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Some new quicker approximations of Glaisher–Kinkelin's and Bendersky–Adamchik's constants

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ABSTRACT

In this paper, some new polynomial approximations, inequalities and rates of convergence of Glaisher–Kinkelin's and Bendersky–Adamchik's constants are provided. Finally, for demonstrating the superiority of our new convergent sequences over the classical sequences, some numerical computations are also given.

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1. Introduction

In the theory of mathematical constants, it is very important to construct new sequences which converge to these fundamental constants with increasingly higher speed. To the best of our knowledge, one of the most useful convergent sequences in mathematics is

$$w_n = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln(n) + \frac{n^2}{4}, \tag{1.1}$$

which converges towards the well-known Glaisher–Kinkelin's constant $\ln(A)$ and $A \approx 1.282427130\ldots$. This constant appeared in Barnes [1], and is also related to the Riemann zeta function ζ, or the Euler's constant $\gamma = 0.577215664\ldots$.

Related to Glaisher–Kinkelin's constant, the following sequences are defined,

$$s_n = \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln(n) + \frac{n^3}{9} - \frac{n}{12} \tag{1.2}$$

and

$$t_n = \sum_{k=1}^{n} k^3\ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\ln(n) + \frac{n^4}{16} - \frac{n^2}{12}, \tag{1.3}$$

which converge towards the well-known Bendersky–Adamchik's constant $\ln(B)$ and $\ln(C)$, where $B \approx 1.03091675\ldots$ and $C \approx 0.97955746\ldots$. These two constants were considered by Choi and Srivastava in [3,4,2] in the theory of multiple gamma functions.

These convergent sequences and constants play a key role in many areas of mathematics and science in general, as theory of probability, applied statistics, physics, special functions, number theory, or analysis.

Up until now, many researchers made great efforts in the area of concerning the rate of convergence of these sequences, and establishing faster sequences to converge to Glaisher–Kinkelin's and Bendersky–Adamchik's constants. For example, Mortici [12] provided some new inequalities for these constants as follows:

$$w_n - \frac{1}{720n^2} + \frac{1}{5040n^4} - \frac{1}{10\,080n^6} < \ln(A) < w_n - \frac{1}{720n^2} + \frac{1}{5040n^4}, \tag{1.4}$$

$$s_n + \frac{1}{360n} - \frac{1}{7560n^3} < \ln(B) < s_n + \frac{1}{360n}, \tag{1.5}$$

$$t_n + \frac{1}{5040n^2} - \frac{1}{33\,600n^4} < \ln(C) < t_n + \frac{1}{5040n^2}. \tag{1.6}$$

It is their work that motivate our study. In this paper, we provide some quicker convergent sequences for Glaisher–Kinkelin's and Bendersky–Adamchik's constants as follows:

Theorem 1.1. *For Glaisher–Kinkelin's constant, we have*

$$\ln(A) \approx \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{a_3}{n^3} + \frac{a_4}{n^4} + \frac{a_5}{n^5} + \cdots\right) + \frac{n^2}{4}, \tag{1.7}$$

where

$$a_3 = \frac{1}{360}, \qquad a_4 = -\frac{1}{360}, \qquad a_5 = \frac{29}{15\,120}, \ldots.$$

For Bendersky–Adamchik's constants, we have

$$\ln(B) \approx \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln\left(n + \frac{b_3}{n^3} + \frac{b_4}{n^4} + \frac{b_5}{n^5} + \cdots\right) + \frac{n^3}{9} - \frac{n}{12}, \tag{1.8}$$

$$\ln(C) \approx \sum_{k=1}^{n} k^3\ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\ln\left(n + \frac{c_5}{n^5} + \frac{c_6}{n^6} + \frac{c_7}{n^7} + \cdots\right) + \frac{n^4}{16} - \frac{n^2}{12}, \tag{1.9}$$

where

$$b_3 = -\frac{1}{120}, \qquad b_4 = \frac{1}{80}, \qquad b_5 = -\frac{143}{10\,080}, \ldots$$

and

$$c_5 = -\frac{1}{1260}, \qquad c_6 = \frac{1}{630}, \qquad c_7 = -\frac{19}{8400}, \ldots.$$

