j} q_1^{f_1}\cdots q_k^{f_k}$, where the p_i are congruent to ± 1 modulo 8 and the q_i are congruent to ± 3 modulo 8. Let $a(m)$ denote the number of overpartitions of m into distinct parts which differ by at least two if the smaller part is overlined. Let $a^{\pm}(m)$ denote the number of overpartitions counted by $a(m)$ with largest part even (odd). Then $a^+(m) - a^-(m)$ is equal to 0, if some f_i is odd, and $-2i^{m^2-m}(e_1+1)\cdots(e_j+1)$ otherwise.*

Theorem 1.2. *Let m have the prime factorization $m = 2^a 3^b p_1^{e_1}\cdots p_j^{e_j} q_1^{f_1}\cdots q_k^{f_k} r_1^{g_1}\cdots r_\ell^{g_\ell}$, where the p_i are congruent to 1 modulo 12, the q_i are congruent to ± 5 modulo 12, and the r_i are congruent to 11 modulo 12. Let $b(m)$ denote the number of overpartitions $b_1 + b_2 + \cdots + b_{2n}$ of m such that b_i can be overlined only if i is odd and such that*

$$b_{2j} - b_{2j+1} \geqslant \begin{cases} 1, & b_{2j+1} \text{ non-overlined}, \\ 2, & b_{2j+1} \text{ overlined}. \end{cases}$$

Let $b^{\pm}(m)$ denote the number of overpartitions counted by $b(m)$ with largest part even (odd). Then $b^{+}(m) - b^{-}(m)$ is equal to 0, if some f_i is odd or $a + b + \sum g_i$ is even, and $2(-1)^{m+1}(e_1+1)\cdots(e_j+1)(g_1+1)\cdots(g_\ell+1)$ otherwise.

Theorem 1.3. *Let m have the prime factorization $m = 2^a 3^b p_1^{e_1}\cdots p_j^{e_j} q_1^{f_1}\cdots q_k^{f_k} r_1^{g_1}\cdots r_\ell^{g_\ell}$, where the p_i are congruent to ± 7 or ± 11 modulo 24, the q_i are congruent to 1 or 19 modulo 24, and the r_i are congruent to 5 or 23 modulo 24. Let $c(m)$ denote the number of overpartitions $c_1 + c_2 + \cdots + c_n$ of m such that*

$$c_j - c_{j+1} \geqslant \begin{cases} 1, & c_{j+1} \text{ even}, \\ 2, & c_{j+1} \text{ odd and overlined}, \\ 3, & c_{j+1} \text{ even and overlined}. \end{cases}$$

Let $c^{\pm}(m)$ denote the number of overpartitions counted by $c(m)$ with largest part even (odd). Then $c^{-}(m) - c^{+}(m)$ is equal to 0, if some e_i is odd or $a + \sum g_i$ is odd, and $-2i^{m^2+m}(f_1+1)\cdots(f_k+1)(g_1+1)\cdots(g_\ell+1)$ otherwise.

The generating functions for $a(m)$, $b(m)$, and $c(m)$ are the series in the Rogers–Ramanujan type identities

$$\sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n} = \frac{(-q;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$\sum_{n\geqslant 0} \frac{(-1;q^2)_n q^{n^2+n}}{(q)_{2n}} = \frac{(-q^2;q^2)_\infty(-q^3,q^6;q^6)_\infty}{(q^2;q^2)_\infty(q^3,-q^6;q^6)_\infty} \tag{1.5}$$

and

$$\sum_{n\geqslant 0} \frac{(-1)_{2n} q^n}{(q^2;q^2)_n} = \frac{(-q)_\infty(q^3;q^3)_\infty}{(q)_\infty(-q^3;q^3)_\infty}. \tag{1.6}$$

As usual, we have employed the notation

$$(a)_n := (a;q)_n := \prod_{j=0}^{n-1}(1 - aq^j). \tag{1.7}$$

The difference functions occurring in Theorems 1.1–1.3 correspond to the related series

$$\sum_{n=1}^{\infty} \frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q)_n}, \tag{1.8}$$

$$\sum_{n=1}^{\infty} \frac{(q^2;q^2)_{n-1}(-1)^n q^{n^2+n}}{(-q)_{2n}} \tag{1.9}$$

and

$$\sum_{n=1}^{\infty} \frac{(q)_{2n-1}q^n}{(-q^2;q^2)_n}, \tag{1.10}$$

which we shall link to $\mathbb{Z}(\sqrt{2})$, $\mathbb{Z}(\sqrt{3})$, and $\mathbb{Z}(\sqrt{6})$, respectively.

