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## Binary quadratic forms and the Fourier coefficients of certain weight 1 *eta*-quotients

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### ABSTRACT

We state and prove an identity which represents the most general *eta*-products of weight 1 by binary quadratic forms. We discuss the utility of binary quadratic forms in finding a multiplicative completion for certain *eta*-quotients. We then derive explicit formulas for the Fourier coefficients of certain *eta*-quotients of weight 1 and level 47, 71, 135, 648, 1024, and 1872.

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## 1. Introduction and notation

Throughout the paper we assume  $q$  is a complex number with  $|q| < 1$ . We use the standard notations

$$(a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n) \quad (1.1)$$

and

$$E(q) := (q; q)_\infty. \quad (1.2)$$

Next, we recall the Ramanujan theta function

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1. \quad (1.3)$$

The function  $f(a, b)$  satisfies the Jacobi triple product identity [3, Entry 19]

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty, \quad (1.4)$$

along with

$$f(a, b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}), \quad (1.5)$$

where  $n \in \mathbb{Z}$  [3, Entry 18]. One may use (1.4) to derive the following special cases:

$$E(q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}, \quad (1.6)$$

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{E^5(q^2)}{E^2(q^4)E^2(q)}, \quad (1.7)$$

$$\psi(q) := f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2-n} = \frac{E^2(q^2)}{E(q)}, \quad (1.8)$$

$$f(q, q^2) = \frac{E^2(q^3)E(q^2)}{E(q^6)E(q)}, \quad (1.9)$$

$$f(q, q^5) = \frac{E(q^{12})E^2(q^2)E(q^3)}{E(q^6)E(q^4)E(q)}. \quad (1.10)$$

Note that (1.6) is the famous Euler pentagonal number theorem. Splitting (1.6) according to the parity of the index of summation, we find

$$E(q) = \sum_{n=-\infty}^{\infty} q^{6n^2-n} - q \sum_{n=-\infty}^{\infty} q^{6n^2+5n} = f(q^5, q^7) - qf(q, q^{11}). \quad (1.11)$$

We will employ the similarly derived relations

$$\phi(q) = \phi(q^9) + 2qf(q^3, q^{15}), \quad (1.12)$$

$$\phi(q) = \phi(q^4) + 2q\psi(q^8), \quad (1.13)$$

$$\psi(-q) = f(q^6, q^{10}) - qf(q^2, q^{14}), \quad (1.14)$$

and

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \quad (1.15)$$

We comment that the relation

$$E(-q) = \frac{E^3(q^2)}{E(q^4)E(q)}$$

can be used to deduce

$$\phi(-q) = \frac{E^2(q)}{E(q^2)},$$

and

$$\psi(-q) = \frac{E(q)E(q^4)}{E(q^2)}.$$

We now recall some notation from elementary number theory. Given an integer  $n$ , we use the term  $\text{ord}_p(n)$  to represent the unique integer with the properties  $p^{\text{ord}_p(n)} \mid n$  and  $p^{1+\text{ord}_p(n)} \nmid n$ .

For an odd prime  $p$ , Legendre's symbol is defined by

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue modulo } p \text{ and } p \nmid n, \\ -1 & \text{if } n \text{ is a quadratic nonresidue modulo } p \text{ and } p \nmid n, \\ 0 & \text{if } p \mid n. \end{cases}$$

Kronecker's symbol  $\left(\frac{n}{m}\right)$  is defined by

$$\left(\frac{n}{m}\right) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m \text{ is a prime dividing } n, \\ \text{Legendre's symbol} & \text{if } m \text{ is an odd prime,} \end{cases}$$

$$\left(\frac{n}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & n \equiv \pm 1 \pmod{8}, \\ -1 & n \equiv \pm 3 \pmod{8}, \end{cases}$$

and in general  $\left(\frac{n}{m}\right) = \prod_{i=1}^s \left(\frac{n}{p_i}\right)$ , where  $m = \prod_{i=1}^s p_i$  is a prime factorization of  $m$ .

We recall some well-known facts from the theory of binary quadratic forms. For  $a, b, c \in \mathbb{Z}$ , we call  $(a, b, c) := ax^2 + bxy + cy^2$  a binary quadratic form of discriminant  $d := b^2 - 4ac$ . We are only interested in the case  $d < 0$  and  $a > 0$ , and the forms in this case are called positive definite. From now on we will use the abbreviated term “form” to refer to a positive definite binary quadratic form. If we set  $g(x, y) = ax^2 + bxy + cy^2$ , then the binary quadratic form  $h(x, y) := g(\alpha x + \beta y, \gamma x + \delta y)$  is called equivalent to  $g(x, y)$ , when  $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  is in  $\mathrm{SL}(2, \mathbb{Z})$ . We write  $g(x, y) \sim h(x, y)$  and note that  $\sim$  is an equivalence relation on the set of binary quadratic forms of discriminant  $d$ . We define  $H(d)$  to be the set of all binary quadratic forms of discriminant  $d$  modulo the equivalence relation  $\sim$ . It is well-known that  $H(d)$  has a group structure, and is called the class group of discriminant  $d$  [5]. The conductor of  $d$  is the largest integer  $f$  such that  $\frac{d}{f^2}$  is also a discriminant of binary quadratic forms.

The form  $(a, b, c)$  has the associated theta series

$$B(a, b, c, q) := \sum_{x, y} q^{ax^2 + bxy + cy^2} = \sum_{n \geq 0} (a, b, c, n) q^n. \quad (1.16)$$

Note that  $(a, b, c, n) = 0$  whenever  $n \notin \mathbb{Z}_{\geq 0}$ .

Let  $w(d) = 6, 4$ , or  $2$  if  $d = -3, -4$  or  $d < -4$ , respectively. The prime  $p$  is represented by a form of discriminant  $d$  if and only if  $p \nmid f$  and  $\left(\frac{d}{p}\right) = 0, 1$ . Let us take  $(a, b, c)$  to be a form of discriminant  $d$  which represents  $p$ . Since  $(a, b, c)$  and  $(a, -b, c)$  represent the same integers, we see  $(a, -b, c)$  must also represent  $p$ . Apart from  $(a, b, c)$  and  $(a, -b, c)$ , no other class of forms of discriminant  $d$  will represent  $p$ . If  $p \nmid f$  and  $\left(\frac{d}{p}\right) = 0$ , then  $(a, b, c, p) = w(d)$ . If  $\left(\frac{d}{p}\right) = 1$ , then  $(a, b, c, p) = w(d)$  or  $2w(d)$  when  $(a, b, c) \approx (a, -b, c)$  or  $(a, b, c) \sim (a, -b, c)$ , respectively [8].

We will need a method to determine which primes  $p$  are represented by a given form. Besides using basic congruence conditions from genus theory, see [8], we follow [7] and [20] by employing the Weber class polynomial. For a form  $(a, b, c)$  with discriminant  $d$ , we let  $\tau = \frac{-b + \sqrt{d}}{a}$ , and recall that the Weber class polynomial,  $W_d$ , is defined as the minimal polynomial of  $f(\tau)$ , where  $f$  is a particular normalized Weber function generating the same class field as Klein’s modular function  $j(\tau)$ . (See [7, 20] for details.)

We will give examples where the factorization pattern of  $W_d \pmod{p}$  suffices to determine if  $(a, b, c)$  represents  $p$ . When the factorization pattern of  $W_d \pmod{p}$  does not suffice, we use  $\mathrm{rem}(z^p, W_d(z)) \pmod{p}$ . Here and throughout the manuscript,  $\mathrm{rem}(z^p, W_d(z)) \pmod{p}$  denotes the remainder of  $z^p$  upon division by  $W_d(z)$  modulo  $p$ . If  $\left(\frac{d}{p}\right) = 1$ , an analysis of  $\mathrm{rem}(z^p, W_d(z)) \pmod{p}$  always enables us to determine which form of discriminant  $d$  represents  $p$ .

An *eta*-quotient is a finite product

$$H(q) := q^j \prod_{i=1}^M E^{r_i}(q^{s_i}),$$

where  $s_1, \dots, s_M$  are positive integers,  $r_1, \dots, r_M$  are non-zero integers, and  $j = \sum_{i=1}^M \frac{r_i s_i}{24} \in \mathbb{Z}$ . We call  $H$  an *eta-quotient* since  $q^{\frac{j}{24}} E(q) = \eta(q)$ , where  $\eta(q)$  is the Dedekind eta function. The weight of  $H$  is defined to be  $\sum_{i=1}^M \frac{r_i}{2}$ . The level  $N$  of  $H$  is defined to be the smallest multiple of  $\text{lcm}(s_1, \dots, s_M)$  such that  $\sum_{i=1}^M \frac{r_i N}{s_i} \equiv 0 \pmod{24}$ .  $H$  has a Fourier expansion, and we use the notation  $[q^n]H(q)$  to denote the coefficient of  $q^n$  in the expansion of  $H$ . We call a series  $\sum_{n>0} a(n)q^n$  multiplicative when  $a(n)$  is multiplicative. It is a continuing area of research to determine the Fourier coefficients of certain *eta-quotients*. (See [1,2,6,22,12,21].)

Let  $Q$  be a form of discriminant  $d = d_0 \cdot f^2$  with conductor  $f$ . Let  $\vartheta_Q = \sum_{n \geq 0} h(n)q^n$  be the associated theta series for  $Q$ . In [11] Hecke defines the operator  $T_p$  by

$$T_p \left( \sum_{n=0}^{\infty} h(n)q^n \right) = \sum_{n=0}^{\infty} \left[ h(pn) + \left( \frac{d}{p} \right) h(n/p) \right] q^n. \quad (1.17)$$

Let  $S$  be the set of all linear combinations of theta series associated to the forms of discriminant  $d$ . It is well-known that for  $s \in S$ , we have  $T_p(s) \in S$  [11]. Moreover, Hecke gives explicit information regarding which theta series are involved in the linear combination  $T_p(s)$  [11, p. 794; (7)]. In the case that  $p$  is represented by a form of discriminant  $d$ , we define  $Q_0$  to be such a form. It can be shown that

$$T_p(\vartheta_Q) = \begin{cases} \vartheta_{Q \cdot Q_0} + \vartheta_{Q \cdot Q_0^{-1}}, & \left( \frac{d}{p} \right) = 1, \\ \vartheta_{Q \cdot Q_0}, & p \mid d, p \nmid f, \\ F(q^p), & p \mid f, \\ 0, & \left( \frac{d}{p} \right) = -1, \end{cases} \quad (1.18)$$

where the binary operation  $\cdot$  is Gaussian composition of forms, and  $F(q) = \vartheta_{Q_1}$  where  $Q_1$  is a particular form of discriminant  $\frac{d}{p^2}$ .

When  $T_p(s) = s \cdot \lambda_p$  for some constant  $\lambda_p$ , we call  $s$  an eigenform of  $T_p$  with eigenvalue  $\lambda_p$ . If  $\sum h(n)q^n$  is an eigenform for all Hecke operators, then (1.17) implies

$$\lambda_p h(n) = h(pn) + \left( \frac{d}{p} \right) h(n/p) \quad (1.19)$$

for any positive integer  $n$  and any prime  $p$ . Taking  $n = 1$  yields  $\lambda_p \cdot h(1) = h(p)$ , and we will always take  $h$  appropriately normalized so that  $h(1) = 1$ . Hence the eigenvalues of all  $T_p$  are exactly the evaluations of  $h$  at the primes. Eq. (1.19) shows  $h$  is multiplicative. Moreover, taking  $n = p^k$  in (1.19) yields

$$h(p^{k+1}) = h(p)h(p^k) - \left( \frac{d}{p} \right) h(p^{k-1}). \quad (1.20)$$

**Table 1**  
Multiplicative combinations.

$C_5 \cong \langle A \rangle$	$\frac{1}{2}(I(q) - \mu A(q) - \lambda A^2(q))$ $\frac{1}{2}(I(q) - \lambda A(q) - \mu A^2(q))$
$C_7 \cong \langle A \rangle$	$\frac{1}{2}(I(q) + \alpha A(q) + \beta A^2(q) + \gamma A^3(q))$ $\frac{1}{2}(I(q) + \beta A(q) + \gamma A^2(q) + \alpha A^3(q))$ $\frac{1}{2}(I(q) + \gamma A(q) + \alpha A^2(q) + \beta A^3(q))$
$C_6 \cong \langle A \rangle$	$\frac{1}{2}(I(q) - A^2(q) + A(q) - A^3(q))$
$C_8 \cong \langle A \rangle$	$\frac{1}{2}(I(q) - A^4(q) + \sqrt{2}A(q) - \sqrt{2}A^3(q))$
$C_4 \times C_4 \cong \langle A, B \rangle$	$\frac{1}{2}(I(q) + A^2(q) - B^2(q) - A^2B^2(q) + 2A(q) - 2AB^2(q))$

Given  $s, s' \in S$  we say that  $s + s'$  is a completion of  $s$  if  $s + s'$  is an eigenform for all Hecke operators. We say that  $s$  and  $s'$  are congruentially disjoint if there exists a modulus  $M$ , such that for any  $n, m$  with  $[q^n]s \neq 0$ ,  $[q^m]s' \neq 0$ , we have  $n \not\equiv m \pmod{M}$ .