Next, using Theorem 1.1, we provide some inequalities for Glaisher–Kinkelin's and Bendersky–Adamchik's constants

Theorem 1.2. *For all natural numbers $n \geq 1$, we have*

$$\tilde{w}_n^{(1)} < \ln(A) < \tilde{w}_n^{(2)}, \tag{1.10}$$

$$\tilde{s}_n^{(2)} < \ln(B) < \tilde{s}_n^{(1)}, \tag{1.11}$$

$$\tilde{t}_n^{(2)} < \ln(C) < \tilde{t}_n^{(1)}, \tag{1.12}$$

where

$$\tilde{w}_n^{(1)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{1}{360n^3}\right) + \frac{n^2}{4},$$

$$\tilde{w}_n^{(2)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{1}{360n^3} - \frac{1}{360n^4}\right) + \frac{n^2}{4},$$

$$\tilde{s}_n^{(1)} = \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln\left(n - \frac{1}{120n^3}\right) + \frac{n^3}{9} - \frac{n}{12},$$

$$\tilde{s}_n^{(2)} = \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln\left(n - \frac{1}{120n^3} + \frac{1}{80n^4}\right) + \frac{n^3}{9} - \frac{n}{12},$$

$$\tilde{t}_n^{(1)} = \sum_{k=1}^{n} k^3\ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\ln\left(n - \frac{1}{1260n^5}\right) + \frac{n^4}{16} - \frac{n^2}{12},$$

$$\tilde{t}_n^{(2)} = \sum_{k=1}^{n} k^3\ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\ln\left(n - \frac{1}{1260n^5} + \frac{1}{630n^6}\right) + \frac{n^4}{16} - \frac{n^2}{12}.$$

Finally, to show that the three new approximations convergence faster, combining Theorems 1.1 and 1.2, we provide the rates of convergence of these three sequences as follows:

Theorem 1.3. *For all natural numbers $n \geq 2$, we have*

$$\frac{29}{30\,240(n+1)^4} < \tilde{w}_n^{(2)} - \ln(A) < \frac{29}{30\,240(n-1)^4}; \tag{1.13}$$

$$\frac{143}{30\,240(n+1)^3} < \ln(B) - \tilde{s}_n^{(2)} < \frac{143}{30\,240(n-1)^3}; \tag{1.14}$$

$$\frac{19}{33\,600(n+1)^4} < \ln(C) - \tilde{t}_n^{(2)} < \frac{19}{33\,600(n-1)^4}. \tag{1.15}$$

To obtain Theorem 1.1, we need the following lemma which was used in [7–11] and very useful for constructing asymptotic expansions.

Lemma 1.1. *If $(x_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty], \tag{1.16}$$

with $s > 1$, then

$$\lim_{n\to\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{1.17}$$

Lemma 1.1 was first proved by Mortici in [10]. From Lemma 1.1, we can see that the speed of convergence of the sequence $(x_n)_{n\geq 1}$ increases together with the value s satisfying (1.16).

The rest of this paper is arranged as follows: In Section 2, we provide the proof of Theorem 1.1. In Section 3, the proof of Theorem 1.2 is given. In Section 4, we complete the proof of Theorem 1.3. In Section 5, we give some numerical computations which demonstrate the superiority of our new convergent sequences over the classical sequences.