2. Proofs of the main theorems

We require some background on Bailey pairs. Two sequences (α_n, β_n) are said to form a Bailey pair with respect to a if for all $n \geqslant 0$ we have

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_n}{(q)_{n-r}(aq)_{n+r}}. \tag{2.1}$$

The following weakened form of the Bailey lemma allows us to prove identities using Bailey pairs:

Lemma 2.1 (Andrews and Hickerson [5, Corollary 2.1]). *If (α_n, β_n) is a Bailey pair with respect to a, then*

$$\sum_{n=0}^{\infty}(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\beta_n = \frac{(aq/\rho_1)_\infty(aq/\rho_2)_\infty}{(aq)_\infty(aq/\rho_1\rho_2)_\infty}\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n(aq/\rho_1\rho_2)^n\alpha_n}{(aq/\rho_1)_n(aq/\rho_2)_n}, \tag{2.2}$$

provided both sides converge absolutely.

Essential to creating Bailey pairs linked to quadratic forms will be the following corollary of a lattice structure of Bailey pairs described in [9]:

Lemma 2.2. *If (α_n, β_n) is a Bailey pair with respect to 1, then (α_n^*, β_n^*) is a Bailey pair with respect to q, where $\alpha_0^* = \beta_0^* = 0$, and for $n \geqslant 1$,*

$$\beta_n^* = \frac{-1}{(1-q^n)}\beta_n \tag{2.3}$$

and

$$\alpha_n^* = \frac{(-1)^n q^{n(n-1)/2}(1-q^{2n+1})}{(1-q)}\left(n + \sum_{r=1}^{n}\frac{(-1)^{r+1}q^{-r(r-1)/2}\alpha_r}{(1-q^r)} + 2\sum_{r=1}^{n}\frac{q^r}{(1-q^r)}\right). \tag{2.4}$$

Proof. In [9] it is observed that if (α_n, β_n) is a Bailey with respect to a/q, then (α_n^*, β_n^*) is a Bailey pair with respect to a, where

$$\alpha_n^* = \frac{(a/b)_n(-b)^n q^{n(n-1)/2}(1-aq^{2n})}{(bq)_n(1-a)} \sum_{r=0}^{n} \frac{(b)_r(-b)^{-r}\alpha_r}{(a/b)_r q^{r(r-1)/2}} \tag{2.5}$$

and

$$\beta_n^* = \frac{(b)_n \beta_n}{(bq)_n}. \tag{2.6}$$

Let $a = q$, differentiate with respect to b, and set $b = 1$ to obtain the desired statement. □

Proof of Theorem 1.1. We begin by recalling a Bailey pair with respect to 1 from [3],

$$\alpha_n = \begin{cases} q^{n(3n+1)/2} \sum\limits_{j=-n}^{n} (-1)^j q^{-j^2} - q^{n(3n-1)/2} \sum\limits_{j=-n+1}^{n-1} (-1)^j q^{-j^2}, & n \geqslant 1, \\ 1, & n = 0 \end{cases} \tag{2.7}$$

and

$$\beta_n = \frac{1}{(-q)_n}. \tag{2.8}$$

Inserting this pair into Lemma 2.2 and simplifying yields the Bailey pair with respect to q,

$$\alpha_n = \frac{(-1)^n q^{n(n-1)/2}(1-q^{2n+1})}{(1-q)} \sum_{r=1}^{n} \sum_{j=-r+1}^{r} (-1)^{r+j} q^{r^2-j^2} \tag{2.9}$$

and

$$\beta_n = \frac{-1}{(-q)_n(1-q^n)}. \tag{2.10}$$

Although it appears that we will end up with a ternary quadratic form in our sum, there is some collapsing that takes place. Namely, if we insert this new pair into Lemma 2.1 with $\rho_1 = q$ and $\rho_2 \to \infty$, the result is

$$2\sum_{n=1}^{\infty} \frac{(q)_{n-1}(-1)^n q^{n(n+1)/2}}{(-q)_n} = 2\sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^{n+j+1} q^{2n^2-j^2}. \tag{2.11}$$

The next step is to interpret the left-hand side of the above equation as a generating function. From [6], the term $2(q)_{n-1}/(-q)_n$ in the sum generates an overpartition into n non-negative parts, weighted by -1 raised to the largest part.

Then the term $q^{n(n+1)/2}$ adds one to the smallest part, two to the next smallest, and so on, resulting in the claimed difference conditions. The term $(-1)^n$ keeps the weight -1 raised to the largest part.

