Journal of Number Theory 153 (2015) 107–116

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Applications of stuffle product of multiple zeta values

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A R T I C L E   I N F O

Article history:
Received 17 November 2014
Received in revised form 26 January 2015
Accepted 27 January 2015
Available online 5 March 2015
Communicated by David Goss

MSC:
11M32
05A15

Keywords:
Multiple zeta value
Stuffle product
Delannoy number

A B S T R A C T

We obtain some formulas for the stuffle product and apply them to derive a decomposition formula for multiple zeta values. Moreover, we give an application to combinatorics and get the following identity:

$$D(n+1,t+1)+D(n,t) = 2\sum_{\ell=0}^{n} D(t,n-\ell)+2\sum_{\ell=0}^{t} D(n,t-\ell),$$

where $D(n,t)$ is the Delannoy number.
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1. Introduction

The multiple zeta value (MZV) is defined by

$$\zeta(\boldsymbol{\alpha}) = \sum_{n_1>n_2>\cdots>n_r>0} n_1^{-\alpha_1} n_2^{-\alpha_2} \cdots n_r^{-\alpha_r},$$

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http://dx.doi.org/10.1016/j.jnt.2015.01.003
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where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is an r-tuple of positive integers with $\alpha_1 \geq 2$. The number r is called the depth of $\zeta(\boldsymbol{\alpha})$ and $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ is called the weight of $\zeta(\boldsymbol{\alpha})$. For convenience, we write $\{s\}^k$ to be k repetitions of s, for example, $\zeta(\{s\}^3) = \zeta(s, s, s)$, and in particular $\zeta(t, \{s\}^0) = \zeta(t)$.

Recently multiple zeta values (MZVs) and their generalizations have attracted much attention, both in pure mathematics and theoretical physics (see [3]). A systematic study only started in the early 1990s, although the prehistory can be traced back to Euler in the 18th century.

A principal goal in the theoretical study of MZVs is to determine all possible algebraic relations among them. Several explicit values are interesting and known for special index sets (e.g. [1,2,9,10]). For example, Zagier [10] evaluated the value of $\zeta(\{2\}^a, 3, \{2\}^b)$ and T. Arakawa and M. Kaneko (ref. [9, Theorem 1]) evaluated the value of $\zeta(\{2n\}^m)$.

Let us consider the coding of multi-indices $\vec{s} = (s_1, \ldots, s_k)$, s_i are positive integers and $s_1 > 1$, by words (that is, by monomials in non-commutative variables) over the alphabet $X = \{x, y\}$ by the rule

$$\vec{s} \mapsto x_{\vec{s}} = x^{s_1-1} y x^{s_2-1} y \cdots x^{s_k-1} y.$$

We set

$$\zeta(x_{\vec{s}}) := \zeta(\vec{s})$$

for all admissible words (that is, beginning with x and ending with y); then the weight (or the degree) $|x_{\vec{s}}| := |\vec{s}|$ coincides with the total degree of the monomial $x_{\vec{s}}$, whereas the length (or the depth) $l(x_{\vec{s}}) := l(\vec{s})$ is the degree with respect to the variable y.

Let $\mathbb{Q}\langle X\rangle = \mathbb{Q}\langle x, y\rangle$ be the $\mathbb{Q}$-algebra of polynomials in two non-commutative variables which is graded by the degree (where each of the variables x and y is assumed to be of degree 1); we identify the algebra $\mathbb{Q}\langle X\rangle$ with the graded $\mathbb{Q}$-vector space $\mathfrak{H}$ spanned by the monomials in the variables x and y (see [7]).

We also introduce the graded $\mathbb{Q}$-vector spaces $\mathfrak{H}^1 = \mathbb{Q}\mathbf{1} \bigoplus \mathfrak{H}y$ and $\mathfrak{H}^0 = \mathbb{Q}\mathbf{1} \bigoplus x\mathfrak{H}y$, where $\mathbf{1}$ denotes the unit (the empty word of weight 0 and length 0) of the algebra $\mathbb{Q}\langle X\rangle$. Then the space $\mathfrak{H}^1$ can be regarded as the subalgebra of $\mathbb{Q}\langle X\rangle$ generated by the words $z_s = x^{s-1}y$, whereas $\mathfrak{H}^0$ is the $\mathbb{Q}$-vector space spanned by all admissible words.