2. Proof of Theorem 1.1

First, we deal with (1.7). Based on the argument of Theorem 2.1 in [11] or Theorem 5 in [13], we need to find the value $a_3 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$\tilde{w}_n^{(1)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{a_3}{n^3}\right) + \frac{n^2}{4}. \tag{2.1}$$

To measure the accuracy of this approximation, a method is to say that an approximation (2.1) is better if $\tilde{w}_n^{(1)} - \ln(A)$ converges to zero faster. Using (2.1) and developing the power series in $1/n$, we have

$$\tilde{w}_n^{(1)} - \tilde{w}_{n+1}^{(1)} = \frac{1-360a_3}{360n^3} - \frac{1}{240n^4} + \frac{140a_3+1}{210n^5} + O\left(\frac{1}{n^6}\right). \tag{2.2}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(\tilde{w}_n^{(1)} - \ln(A))_{n\geq 1}$ is even higher as the value s satisfying (1.16). Thus, using Lemma 1.1, we have:

(i) If $a_3 \neq 1/360$, then the rate of convergence of the sequence $(\tilde{w}_n^{(1)} - \ln(A))_{n\geq 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2\big(\tilde{w}_n^{(1)} - \ln(A)\big) = \frac{1-360a_3}{720} \neq 0.$$

(ii) If $a_3 = 1/360$, then from (2.2), we have

$$\tilde{w}_n^{(1)} - \tilde{w}_{n+1}^{(1)} = -\frac{1}{240n^4} + O\left(\frac{1}{n^5}\right)$$

and the rate of convergence of the sequence $(\tilde{w}_n^{(1)} - \ln(A))_{n\geq 1}$ is n^{-3}, since

$$\lim_{n\to\infty} n^3\big(\tilde{w}_n^{(1)} - \ln(A)\big) = -\frac{1}{720}.$$

We know that the fastest possible sequence $(\tilde{w}_n^{(1)})_{n\geq 1}$ is obtained only for $a_3 = 1/360$.

Next, we define the sequence $(\tilde{w}_n^{(2)})_{n\geq 1}$ by the relation

$$w_n^{(2)} = \sum_{k=1}^{n} k\ln(k) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\ln\left(n + \frac{1}{360n^3} + \frac{a_4}{n^4}\right) + \frac{n^2}{4}. \tag{2.3}$$

Using the same method from (2.1) to (2.2), we have

$$\tilde{w}_n^{(2)} - \tilde{w}_{n+1}^{(2)} = \frac{-1 - 360a_4}{240n^4} + \frac{5 + 756a_4}{756n^5} + \frac{-29 - 1260a_4}{3024n^6} + O\left(\frac{1}{n^7}\right). \tag{2.4}$$

The fastest possible sequence $(\tilde{w}_n^{(2)})_{n\geq 1}$ is obtained only for $a_4 = -1/360$. Then, from (2.4), we have

$$\tilde{w}_n^{(2)} - \tilde{w}_{n+1}^{(2)} = \frac{29}{7560n^5} + O\left(\frac{1}{n^6}\right)$$

and the rate of convergence of the sequence $(\tilde{w}_n^{(2)} - \ln(A))_{n\geq 1}$ is n^{-4}, since

$$\lim_{n\to\infty} n^4\left(\tilde{w}_n^{(2)} - \ln(A)\right) = \frac{29}{30\,240}.$$

Similarly, we have $a_5 = 29/15\,120, \cdots$, the new sequence (1.7) is obtained.

Next, we deal with (1.8). We need to find the value $b_3 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$\tilde{s}_n^{(1)} = \sum_{k=1}^{n} k^2\ln(k) - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\ln\left(n + \frac{b_3}{n^3}\right) + \frac{n^3}{9} - \frac{n}{12}. \tag{2.5}$$

Using (2.5) and developing the power series in $1/n$, we have

$$\tilde{s}_n^{(1)} - \tilde{s}_{n+1}^{(1)} = \frac{-1 - 120b_3}{360n^2} + \frac{1 - 240b_3}{360n^3} + \frac{280b_3 - 1}{420n^4} + O\left(\frac{1}{n^5}\right). \tag{2.6}$$

From Lemma 1.1, we know that the speed of convergence of the sequence $(\tilde{s}_n^{(1)} - \ln(B))_{n\geq 1}$ is even higher as the value s satisfying (1.16). Thus, using Lemma 1.1, we have

(i) If $b_3 \neq -1/120$, then the rate of convergence of the sequence $(\tilde{s}_n^{(1)} - \ln(B))_{n\geq 1}$ is n^{-1}, since

$$\lim_{n\to\infty} n\left(s_n^{(1)} - \ln(B)\right) = \frac{-1 - 120b_3}{360} \neq 0.$$