In later sections, we find the Fourier coefficients of certain *eta*-quotients  $H$ , by finding a completion  $s' + H$  with the additional property that  $s'$  and  $H$  are congruentially disjoint. The ability to extract the coefficients of  $H$  by employing congruences, contributes to the simplicity of our results.

To find a completion, we use certain multiplicative combinations discussed in [18]. Let  $A^n(q)$  be the associated theta series of the binary quadratic form  $A^n$  ( $A$  composed with itself  $n$  times). We employ a result of Sun and Williams [18] to find the multiplicative linear combinations given in Table 1, where  $I(q) := A^0(q)$  and we have set  $\alpha := 2\cos(\frac{2\pi}{7})$ ,  $\beta := 2\cos(\frac{4\pi}{7})$ ,  $\gamma := 2\cos(\frac{6\pi}{7})$ ,  $\mu := \frac{1-\sqrt{5}}{2}$ ,  $\lambda := \frac{1+\sqrt{5}}{2}$ . The notation  $\langle A_1, A_2, \dots, A_n \rangle$  is used to denote the group generated by  $A_1, \dots, A_n$  with  $|A_1| \geq |A_2| \geq \dots \geq |A_n|$ , where  $|A|$  denotes the order of  $A$ . Here and throughout, we use  $C_n$  to denote the cyclic group of order  $n$ . Table 1 lists multiplicative combinations that apply directly to the examples appearing in later sections. We remind the reader that the property of being an eigenform for all Hecke operators is a stronger condition than multiplicativity.

In [14], Schoeneberg proved the identity

$$\frac{B(6, 1, \frac{k+1}{24}, q) - B(6, 5, \frac{k+25}{24}, q)}{2} = q^{\frac{k+1}{24}} E(q)E(q^k), \quad (1.21)$$

for  $k \equiv -1 \pmod{24}$ .

In Section 2 we state and prove an extra parameter generalization of (1.21). This generalization is Theorem 2.1. Theorem 2.1 provides a representation of the most general *eta*-products of weight 1, by the difference of two theta series arising from binary quadratic forms. We then state and prove Theorem 2.2, Theorem 2.3, and Theorem 2.4 which provide representations for two families of weight 1 *eta*-quotients as a difference of theta series. We would like to point out that many special cases of Theorem 2.1 were previously discussed in the literature. (See [2,4,9,13–15,17,19].)

The proofs of [Theorems 2.1, 2.2, 2.3, and 2.4](#) are all elementary, in the sense that they make essential use of dissections of  $q$ -series. Lastly, we remark that the *eta*-products arising from [Theorem 2.1](#) will always yield cusp forms, while the *eta*-quotients arising from [Theorem 2.2](#), [Theorem 2.3](#), and [2.4](#) are not generally cusp forms.

In [Sections 3 and 4](#) we employ [Theorem 2.1](#) and give explicit formulas for the Fourier coefficients of *eta*-products of level 47 and 71. In [Section 5](#) we consider *eta*-quotients of level 135, 648, 1024, and 1872, and employ [Theorems 2.1, 2.2, 2.3, and 2.4](#) to deduce formulas for their Fourier coefficients. Also in [Section 5](#), we contrast our multiplicative completion of  $q^7 E(q^{12}) E(q^{156})$  with that of Gordon and Hughes (see [\[10\]](#)). In [Section 6](#) we give concluding remarks.

## 2. Elementary proofs of [Theorems 2.1, 2.2, 2.3, and 2.4](#)

**Theorem 2.1.** *If  $m, s$  are positive integers with  $24s - m > 0$ , then*

$$\frac{B(6m, m, s, q) - B(6m, 5m, s + m, q)}{2} = q^s E(q^m) E(q^{24s-m}).$$

Without loss of generality we assume that  $m$  and  $s$  are coprime, and hence  $(6m, m, s)$  and  $(6m, 5m, s+m)$  are primitive forms, but not necessarily reduced. The transformations  $(x, y) \rightarrow (x + \frac{y}{2}, -y)$  and  $(x, y) \rightarrow (x + \frac{y}{3}, y)$ , send  $(6m, m, s)$  to  $(6m, 5m, s+m)$ . Using these transformations we see  $(6m, m, s)$  and  $(6m, 5m, s+m)$  are equivalent over the  $p$ -adic integers for all primes  $p$ . Equivalence over the  $p$ -adic integers implies that  $(6m, m, s)$  and  $(6m, 5m, s+m)$  share the same genus [\[7,20\]](#). Hence  $q^s E(q^m) E(q^{24s-m})$  is a cusp form with simple congruential properties [\[16, p. 577\]](#).

We mention that [Theorem 2.1](#) is explicitly used in [Section 3](#), see [\(3.1\)](#); [Section 4](#), see [\(4.1\)](#); [Section 5](#), see [Sections 5.1 and 5.4](#). We now proceed with the proof of [Theorem 2.1](#).

**Proof of Theorem 2.1.** For a fixed  $j$  we have

$$\begin{aligned} \sum_{\substack{x \\ y \equiv j \pmod{12}}} q^{6mx^2 + mxy + sy^2} &= \sum_{\substack{x \\ y \equiv -j \pmod{12}}} q^{6mx^2 + mxy + sy^2} \\ &= \sum_{x,y} q^{6mx^2 + mx(12y+j) + s(12y+j)^2}. \end{aligned} \quad (2.1)$$

Letting  $x \rightarrow -x + y$  and  $y \rightarrow -y$  on the right-hand side of [\(2.1\)](#) yields

$$\begin{aligned} \sum_{x,y} q^{j^2 s - jmx + 6mx^2 - j(24s-m)y + 6(24s-m)y^2} \\ &= q^{sj^2} \left( \sum_x q^{m(6x^2 - jx)} \right) \left( \sum_y Q^{(6y^2 - jy)} \right) \\ &= q^{sj^2} f(q^{m(6-j)}, q^{m(6+j)}) f(Q^{6-j}, Q^{6+j}), \end{aligned} \quad (2.2)$$

with  $Q := q^{24s-m}$ .

Employing (2.1) and (2.2) we obtain

$$\begin{aligned}
 B(6m, m, s, q) &= \sum_{j=0}^{11} q^{sj^2} f(q^{m(6-j)}, q^{m(6+j)}) f(Q^{6-j}, Q^{6+j}) \\
 &= f(q^{6m}, q^{6m}) f(Q^6, Q^6) \\
 &\quad + 2 \sum_{j=1}^5 q^{sj^2} f(q^{m(6-j)}, q^{m(6+j)}) f(Q^{6-j}, Q^{6+j}) \\
 &\quad + q^{36s} f(1, q^{12m}) f(1, Q^{12}).
 \end{aligned} \tag{2.3}$$

We repeat the above process with the form  $(6m, 5m, s+m)$  to get the analogous seven term expansion for  $B(6m, 5m, s+m, q)$ :

$$\begin{aligned}
 B(6m, 5m, s+m, q) &= \sum_{j=0}^{11} q^{(s+m)j^2} f(q^{m(6-5j)}, q^{m(6+5j)}) f(Q^{6-j}, Q^{6+j}) \\
 &= f(q^{6m}, q^{6m}) f(Q^6, Q^6) \\
 &\quad + 2 \sum_{j=1}^5 q^{(s+m)j^2} f(q^{m(6-5j)}, q^{m(6+5j)}) f(Q^{6-j}, Q^{6+j}) \\
 &\quad + q^{36(s+m)} f(q^{-24m}, q^{36m}) f(1, Q^{12}).
 \end{aligned} \tag{2.4}$$

Appropriately applying (1.5) to the right-hand side of (2.4), we write (2.4) as

$$\begin{aligned}
 B(6m, 5m, s+m, q) &= f(q^{6m}, q^{6m}) f(Q^6, Q^6) \\
 &\quad + 2q^{s+m} f(q^m, q^{11m}) f(Q^5, Q^7) \\
 &\quad + 2 \sum_{j=2}^4 q^{sj^2} f(q^{m(6-j)}, q^{m(6+j)}) f(Q^{6-j}, Q^{6+j}) \\
 &\quad + 2q^{25s-m} f(q^{5m}, q^{7m}) f(Q, Q^{11}) \\
 &\quad + q^{36s} f(1, q^{12m}) f(1, Q^{12}).
 \end{aligned} \tag{2.5}$$

It is clear that many of the terms in (2.5) also appear in (2.3). Subtracting (2.5) from (2.3) we perform numerous cancellations to obtain

$$\begin{aligned}
 &\frac{B(6m, m, s, q) - B(6m, 5m, s+m, q)}{2} \\
 &= q^s f(q^{5m}, q^{7m}) f(Q^5, Q^7) + q^{25s} f(q^m, q^{11m}) f(Q, Q^{11}) \\
 &\quad - q^{s+m} f(q^m, q^{11m}) f(Q^5, Q^7) - q^s f(q^{5m}, q^{7m}) f(Q, Q^{11}).
 \end{aligned} \tag{2.6}$$



The four terms on the right-hand side of (2.6) can be written as the product

$$q^s \cdot (f(q^{5m}, q^{7m}) - q^m f(q^m, q^{11m})) \cdot (f(Q^5, Q^7) - Qf(Q, Q^{11})). \quad (2.7)$$

Employing (1.11), we conclude that

$$\frac{B(6m, m, s, q) - B(6m, 5m, s + m, q)}{2} = q^s E(q^m) E(q^{24s-m}). \quad \square \quad (2.8)$$

We remark that a result of Sun given in [17] is closely related to our Theorem 2.1. Sun's result makes essential use of a lemma, found in [19, p. 356], which employs representations by four binary quadratic forms. It can be shown that the result of [17, p. 16, Theorem 3.1] follows as a corollary to our Theorem 2.1.

**Theorem 2.2.** *If  $m, s$  are positive integers with  $8s - m > 0$ , then*

$$\begin{aligned} \frac{B(8m, 2m, s, q) - B(8m, 6m, s + m, q)}{2} &= q^s \frac{E(q^m) E(q^{4m}) E(q^{8s-m}) E(q^{4(8s-m)})}{E(q^{2m}) E(q^{2(8s-m)})} \\ &= q^s \psi(-q^{8s-m}) \psi(-q^m). \end{aligned}$$

We give an application of Theorem 2.2 in Section 5. Specifically, the example involving level 1024 (see Section 5.3) makes explicit use of Theorem 2.2. The proof of Theorem 2.2 is similar to the proof of Theorem 2.1, and so we sketch the proof.

**Proof of Theorem 2.2.** One can show that we have the 5 term expansions

$$\begin{aligned} B(8m, 2m, s, q) &= \sum_{j=0}^7 q^{sj^2} f(q^{m(8-2j)}, q^{m(8+2j)}) f(Q^{8-2j}, Q^{8+2j}) \\ &= f(q^{8m}, q^{8m}) f(Q^8, Q^8) \\ &\quad + 2q^s f(q^{6m}, q^{10m}) f(Q^6, Q^{10}) \\ &\quad + 2q^{4s} f(q^{4m}, q^{12m}) f(Q^4, Q^{12}) \\ &\quad + 2q^{9s} f(q^{2m}, q^{14m}) f(Q^2, Q^{14}) \\ &\quad + q^{16s} f(1, q^{16m}) f(1, Q^{16}), \end{aligned} \quad (2.9)$$

and

$$\begin{aligned} B(8m, 6m, s + m, q) &= \sum_{j=0}^7 q^{(s+m)j^2} f(q^{m(8-6j)}, q^{m(8+6j)}) f(Q^{8-2j}, Q^{8+2j}) \\ &= f(q^{8m}, q^{8m}) f(Q^8, Q^8) \\ &\quad + 2q^{(s+m)} f(q^{2m}, q^{14m}) f(Q^6, Q^{10}) \end{aligned}$$

$$\begin{aligned}
& + 2q^{4s} f(q^{4m}, q^{12m}) f(Q^4, Q^{12}) \\
& + 2q^{9s-m} f(q^{6m}, q^{10m}) f(Q^2, Q^{14}) \\
& + q^{16s} f(1, q^{16m}) f(1, Q^{16}), \tag{2.10}
\end{aligned}$$

where  $Q := q^{8s-m}$ . Subtracting (2.10) from (2.9) we obtain

$$\begin{aligned}
& B(8m, 2m, s, q) - B(8m, 6m, s + m, q) \\
& = 2q^s \cdot (f(q^{6m}, q^{10m}) - q^m f(q^{2m}, q^{14m})) \cdot (f(Q^6, Q^{10}) - Qf(Q^2, Q^{14})). \tag{2.11}
\end{aligned}$$

Employing (1.14), we conclude that

$$\frac{B(8m, 2m, s, q) - B(8m, 6m, s + m, q)}{2} = q^s \psi(-q^{8s-m}) \psi(-q^m). \quad \square \tag{2.12}$$

**Theorem 2.3.** *If  $m, k$  are positive integers, then*

$$\begin{aligned}
\frac{B(m, 0, 4k, q) - B(4m, 4m, k + m, q)}{2} &= q^m \frac{E^2(q^s) E^2(q^{16m})}{E(q^{2s}) E(q^{8m})} \\
&= q^m \psi(q^{8m}) \phi(-q^k).
\end{aligned}$$

We give an application of Theorem 2.3 in Section 5. The example involving level 1024 (see Section 5.3) makes explicit use of both Theorem 2.2 and Theorem 2.3.