Let us define the shuffle product ш on $\mathfrak{H}$ and the stuffle product $*$ (the *harmonic product*) on $\mathfrak{H}^1$ by the rules

$$\mathbf{1} ш w = w ш \mathbf{1} = w, \quad \mathbf{1} * w = w * \mathbf{1} = w \tag{1}$$

for any word w, and

$$x_1 u ш x_2 v = x_1(u ш x_2 v) + x_2(x_1 u ш v), \tag{2}$$

$$z_j u * z_k v = z_j(u * z_k v) + z_k(z_j u * v) + z_{j+k}(u * v) \tag{3}$$

for any words u, v, any letters $x_i = x$ or y $(i = 1, 2)$, and any generators z_j, z_k of the subalgebra $\mathfrak{H}^1$, and then extend the above rules to the whole algebra $\mathfrak{H}$ and the whole subalgebra $\mathfrak{H}^1$ by linearity. Sometimes it becomes useful to consider the stuffle product on the whole algebra $\mathfrak{H}$ by formally adding to the last rule the rule

$$x^j * w = w * x^j = wx^j$$

for any word w and any integer $j \geq 1$. We note that induction arguments enable us to prove that each of the above products is commutative and associative. Then the map ζ is a morphism with respect to both products ш and $*$, and it satisfies the following property:

$$\zeta(w_1 ш w_2) = \zeta(w_1)\zeta(w_2) = \zeta(w_1 * w_2) \qquad \text{for all } w_1, w_2 \in \mathfrak{H}^0.$$

Recently, Eie, Wei [5] and Lei, Guo, Ma [8] are both based on the shuffle product to get some restricted decomposition formulas for MZVs. For example,

Proposition 1. *(See Theorem 1.1 of [8].) For positive integers m, n, j, and k, we have*

$$\begin{aligned}
&\zeta(m+1, \{1\}^{j-1})\zeta(n+1, \{1\}^{k-1}) \\
&\quad = \sum_{\substack{0 \leq n_1 \leq n \\ j_1 + j_2 = j, j_i \geq 0 \\ |\boldsymbol{\alpha}| = n - n_1 + j_1 + 1}} \binom{m-1+n_1}{m-1} \binom{j_2+k-1}{k-1} \\
&\qquad\qquad \times \zeta(m + n_1 + \alpha_1, \alpha_2, \ldots, \alpha_{j_1}, \alpha_{j_1+1}, \{1\}^{j_2+k-1}) \\
&\quad + \sum_{\substack{0 \leq t \leq k-1 \\ m_1 + m_2 = m-1, m_i \geq 0 \\ |\boldsymbol{\beta}| = m_2 + k - t + 1}} \binom{m_1+n-1}{n-1} \binom{j+t}{j} \\
&\qquad\qquad \times \zeta(m_1 + n + \beta_1, \beta_2, \ldots, \beta_{k-t}, \beta_{k-t+1} + 1, \{1\}^{j+t-1}),
\end{aligned}$$

where $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_{j_1} + \alpha_{j_1+1}$ with $\alpha_i \geq 1$ and $|\boldsymbol{\beta}| = \beta_1 + \beta_2 + \cdots + \beta_{k-t} + \beta_{k-t+1}$ with $\beta_i \geq 1$.

In this paper we derive some formulas for the stuffle product $*$ and apply them to obtain a corresponding decomposition formula for $\zeta(m, \{p\}^n)\zeta(s, \{p\}^t)$, where $m \geq 2$, $s \geq 2$, $p \geq 1$, $n \geq 0$, and $t \geq 0$ are integers.