(ii) If $b_3 = -1/120$, then from (2.6), we have

$$\tilde{s}_n^{(1)} - \tilde{s}_{n+1}^{(1)} = \frac{1}{120n^3} + O\left(\frac{1}{n^4}\right)$$

and the rate of convergence of the sequence $(\tilde{s}_n^{(1)} - \ln(B))_{n\geq 1}$ is n^{-2}, since

$$\lim_{n\to\infty} n^2\big(\tilde{s}_n^{(1)} - \ln(B)\big) = \frac{1}{240}.$$

We know that the fastest possible sequence $(\tilde{s}_n^{(1)})_{n\geq 1}$ is obtained only for $b_3 = -1/120$.

Similarly, we have $b_4 = 1/80$, $b_5 = -143/10\,080, \cdots$, and the new sequence (1.8) is obtained.

Finally, we deal with (1.9). We need to find the value $c_5 \in \mathbb{R}$ which produces the most accurate approximation of the form

$$\tilde{t}_n^{(1)} = \sum_{k=1}^{n} k^3 \ln(k) - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right) \ln\left(n + \frac{c_5}{n^5}\right) + \frac{n^4}{16} - \frac{n^2}{12}. \tag{2.7}$$

Using the same method from (2.5) to (2.6), we have

$$\tilde{t}_n^{(1)} - \tilde{t}_{n+1}^{(1)} = \frac{-1 - 1260c_5}{2520n^3} + \frac{1 - 1260c_5}{1680n^4} + \frac{25\,200c_5 - 17}{25\,200n^5} + O\left(\frac{1}{n^6}\right). \tag{2.8}$$

The fastest possible sequence $(\tilde{t}_n^{(1)})_{n\geq 1}$ is obtained only for $c_5 = -1/1260$. Then, from (2.8), we have

$$\tilde{t}_n^{(1)} - \tilde{t}_{n+1}^{(1)} = \frac{1}{840n^4} + O\left(\frac{1}{n^5}\right)$$

and the rate of convergence of the sequence $(\tilde{t}_n^{(1)} - \ln(C))_{n\geq 1}$ is n^{-3}, since

$$\lim_{n\to\infty} n^3\big(\tilde{t}_n^{(1)} - \ln(C)\big) = \frac{1}{2520}.$$

Similarly, we have $c_6 = 1/630$, $c_7 = -19/8400, \cdots$, and the new sequence (1.9) is obtained.

3. Proof of Theorem 1.2

First, we deal with (1.10). Since $\tilde{w}_n^{(1)}, \tilde{w}_n^{(2)}$ converge to $\ln(A)$, we only need to show that $(\tilde{w}_n^{(1)})_{n\geq 1}$ is strictly increasing and $(\tilde{w}_n^{(2)})_{n\geq 1}$ is strictly decreasing.

Let $f_A(x) = \tilde{w}_x^{(1)} - \tilde{w}_{x+1}^{(1)}$, $g_A(x) = \tilde{w}_x^{(2)} - \tilde{w}_{x+1}^{(2)}$. By some calculations, we have

$$f_A'''(x) = \frac{F_A(x)}{2x^3(x+1)^3(360x^4+1)^3(360x^4+1440x^3+2160x^2+1440x+361)^3} > 0,$$

$$g_A'''(x) = -\frac{G_A(x)}{6x^3(x+1)^3(360x^5+x-1)^3(360x^5+1800x^4+3600x^3+3600x^2+1801x+360)^3} < 0,$$