**Proof of Theorem 2.3.** We have

$$\begin{aligned}
& B(m, 0, 4k, q) - B(4m, 4m, k + m, q) \\
&= \sum_{\substack{x \\ y \equiv 0 \pmod{2}}} q^{mx^2 + ky^2} - \sum_{x, y} q^{m(2x+y)^2 + ky^2} \\
&= \sum_{\substack{x \\ y \equiv 0 \pmod{2}}} q^{mx^2 + ky^2} - \sum_{x \equiv y \pmod{2}} q^{mx^2 + ky^2} \\
&= \sum_{\substack{x \equiv 1 \pmod{2} \\ y \equiv 0 \pmod{2}}} q^{mx^2 + ky^2} - \sum_{x \equiv y \equiv 1 \pmod{2}} q^{mx^2 + ky^2} \\
&= \sum_{x \equiv 1 \pmod{2}} q^{mx^2} \left( \sum_{y \equiv 0 \pmod{2}} q^{ky^2} - \sum_{y \equiv 1 \pmod{2}} q^{ky^2} \right) \\
&= \left( \sum_x q^{m(2x+1)^2} \right) \left( \sum_y (-1)^y q^{ky^2} \right) \\
&= 2q^m \psi(q^{8m}) \phi(-q^k). \quad \square
\end{aligned}$$

**Theorem 2.4.** *If  $m, s$  are positive integers, then*

$$\begin{aligned} & \frac{B(m, 0, 9s, q) - B(9m, 6m, s + m, q)}{2} \\ &= q^m \frac{E(q^s)E(q^{4s})E^2(q^{6s})}{E(q^{2s})E(q^{3s})E(q^{12s})} \cdot \frac{E^2(q^{6m})E(q^{9m})E(q^{36m})}{E(q^{3m})E(q^{12m})E(q^{18m})}. \end{aligned}$$

**Proof.** We employ (1.12) to find

$$B(m, 0, 9s, q) = \phi(q^{9m})\phi(q^{9s}) + 2q^m f(q^{3m}, q^{15m})\phi(q^{9s}). \quad (2.13)$$

We also have

$$\begin{aligned} B(9m, 6m, s + m, q) &= \sum_{x, 3|y} q^{m(3x+y)^2+sy^2} + \sum_{x, 3 \nmid y} q^{m(3x+y)^2+sy^2} \\ &= \sum_{x, y} q^{9m(x+y)^2+9sy^2} + 2 \sum_{x, y} q^{m(3x+3y+1)^2+s(3y+1)^2} \\ &= \phi(q^{9m})\phi(q^{9s}) + 2q^{s+m} \sum_{x, y} q^{6mx+9mx^2+6sy+9sy^2} \\ &= \phi(q^{9m})\phi(q^{9s}) + 2q^{s+m} f(q^{3m}, q^{15m}) f(q^{3s}, q^{15s}). \end{aligned}$$

Thus we come to

$$\begin{aligned} & \frac{B(m, 0, 9s, q) - B(9m, 6m, s + m, q)}{2} \\ &= q^m f(q^{3m}, q^{15m}) (\phi(q^{9s}) - q^s f(q^{3s}, q^{15s})). \end{aligned} \quad (2.14)$$

Letting  $q \rightarrow -q^s$  in (1.15) yields

$$\psi(-q^s) = f(-q^{3s}, q^{6s}) - q^s \psi(-q^{9s}). \quad (2.15)$$

Multiplying both sides of (2.15) by  $\frac{E^2(q^{6s})}{E(q^{3s})E(q^{12s})}$  gives

$$\frac{E(q^s)E(q^{4s})E^2(q^{6s})}{E(q^{2s})E(q^{3s})E(q^{12s})} = \phi(q^{9s}) - q^s f(q^{3s}, q^{15s}). \quad (2.16)$$

Letting  $q \rightarrow q^{3m}$  in (1.10) and employing (2.16), allows us to write the right-hand side of (2.14) as an *eta*-quotient. Hence the proof is complete.  $\square$

We remark that unlike Theorem 2.1, the forms of Theorem 2.2, Theorem 2.3, and Theorem 2.4 are not necessarily in the same genus.

### 3. Weight 1 eta-product of level 47

In this section we determine the Fourier coefficients of  $q^2 E(q)E(q^{47})$ . We have  $\text{CL}(-47) \cong C_5$  and the reduced forms are

$\text{CL}(-47) \cong C_5$	$(\frac{-47}{p})$
Principal genus	$(1, 1, 12), (2, \pm 1, 6), (3, \pm 1, 4)$
	$+1$

In the above table,  $p$  is taken to be coprime to  $-47$  and represented by the given genus.

Taking  $m = 1$  and  $s = 2$  in [Theorem 2.1](#) yields

$$\begin{aligned} \sum_{n>0} a(n)q^n &:= \frac{B(6, 1, 2, q) - B(6, 5, 3, q)}{2} = \frac{B(2, 1, 6, q) - B(3, 1, 4, q)}{2} \\ &= q^2 E(q)E(q^{47}). \end{aligned} \quad (3.1)$$

One can check  $q^2 E(q)E(q^{47})$  is not an eigenform for all Hecke operators; hence, we must find a completion for  $q^2 E(q)E(q^{47})$ . Throughout this examples we will let  $\mu := \frac{1-\sqrt{5}}{2}$  and  $\lambda := \frac{1+\sqrt{5}}{2}$ . Using the first and second row of [Table 1](#), with  $A = (2, 1, 6)$ , we find

$$A_1(q) := \frac{B(1, 1, 12, q) - \mu B(2, 1, 6, q) - \lambda B(3, 1, 4, q)}{2}, \quad (3.2)$$

and

$$A_2(q) := \frac{B(1, 1, 12, q) - \lambda B(2, 1, 6, q) - \mu B(3, 1, 4, q)}{2} \quad (3.3)$$

are multiplicative. We show [\(3.2\)](#) and [\(3.3\)](#) are eigenforms for all Hecke operators, and we find their eigenvalues. We then derive a formula for the coefficients of  $A_1(q)$  and  $A_2(q)$ , and exploit

$$[q^n] q^2 E(q)E(q^{47}) = \frac{[q^n]A_1(q) - [q^n]A_2(q)}{\sqrt{5}}, \quad (3.4)$$

to find the Fourier coefficients of  $q^2 E(q)E(q^{47})$ .

To show [\(3.2\)](#) and [\(3.3\)](#) are eigenforms for  $T_p$ , we consider the action of  $T_p$  on the forms of discriminant  $-47$ . We separate cases according to the value of  $(\frac{-47}{p})$ .

**Case 1.**  $(\frac{-47}{p}) = 1$ .

[Table 2](#) gives the explicit action of  $T_p$  on the theta series associated with forms of discriminant  $-47$ .

We comment that [Table 2](#) is consistent with the formulas of Hecke [[11](#), p. 794].

With the aid of [Table 2](#), we compute the action of  $T_p$  on  $A_1(q)$  and  $A_2(q)$  in [Table 3](#). Hence, [\(3.2\)](#) and [\(3.3\)](#) are eigenforms for  $T_p$ , with eigenvalues  $2, -\mu, -\lambda$  when  $(\frac{-47}{p}) = 1$ .

**Table 2**

Hecke operator action on forms with discriminant  $-47$ .

$B(a, b, c, q)$	$(1, 1, 12, p) > 0$	$(3, 1, 4, p) > 0$	$(2, 1, 6, p) > 0$
$B(1, 1, 12, q) \xrightarrow{T_p}$	$2B(1, 1, 12, q)$	$2B(3, 1, 4, q)$	$2B(2, 1, 6, q)$
$B(3, 1, 4, q) \xrightarrow{T_p}$	$2B(3, 1, 4, q)$	$B(1, 1, 12, q) + B(2, 1, 6, q)$	$B(3, 1, 4, q) + B(2, 1, 6, q)$
$B(2, 1, 6, q) \xrightarrow{T_p}$	$2B(2, 1, 6, q)$	$B(3, 1, 4, q) + B(2, 1, 6, q)$	$B(1, 1, 12, q) + B(3, 1, 4, q)$

**Table 3**

Hecke operator action on  $A_1(q)$ ,  $A_2(q)$ .

	$(1, 1, 12, p) > 0$	$(3, 1, 4, p) > 0$	$(2, 1, 6, p) > 0$
$A_1(q) \xrightarrow{T_p}$	$2A_1(q)$	$-\lambda A_1(q)$	$-\mu A_1(q)$
$A_2(q) \xrightarrow{T_p}$	$2A_2(q)$	$-\mu A_2(q)$	$-\lambda A_2(q)$

**Case 2.**  $\left(\frac{-47}{p}\right) = -1$ .

A prime  $p$  with  $\left(\frac{-47}{p}\right) = -1$  implies  $(a, b, c, p) = 0$  for any form of discriminant  $-47$ . Thus, (3.2) and (3.3) are eigenforms for such  $T_p$  with eigenvalue 0.

**Case 3.**  $\left(\frac{-47}{p}\right) = 0$ .

We find that (3.2) and (3.3) are eigenforms for  $T_{47}$  with eigenvalue 1.

We have shown (3.2) and (3.3) are eigenforms for all Hecke operators, and have found their corresponding eigenvalues.

We now state criteria to determine when  $(a, b, c, p) > 0$  for a form of discriminant  $-47$ . We are able to distinguish each case by examining the Weber class polynomial for discriminant  $-47$ ,  $W_{-47}(z) := z^5 + 2z^4 + 2z^3 + z^2 - 1$ . We only consider primes  $p$  with  $\left(\frac{-47}{p}\right) \neq -1$ . As mentioned in Section 1, we follow [7] and [20] to find:

- (1)  $(1, 1, 12, p) > 0$  iff  $p = 47$  or  $\text{rem}(z^p, W_{-47}(z)) \equiv z \pmod{p}$ ,
- (2)  $(2, 1, 6, p) > 0$  iff  $\text{rem}(94z^p, W_{-47}(z)) \equiv (-47 + 3r_p)z^4 + (-5r_p - 47)z^3 + (-11r_p - 47)z^2 + (-5r_p - 47)z - 47 - r_p \pmod{p}$ ,
- (3)  $(3, 1, 4, p) > 0$  iff  $\text{rem}(94z^p, W_{-47}(z)) \equiv (r_p + 47)z^4 + (47 - r_p)z^3 + (7r_p + 47)z^2 + 12zr_p - 47 + 5r_p \pmod{p}$ ,

where  $r_p$  is defined by  $(r_p)^2 \equiv -47 \pmod{p}$ .

Define  $S_1, S_2, S_3$  to be the set of primes  $p \neq 47$  represented by  $(1, 1, 12)$ ,  $(2, 1, 6)$ , and  $(3, 1, 4)$ , respectively. Let  $S_4$  be the set of primes  $p$  with  $\left(\frac{-47}{p}\right) = -1$ .

We factor a positive integer  $n$  as

$$n = 47^{\text{ord}_{47}(n)} \prod_{p_1 \in S_1} p_1^{\text{ord}_{p_1}(n)} \prod_{p_2 \in S_2} p_2^{\text{ord}_{p_2}(n)} \prod_{p_3 \in S_3} p_3^{\text{ord}_{p_3}(n)} \prod_{p_4 \in S_4} p_4^{\text{ord}_{p_4}(n)}. \quad (3.5)$$

Employing (1.20), along with our computed eigenvalues, we obtain

$$[q^{p^\nu}]A_1(q) = \begin{cases} 1, & p = 47, \\ 1 + \nu, & p \in S_1, \\ U(\nu), & p \in S_2, \\ V(\nu), & p \in S_3, \\ \frac{(-1)^\nu + 1}{2}, & p \in S_4, \end{cases} \quad (3.6)$$

and

$$[q^{p^\nu}]A_2(q) = \begin{cases} 1, & p = 47, \\ 1 + \nu, & p \in S_1, \\ V(\nu), & p \in S_2, \\ U(\nu), & p \in S_3, \\ \frac{(-1)^\nu + 1}{2}, & p \in S_4, \end{cases} \quad (3.7)$$

where

$$U(n) := \frac{\sin(2\pi(n+1)/5)}{\sin(2\pi/5)}, \quad (3.8)$$

and

$$V(n) := \frac{\sin(4\pi(n+1)/5)}{\sin(4\pi/5)}, \quad (3.9)$$

are arithmetic functions of period 5. Using the multiplicativity of  $A_1(q)$  and  $A_2(q)$ , along with (3.6) and (3.7), we have

$$[q^n]A_1(q) = \Delta(n) \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)), \quad (3.10)$$

and

$$[q^n]A_2(q) = \Delta(n) \prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)), \quad (3.11)$$

where

$$\Delta(n) := \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_4 \in S_4} \frac{1 + (-1)^{\text{ord}_{p_4}(n)}}{2}.$$

Employing (3.4), (3.10), and (3.11), we obtain

$$a(n) = \Delta(n) \frac{P(n)}{\sqrt{5}}, \quad (3.12)$$

**Table 4**  
Values of  $U(n)$ ,  $V(n)$ .

$n$	$U(n)$	$V(n)$
$n \equiv 0 \pmod{5}$	1	1
$n \equiv 1 \pmod{5}$	$-\mu$	$-\lambda$
$n \equiv 2 \pmod{5}$	$\mu$	$\lambda$
$n \equiv 3 \pmod{5}$	$-1$	$-1$
$n \equiv 4 \pmod{5}$	0	0

where

$$P(n) = \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) - \prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)). \quad (3.13)$$

We explore the functions  $U(n)$  and  $V(n)$ , and find a different representation for both  $P(n)$  and  $a(n)$ . Since  $U(n)$  and  $V(n)$  are periodic, we can tabulate their explicit values in Table 4.