First, we introduce some notations which we need. Let A_b^a denote the set of all possible sequences containing a times p and b times $2p$. For example,

$$\begin{aligned}
A_2^3 = \{&(\{p\}^3, \{2p\}^2), (\{p\}^2, 2p, p, 2p), (\{p\}^2, \{2p\}^2, p), (p, 2p, p, 2p, p), \\
&(p, \{2p\}^2, \{p\}^2), (p, 2p, \{p\}^2, 2p), (2p, \{p\}^3, 2p), (2p, \{p\}^2, \{2p\}, p), \\
&(2p, p, 2p, \{p\}^2), (\{2p\}^3, \{p\}^3)\}.
\end{aligned}$$

Then our main theorem can be stated as

Theorem 1. *Let $m \geq 2$, $s \geq 2$, $p \geq 1$, $n \geq 0$, and $t \geq 0$ be integers. Then*

$$\begin{aligned}
&\zeta(m, \{p\}^n)\zeta(s, \{p\}^t) \\
&= \sum_{\substack{1\leq \ell\leq n \\ 0\leq k\leq \min(t,n-\ell) \\ \boldsymbol{\alpha}\in A_k^{t+n-\ell-2k}}} \binom{t+n-\ell-2k}{t-k} \Big[\zeta(m, \{p\}^{\ell-1}, s+p, \alpha_1, \ldots, \alpha_{n+t-\ell-k}) \\
&\qquad\qquad + \zeta(m, \{p\}^{\ell}, s, \alpha_1, \ldots, \alpha_{n+t-\ell-k})\Big] \\
&+ \sum_{\substack{1\leq \ell\leq t \\ 0\leq k\leq \min(n,t-\ell) \\ \boldsymbol{\alpha}\in A_k^{t+n-\ell-2k}}} \binom{t+n-\ell-2k}{n-k} \Big[\zeta(s, \{p\}^{\ell-1}, m+p, \alpha_1, \ldots, \alpha_{n+t-\ell-k}) \\
&\qquad\qquad + \zeta(s, \{p\}^{\ell}, m, \alpha_1, \ldots, \alpha_{n+t-\ell-k})\Big] \\
&+ \sum_{\substack{0\leq k\leq \min(t,n) \\ \boldsymbol{\alpha}\in A_k^{t+n-2k}}} \binom{t+n-2k}{n-k} \Big[\zeta(m, s, \alpha_1, \ldots, \alpha_{n+t-k}) \\
&\qquad\qquad + \zeta(s, m, \alpha_1, \ldots, \alpha_{n+t-k}) + \zeta(m+s, \alpha_1, \ldots, \alpha_{n+t-k})\Big].
\end{aligned} \tag{4}$$

The stuffle product of two multiple zeta values of depth m and n, respectively, will product $D(m,n)$ numbers of MZVs, where $D(m,n)$ is the Delannoy number. The Delannoy number $D(m,n)$ (see page 81 of [4]) is defined for nonnegative integers m and n by

$$D(m,n) = \begin{cases} 1, & \text{if } m \cdot n = 0, \\ D(m-1,n) + D(m-1,n-1) + D(m,n-1), & \text{if } m \cdot n \neq 0. \end{cases} \tag{5}$$

By counting the number of MZVs in Eq. (4) produced from the stuffle product, we obtain an interesting identity:

Theorem 2. *For nonnegative integers n and t, we have*

$$D(n+1,t+1) + D(n,t) = 2\sum_{\ell=0}^{n} D(t, n-\ell) + 2\sum_{\ell=0}^{t} D(n, t-\ell). \tag{6}$$

2. Simple stuffle products

Let G_b^a denote the set of all possible words containing a times z_p and b times z_{2p}. For example,

$$G_2^3 = \{z_p^3 z_{2p}^2,\ z_p^2 z_{2p} z_p z_{2p},\ z_p^2 z_{2p}^2 z_p,\ z_p z_{2p} z_p z_{2p} z_p,\ z_p z_{2p}^2 z_p^2,$$
$$z_p z_{2p} z_p^2 z_{2p},\ z_{2p} z_p^3 z_{2p},\ z_{2p} z_p^2 z_{2p} z_p,\ z_{2p} z_p z_{2p} z_p^2,\ z_{2p}^3 z_p^3\}.$$

We use $\sum_{\boldsymbol{w}\in G_b^a} \boldsymbol{w}$ to indicate

$$\sum_{\boldsymbol{w}\in G_b^a} \boldsymbol{w} := \sum_{\substack{w=w_1w_2\cdots w_{a+b}\in G_b^a \\ w_i\in\{z_p,z_{2p}\} \\ \#\{w_i\,:\,w_i=z_p\}=a}} w_1w_2\cdots w_{a+b}.$$

For our convenience, we let $z_p^0 = \mathbf{1}$.