where

$$
\begin{aligned}
F_A(x) = {} & 562\,986\,720x + 3\,465\,352\,468\,320\,768\,000x^{16} + 1\,321\,288\,738\,099\,200\,000x^{19} \\
& + 2\,176\,782\,336\,000\,000x^{23} + 26\,605\,117\,440\,000\,000x^{22} \\
& + 151\,891\,034\,112\,000\,000x^{21} + 537\,784\,960\,020\,480\,000x^{20} \\
& + 3\,272\,090\,834\,534\,400\,000x^{17} + 2\,386\,418\,568\,192\,000\,000x^{18} \\
& + 14\,511\,340\,088x^4 - 613\,501\,543\,221\,120x^8 + 326\,634\,457\,115\,520\,000x^{12} \\
& + 1\,824\,559\,560\,069\,120\,000x^{14} + 894\,162\,859\,591\,680\,000x^{13} \\
& + 83\,383\,032\,198\,988\,800x^{11} + 12\,117\,703\,331\,635\,200x^{10} \\
& - 284\,305\,012\,992\,000x^9 - 155\,011\,978\,272\,960x^7 - 20\,743\,871\,103\,360x^6 \\
& - 1\,332\,784\,212\,480x^5 + 3\,090\,188\,880x^2 + 10\,899\,957\,608x^3 + 47\,045\,881 \\
& + 2\,854\,285\,964\,116\,992\,000x^{15},
\end{aligned}
$$

$$
\begin{aligned}
G_A(x) = {} & 3\,584\,444\,799\,343\,426\,560x^{14} + 108\,626\,650\,193\,138\,688\,000x^{20} + 2\,241\,043\,200x \\
& + 91\,231\,436\,172\,676\,608\,000x^{21} + 61\,969\,625\,709\,866\,496\,000x^{22} \\
& + 22\,365\,172\,491\,587\,256x^{10} + 82\,768\,456\,160\,449\,920\,000x^{18} \\
& + 954\,059\,363\,880\,370\,560x^{13} + 10\,521\,114\,624\,000\,000x^{28} + 110\,380\,020\,819x^4 \\
& + 38\,049\,203\,537x^3 + 11\,768\,202\,720x^2 + 33\,649\,521\,271\,234\,560\,000x^{23} \\
& + 105\,150\,184\,424\,994\,816\,000x^{19} + 152\,374\,763\,520\,000\,000x^{27} \\
& + 4\,596\,921\,681\,223\,680\,000x^{25} + 6\,212\,992\,160\,247\,888x^9 \\
& + 14\,305\,316\,480\,225\,280\,000x^{24} + 238\,786\,513\,777\,236\,960x^{12} \\
& + 169\,407\,912\,114\,744x^7 + 52\,771\,118\,954\,658\,508\,800x^{17} + 707\,643\,146\,859x^5 \\
& + 13\,516\,465\,204\,881x^6 + 11\,047\,988\,148\,370\,444\,800x^{15} \\
& + 27\,050\,892\,805\,333\,555\,200x^{16} + 69\,372\,578\,784\,375\,360x^{11} \\
& + 1\,261\,219\,906\,565\,640x^8 + 1\,051\,026\,699\,202\,560\,000x^{26} + 186\,624\,000.
\end{aligned}
$$

Combining $f_A''(\infty) = 0$, $g_A''(\infty) = 0$ and $f_A'''(x) > 0$, $g_A'''(x) < 0$, we have $f_A''(x) < 0$, $g_A''(x) > 0$ for $x \geq 1$. Thus, $f_A(x)$ is strictly concave, and $g_A(x)$ is strictly convex. Combining $f_A(\infty) = 0$ and $g_A(\infty) = 0$, we obtain $f_A(x) < 0$ and $g_A(x) > 0$ for $x \geq 1$. The proof of (1.10) is completed.

Next, we deal with (1.11). We only need to show that $(\tilde{s}_n^{(2)})_{n\geq 1}$ is strictly increasing and $(\tilde{s}_n^{(1)})_{n\geq 1}$ is strictly decreasing. Let $f_B(x) = \tilde{s}_x^{(2)} - \tilde{s}_{x+1}^{(2)}$, $g_B(x) = \tilde{s}_x^{(1)} - \tilde{s}_{x+1}^{(1)}$. By similar calculations, we have $f_B'''(x) > 0$ and $g_B'''(x) < 0$. Combining $f_B''(\infty) = 0$, $g_B''(\infty) = 0$ and $f_B'''(x) > 0$, $g_B'''(x) < 0$, we have $f_B''(x) < 0$, $g_B''(x) > 0$ for $x \geq 1$. Thus, $f_B(x)$ is strictly concave, and $g_B(x)$ is strictly convex. Combining $f_B(\infty) = 0$ and $g_B(\infty) = 0$, we obtain $f_B(x) < 0$ and $g_B(x) > 0$ for $x \geq 1$. The proof of (1.11) is completed.