Given a positive integer  $n$ , we define  $r_i$  to be the number of primes  $p_2 \in S_2$  such that  $\text{ord}_{p_2}(n) \equiv i \pmod{5}$ . Similarly,  $s_i$  is the number of primes  $p_3 \in S_3$  such that  $\text{ord}_{p_3}(n) \equiv i \pmod{5}$ .

We have

$$\prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) = \delta_{r_4,0} \cdot (-1)^{r_1+r_3} \cdot \mu^{r_1+r_2}, \quad (3.14)$$

and

$$\prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) = \delta_{s_4,0} \cdot (-1)^{s_1+s_3} \cdot \lambda^{s_1+s_2}, \quad (3.15)$$

where  $\delta_{i,j}$  is 1 if  $i = j$  and 0 otherwise.

Then

$$\begin{aligned} & \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) \\ &= \delta_{r_4+s_4,0} \cdot (-1)^{r_2+r_3+s_1+s_3} \cdot \lambda^{s_1+s_2-r_1-r_2}, \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} & \prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) \\ &= \delta_{r_4+s_4,0} \cdot (-1)^{r_2+r_3+s_1+s_3} \cdot \mu^{s_1+s_2-r_1-r_2}. \end{aligned} \quad (3.17)$$

Subtracting (3.17) from (3.16) yields

$$P(n) = (-1)^{r_2+r_3+s_1+s_3} \cdot \delta_{r_4+s_4,0} \cdot (\lambda^{s_1+s_2-r_1-r_2} - \mu^{s_1+s_2-r_1-r_2}), \quad (3.18)$$

and plugging (3.18) into (3.12) yields an alternate representation for  $a(n)$ .

We can simplify further by employing the identity

$$\frac{\lambda^x - \mu^x}{\sqrt{5}} = b(x-1), \quad (3.19)$$

where

$$b(L) := \sum_{j=-L}^L (-1)^j \binom{L}{\lceil \frac{L+5j}{2} \rceil},$$

$\lceil \cdot \rceil$  is the ceiling function, and  $x$  is an integer greater than 0. For the case  $x < 0$  we can use

$$\frac{\lambda^{-|x|} - \mu^{-|x|}}{\sqrt{5}} = (-1)^{x+1} \frac{\lambda^{|x|} - \mu^{|x|}}{\sqrt{5}},$$

in order to employ (3.19). Noting that  $x = 0$  implies  $\frac{\lambda^x - \mu^x}{\sqrt{5}} = 0$ , we come to

$$a(n) = \begin{cases} \delta_{r_4+s_4,0} \cdot (-1)^{r_2+r_3+s_1+s_3} \\ \quad \cdot \Delta(n) \cdot b(|s_1+s_2-r_1-r_2|-1), & s_1+s_2-r_1-r_2 > 0, \\ \delta_{r_4+s_4,0} \cdot (-1)^{r_1+r_3+s_2+s_3+1} \\ \quad \cdot \Delta(n) \cdot b(|s_1+s_2-r_1-r_2|-1), & s_1+s_2-r_1-r_2 < 0, \\ 0, & s_1+s_2-r_1-r_2 = 0. \end{cases} \quad (3.20)$$

#### 4. Weight 1 eta-product of level 71

In this section we determine the Fourier coefficients of  $q^3 E(q) E(q^{71})$ . We have  $\text{CL}(-71) \cong C_7$  and the reduced forms are

$\text{CL}(-71) \cong C_7$			$(\frac{-71}{p})$
Principal genus	$(1, 1, 18), (2, \pm 1, 9), (3, \pm 1, 6), (4, \pm 3, 5)$		+1

In the above table,  $p$  is taken to be coprime to  $-71$  and represented by the given genus.

Taking  $m = 1$  and  $s = 3$  in Theorem 2.1 yields

$$\begin{aligned} \sum_{n>0} a(n) q^n &:= \frac{B(6, 1, 3, q) - B(6, 5, 4, q)}{2} = \frac{B(3, 1, 6, q) - B(4, 3, 5, q)}{2} \\ &= q^3 E(q) E(q^{71}). \end{aligned} \quad (4.1)$$



**Table 5**Hecke operator action on forms with discriminant  $-71$ .

$B(a, b, c, q)$	$(1, 1, 18, p) > 0$	$(2, 1, 9, p) > 0$	$(4, 3, 5, p) > 0$	$(3, 1, 6, p) > 0$
$B(1, 1, 18, q) \xrightarrow{T_p}$	$2B(1, 1, 18, q)$	$2B(2, 1, 9, q)$	$2B(4, 3, 5, q)$	$2B(3, 1, 6, q)$
$B(2, 1, 9, q) \xrightarrow{T_p}$	$2B(2, 1, 9, q)$	$B(1, 1, 18, q)$ $+ B(4, 3, 5, q)$	$B(2, 1, 9, q)$ $+ B(3, 1, 6, q)$	$B(3, 1, 6, q)$ $+ B(4, 3, 5, q)$
$B(4, 3, 5, q) \xrightarrow{T_p}$	$2B(4, 3, 5, q)$	$B(2, 1, 9, q)$ $+ B(3, 1, 6, q)$	$B(1, 1, 18, q)$ $+ B(3, 1, 6, q)$	$B(2, 1, 9, q)$ $+ B(4, 3, 5, q)$
$B(3, 1, 6, q) \xrightarrow{T_p}$	$2B(3, 1, 6, q)$	$B(3, 1, 6, q)$ $+ B(4, 3, 5, q)$	$B(2, 1, 9, q)$ $+ B(4, 3, 5, q)$	$B(1, 1, 18, q)$ $+ B(2, 1, 9, q)$

Note that  $q^3 E(q) E(q^{71})$  is not multiplicative. Throughout this example we will let  $\alpha := 2 \cos(\frac{2\pi}{7})$ ,  $\beta := 2 \cos(\frac{4\pi}{7})$ ,  $\gamma := 2 \cos(\frac{6\pi}{7})$ . Using [Table 1](#), we find

$$A_1(q) := \frac{B(1, 1, 18, q) + \gamma B(2, 1, 9, q) + \alpha B(4, 3, 5, q) + \beta B(3, 1, 6, q)}{2}, \quad (4.2)$$

$$A_2(q) := \frac{B(1, 1, 18, q) + \alpha B(2, 1, 9, q) + \beta B(4, 3, 5, q) + \gamma B(3, 1, 6, q)}{2}, \quad (4.3)$$

and

$$A_3(q) := \frac{B(1, 1, 18, q) + \beta B(2, 1, 9, q) + \gamma B(4, 3, 5, q) + \alpha B(3, 1, 6, q)}{2}, \quad (4.4)$$

are multiplicative. Note that  $\alpha, \beta, \gamma$  are the roots of  $F(x) := x^3 + x^2 - 2x - 1$ .

We show  $A_1(q), A_2(q), A_3(q)$  are eigenforms for all Hecke operators, and we find their eigenvalues. We then derive a formula for their coefficients, and exploit

$$\begin{aligned}
 & (\beta - \alpha) \cdot \frac{A_1(q)}{7} \\
 & + (\gamma - \beta) \cdot \frac{A_2(q)}{7} \\
 & + (\alpha - \gamma) \cdot \frac{A_3(q)}{7} \\
 & = \sum_{n>0} a(n) q^n = q^3 E(q) E(q^{71}),
 \end{aligned}$$

to find the Fourier coefficients of  $q^3 E(q) E(q^{71})$ .

We consider the action of  $T_p$  on the associated theta series of forms of discriminant  $-71$ . We separate cases according to the value of  $(\frac{-71}{p})$ .

**Case 1.**  $(\frac{-71}{p}) = 1$ .

[Table 5](#) gives the explicit action of  $T_p$  on the theta series associated with forms of discriminant  $-71$ .

We comment that [Table 5](#) is consistent with the formulas of Hecke [[11](#), p. 794].

**Table 6**Hecke operator action on  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$ .

	$(1, 1, 18, p) > 0$	$(2, 1, 9, p) > 0$	$(4, 3, 5, p) > 0$	$(3, 1, 6, p) > 0$
$A_1(q) \xrightarrow{T_p}$	$2A_1(q)$	$\gamma A_1(q)$	$\alpha A_1(q)$	$\beta A_1(q)$
$A_2(q) \xrightarrow{T_p}$	$2A_2(q)$	$\alpha A_2(q)$	$\beta A_2(q)$	$\gamma A_2(q)$
$A_3(q) \xrightarrow{T_p}$	$2A_3(q)$	$\beta A_3(q)$	$\gamma A_3(q)$	$\alpha A_3(q)$

With the aid of Table 5, we compute the action of  $T_p$  on  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$  in Table 6. Hence,  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$  are eigenforms for  $T_p$ , with eigenvalues 2,  $\alpha$ ,  $\beta$ ,  $\gamma$  when  $(\frac{-71}{p}) = 1$ .

**Case 2.**  $(\frac{-71}{p}) = -1$ .

A prime  $p$  with  $(\frac{-71}{p}) = -1$  implies  $(a, b, c, p) = 0$  for any form of discriminant  $-71$ . Thus,  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$  are eigenforms for such  $T_p$  with eigenvalue 0.

**Case 3.**  $(\frac{-71}{p}) = 0$ .

We find that  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$  are eigenforms for  $T_{71}$  with eigenvalue 1.

We have shown  $A_1(q)$ ,  $A_2(q)$ ,  $A_3(q)$  are eigenforms for all Hecke operators, and have found their corresponding eigenvalues.

We now state criteria to determine when  $(a, b, c, p) > 0$  for a form of discriminant  $-71$ . We are able to distinguish each case by examining the Weber class polynomial for discriminant  $-71$ ,  $W_{-71}(z) := z^7 + z^6 - z^5 - z^4 - z^3 + z^2 + 2z - 1$ . We only consider primes  $p$  with  $(\frac{-71}{p}) \neq -1$ . We follow the method of [7] and [20] to find:

- (1)  $(1, 1, 18, p) > 0$  iff  $p = 71$  or  $\text{rem}(z^p, W_{-71}(z)) \equiv z \pmod{p}$ ,
- (2)  $(2, 1, 9, p) > 0$  iff  $\text{rem}(142z^p, W_{-71}(z)) \equiv (2r_p - 142)z^6 + (6r_p - 142)z^5 + (-5r_p + 71)z^4 + (-6r_p + 142)z^3 + (-5r_p + 213)z^2 + (-15r_p - 71)z - 213 + 11r_p \pmod{p}$ ,
- (3)  $(4, 3, 5, p)$  iff  $\text{rem}(142z^p, W_{-71}(z)) \equiv 20r_p z^6 + 16r_p z^5 - 10r_p z^4 - 20r_p z^3 + (-27r_p - 71)z^2 + (13r_p - 71)z + 20r_p \pmod{p}$ ,
- (4)  $(3, 1, 6, p) > 0$  iff  $\text{rem}(142z^p, W_{-71}(z)) \equiv (10r_p + 142)z^6 + (10r_p + 142)z^5 + (r_p - 71)z^4 + (-2r_p - 142)z^3 + (-4r_p - 142)z^2 + (5r_p + 71)z + 142 + 4r_p \pmod{p}$ ,

where  $r_p$  is defined by  $(r_p)^2 \equiv -71 \pmod{p}$ .

Define  $S_1, S_2, S_3, S_4$  to be the set of primes  $p \neq 71$  represented by  $(1, 1, 18)$ ,  $(2, 1, 9)$ ,  $(4, 3, 5)$ , and  $(3, 1, 6)$ , respectively. Let  $S_5$  be the set of primes  $p$  with  $(\frac{-71}{p}) = -1$ .