Lemma 1. *For integers $m \geq 0$, $n \geq 0$, and $p \geq 1$, we have*

$$z_p^m * z_p^n = \sum_{k=0}^{\min(m,n)} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w}\in G_k^{m+n-2k}} \boldsymbol{w}. \tag{7}$$

Proof. We use induction on $m+n$. Assume that Eq. (7) holds for $s+t$ with $s+t < m+n$, that is,

$$z_p^s * z_p^t = \sum_{k=0}^{\min(s,t)} \binom{s+t-2k}{s-k} \sum_{\boldsymbol{w}\in G_k^{s+t-2k}} \boldsymbol{w}. \tag{8}$$

Now applying Eq. (3), we have

$$z_p^m * z_p^n = z_p(z_p^{m-1} * z_p^n) + z_p(z_p^m * z_p^{n-1}) + z_{2p}(z_p^{m-1} * z_p^{n-1}).$$

The sums of the degree of factors $z_p^{m-1} * z_p^n$, $z_p^m * z_p^{n-1}$, and $z_p^{m-1} * z_p^{n-1}$ are all less than $m+n$. By the inductive hypothesis (8) we can rewrite the above identity as

$$\begin{aligned} z_p^m * z_p^n = z_p &\sum_{k=0}^{\min(m-1,n)} \binom{m+n-1-2k}{m-1-k} \sum_{\boldsymbol{w}\in G_k^{m+n-1-2k}} \boldsymbol{w} \\ &+ z_p \sum_{k=0}^{\min(m,n-1)} \binom{m+n-1-2k}{m-k} \sum_{\boldsymbol{w}\in G_k^{m+n-1-2k}} \boldsymbol{w} \\ &+ z_{2p} \sum_{k=0}^{\min(m-1,n-1)} \binom{m+n-2-2k}{m-1-k} \sum_{\boldsymbol{w}\in G_k^{m+n-2-2k}} \boldsymbol{w}. \end{aligned} \tag{9}$$

Since the stuffle product $*$ is commutative, we can assume that $m \geq n$. Firstly, if $m = n$, Eq. (9) becomes

$$\begin{aligned}
z_p^n * z_p^n &= z_p \sum_{k=0}^{n-1} \binom{2n-1-2k}{n-1-k} \sum_{\boldsymbol{w}\in G_k^{2n-1-2k}} \boldsymbol{w} + z_p \sum_{k=0}^{n-1} \binom{2n-1-2k}{n-k} \sum_{\boldsymbol{w}\in G_k^{2n-1-2k}} \boldsymbol{w} \\
&\quad + z_{2p} \sum_{k=0}^{n-1} \binom{2n-2-2k}{n-1-k} \sum_{\boldsymbol{w}\in G_k^{2n-2-2k}} \boldsymbol{w} \\
&= z_p \sum_{k=0}^{n-1} \left[\binom{2n-1-2k}{n-1-k} + \binom{2n-2k-1}{n-k} \right] \sum_{\boldsymbol{w}\in G_k^{2n-1-2k}} \boldsymbol{w} \\
&\quad + z_{2p} \sum_{k=1}^{n} \binom{2n-2k}{n-k} \sum_{\boldsymbol{w}\in G_{k-1}^{2n-2k}} \boldsymbol{w} \\
&= z_p \sum_{k=0}^{n-1} \binom{2n-2k}{n-k} \sum_{\boldsymbol{w}\in G_k^{2n-1-2k}} \boldsymbol{w} + z_{2p} \sum_{k=1}^{n} \binom{2n-2k}{n-k} \sum_{\boldsymbol{w}\in G_{k-1}^{2n-2k}} \boldsymbol{w}.
\end{aligned}$$