Finally, we deal with (1.12). We only need to show that $(\tilde{t}_n^{(2)})_{n\geq 1}$ is strictly increasing and $(\tilde{t}_n^{(1)})_{n\geq 1}$ is strictly decreasing. Let $f_C(x) = \tilde{t}_x^{(2)} - \tilde{t}_{x+1}^{(2)}$, $g_C(x) = \tilde{t}_x^{(1)} - \tilde{t}_{x+1}^{(1)}$. By similar calculations, we have $f_C'''(x) > 0$ and $g_C'''(x) < 0$. Combining $f_C''(\infty) = 0$, $g_C''(\infty) = 0$ and $f_C'''(x) > 0$, $g_C'''(x) < 0$, we have $f_C''(x) < 0$, $g_C''(x) > 0$ for $x \geq 1$. Thus, $f_C(x)$ is strictly concave, and $g_C(x)$ is strictly convex. Combining $f_C(\infty) = 0$ and $g_C(\infty) = 0$, we obtain $f_C(x) < 0$ and $g_C(x) > 0$ for $x \geq 1$. The proof of (1.12) is completed.

4. Proof of Theorem 1.3

First, we prove (1.13). Based on the argument of Theorem in [5] or the method in [6], it is easy to have

$$\tilde{w}_n^{(2)} - \ln(A) = \sum_{k=n}^{\infty} \left(\tilde{w}_n^{(2)} - \tilde{w}_{n+1}^{(2)}\right) = \sum_{k=n}^{\infty} f_{\tilde{w}}(k). \tag{4.1}$$

By some calculations, we have

$$\frac{29}{1512(x+1)^6} < -f_{\tilde{w}}'(x) < \frac{29}{1512x^6}, \tag{4.2}$$

as $x \geq 1$. For the upper bound in (1.13), using $f_{\tilde{w}}(\infty) = 0$, we have

$$f_{\tilde{w}}(k) = -\int_k^{\infty} f_{\tilde{w}}'(x)dx < \frac{29}{1512}\int_k^{\infty} x^{-6}dx = \frac{29}{7560}k^{-5} < \frac{29}{7560}\int_{k-1}^{k} x^{-5}dx. \tag{4.3}$$

Combining (4.1) and (4.3), for all nature number $n \geq 2$, we have

$$\tilde{w}_n^{(2)} - \ln(A) < \sum_{k=n}^{\infty} \frac{29}{7560}\int_{k-1}^{k} x^{-5}dx = \frac{29}{7560}\int_{n-1}^{\infty} x^{-5}dx = \frac{29}{30\,240(n-1)^4}. \tag{4.4}$$

For the lower bound, combining (4.2), we have

$$\begin{aligned} f_w(k) &= -\int_k^{\infty} f_w'(x)dx > \frac{29}{1512}\int_k^{\infty} (x+1)^{-6}dx = \frac{29}{7560}(k+1)^{-5} \\ &> \frac{29}{7560}\int_{k+1}^{k+2} x^{-5}dx. \end{aligned} \tag{4.5}$$

Combining (4.1) and (4.5), we have

$$\tilde{w}_n^{(2)} - \ln(A) > \sum_{k=n}^{\infty} \frac{29}{7560} \int_{k+1}^{k+2} x^{-5} dx = \frac{29}{7560} \int_{n+1}^{\infty} x^{-5} dx = \frac{29}{30\,240(n+1)^4}. \tag{4.6}$$

Combining (4.4) and (4.6), we complete the proof of (1.13).