We factor a positive integer  $n$  as

$$n = 71^{\text{ord}_{71}(n)} \prod_{p_1 \in S_1} p_1^{\text{ord}_{p_1}(n)} \prod_{p_2 \in S_2} p_2^{\text{ord}_{p_2}(n)} \prod_{p_3 \in S_3} p_3^{\text{ord}_{p_3}(n)} \prod_{p_4 \in S_4} p_4^{\text{ord}_{p_4}(n)} \prod_{p_5 \in S_5} p_5^{\text{ord}_{p_5}(n)}. \quad (4.5)$$

**Table 7**  
Values of  $U(n)$ ,  $V(n)$ ,  $W(n)$ .

$n$	$U(n)$	$V(n)$	$W(n)$
$n \equiv 0 \pmod{7}$	1	1	1
$n \equiv 1 \pmod{7}$	$\alpha$	$\beta$	$\gamma$
$n \equiv 2 \pmod{7}$	$-\gamma^{-1}$	$-\alpha^{-1}$	$-\beta^{-1}$
$n \equiv 3 \pmod{7}$	$\gamma^{-1}$	$\alpha^{-1}$	$\beta^{-1}$
$n \equiv 4 \pmod{7}$	$-\alpha$	$-\beta$	$-\gamma$
$n \equiv 5 \pmod{7}$	$-1$	$-1$	$-1$
$n \equiv 6 \pmod{7}$	0	0	0

Letting

$$\Delta(n) := \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_5 \in S_5} \frac{1 + (-1)^{\text{ord}_{p_5}(n)}}{2},$$

we obtain

$$[q^n]A_1(q) = \Delta(n) \prod_{p_2 \in S_2} W(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} V(\text{ord}_{p_4}(n)), \quad (4.6)$$

$$[q^n]A_2(q) = \Delta(n) \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} W(\text{ord}_{p_4}(n)), \quad (4.7)$$

$$[q^n]A_3(q) = \Delta(n) \prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} W(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} U(\text{ord}_{p_4}(n)), \quad (4.8)$$

where

$$U(n) := \frac{\sin(2\pi(n+1)/7)}{\sin(2\pi/7)}, \quad (4.9)$$

$$V(n) := \frac{\sin(4\pi(n+1)/7)}{\sin(4\pi/7)}, \quad (4.10)$$

and

$$W(n) := \frac{\sin(6\pi(n+1)/7)}{\sin(6\pi/7)}. \quad (4.11)$$

Since  $U(n)$ ,  $V(n)$  and  $W(n)$  are periodic, we can tabulate their explicit values in [Table 7](#).

To ease notation we define

$$\Delta_1(n) := \prod_{p_2 \in S_2} W(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} V(\text{ord}_{p_4}(n)).$$

$$\Delta_2(n) := \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} W(\text{ord}_{p_4}(n)),$$

$$\Delta_3(n) := \prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} W(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} U(\text{ord}_{p_4}(n)).$$

Thus we come to

$$a(n) = \frac{\Delta(n)}{7} P(n),$$

where

$$\begin{aligned} P(n) &= (\beta - \alpha)\Delta_1(n) \\ &\quad + (\gamma - \beta)\Delta_2(n) \\ &\quad + (\alpha - \gamma)\Delta_3(n). \end{aligned}$$

Similar to our approach for discriminant  $-47$  we can find another representation of  $a(n)$  by using the tabulated values of  $U, V$  and  $W$ .

Given a positive integer  $n$ , we define  $r_i$  to be the number of primes  $p_2 \in S_2$  such that  $\text{ord}_{p_2}(n) \equiv i \pmod{7}$ . Similarly,  $s_i$  is the number of primes  $p_3 \in S_3$  such that  $\text{ord}_{p_3}(n) \equiv i \pmod{7}$ , and  $t_i$  is the number of primes  $p_4 \in S_4$  such that  $\text{ord}_{p_4}(n) \equiv i \pmod{7}$ . We can now write  $\Delta_1, \Delta_2, \Delta_3$  in terms of the new notation.

We use the tabulated values of  $U(n), V(n), W(n)$  to find

$$\begin{aligned} \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) &= \delta_{r_6,0}(-1)^{r_2+r_4+r_5} \alpha^{r_1+r_4} \gamma^{-r_2-r_3}, \\ \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) &= \delta_{s_6,0}(-1)^{s_2+s_4+s_5} \beta^{s_1+s_4} \alpha^{-s_2-s_3}, \\ \prod_{p_4 \in S_4} W(\text{ord}_{p_4}(n)) &= \delta_{t_6,0}(-1)^{t_2+t_4+t_5} \gamma^{t_1+t_4} \beta^{-t_2-t_3}, \end{aligned}$$

and we note that the symmetry between the three products is the same symmetry seen in the table of values for  $U(n), V(n), W(n)$ . Thus we come to

$$\begin{aligned} \Delta_1(n) &= \delta_{r_6+s_6+t_6,0} \cdot (-1)^{r_2+r_4+r_5+s_2+s_4+s_5+t_2+t_4+t_5} \\ &\quad \cdot \alpha^{s_1+s_4-t_2-t_3} \beta^{t_1+t_4-r_2-r_3} \gamma^{r_1+r_4-s_2-s_3}. \end{aligned} \quad (4.12)$$

We exploit symmetry to find that  $\Delta_2(n)$  is equal to  $\Delta_1(n)$  under the permutation  $\sigma := (\alpha, \beta, \gamma)$ , and  $\Delta_3(n)$  is equal to  $\Delta_1(n)$  under the permutation  $\sigma^2$ . Explicitly

$$\begin{aligned} \Delta_2(n) &= \sigma(\Delta_1(n)), \\ \Delta_3(n) &= \sigma^2(\Delta_1(n)), \end{aligned}$$

and thus we may write

$$\begin{aligned} P(n) &= (\beta - \alpha)\Delta_1(n) \\ &\quad + \sigma((\beta - \alpha)\Delta_1(n)) \\ &\quad + \sigma^2((\beta - \alpha)\Delta_1(n)). \end{aligned}$$

**Table 8**

Values of  $a(n)$  at prime powers.

	$(1, 1, 18, p) > 0$	$(2, 1, 9, p) > 0$	$(4, 3, 5, p) > 0$	$(3, 1, 6, p) > 0$
$a(p^\nu), \nu \equiv 0 \pmod{7}$	0	0	0	0
$a(p^\nu), \nu \equiv 1 \pmod{7}$	0	0	-1	1
$a(p^\nu), \nu \equiv 2 \pmod{7}$	0	-1	1	0
$a(p^\nu), \nu \equiv 3 \pmod{7}$	0	1	-1	0
$a(p^\nu), \nu \equiv 4 \pmod{7}$	0	0	1	-1
$a(p^\nu), \nu \equiv 5 \pmod{7}$	0	0	0	0
$a(p^\nu), \nu \equiv 6 \pmod{7}$	0	0	0	0

Using the relation

$$a(n) = \frac{\Delta(n)}{7} P(n),$$

along with the formula for  $P(n)$ , yields a general formula for  $a(n) = [q^n]q^3 E(q)E(q^{71})$ .

We now consider some special values of  $n$ , and the corresponding formula for  $a(n)$ .

Given a prime  $p$  which is represented by a form of discriminant  $-71$ , we list the values  $a(p^\nu)$  in Table 8.

It is interesting to examine the special case  $n = p_1^{\nu_1} p_2^{\nu_2} \dots p_r^{\nu_r}$  with  $p_1, \dots, p_r \in S_2$ . It can be shown by induction that we have

$$a(n) = \delta_{r_6,0}(-1)^{r_1+r_3+r_5} G(r_3 + r_2, r_1 + r_4), \quad (4.13)$$

where  $G(L, M)$  is given by

$$G(L, M) = \sum_{j=-L-M}^{L+M} T(L, M, 2+7j) - T(L, M, 1+7j), \quad (4.14)$$

and the trinomial coefficient  $T(L, M, a)$  is defined by

$$\sum_{a=-L-\lceil M/2 \rceil}^{L+\lceil M/2 \rceil} T(L, M, a) x^a = \frac{(x^2 + x + 1)^L (x + 1)^M}{x^{L+\lceil M/2 \rceil}}. \quad (4.15)$$

Analogous formulas can be derived when all the prime divisors of  $n$  are contained in a single  $S_i$  for  $1 \leq i \leq 5$ . The general formula is more involved and will be discussed elsewhere.

## 5. Weight 1 eta-quotients of level 135, 648, 1024, 1872

### 5.1. Eta-products of level 135

Applying Theorem 2.1 with  $m = 9, s = 1$  yields

$$\begin{aligned} \frac{B(54, 9, 1, q) - B(54, 45, 10, q)}{2} &= \frac{B(1, 1, 34, q) - B(4, 3, 9, q)}{2} \\ &= qE(q^9)E(q^{15}), \end{aligned} \quad (5.1)$$

**Table 9**Hecke operator action on forms with discriminant  $-135$ .

$B(a, b, c, q)$	$(1, 1, 34, p) > 0$	$(4, 3, 9, p) > 0$	$(2, 1, 17, p) > 0$	$(5, 5, 8, p) > 0$
$B(1, 1, 34, q) \xrightarrow{T_p}$	$2B(1, 1, 34, q)$	$2B(4, 3, 9, q)$	$2B(2, 1, 17, q)$	$2B(5, 5, 8, q)$
$B(4, 3, 9, q) \xrightarrow{T_p}$	$2B(4, 3, 9, q)$	$B(1, 1, 34, q)$ $+ B(4, 3, 9, q)$	$B(2, 1, 17, q)$ $+ B(5, 5, 8, q)$	$2B(2, 1, 17, q)$
$B(2, 1, 17, q) \xrightarrow{T_p}$	$2B(2, 1, 17, q)$	$B(5, 5, 8, q)$ $+ B(2, 1, 17, q)$	$B(1, 1, 34, q)$ $+ B(4, 3, 9, q)$	$2B(4, 3, 9, q)$
$B(5, 5, 8, q) \xrightarrow{T_p}$	$2B(5, 5, 8, q)$	$2B(2, 1, 17, q)$	$2B(4, 3, 9, q)$	$2B(1, 1, 34, q)$

and  $m = 3$ ,  $s = 2$  gives

$$\begin{aligned} \frac{B(18, 3, 2, q) - B(18, 15, 5, q)}{2} &= \frac{B(2, 1, 17, q) - B(5, 5, 8, q)}{2} \\ &= q^2 E(q^3) E(q^{45}). \end{aligned} \quad (5.2)$$

We have  $\text{CL}(-135) \cong C_6$  and the reduced forms listed according to genus are

$\text{CL}(-135) \cong C_6$	$(\frac{p}{5})$	$(\frac{p}{3})$
Principal genus $(1, 1, 34), (4, 3, 9), (4, -3, 9)$	$+1$	$+1$
Second genus $(5, 5, 8), (2, 1, 17), (2, -1, 17)$	$-1$	$-1$

In the above table,  $p$  is taken to be coprime to  $-135$  and represented by the given genus.

With the aid of Table 1 with  $A = (2, 1, 17)$  and  $\langle A \rangle \cong C_6$ , we find

$$A(q) := \frac{B(1, 1, 34, q) - B(4, 3, 9, q) + B(2, 1, 17, q) - B(5, 5, 8, q)}{2} \quad (5.3)$$

is multiplicative. Note that (5.3) is the sum of (5.1) and (5.2). Similar to the previous example, we show (5.3) is an eigenform for all Hecke operators by examining the action of  $T_p$  on the forms of discriminant  $-135$ .

We remark that both  $qE(q^9)E(q^{15}) \pm q^2E(q^3)E(q^{45})$  are eigenforms for all  $T_p$ , and that  $qE(q^9)E(q^{15})$ ,  $q^2E(q^3)E(q^{45})$  are related to the quadratic field  $\mathbb{Q}(\sqrt{-15})$ , as explained by Köhler in [12, p. 252].

**Case 1.**  $(\frac{-135}{p}) = 1$ .

Table 9 gives the explicit action of  $T_p$  on the theta series associated with forms of discriminant  $-135$ .

We comment that Table 9 is consistent with the formulas of Hecke [11, p. 794].

Using Table 9, we find the action of  $T_p$  on  $A(q)$  in Table 10.

**Table 10**

Hecke operator action on  $A(q)$ .

	$(1, 1, 34, p) > 0$	$(4, 3, 9, p) > 0$	$(2, 1, 17, p) > 0$	$(5, 5, 8, p) > 0$
$A(q) \xrightarrow{T_p}$	$2A(q)$	$-A(q)$	$A(q)$	$-2A(q)$

**Table 11**

Hecke operator action on forms with discriminant  $-135$ .

$B(1, 1, 34, q) \xrightarrow{T_3}$	$B(1, 1, 4, q^3)$	$B(1, 1, 34, q) \xrightarrow{T_5}$	$B(5, 5, 8, q)$
$B(4, 3, 9, q) \xrightarrow{T_3}$	$B(1, 1, 4, q^3)$	$B(4, 3, 9, q) \xrightarrow{T_5}$	$B(2, 1, 17, q)$
$B(2, 1, 17, q) \xrightarrow{T_3}$	$B(2, 1, 2, q^3)$	$B(2, 1, 17, q) \xrightarrow{T_5}$	$B(4, 3, 9, q)$
$B(5, 5, 8, q) \xrightarrow{T_3}$	$B(2, 1, 2, q^3)$	$B(5, 5, 8, q) \xrightarrow{T_5}$	$B(1, 1, 34, q)$

**Case 2.**  $\left(\frac{-135}{p}\right) = -1$ .

A form  $(a, b, c)$  of discriminant  $-135$  has  $(a, b, c, p) = 0$  when  $\left(\frac{-135}{p}\right) = -1$ , and hence (5.3) is an eigenform for such  $T_p$  with eigenvalue 0.