Now we relate the right-hand side of (2.11) to $\mathbb{Z}(\sqrt{2})$. To this end, recall (by Andrews et al. [2, Lemma 3], for example) that for each equivalence class of elements of negative norm in $\mathbb{Z}(\sqrt{2})$ there is a unique representative $j + n\sqrt{2}$ having $n > 0$ and $-n < j \leqslant n$. It is easy to check that $(-1)^{n+j+1}$ is -1 if $2n^2 - j^2$ is 0 or 1 modulo 4, and $+1$ otherwise. Hence the coefficient of q^m in (2.11) is equal to $-2i^{m^2-m}$ times the number of elements of norm $-m$. Since $\mathbb{Z}(\sqrt{2})$ has a unit of norm -1, the number of elements of norm $-m$ is the same as the number with norm m.

To finish the proof, we shall count the number of such elements. By unique factorization, the number of elements with norm mn is equal to the number of elements with norm m times the number of elements with norm m. It suffices, then, to consider powers of primes. We recall (from [8, p. 190], for example) that primes p equivalent to ± 1 modulo 8 split in $\mathbb{Z}(\sqrt{2})$, while those equivalent to ± 3 modulo 8 are inert. Hence, in the first case there are $(e+1)$ elements of norm p^e and in the second case there is one element of norm p^e if e is even and none otherwise. Finally, the prime 2 ramifies so there is one element of norm 2^a for every a. Putting everything together gives the desired formula. $\square$

Proof of Theorem 1.2. We recall another Bailey pair with respect to 1 from [3],

$$\alpha_n = \begin{cases} q^{n(2n+1)} \sum_{j=-n}^{n} (-1)^j q^{-j^2} - q^{n(2n-1)} \sum_{j=-n+1}^{n-1} (-1)^j q^{-j^2}, & n \geqslant 1, \\ 1, & n = 0 \end{cases} \tag{2.12}$$

and

$$\beta_n = \frac{1}{(-q)_{2n}}. \tag{2.13}$$

Here q has been replaced by q^2 in the definition of a Bailey pair. Inserting this pair into Lemma 2.2 (remembering to replace q by q^2) and simplifying gives the Bailey pair with respect to q^2,

$$\alpha_n = \frac{(-1)^n q^{n^2-n}(1-q^{4n+2})}{(1-q^2)} \sum_{r=1}^{n} \sum_{j=-r+1}^{r} (-1)^{r+j} q^{r^2-j^2} \tag{2.14}$$

and

$$\beta_n = \frac{-1}{(-q)_{2n}(1-q^{2n})}. \tag{2.15}$$

Inserting this new pair into Lemma 2.1 (with $q \to q^2$) and setting $\rho_1 = q^2$, $\rho_2 \to \infty$ yields the identity

$$2\sum_{n=1}^{\infty} \frac{(q^2;q^2)_{n-1}(-1)^n q^{n^2+n}}{(-q)_{2n}} = 2\sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^{n+j+1} q^{3n^2-j^2}. \tag{2.16}$$

To determine the overpartitions generated by the left-hand side, we begin with the term $1/(-q)_{2n}$, which generates a partition λ into exactly $2n$ non-negative parts, weighted by -1 raised to the largest part. Next, the term $(-1)^n q^{n^2+n}$ adds n to the first two parts, $n-1$ to the next two, and so on, the weight still being -1 raised to the largest part. Then, $(q^2;q^2)_{n-1}$ generates a partition μ into distinct even parts less than $2n$. For each of these parts $2k$, beginning with the largest, we add 1 to the first $2k$ parts of λ and overline the $2k+1$st part. The factor of 2 allows the largest part to possibly be overlined. One easily checks that the difference conditions in the theorem are satisfied and that the weight is still -1 raised to the largest part.

For the right-hand side, we again invoke [2, Lemma 3] to see that the inequalities in the double sum guarantee that we are choosing exactly one element from each equivalence class of numbers with a given negative norm in $\mathbb{Z}(\sqrt{3})$. It easy to check that $(-1)^{n+j+1}$ is $(-1)^{m+1}$ if $3n^2 - j^2 = m$, so that the coefficient of q^m in (2.16) is $2(-1)^m$ times the number of elements with norm $-m$ in $\mathbb{Z}(\sqrt{3})$.