Next, we prove (1.14). It is easy to have

$$\ln(B) - \tilde{s}_n^{(2)} = \sum_{k=n}^{\infty} \left(\tilde{s}_{k+1}^{(2)} - \tilde{s}_k^{(2)}\right) = \sum_{k=n}^{\infty} f_{\tilde{s}}(k). \tag{4.7}$$

By similar calculation, we have

$$\frac{143}{2520(x+1)^5} < -f'_{\tilde{s}}(x) < \frac{143}{2520x^5}, \tag{4.8}$$

for $x \geq 1$. For the upper bound in (1.14), using $f_{\tilde{s}}(\infty) = 0$, we have

$$f_{\tilde{s}}(k) = -\int_k^{\infty} f'_{\tilde{s}}(x) dx < \frac{143}{2520} \int_k^{\infty} x^{-5} dx = \frac{143}{10\,080} k^{-4} < \frac{143}{10\,080} \int_{k-1}^{k} x^{-4} dx. \tag{4.9}$$

Combining (4.7) and (4.9), for all nature number $n \geq 2$, we have

$$\ln(B) - \tilde{s}_n^{(2)} < \sum_{k=n}^{\infty} \frac{143}{10\,080} \int_{k-1}^{k} x^{-4} dx = \frac{143}{10\,080} \int_{n-1}^{\infty} x^{-4} dx = \frac{143}{30\,240(n-1)^3}. \tag{4.10}$$

For the lower bound, combining (4.8), we have

$$f_{\tilde{s}}(k) = -\int_k^{\infty} f'_{\tilde{s}}(x) dx > \frac{143}{2520} \int_k^{\infty} (x+1)^{-5} dx = \frac{143}{10\,080} (k+1)^{-4} > \frac{143}{10\,080} \int_{k+1}^{k+2} x^{-4} dx. \tag{4.11}$$

Combining (4.7) and (4.11), we have

$$\ln(B) - \tilde{s}_n^{(2)} > \sum_{k=n}^{\infty} \frac{143}{10\,080} \int_{k+1}^{k+2} x^{-4} dx = \frac{143}{10\,080} \int_{n+1}^{\infty} x^{-4} dx = \frac{143}{30\,240(n+1)^3}. \tag{4.12}$$

Combining (4.10) and (4.12), we complete the proof of (1.14).

Finally, we prove (1.15).

$$\ln(C) - \tilde{t}_n^{(2)} = \sum_{k=n}^{\infty} \left(\tilde{t}_{k+1}^{(2)} - \tilde{t}_k^{(2)}\right) = \sum_{k=n}^{\infty} f_{\tilde{t}}(k). \tag{4.13}$$

By some calculation, we have

$$\frac{19}{1680(x+1)^6} < -f_{\tilde{t}}'(x) < \frac{19}{1680x^6}, \tag{4.14}$$

for $x \geq 1$. For the upper bound in (1.15), combining $f_{\tilde{t}}(\infty) = 0$, we have

$$f_{\tilde{t}}(k) = -\int_k^\infty f_{\tilde{t}}'(x)dx < \frac{19}{1680}\int_k^\infty x^{-6}dx = \frac{19}{8400}k^{-5} < \frac{19}{8400}\int_{k-1}^k x^{-5}dx. \tag{4.15}$$

Combining (4.13) and (4.15), for all natural number $n \geq 2$, we have

$$\ln(C) - t_n^{(2)} < \sum_{k=n}^\infty \frac{19}{8400}\int_{k-1}^k x^{-5}dx = \frac{19}{8400}\int_{n-1}^\infty x^{-5}dx = \frac{19}{33\,600(n-1)^4}. \tag{4.16}$$

For the lower bound, combining (4.14), we have

$$f_{\tilde{t}}(k) = -\int_k^\infty f_{\tilde{t}}'(x)dx > \frac{19}{1680}\int_k^\infty (x+1)^{-6}dx = \frac{19}{8400}(k+1)^{-5} > \frac{19}{8400}\int_{k+1}^{k+2} x^{-5}dx. \tag{4.17}$$

Combining (4.13) and (4.17), we have

$$\ln(C) - t_n^{(2)} > \sum_{k=n}^\infty \frac{19}{8400}\int_{k+1}^{k+2} x^{-5}dx = \frac{19}{8400}\int_{n+1}^\infty x^{-5}dx = \frac{19}{33\,600(n+1)^4}. \tag{4.18}$$

Combining (4.16) and (4.18), we complete the proof of (1.15).