**Case 3.**  $\left(\frac{-135}{p}\right) = 0$ .

Table 11 gives the explicit action of  $T_3$  and  $T_5$  on the theta series associated with forms of discriminant  $-135$ .

Using Table 11, we see that  $A(q)$  is an eigenform for  $T_5$  with eigenvalue  $-1$ , and an eigenform for  $T_3$  with eigenvalue 0.

We have shown (5.3) is an eigenform for all Hecke operators, and have found the corresponding eigenvalues.

We now state criteria to determine when  $(a, b, c, p) > 0$  for a form of discriminant  $-135$ . We examine the factorization of the Weber class polynomial  $W_{-135}(x) = x^6 - x^3 - 1$  modulo  $p$ . Following the method of [7] and [20], we find that for a prime  $p$  with  $\left(\frac{-135}{p}\right) = 1$ , we have

- (1)  $p$  is represented by the form  $(1, 1, 34)$  if and only if  $W_{-135}(x)$  splits completely modulo  $p$ ,
- (2)  $p$  is represented by the form  $(4, 3, 9)$  if and only if  $W_{-135}(x)$  factors into two irreducible cubic polynomials modulo  $p$ ,
- (3)  $p$  is represented by the form  $(5, 5, 8)$  if and only if  $W_{-135}(x)$  factors into three irreducible quadratic polynomials modulo  $p$ ,
- (4)  $p$  is represented by the form  $(2, 1, 17)$  if and only if  $W_{-135}(x)$  remains irreducible modulo  $p$ .

We define  $S_1, S_2, S_3, S_4$  to be the set of primes  $p \neq 5$  represented by  $(1, 1, 34), (5, 5, 8), (4, 3, 9), (2, 1, 17)$ , respectively. We also take  $S_5$  to be the set of primes  $p$  with  $\left(\frac{-135}{p}\right) = -1$ . Employing (1.20) along with the previously computed eigenvalues, we obtain

**Table 12**  
Values of  $U(n)$ ,  $V(n)$ .

$n$	$U(n)$	$V(n)$
$n \equiv 0 \pmod{6}$	1	1
$n \equiv 1 \pmod{6}$	-1	1
$n \equiv 2 \pmod{6}$	0	0
$n \equiv 3 \pmod{6}$	1	-1
$n \equiv 4 \pmod{6}$	-1	-1
$n \equiv 5 \pmod{6}$	0	0

$$[q^{\nu}]A(q) = \begin{cases} 0, & p = 3, \nu > 0, \\ (-1)^{\nu}, & p = 5, \\ 1 + \nu, & p \in S_1, \\ (-1)^{\nu}(1 + \nu), & p \in S_2, \\ U(\nu), & p \in S_3, \\ V(\nu), & p \in S_4, \\ \frac{1+(-1)^{\nu}}{2}, & p \in S_5, \end{cases} \quad (5.4)$$

where

$$U(n) := \frac{\sin(2\pi(n+1)/3)}{\sin(2\pi/3)}, \quad (5.5)$$

and

$$V(n) := \frac{\sin(\pi(n+1)/3)}{\sin(\pi/3)}. \quad (5.6)$$

The functions  $U(n)$  and  $V(n)$  are periodic and we tabulate their values in Table 12.

Given a positive integer  $n$  we write

$$n = 3^a 5^b \prod_{p_1 \in S_1} p_1^{\text{ord}_{p_1}(n)} \prod_{p_2 \in S_2} p_2^{\text{ord}_{p_2}(n)} \prod_{p_3 \in S_3} p_3^{\text{ord}_{p_3}(n)} \prod_{p_4 \in S_4} p_4^{\text{ord}_{p_4}(n)} \prod_{p_5 \in S_5} p_5^{\text{ord}_{p_5}(n)}, \quad (5.7)$$

where  $a = \text{ord}_3(n)$  and  $b = \text{ord}_5(n)$ . Given the factorization of  $n$  in (5.7), we employ (5.4) to find the coefficient  $[q^n]A(q)$  to be

$$\begin{aligned} & (-1)^{b+t} \cdot \delta_{a,0} \\ & \cdot \prod_{p \in S_1 \cup S_2} (1 + \text{ord}_p(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} V(\text{ord}_{p_4}(n)) \prod_{p_5 \in S_5} \frac{1 + (-1)^{\text{ord}_{p_5}(n)}}{2}, \end{aligned} \quad (5.8)$$

where  $t$  is the number of primes factors of  $n$ , counting multiplicity, that are contained in  $S_2$ .



Similar to the previous example, we are able to rewrite (5.8) by exploiting the periodicity of  $U(n), V(n)$ . Define  $r_i$  to be the number of primes  $p_3 \in S_3$  such that  $\text{ord}_{p_3}(n) \equiv i \pmod{3}$ , and  $s_i$  to be the number of primes  $p_4 \in S_4$  such that  $\text{ord}_{p_4}(n) \equiv i \pmod{6}$ .

Given the factorization of  $n$  in (5.7) we have

$$\prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) = \delta_{r_2,0} \cdot (-1)^{r_1}, \quad (5.9)$$

and

$$\prod_{p_4 \in S_4} V(\text{ord}_{p_4}(n)) = \delta_{s_2+s_5,0} \cdot (-1)^{s_3+s_4}. \quad (5.10)$$

We rewrite (5.8) as

$$\begin{aligned} [q^n]A(q) &= (-1)^{b+t+r_1+s_3+s_4} \cdot \delta_{a+r_2+s_2+s_5,0} \\ &\cdot \prod_{p \in S_1 \cup S_2} (1 + \text{ord}_p(n)) \prod_{p_5 \in S_5} \frac{1 + (-1)^{\text{ord}_{p_5}(n)}}{2}. \end{aligned} \quad (5.11)$$

(5.11) gives the Fourier coefficients of  $qE(q^9)E(q^{15}) + q^2E(q^3)E(q^{45})$ . We note that  $[q^n]qE(q^9)E(q^{15}) \neq 0$  implies  $n \equiv 1 \pmod{3}$ , and  $[q^n]q^2E(q^3)E(q^{45}) \neq 0$  implies  $n \equiv 2 \pmod{3}$ . Thus we can employ congruences to extract the Fourier coefficients of each product from  $qE(q^9)E(q^{15}) + q^2E(q^3)E(q^{45})$ . We have

$$[q^n]qE(q^9)E(q^{15}) = \begin{cases} [q^n]A(q), & n \equiv 1 \pmod{3}, \\ 0, & n \equiv 0, 2 \pmod{3}, \end{cases}$$

and

$$[q^n]q^2E(q^3)E(q^{45}) = \begin{cases} [q^n]A(q), & n \equiv 2 \pmod{3}, \\ 0, & n \equiv 0, 1 \pmod{3}. \end{cases}$$

We can also write the Fourier coefficients of the products in the following form

$$[q^n]2qE(q^9)E(q^{15}) = (1 + (-1)^{b+t+s_1+s_3})[q^n]A(q), \quad (5.12)$$

and

$$[q^n]2q^2E(q^3)E(q^{45}) = (1 - (-1)^{b+t+s_1+s_3})[q^n]A(q). \quad (5.13)$$

**Table 13**Hecke operator action on forms with discriminant  $-648$ .

$B(a, b, c, q)$	$(1, 0, 162, p) > 0$	$(9, 6, 19, p) > 0$	$(2, 0, 81, p) > 0$	$(11, 10, 17, p) > 0$
$B(1, 0, 162, q) \xrightarrow{T_p}$	$2B(1, 0, 162, q)$	$2B(9, 6, 19, q)$	$2B(2, 0, 81, q)$	$2B(11, 10, 17, q)$
$B(9, 6, 19, q) \xrightarrow{T_p}$	$2B(9, 6, 19, q)$	$B(1, 0, 162, q)$ + $B(9, 6, 19, q)$	$2B(11, 10, 17, q)$	$B(2, 0, 81, q)$ + $B(11, 10, 17, q)$
$B(2, 0, 81, q) \xrightarrow{T_p}$	$2B(2, 0, 81, q)$	$2B(11, 10, 17, q)$	$2B(1, 0, 162, q)$	$2B(9, 6, 19, q)$
$B(11, 10, 17, q) \xrightarrow{T_p}$	$2B(11, 10, 17, q)$	$B(11, 10, 17, q)$ + $B(2, 0, 81, q)$	$2B(9, 6, 19, q)$	$B(1, 0, 162, q)$ + $B(9, 6, 19, q)$

### 5.2. Eta-products of level $648$

Applying [Theorem 2.4](#) with  $m = 1, s = 18$  yields

$$g(q) := \frac{B(1, 0, 162, q) - B(9, 6, 19, q)}{2} = q \frac{E^2(q^6)E(q^9)E(q^{72})E^2(q^{108})}{E(q^3)E(q^{12})E(q^{54})E(q^{216})}, \quad (5.14)$$

and  $m = 2, s = 9$  gives

$$h(q) := \frac{B(2, 0, 81, q) - B(11, 10, 17, q)}{2} = q^2 \frac{E(q^9)E^2(q^{12})E^2(q^{54})E(q^{72})}{E(q^6)E(q^{24})E(q^{27})E(q^{108})}. \quad (5.15)$$

We remark that both  $g(q)$  and  $h(q)$  are cusp forms. We have  $\text{CL}(-648) \cong C_6$  and the reduced forms listed according to genus are

$\text{CL}(-648) \cong C_6$	$(\frac{2}{3})$	$(\frac{-2}{p})$
Principal genus	$(1, 0, 162), (9, 6, 19), (9, -6, 19)$	+1
Second genus	$(2, 0, 81), (11, 10, 17), (11, -10, 17)$	-1

In the above table,  $p$  is taken to be coprime to  $-648$  and represented by the given genus. Let

$$A(q) := g(q) + h(q) = \frac{B(1, 0, 162, q) - B(9, 6, 19, q) + B(2, 0, 81, q) - B(11, 10, 17, q)}{2}. \quad (5.16)$$

Similar to the previous examples, we show (5.16) is an eigenform for all Hecke operators by examining the action of  $T_p$  on the forms of discriminant  $-648$ .

**Case 1.**  $(\frac{-648}{p}) = 1$ .

[Table 13](#) gives the explicit action of  $T_p$  on the theta series associated with forms of discriminant  $-648$ .

We comment that [Table 13](#) is consistent with the formulas of Hecke.

Using [Table 13](#), we find the action of  $T_p$  on  $A(q)$  in [Table 14](#).

**Table 14**

Hecke operator action on  $A(q)$ .

	$(1, 0, 162, p) > 0$	$(9, 6, 19, p) > 0$	$(2, 0, 81, p) > 0$	$(11, 10, 17, p) > 0$
$A(q) \xrightarrow{T_p}$	$2A(q)$	$-A(q)$	$2A(q)$	$-A(q)$

**Table 15**

Hecke operator action on forms with discriminant  $-648$ .

$B(1, 0, 162, q) \xrightarrow{T_2}$	$B(2, 0, 81, q)$	$B(1, 0, 162, q) \xrightarrow{T_3}$	$B(1, 0, 18, q^3)$
$B(9, 6, 19, q) \xrightarrow{T_2}$	$B(11, 10, 17, q)$	$B(9, 6, 19, q) \xrightarrow{T_3}$	$B(1, 0, 18, q^3)$
$B(2, 0, 81, q) \xrightarrow{T_2}$	$B(1, 0, 162, q)$	$B(2, 0, 81, q) \xrightarrow{T_3}$	$B(2, 0, 9, q^3)$
$B(11, 10, 17, q) \xrightarrow{T_2}$	$B(9, 6, 19, q)$	$B(11, 10, 17, q) \xrightarrow{T_3}$	$B(2, 0, 9, q^3)$

**Case 2.**  $\left(\frac{-648}{p}\right) = -1$ .

A form  $(a, b, c)$  of discriminant  $-648$  has  $(a, b, c, p) = 0$  when  $\left(\frac{-648}{p}\right) = -1$ , and hence (5.16) is an eigenform for such  $T_p$  with eigenvalue 0.

**Case 3.**  $\left(\frac{-648}{p}\right) = 0$ .

Table 15 gives the explicit action of  $T_2$  and  $T_3$  on the theta series associated with forms of discriminant  $-648$ .

Using Table 15, we see that  $A(q)$  is an eigenform for  $T_2$  with eigenvalue 1, and an eigenform for  $T_3$  with eigenvalue 0.

We have shown  $A(q)$  is an eigenform for all Hecke operators, and have found the corresponding eigenvalues. Lastly we note that Table 15 also shows  $g(q) - h(q)$  is an eigenform for all  $T_p$ .