We count the number of such elements as in the proof of Theorem 1.1 above, except we need to pay attention to the signs, since $\mathbb{Z}(\sqrt{3})$ does not contain a unit with norm -1. To do this correctly, we replace primes that are 2, 3, or 11 modulo 12 by their negatives. In $\mathbb{Z}(\sqrt{3})$, the primes q equivalent to ± 5 modulo 12 are inert, the primes p equivalent to ± 1 modulo 12 split, and the primes 2 and 3 ramify. For the inert primes, there is one element of norm q^e if e is even, and none otherwise. In the case of primes that split, the number of elements of norm p^e is $e+1$ for primes p that are 1 modulo 12 and the number of elements of norm $(-r)^e$ is $(e+1)$ for primes r that are 11 modulo 12. As 2 and 3 ramify, there is one element of norm $(-2)^a$ and one of norm $(-3)^b$. Remembering that we want elements of negative norm, we use multiplicativity to assemble the desired formula. □

Proof of Theorem 1.3. Here we return to the Bailey pair in (2.9) and (2.10). Replace q by q^2 and substitute the pair into Lemma 2.1 with $\rho_1 = q$ and $\rho_2 = q^2$ to get

$$2\sum_{n=1}^{\infty} \frac{(q)_{2n-1} q^n}{(-q^2;q^2)_n} = 2\sum_{n=1}^{\infty} \sum_{j=-n+1}^{n} (-1)^{j+1} q^{3n^2-2j^2}. \tag{2.17}$$

It turns out that the sum on the right-hand side is also equal to

$$\sum_{x=1}^{\infty} \sum_{-x<3y\leqslant x} (-1)^{x+y+1} q^{x^2-6y^2}. \tag{2.18}$$

To see this, rewrite the above as

$$\sum_{x=1}^{\infty} (-1)^{x+1} q^{x^2} - \sum_{x=1}^{\infty} q^{3x^2} + 2 \sum_{x \geqslant 3y \geqslant 1} (-1)^{x+y+1} q^{x^2-6y^2} \tag{2.19}$$

and rewrite the right-hand side of (2.17) as

$$\sum_{n=1}^{\infty} (-1)^{n+1} q^{n^2} - \sum_{n=1}^{\infty} q^{3n^2} + 2 \sum_{n-1 \geqslant j \geqslant 1} (-1)^{j+1} q^{3n^2-2j^2}. \tag{2.20}$$

The rightmost sums in the above two expressions can be equated by using the transformation $n = x - 2y$ and $j = x - 3y$.

On the left-hand side of (2.17), $2q^n(q^2;q^2)_{n-1}/(-q^2;q^2)_n$ generates an overpartition λ into n positive odd parts, weighted by the negative of (-1) raised to the largest part. Also, $(q;q^2)_n$ generates a partition v into distinct odd parts less than $2n+1$, weighted by (-1) raised to the number of parts. Given two such objects, write the parts of μ in non-increasing order. Remove the largest part of v, say $2k-1$, then add 2 to the first $k-1$ parts of μ and add 1 to the kth part. Repeat this process until v is empty. The result is easily seen to be an overpartition which satisfies the difference conditions in the theorem, and the weight is still the negative of (-1) raised to the largest part. The location of the even parts indicates clearly how to reverse the process.

Now we appeal one more time to [2, Lemma 3] to see that in (2.18) we are counting exactly one element from each equivalence class of numbers of a given positive norm m in $\mathbb{Z}(\sqrt{6})$. It is easily verified that the sign is 1, for those m that are 1 or 2 modulo 4, and -1 otherwise. Hence, the coefficient of q^m in (2.18) is $-2i^{m^2+m}$ times the number of elements with norm m in $\mathbb{Z}(\sqrt{6})$.

To count the number of such elements, we note that in $\mathbb{Z}(\sqrt{6})$, the primes p that are ± 7 or ± 11 modulo 24 are inert, the primes q that are ± 1 or ± 5 modulo 24 split, and the primes 2 and 3 ramify. To correctly keep track of the sign in this case, we replace the primes that are 2, 5, or 23 modulo 24 by their negatives. Now, as above, there are no elements of norm p^e if e is odd and one element of norm p^e if e is even. There are $(f+1)$ elements of norm q^f if q is 1 or 19 modulo 24, and there are $(g+1)$ elements of norm $(-r)^g$ if r is 5 or 23 modulo 24. Finally there is one element of the norm $(-2)^a$ and one of norm 2^b. Using multiplicativity and keeping in mind that we want elements of positive norm, one assembles the desired formula. $\square$

3. Concluding remarks

It is worth pointing out that, with the formulas in Theorems 1.1–1.3 in hand, one concludes that the corresponding difference functions are almost always 0 and are infinitely often equal to any given $2k \in \mathbb{Z}$. It is also worth asking, though a positive response seems unlikely, whether there is any way to approach the main theorems using combinatorial arguments. Finally, we observe that while there is an extensive literature on finding q-series expansions for modular forms, the interaction between q-series and other number-theoretic objects is relatively unexplored. It would be nice to see some systematic results like those here and/or in [2,4,5].

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