5. Numerical computation

In this section, we give three tables to demonstrate the superiority of our new convergent sequences $\tilde{w}_n^{(1)}$, $\tilde{w}_n^{(2)}$, $\tilde{s}_n^{(1)}$, $\tilde{s}_n^{(2)}$, $\tilde{t}_n^{(1)}$ and $\tilde{t}_n^{(2)}$ over the classical sequences w_n, s_n, t_n, respectively.

Combining Theorem 1.1, Theorem 1.2 and Theorem 1.3, we have Table 1, Table 2 and Table 3.

In conclusion, we assert that the use of polynomial in the problem of approximating the constants of Glaisher–Kinkelin type is more adequate than the use of classical asymptotic series, since more accurate approximations are obtained.

Table 1
Simulations for w_n, $\tilde{w}_n^{(1)}$ and $\tilde{w}_n^{(2)}$.

n	$\frac{w_n-\ln(A)}{\ln(A)}$	$\frac{\ln(A)-\tilde{w}_n^{(1)}}{\ln(A)}$	$\frac{\tilde{w}_n^{(2)}-\ln(A)}{\ln(A)}$
10	5.5754×10^{-5}	5.7558×10^{-6}	3.9523×10^{-7}
25	8.9314×10^{-6}	3.6176×10^{-7}	9.9662×10^{-9}
50	2.2332×10^{-6}	4.4943×10^{-8}	6.1983×10^{-10}
100	5.5833×10^{-7}	5.6007×10^{-9}	3.8645×10^{-11}
250	8.9334×10^{-8}	3.5778×10^{-10}	9.8788×10^{-13}
1000	5.5834×10^{-9}	5.5851×10^{-12}	3.8561×10^{-15}

Table 2
Simulations for s_n, $\tilde{s}_n^{(1)}$ and $\tilde{s}_n^{(2)}$.

n	$\frac{\ln(B)-s_n}{\ln(B)}$	$\frac{\tilde{s}_n^{(1)}-\ln(B)}{\ln(B)}$	$\frac{\ln(B)-\tilde{s}_n^{(2)}}{\ln(B)}$
10	9.1186×10^{-3}	1.4184×10^{-3}	1.6216×10^{-4}
25	3.6489×10^{-3}	2.2215×10^{-4}	1.0115×10^{-5}
50	1.8245×10^{-3}	5.5137×10^{-5}	1.2534×10^{-6}
100	9.1228×10^{-4}	1.3734×10^{-5}	1.5599×10^{-7}
250	3.6492×10^{-4}	2.1927×10^{-6}	9.9571×10^{-9}
1000	9.1229×10^{-5}	1.3689×10^{-7}	1.5537×10^{-10}

Table 3
Simulations for t_n, $\tilde{t}_n^{(1)}$ and $\tilde{t}_n^{(2)}$.

n	$\frac{\ln(C)-t_n}{-\ln(C)}$	$\frac{\tilde{t}_n^{(1)}-\ln(C)}{-\ln(C)}$	$\frac{\ln(C)-\tilde{t}_n^{(2)}}{-\ln(C)}$
10	9.5911×10^{-5}	2.0314×10^{-5}	2.9306×10^{-6}
25	1.5365×10^{-5}	1.2578×10^{-6}	7.20525×10^{-8}
50	3.8419×10^{-6}	1.5545×10^{-7}	4.4416×10^{-9}
100	9.6053×10^{-7}	1.9321×10^{-8}	2.7568×10^{-10}
250	1.5369×10^{-7}	1.2323×10^{-9}	7.0278×10^{-12}
1000	9.6054×10^{-9}	1.9222×10^{-11}	2.7395×10^{-14}

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