Since $G_b^a = \{z_p \boldsymbol{w}_1 : \boldsymbol{w}_1 \in G_b^{a-1}\} \bigcup \{z_{2p} \boldsymbol{w}_2 : \boldsymbol{w}_2 \in G_{b-1}^a\}$, we have

$$z_p \sum_{\boldsymbol{w}\in G_b^{a-1}} \boldsymbol{w} + z_{2p} \sum_{\boldsymbol{w}\in G_{b-1}^{a}} = \sum_{\boldsymbol{w}\in G_b^a} \boldsymbol{w}.$$

Thus the last equation in the above identity can be calculated as

$$\begin{aligned}
z_p^n * z_p^n &= \binom{2n}{n} z_p^{2n} + z_{2p}^n + \sum_{k=1}^{n-1} \binom{2n-2k}{n-k} \left(z_p \sum_{\boldsymbol{w}\in G_k^{2n-2k-1}} \boldsymbol{w} + z_{2p} \sum_{\boldsymbol{w}\in G_{k-1}^{2n-2k}} \boldsymbol{w} \right) \\
&= \binom{2n}{n} z_p^{2n} + z_{2p}^n + \sum_{k=1}^{n-1} \binom{2n-2k}{n-k} \sum_{\boldsymbol{w}\in G_k^{2n-2k}} \boldsymbol{w} \\
&= \sum_{k=0}^{n} \binom{2n-2k}{n-k} \sum_{w\in G_k^{2n-2k}} \boldsymbol{w}.
\end{aligned}$$

Therefore the case of $m = n$ holds. Now we let $m > n$, Eq. (9) becomes

$$\begin{aligned}
z_p^m * z_p^n &= z_p \sum_{k=0}^{n} \binom{m+n-1-2k}{m-1-k} \sum_{\boldsymbol{w}\in G_k^{m+n-1-2k}} \boldsymbol{w} \\
&\quad + z_p \sum_{k=0}^{n-1} \binom{m+n-1-2k}{m-k} \sum_{\boldsymbol{w}\in G_k^{m+n-1-2k}} \boldsymbol{w} \\
&\quad + z_{2p} \sum_{k=0}^{n-1} \binom{m+n-2-2k}{m-1-k} \sum_{\boldsymbol{w}\in G_k^{m+n-2-2k}} \boldsymbol{w}
\end{aligned}$$

$$
\begin{aligned}
&= z_p \sum_{\boldsymbol{w} \in G_n^{m-n-1}} \boldsymbol{w} + z_{2p} \sum_{k=1}^{n} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_{k-1}^{m+n-2k}} \boldsymbol{w} \\
&\quad + z_p \sum_{k=0}^{n-1} \left[\binom{m+n-1-2k}{m-1-k} + \binom{m+n-1-2k}{m-k} \right] \sum_{\boldsymbol{w} \in G_k^{m+n-1-2k}} \boldsymbol{w} \\
&= z_p \sum_{\boldsymbol{w} \in G_n^{m-n-1}} \boldsymbol{w} + z_{2p} \sum_{k=1}^{n} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_{k-1}^{m+n-2k}} \boldsymbol{w} \\
&\quad + z_p \sum_{k=0}^{n-1} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_k^{m+n-1-2k}} \boldsymbol{w} \\
&= z_p \sum_{\boldsymbol{w} \in G_n^{m-n-1}} \boldsymbol{w} + \binom{m+n}{m} z_p^{m+n} + z_{2p} \sum_{\boldsymbol{w} \in G_{n-1}^{m-n}} \boldsymbol{w} \\
&\quad + \sum_{k=1}^{n-1} \binom{m+n-2k}{m-k} \left(z_p \sum_{\boldsymbol{w} \in G_k^{m+n-1-2k}} \boldsymbol{w} + z_{2p} \sum_{\boldsymbol{w} \in G_{k-1}^{m+n-2k}} \boldsymbol{w} \right) \\
&= \binom{m+n}{m} z_p^{m+n} + \sum_{k=1}^{n-1} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_k^{m+n-2k}} \boldsymbol{w} \\
&\quad + \left(z_p \sum_{\boldsymbol{w} \in G_n^{m-n-1}} \boldsymbol{w} + z_{2p} \sum_{\boldsymbol{w} \in G_{n-1}^{m-n}} \boldsymbol{w} \right) \\
&= \binom{m+n}{m} z_p^{m+n} + \sum_{\boldsymbol{w} \in G_n^{m-n}} \boldsymbol{w} + \sum_{k=1}^{n-1} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_k^{m+n-2k}} \boldsymbol{w} \\
&= \sum_{k=0}^{n} \binom{m+n-2k}{m-k} \sum_{\boldsymbol{w} \in G_k^{m+n-2k}} \boldsymbol{w}.
\end{aligned}
$$