We now state criteria to determine when  $(a, b, c, p) > 0$  for a form of discriminant  $-648$ . We examine the factorization of the Weber class polynomial  $W_{-648}(x) = x^6 - 7758x^5 - 17217x^4 - 25316x^3 - 17217x^2 - 7758x + 1$  modulo  $p$ . Following the method of [7] and [20], we find that for a prime  $p$  with  $\left(\frac{-648}{p}\right) = 1$ , we have

- (1)  $p$  is represented by the form  $(1, 0, 162)$  if and only if  $W_{-648}(x)$  splits completely modulo  $p$ ,
- (2)  $p$  is represented by the form  $(9, 6, 19)$  if and only if  $W_{-648}(x)$  factors into two irreducible cubic polynomials modulo  $p$ ,
- (3)  $p$  is represented by the form  $(2, 0, 81)$  if and only if  $W_{-648}(x)$  factors into three irreducible quadratic polynomials modulo  $p$ ,
- (4)  $p$  is represented by the form  $(11, 10, 17)$  if and only if  $W_{-648}(x)$  remains irreducible modulo  $p$ .

We define  $S_1$  to be the set of primes  $p \neq 2$  with  $(1, 0, 162, p) + (2, 0, 81, p) > 0$ . Similarly, we let  $S_2$  be the set of primes  $p$  with  $(9, 6, 19, p) + (11, 10, 17, p) > 0$ . We also

take  $S_3$  to be the set of primes  $p$  with  $(\frac{-648}{p}) = -1$ . Employing (1.20) along with the previously computed eigenvalues, we obtain

$$[q^\nu]A(q) = \begin{cases} 0, & p = 3, \nu > 0, \\ 1, & p = 2, \\ 1 + \nu, & p \in S_1, \\ U(\nu), & p \in S_2, \\ \frac{1+(-1)^\nu}{2}, & p \in S_3, \end{cases} \quad (5.17)$$

where

$$U(n) := \frac{\sin(2\pi(n+1)/3)}{\sin(2\pi/3)}. \quad (5.18)$$

The function  $U(n)$  is discussed in Section 5.1, and the explicit values of  $U(n)$  can be found in Table 12.

Given a positive integer  $n$  we write

$$n = 2^a 3^b \prod_{p_1 \in S_1} p_1^{\text{ord}_{p_1}(n)} \prod_{p_2 \in S_2} p_2^{\text{ord}_{p_2}(n)} \prod_{p_3 \in S_3} p_3^{\text{ord}_{p_3}(n)}, \quad (5.19)$$

where  $a = \text{ord}_2(n)$  and  $b = \text{ord}_3(n)$ . Given the factorization of  $n$  in (5.19), we employ (5.17) to find the coefficient  $[q^n]A(q)$  to be

$$\delta_{b,0} \cdot \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} \frac{1 + (-1)^{\text{ord}_{p_3}(n)}}{2}. \quad (5.20)$$

Define  $r_i$  to be the number of primes  $p_2 \in S_2$  such that  $\text{ord}_{p_2}(n) \equiv i \pmod{3}$ .

Given the factorization of  $n$  in (5.19) we have

$$\prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) = \delta_{r_2,0} \cdot (-1)^{r_1}, \quad (5.21)$$

and we rewrite (5.20) as

$$[q^n]A(q) = (-1)^{r_1} \cdot \delta_{b+r_2,0} \cdot \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_3 \in S_3} \frac{1 + (-1)^{\text{ord}_{p_3}(n)}}{2}. \quad (5.22)$$

Both (5.20) and (5.22) give a formula for the Fourier coefficients of  $A(q)$ , and to extract the Fourier coefficients of the cusp forms  $g(q)$  and  $h(q)$  we employ congruences. Note that  $[q^n]g(q) \neq 0$  implies  $n \equiv 1 \pmod{3}$ , and  $[q^n]h(q) \neq 0$  implies  $n \equiv 2 \pmod{3}$ . We have

$$[q^n]g(q) = \begin{cases} [q^n]A(q), & n \equiv 1 \pmod{3}, \\ 0, & n \equiv 0, 2 \pmod{3}, \end{cases}$$

**Table 16**

Hecke operator action on  $A(q)$ .

	$(1, 0, 256, p) > 0$	$(4, 4, 65, p) > 0$	$(16, 8, 17, p) > 0$	$(5, 4, 52, p) > 0$	$(13, 4, 20, p) > 0$
$A(q) \xrightarrow{T_p}$	$2A(q)$	$-2A(q)$	0	$\sqrt{2}A(q)$	$-\sqrt{2}A(q)$

and

$$[q^n]h(q) = \begin{cases} [q^n]A(q), & n \equiv 2 \pmod{3}, \\ 0, & n \equiv 0, 1 \pmod{3}. \end{cases}$$

### 5.3. Eta-quotients of level 1024

Taking  $m = 1, k = 64$  in Theorem 2.3 and  $m = 8, s = 5$  in Theorem 2.2 yields

$$\frac{B(1, 0, 256, q) - B(4, 4, 65, q)}{2} = q\psi(q^8)\phi(-q^{64}), \quad (5.23)$$

and

$$\frac{B(5, 4, 52, q) - B(13, 4, 20, q)}{2} = q^5\psi(-q^8)\psi(-q^{32}). \quad (5.24)$$

We find  $\text{CL}(-1024) \cong C_8$  and we list the reduced forms according to their genus

$\text{CL}(-1024) \cong C_8$		$(\frac{-1}{p})$	$(\frac{2}{p})$
Principal genus	$(1, 0, 256), (4, 4, 65), (16, 8, 17), (16, -8, 17)$	+1	+1
Second genus	$(5, 4, 52), (5, -4, 52), (13, 4, 20), (13, -4, 20)$	+1	-1

In the above table,  $p$  is taken to be odd so that it is coprime to  $-1024 = 2^{10}$ . Before moving on, we remark that the *eta*-quotients  $q\psi(q^8)\phi(-q^{64})$  and  $q^5\psi(-q^8)\psi(-q^{32})$  are considered in [12, p. 232]. Using Table 1, we find

$$A(q) := \frac{B(1, 0, 256, q) - B(4, 4, 65, q) + \sqrt{2}(B(5, 4, 52, q) - B(13, 4, 20, q))}{2} \quad (5.25)$$

to be multiplicative. The action of  $T_p$  on  $A(q)$  is given in Table 16 for primes  $p$  with  $(\frac{-1024}{p}) = 1$ .

If  $(\frac{-1024}{p}) = -1$ , then  $A(q)$  is an eigenform for  $T_p$  with eigenvalue 0.  $A(q)$  is an eigenform for  $T_2$  with eigenvalue 0.

**Table 17**  
Values of  $U(n)$ ,  $V(n)$ .

$n$	$U(n)$	$V(n)$
$n \equiv 0 \pmod{8}$	1	1
$n \equiv 1 \pmod{8}$	$\sqrt{2}$	$-\sqrt{2}$
$n \equiv 2 \pmod{8}$	1	1
$n \equiv 3 \pmod{8}$	0	0
$n \equiv 4 \pmod{8}$	-1	-1
$n \equiv 5 \pmod{8}$	$-\sqrt{2}$	$\sqrt{2}$
$n \equiv 6 \pmod{8}$	-1	-1
$n \equiv 7 \pmod{8}$	0	0

Using (1.20) with our computed eigenvalues, we find

$$[p^\nu]A(q) = \begin{cases} 0, & p = 2, \nu > 0, \\ 1 + \nu, & (1, 0, 256, p) > 0, \\ (-1)^\nu(1 + \nu), & (4, 4, 65, p) > 0, \\ U(\nu), & (5, 4, 52, p) > 0, \\ V(\nu), & (13, 4, 20, p) > 0, \\ (-1)^{\frac{\nu}{2}} \frac{(-1)^\nu + 1}{2}, & (16, 8, 17, p) > 0, \\ \frac{(-1)^\nu + 1}{2}, & \left(\frac{-1024}{p}\right) = -1, \end{cases} \quad (5.26)$$

with

$$U(n) := \frac{\sin(\pi(n+1)/4)}{\sin(\pi/4)}, \quad (5.27)$$

and

$$V(n) := \frac{\sin(3\pi(n+1)/4)}{\sin(3\pi/4)}. \quad (5.28)$$

The functions  $U(n)$  and  $V(n)$  are periodic and we tabulate their values in Table 17.

We mention that the Weber class polynomial for discriminant  $-1024$  is

$$\begin{aligned} W_{-1024}(x) := & x^8 - 2363648x^7 - 14141504x^6 - 33443840x^5 - 9272384x^4 \\ & - 6554624x^3 - 493568x^2 - 278528x - 128. \end{aligned}$$

The factorization pattern of  $W_{-1024}(x) \pmod{p}$  does not distinguish between primes represented by  $(5, 4, 52)$  and  $(13, 4, 20)$ . As in previous examples, one can use the remainder criteria of [7] and [20] to distinguish between primes that are represented by different forms of discriminant  $-1024$ .

Define  $S_1$  to be the set of primes  $p$  represented by  $(1, 0, 256)$  or  $(4, 4, 65)$ . We also define  $S_2, S_3$  to be the set of primes  $p$  represented by  $(5, 4, 52)$ ,  $(13, 4, 20)$ , respectively. Lastly, we let  $S_4$  be the set of primes  $p$  with  $(16, 8, 17, p) > 0$  or  $\left(\frac{-1024}{p}\right) = -1$ .

We find that the formula for  $[q^n]A(q)$  is given by

$$\begin{aligned} & (-1)^{t+\frac{s}{2}} \cdot \delta_{a,0} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \\ & \cdot \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) \prod_{p_4 \in S_4} \frac{(-1)^{\text{ord}_{p_4}(n)} + 1}{2}, \end{aligned} \quad (5.29)$$

where  $a = \text{ord}_2(n)$ ,  $t$  is the number of primes factors  $p$  of  $n$ , counting multiplicity, that are represented by  $(4, 4, 65)$ , and  $s$  is the number of primes factors of  $n$ , counting multiplicity, that are represented by  $(16, 8, 17)$ .

As mentioned earlier, we are able to employ congruences to extract the coefficients of  $q\psi(q^8)\phi(-q^{64})$  and  $q^5\psi(-q^8)\psi(-q^{32})$ . We obtain

$$[q^n]q\psi(q^8)\phi(-q^{64}) = \begin{cases} [q^n]A(q), & n \equiv 1 \pmod{8}, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$[q^n]q^5\psi(-q^8)\psi(-q^{32}) = \begin{cases} [q^n]A(q)/\sqrt{2}, & n \equiv 5 \pmod{8}, \\ 0, & \text{otherwise.} \end{cases}$$

Given a positive integer  $n$ , we define  $r_i$  to be the number of prime factors of  $n$  with  $p_2 \in S_2$  such that  $\text{ord}_{p_2}(n) \equiv i \pmod{8}$ . Similarly,  $s_i$  is the number of prime factors of  $n$  with  $p_3 \in S_3$  and  $\text{ord}_{p_3}(n) \equiv i \pmod{8}$ .

With the notation of  $r_i$  and  $s_i$  we have

$$\prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) = \delta_{r_3+r_7,0} \cdot (-1)^{r_4+r_5+r_6} \cdot 2^{\frac{r_1+r_5}{2}}, \quad (5.30)$$

and

$$\prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) = \delta_{s_3+s_7,0} \cdot (-1)^{s_1+s_4+s_6} \cdot 2^{\frac{s_1+s_5}{2}}, \quad (5.31)$$

where  $\delta_{i,j}$  is 1 if  $i = j$  and 0 otherwise. Letting  $k_i := r_i + s_i$  for  $i = 1, 2$ , we obtain

$$\prod_{p_2 \in S_2} U(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} V(\text{ord}_{p_3}(n)) = \delta_{k_3+k_7,0} \cdot (-1)^{s_1+r_5+k_4+k_6} \cdot 2^{\frac{k_1+k_5}{2}} \quad (5.32)$$

and

$$\prod_{p_2 \in S_2} V(\text{ord}_{p_2}(n)) \prod_{p_3 \in S_3} U(\text{ord}_{p_3}(n)) = \delta_{k_3+k_7,0} \cdot (-1)^{r_1+s_5+k_4+k_6} \cdot 2^{\frac{k_1+k_5}{2}}. \quad (5.33)$$

Using (5.32) and (5.33) we reformulate (5.29) as

$$[q^n]A(q) = (-1)^{t+\frac{s}{2}+s_1+r_5+k_4+k_6} \cdot \delta_{a+k_3+k_7,0} \cdot 2^{\frac{k_1+k_5}{2}} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_4 \in S_4} \frac{(-1)^{\text{ord}_{p_4}(n)} + 1}{2}. \quad (5.34)$$

We obtain

$$[q^n]2q\psi(q^8)\phi(-q^{64}) = [q^n]A(q)(1 + (-1)^{k_1+k_5}), \quad (5.35)$$

$$[q^n]2\sqrt{2}q^5\psi(-q^8)\psi(-q^{32}) = [q^n]A(q)(1 - (-1)^{k_1+k_5}), \quad (5.36)$$

thus the Fourier coefficients of  $2q\psi(q^8)\phi(-q^{64})$  are given by

$$(1 + (-1)^{k_1+k_5})(-1)^{b+t+\frac{s}{2}+s_1+r_5+k_4+k_6} \cdot \delta_{a+k_3+k_7,0} \cdot 2^{\frac{k_1+k_5}{2}} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_4 \in S_4} \frac{(-1)^{\text{ord}_{p_4}(n)} + 1}{2} \quad (5.37)$$

and the Fourier coefficients of  $2\sqrt{2}q^5\psi(-q^8)\psi(-q^{32})$  are given by

$$(1 - (-1)^{k_1+k_5})(-1)^{b+t+\frac{s}{2}+s_1+r_5+k_4+k_6} \cdot \delta_{a+k_3+k_7,0} \cdot 2^{\frac{k_1+k_5}{2}} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_4 \in S_4} \frac{(-1)^{\text{ord}_{p_4}(n)} + 1}{2}. \quad (5.38)$$

#### 5.4. An eta-product of level 1872

Applying [Theorem 2.1](#) with  $m = 12$  and  $s = 7$ , yields

$$\frac{B(72, 12, 7, q) - B(72, 60, 19, q)}{2} = \frac{B(7, 2, 67, q) - B(19, 16, 28, q)}{2} = q^7 E(Q)E(Q^{13}), \quad (5.39)$$

where we define  $Q := q^{12}$ .