Hence by the mathematical induction we complete the proof. □

3. Proof of Theorem 1

For our convenience, we let $z_p^{-1} = \mathbf{0}$.

Lemma 2. *For integers $n \geq 2$, $p \geq 1$, $k \geq 0$, and $m \geq 0$, we have*

$$
z_n z_p^k * z_p^m = \sum_{\ell=0}^{m} (z_p^{\ell-1} z_{n+p} + z_p^{\ell} z_n)(z_p^k * z_p^{m-\ell}). \tag{10}
$$

Proof. We fix the integers n, k, and p. Then we use mathematical induction on m. Assume that Eq. (10) holds for all integer s with $s < m$. The by Eq. (3), we have

$$z_n z_p^k * z_p^m = z_n(z_p^k * z_p^m) + z_p(z_n z_p^k * z_p^{m-1}) + z_{n+p}(z_p^k * z_p^{m-1}).$$

Since $m-1 < m$, we can substitute Eq. (10) into the factor $z_n z_p^k * z_p^{m-1}$ of the above identity.

$$\begin{aligned} z_n z_p^k * z_p^m &= z_n(z_p^k * z_p^m) + z_{n+p}(z_p^k * z_p^{m-1}) \\ &\quad + z_p \sum_{\ell=0}^{m-1} (z_p^{\ell-1} z_{n+p} + z_p^\ell z_n)(z_p^k * z_p^{m-1-\ell}) \\ &= z_n(z_p^k * z_p^m) + (z_{n+p} + z_p z_n)(z_p^k * z_p^{m-1}) \\ &\quad + \sum_{\ell=2}^{m} (z_p^{\ell-1} z_{n+p} + z_p^\ell z_n)(z_p^k * z_p^{m-\ell}) \\ &= \sum_{\ell=0}^{m} (z_p^{\ell-1} z_{n+p} + z_p^\ell z_n)(z_p^k * z_p^{m-\ell}). \end{aligned}$$

This conclusion completes our proof. □

Using the results of Lemma 1, Lemma 2, and

$$z_m z_p^n * z_s z_p^t = z_m(z_p^n * z_s z_p^t) + z_s(z_m z_p^n * z_p^t) + z_{m+s}(z_p^n * z_p^t),$$

we can get the following result.

Theorem 3. *Let $m \geq 2$, $s \geq 2$, $p \geq 1$, $n \geq 0$, and $t \geq 0$ be integers. Then*

$$\begin{aligned} z_m z_p^n * z_s z_p^t = &\sum_{\substack{1 \le \ell \le n \\ 0 \le k \le \min(t, n-\ell) \\ \boldsymbol{w} \in G_k^{t+n-\ell-2k}}} \binom{t+n-\ell-2k}{t-k} z_m (z_p^{\ell-1} z_{s+p} + z_p^\ell z_s) \boldsymbol{w} \\ &+ \sum_{\substack{1 \le \ell \le t \\ 0 \le k \le \min(n, t-\ell) \\ \boldsymbol{w} \in G_k^{t+n-\ell-2k}}} \binom{t+n-\ell-2k}{n-k} z_s (z_p^{\ell-1} z_{m+p} + z_p^\ell z_m) \boldsymbol{w} \\ &+ \sum_{\substack{0 \le k \le \min(t, n) \\ \boldsymbol{w} \in G_k^{t+n-2k}}} \binom{t+n-2k}{n-k} (z_m z_s + z_s z_m + z_{m+s}) \boldsymbol{w}. \end{aligned} \tag{11}$$