We have  $\text{CL}(-1872) \cong C_4 \times C_4$  and

$\text{CL}(-1872) \cong C_4 \times C_4$	$(\frac{p}{3})$	$(\frac{p}{13})$	$(\frac{-1}{p})$
Principal genus	(1, 0, 468), (4, 0, 117), (9, 0, 52), (13, 0, 36)	+1	+1
Second genus	(7, 2, 67), (7, -2, 67), (19, 16, 28), (19, -16, 28)	+1	-1
Third genus	(8, 4, 59), (8, -4, 59), (11, 8, 44), (11, -8, 44)	-1	-1
Fourth genus	(9, 6, 53), (9, -6, 53), (17, 10, 29), (17, -10, 29)	-1	+1

In the above table,  $p$  is taken to be coprime to  $-1872$  and represented by the given genus.



Using the bottom row of Table 1 with  $A = (7, 2, 67)$ ,  $B = (11, 8, 44)$  we find

$$A(q) := \frac{B(1, 0, 468, q) + B(13, 0, 36, q) - B(4, 0, 117, q) - B(9, 0, 52, q)}{2} + 2q^7 E(Q)E(Q^{13}) \quad (5.40)$$

is multiplicative.

Appropriately applying (1.12), (1.13), and (1.15) to

$$\frac{B(1, 0, 468, q) + B(13, 0, 36, q) - B(4, 0, 117, q) - B(9, 0, 52, q)}{2}, \quad (5.41)$$

we see a wonderful cancellation that transforms (5.41) into

$$q[\phi(Q^{39})f(Q^2, Q^4) + Q\phi(Q^3)f(Q^{26}, Q^{52}) - 2Q^5\psi(Q^6)f(Q^{13}, Q^{65}) - 2Q^{10}\psi(Q^{78})f(Q, Q^5)]. \quad (5.42)$$

Since  $Q := q^{12}$ , the Fourier expansion of (5.42) only contains terms with exponents congruent to 1 (mod 12). Similarly, the Fourier expansion of  $q^7 E(Q)E(Q^{13})$  only contains terms with exponents congruent to 7 (mod 12). Hence congruences can be employed to extract coefficients of (5.39) from the completion (5.40).

The expression (5.42) is also discussed in [12, p. 181] where the *eta*-product  $q^7 E(Q)E(Q^{13})$  is related to the quadratic fields  $\mathbb{Q}(\sqrt{-3})$ ,  $\mathbb{Q}(\sqrt{-13})$ , and  $\mathbb{Q}(\sqrt{-39})$ .

In [10], Gordon and Hughes consider the product  $q^7 E(Q)E(Q^{13})$ , and introduce the function

$$h(q) = q[\phi(-Q^{39})f(Q^2, Q^4) + Q\phi(-Q^3)f(Q^{26}, Q^{52}) - Q^5\psi(Q^6)f(Q^{13}, Q^{65}) - Q^{10}\psi(Q^{78})f(Q, Q^5)]. \quad (5.43)$$

One can verify  $h(q) + 2q^7 E(Q)E(Q^{13})$  is an eigenform for some, but not all Hecke operators. Indeed,  $h(q) + 2q^7 E(Q)E(Q^{13})$  is not an eigenform for  $T_7, T_{11}, T_{17}$ , among others. We note the striking similarity between (5.42) and  $h(q)$ . We also comment that Gordon and Hughes were well aware that  $h(q)$  was not an eigenform for all  $T_p$ , and did not make any erroneous claims regarding the action of  $T_p$  on  $h(q)$ .

The action of  $T_p$  on the forms of discriminant  $-1872$  are omitted since the computations are analogous to previous examples. The action of  $T_p$  on  $A(q)$  is given in Table 18 for primes  $p$  with  $(\frac{p}{3}) = (\frac{-13}{p}) = 1$ . Note the property  $(\frac{p}{3}) = (\frac{-13}{p}) = 1$  is equivalent to the prime  $p \neq 13$  being represented by the principal genus or second genus, according to the table of genera at the beginning of this example.

For any prime  $p \neq 13$  that does not have  $(\frac{p}{3}) = (\frac{-13}{p}) = 1$ , we find  $A(q)$  is an eigenform under  $T_p$  with eigenvalue 0.  $A(q)$  is an eigenform for  $T_{13}$  with eigenvalue 1. Hence  $A(q)$  is an eigenform for all Hecke operators. Employing (1.20) we obtain

**Table 18**Hecke operator action on  $A(q)$ .

$(1, 0, 468, p) > 0$	$(4, 0, 117, p) > 0$	$(13, 0, 36, p) > 0$	$(9, 0, 52, p) > 0$	$(7, 2, 67, p) > 0$	$(19, 16, 28, p) > 0$
$A(q) \xrightarrow{T_p} 2A(q)$	$-2A(q)$	$2A(q)$	$-2A(q)$	$2A(q)$	$-2A(q)$

$$[q^{p^\nu}]A(q) = \begin{cases} 0, & p = 2, 3, \nu > 0, \\ 1, & p = 13, \\ 1 + \nu, & p \neq 13, (1, 0, 468, p) + (13, 0, 36, p) + (7, 2, 67, p) > 0, \\ (-1)^\nu(1 + \nu), & (4, 0, 117, p) + (9, 0, 52, p) + (19, 16, 28, p) > 0, \\ (-1)^{\frac{\nu}{2}} \frac{(-1)^\nu + 1}{2}, & -(\frac{p}{3}) = (\frac{-13}{p}) = 1, \\ \frac{(-1)^\nu + 1}{2}, & (\frac{-1872}{p}) = -1. \end{cases} \quad (5.44)$$

Since  $|\text{CL}(-1872)| = 16$ , the associated Weber class polynomial,  $W_{-1872}(x)$ , is a degree 16 polynomial. Explicitly,

$$\begin{aligned} W_{-1872}(x) := & x^{16} - 8x^{15} + 24x^{14} - 34x^{13} + x^{12} + 246x^{11} - 1094x^{10} \\ & + 2574x^9 - 4200x^8 + 5608x^7 - 5144x^6 + 858x^5 + 4189x^4 \\ & - 5166x^3 + 2814x^2 - 750x + 69. \end{aligned}$$

Following [7] and [20], one can use the remainder criteria that we have seen in previous examples to determine which primes are represented by a given form of discriminant  $-1872$ . The coefficients in this criteria become rather large and so we omit the computations. We remark that the procedure is completely analogous to previous examples.

Let  $S_1$  be the set of primes  $p$  with  $(\frac{p}{3}) = (\frac{-13}{p}) = 1$ . Employing (5.44) we obtain

$$[q^n]A(q) = (-1)^{t_1 + \frac{s}{2}} \cdot \delta_{\text{ord}_2(n) + \text{ord}_3(n), 0} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_2 \notin S_1 \cup \{2, 3, 13\}} \frac{(-1)^{\text{ord}_{p_2}(n)} + 1}{2}, \quad (5.45)$$

where  $n$  has  $t_1$  prime factors with  $(4, 0, 117, p) + (9, 0, 52, p) + (19, 16, 28, p) > 0$ ,  $t_2$  prime factors with  $(4, 0, 117, p) + (9, 0, 52, p) + (7, 2, 67, p) > 0$ ,  $s$  prime factors with  $-(\frac{p}{3}) = (\frac{-13}{p}) = 1$ , and the right most product is taken over primes  $p_2 \notin S_1 \cup \{2, 3, 13\}$ . All prime factors are counted with multiplicity, and similar to the previous example,  $s$  odd implies (5.45) vanish.

As mentioned earlier, we can write the Fourier coefficients of  $q^7 E(Q)E(Q^{13})$  by employing congruences. We have

$$[q^n]q^7 E(Q)E(Q^{13}) = \begin{cases} \frac{[q^n]A(q)}{2}, & n \equiv 7 \pmod{12}, \\ 0, & \text{otherwise.} \end{cases}$$

We arrive at the formula

$$\frac{(1 - (-1)^{t_1+t_2})}{4} (-1)^{t_1+\frac{s}{2}} \prod_{p_1 \in S_1} (1 + \text{ord}_{p_1}(n)) \prod_{p_2 \notin S_1 \cup \{2,3,13\}} \frac{(-1)^{\text{ord}_{p_2}(n)} + 1}{2} \quad (5.46)$$

for the Fourier coefficient  $[q^n]q^7 E(Q)E(Q^{13})$ .

## 6. Concluding remarks

We note that the *eta*-product of level 71, discussed in Section 4, has many similarities with the *eta*-product of level 47, discussed in Section 3, and also with the *eta*-product of level 23 which has been previously discussed [4]. In the future we hope to continue the discussion of the *eta*-products  $q^{\frac{p+1}{24}} E(q)E(q^p)$  of level  $p \equiv -1 \pmod{24}$  where  $p$  is prime.

We lastly comment that the authors purposefully did not include any examples which are associated with a class group isomorphic to  $C_4 \times C_2^r$ , with  $0 \leq r \leq 4$ . Much more can be said for these types of examples, and the authors hope to discuss these examples in a separate paper.

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## References

- [1] A. Berkovich, H. Yesilyurt, Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms, *Ramanujan J.* 20 (3) (2009) 375–408.
- [2] A. Berkovich, H. Yesilyurt, On Rogers–Ramanujan functions, binary quadratic forms and eta-quotients, *Proc. Amer. Math. Soc.* (2013), in press, arXiv:1204.1092 [math.NT].
- [3] B.C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.
- [4] F. van der Blij, Binary quadratic forms of discriminant  $-23$ , *Indag. Math.* 55 (1952) 498–503.
- [5] D. Buell, *Binary Quadratic Forms: Classical Theory and Modern Computations*, Springer-Verlag, New York, 1990.

- [6] H.H. Chan, S. Cooper, W.-C. Liaw, On  $\eta^3(a\tau)\eta^3(b\tau)$  with  $a + b = 8$ , *J. Aust. Math. Soc.* 84 (2008) 301–313.
- [7] D.A. Cox, *Primes of the Form  $x^2 + ny^2$ : Fermat, Class Field Theory, and Complex Multiplication*, John Wiley & Sons, New York, 1989.
- [8] L.E. Dickson, *Introduction to the Theory of Numbers*, Dover, New York, 1957.
- [9] R. Fricke, *Lehrbuch der Algebra*, vol. III, Friedr. Vieweg, 1924.
- [10] B. Gordon, K. Hughes, Multiplicative properties of  $\eta$ -products II, in: *Contemp. Math.*, vol. 143, 1993, pp. 415–430.
- [11] E. Hecke, *Mathematische Werke*, Vandenhoeck & Ruprecht, Göttingen, 1970.
- [12] G. Köhler, *Eta Products and Theta Series Identities*, Springer, New York, 2011.
- [13] A. Okamoto, On expressions of theta series by  $\eta$ -products, *Tokyo J. Math.* 34 (2) (2011) 319–326.
- [14] B. Schoeneberg, Bemerkungen über einige Klassen von Modulformen, *Indag. Math.* 70 (1967) 177–182.
- [15] J.-P. Serre, Modular forms of weight one and Galois representations, in: A. Froehlich (Ed.), *Algebraic Number Fields*, 1977, pp. 193–268.
- [16] C.L. Siegel, Über die analytische theorie der quadratische Formen, *Ann. of Math.* 36 (1935) 527–606.
- [17] Z.H. Sun, The expansion of  $\prod_{k=1}^{\infty} (1 - q^{ak})(1 - q^{bk})$ , *Acta Arith.* 134 (1) (2008) 11–29.
- [18] Z.H. Sun, K.S. Williams, On the number of representations of  $n$  by  $ax^2 + bxy + cy^2$ , *Acta Arith.* 122 (2) (2006) 101–171.
- [19] Z.H. Sun, K.S. Williams, Ramanujan identities and Euler products for a type of Dirichlet series, *Acta Arith.* 122 (4) (2006) 349–393.
- [20] J. Voight, Quadratic forms that represent almost the same primes, *Math. Comp.* 76 (2007) 1589–1617.
- [21] K.S. Williams, Fourier series of a class of eta quotients, *Int. J. Number Theory* 8 (2012) 993–1004.
- [22] O.X.M. Yao, E.X.W. Xia, J. Jin, Explicit formulas for the Fourier coefficients of a class of eta quotients, *Int. J. Number Theory* 9 (2013) 487–503.