Note that for $\boldsymbol{w} \in G_b^a$, the map $\zeta(\boldsymbol{w}) = \zeta(\boldsymbol{\alpha})$, where $\boldsymbol{w} = w_1 \cdots w_{a+b}$, $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{a+b})$, and $\alpha_i = p$ if $w_i = z_p$; $\alpha_i = 2p$ if $w_i = z_{2p}$. There is a natural one-to-one correspondence between G_b^a and A_b^a.

Then we apply the map ζ to both sides of Eq. (11). We obtain a decomposition formula for product of two multiple zeta values $\zeta(m, \{p\}^n)\zeta(s, \{p\}^t)$. This is exactly the result of Theorem 1.

4. Application to combinatorics

In combinatorics, the set G_b^a can be identify as the set of lattice paths from $(0,0)$ to (a,b) using only steps east $(1,0)$ (resp. p in G_b^a) and north $(0,1)$ (resp. $2p$ in G_b^a). Hence the number of elements in G_b^a is $\binom{a+b}{a}$.

On the other hand, the definition of the stuffle product $*$ (see Eq. (1) and Eq. (3)) indicates that the stuffle product of two multiple zeta values of depth m and n, respectively, will produce $D(m,n)$ multiple zeta values (see Eq. (5)), where $D(m,n)$ is a Delannoy number.

Again, the Delannoy number $D(m,n)$ can be viewed as the number of lattice paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed. The lattice paths described here are called Delannoy paths which give an alternative characterization of the stuffle product. A detailed discussion can be found in [6].

By counting the number of multiple zeta values in Eq. (4) produced from the stuffle product, we obtain the interesting Delannoy identity (6).

Proof of Theorem 2. Counting the number of multiple zeta values in Eq. (4) we have

$$\begin{aligned} D(n+1,t+1) = {} & 2\sum_{\ell=1}^{n}\sum_{k=0}^{\min(t,n-\ell)}\binom{t+n-\ell-2k}{t-k}\binom{t+n-\ell-k}{k} \\ & + 2\sum_{\ell=1}^{t}\sum_{k=0}^{\min(n,t-\ell)}\binom{t+n-\ell-2k}{n-k}\binom{t+n-\ell-k}{k} \\ & + 3\sum_{k=0}^{\min(n,t)}\binom{n+t-2k}{n-k}\binom{n+t-k}{k}. \end{aligned} \tag{12}$$

Since (see page 81 of [4])

$$D(m,n) = \sum_{k=0}^{\min(m,n)}\binom{m+n-k}{m}\binom{m}{k},$$

we have

$$\sum_{k=0}^{\min(n,q)}\binom{n+q-2k}{n-k}\binom{n+q-k}{k} = \sum_{k=0}^{\min(n,q)}\frac{(n+q-2k)!}{(n-k)!(q-k)!}\frac{(n+q-k)!}{k!(n+q-2k)!}$$

$$= \sum_{k=0}^{\min(n,q)} \frac{n!}{(n-k)!k!} \frac{(n+q-k)!}{(q-k)!n!}$$

$$= \sum_{k=0}^{\min(n,q)} \binom{n}{k} \binom{n+q-k}{n} = D(n,q).$$

Substituting the above identity into Eq. (12) we have

$$D(n+1,t+1) = 2\sum_{\ell=1}^{n} D(t,n-\ell) + 2\sum_{\ell=1}^{t} D(n,t-\ell) + 3D(n,t).$$

Hence we complete the proof. □

Acknowledgments

The author would like to thank the referee for some useful comments and suggestions. This research was completed while visiting the Math Department at the University of Connecticut.

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