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"STRANGE" COMBINATORIAL QUANTUM MODULAR FORMS

AMANDA FOLSOM, CALEB KI, YEN NHI TRUONG VU, AND BOWEN YANG

Abstract. Motivated by the problem of finding explicit q-hypergeometric series which give rise to quantum modular forms, we define a natural generalization of Kontsevich's "strange" function. We prove that our generalized strange function can be used to produce infinite families of quantum modular forms. We do not use the theory of mock modular forms to do so. Moreover, we show how our generalized strange function relates to the generating function for ranks of strongly unimodal sequences both polynomially, and when specialized on certain open sets in $\mathbb{C}$. As corollaries, we reinterpret a theorem due to Folsom-Ono-Rhoades on Ramanujan's radial limits of mock theta functions in terms of our generalized strange function, and establish a related Hecke-type identity.

1. Introduction and statement of results

1.1. **Background and motivation.** Quantum modular forms have been a topic of recent interest. Loosely speaking, as defined by Zagier [19], a quantum modular form is a complex-valued function that exhibits modular-like transformation properties on the rational numbers, as opposed to the upper-half of the complex plane. To be more precise, a *weight k quantum modular form* ($k \in \frac{1}{2}\mathbb{Z}$) is a complex-valued function f on $\mathbb{Q}$ or possibly $\mathbb{P}^1(\mathbb{Q}) \setminus S$ for some appropriate set S, such that for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, where $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is an appropriate subgroup, the function

$$h_\gamma(x) = h_{f,\gamma}(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a 'suitable' property of continuity or analyticity. The $\epsilon(\gamma)$ are appropriate complex numbers, such as those that arise naturally in the theory of half-integral weight modular forms. Here, we have modified Zagier's original definition as in [8] to allow half-integral weights k, subgroups of $\mathrm{SL}_2(\mathbb{Z})$, and multiplier systems $\epsilon(\gamma)$, in accordance with the theory of ordinary modular forms. Zagier's definition, in particular the continuity or analyticity requirement of the "error to modularity" $h_\gamma(x)$, is intentionally vague, so that it may encompass many diverse, interesting, examples.

Among Zagier's pioneering first examples of quantum modular forms is the function $\phi(x) := e(x/24)F(e(x))$ ($e(z) := e^{2\pi i z}$), where $x \in \mathbb{Q} \setminus \{0\}$, and the function $F(q)$ is the "strange" function

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n$$

(where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$ for $n \in \mathbb{N}$, and $(a;q)_0 := 1$) originally studied by Kontsevich [19]. One "strange" aspect of the function $F(q)$ is that it converges on no open subset of $\mathbb{C}$, only when $q = \zeta_k^h := e(h/k)$ ($k \in \mathbb{N}$, $h \in \mathbb{Z}$) is a root of unity. In [19], Zagier proves that the normalized strange function $\phi(x)$ in fact possesses some beautiful analytic properties, which we paraphrase in the following theorem.

Theorem (Zagier, [19])**.** *For $x \in \mathbb{Q} \setminus \{0\}$, we have that $\phi(x)$ is quantum modular form of weight $3/2$ with respect to the group $\mathrm{SL}_2(\mathbb{Z})$. In particular, $h_{\gamma,\phi}(x)$ is a real analytic function.*

Perhaps surprisingly, $F(q)$ has also been connected to a certain function $U(1;q)$ which is of independent interest for its combinatorial properties, and which was also shown in [6] to be both mock modular and quantum modular. (For more on mock modular forms and their numerous applications in recent years, we refer the interested reader to the surveys by Ono [13] and Zagier [17].) To describe this connection more precisely, we introduce some combinatorial functions. A sequence $\{a_j\}_{j=1}^s$ of integers is called a *strongly unimodal sequence of size* n if there exists some integer r such that $0 < a_1 < a_2 < \cdots < a_r > a_{r+1} > \cdots > a_s > 0$, and $a_1 + a_2 + \cdots + a_s = n$. Analogous to the notion of the rank of an integer partition, one also has the notion of the rank of a strongly unimodal sequence; in terms of the definition given above, the *rank* of the strongly unimodal sequence $\{a_j\}_{j=1}^s$ is defined to be $s-2r+1$, the number of terms after the maximal term in the sequence minus the number of terms that precede it. It is not difficult to show that the rank generating function for strongly unimodal sequences satisfies the following [1]

$$U(w;q) := \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} u(m,n)(-w)^m q^n = \sum_{n=0}^{\infty} q^{n+1}(wq;q)_n(w^{-1}q;q)_n, \tag{1.1}$$

where $u(m,n) := \#\{\text{strongly unimodal sequences of size } n, \text{ rank } m\}$. The authors in [6] study this function when $w = 1$, in which case

$$U(1;q) = \sum_{n=1}^{\infty}(u_e(n) - u_0(n))q^n = \sum_{n=0}^{\infty} q^{n+1}(q;q)_n^2,$$

where $u_e(n)$ (resp. $u_0(n)$) counts the number of strongly unimodal sequences of size n and even (resp. odd) rank. The following theorem from [6] exhibits a striking relationship between Kontsevich's strange function $F(q)$ and the unimodal rank function $U(1;q)$.

Theorem ([6, Theorem 1.1])**.** *If q is any root of unity, then $F(q^{-1}) = U(1;q)$.*

The subject of quantum modular forms is relatively young; hence, it has been of recent interest to further explore the theory, and to find explicit examples. A number of recent papers (such as [4, 5, 6, 8, 10, 12, 14]) have explored the connection between quantum modular forms and mock modular forms, and have offered diverse examples. Here, we are motivated not by mock modular forms, but by the problem of finding explicit q-hypergeometric series in the Habiro ring [9, 19] which give rise to quantum modular forms, analogous to Zagier's examination of Kontsevich's strange function $F(q)$. We are also interested in whether such quantum modular forms may be related to $U(w;q)$ at values w other than $w = 1$ as studied in [6].

Indeed, we address these questions in Section 1.2 and Section 1.3. In Section 1.2 we define in (1.2) a natural two-variable generalization of Kontsevich's strange function $F(w;q)$. We show how our strange function $F(w;q)$ is related to the two-variable unimodal rank generating function $U(w;q)$ both polynomially, and, when specialized to certain subsets in $\mathbb{C}$, in Theorem 1.1, Theorem 1.2, and Corollary 1.3. This generalizes the aforementioned theorem [6, Theorem 1.1] stated above. In Section 1.3 we define using our strange function $F(w;q)$ infinite families of functions in (1.9), (1.10) and (1.11), which we show to be quantum modular in Theorem 1.8. In light of this, in Proposition 1.14, we study related asymptotic behaviors.

As a corollary, we show (Corollary 1.4) how our results allow us to reinterpret a recent theorem due to the first author, Ono and Rhoades in [8] related to Ramanujan's radial limits of mock theta functions in terms of our strange function $F(w;q)$. In Theorem 1.5, we also establish a general Hecke-type identity for the two-variable unimodal rank generating function $U(w;q)$, inspired by a conjecture in [6] on congruences associated to the coefficients of $U(-1;q)$, and a theorem in [6] which gives a Hecke-type identity for $U(1;q)$.

Proofs of the results from Section 1.2 are found in Section 2, and proofs of the results from Section 1.3 are found in Section 3.

[1]Note. The function $U(w;q)$ defined in (1.1) is equal to the function $U(-w;q)$ as defined in [6].

1.2. **A generalized strange function and the unimodal rank function.** To this end, we make the following definition of a natural two-variable "strange" function

$$F(w;q) := \sum_{n=0}^{\infty} w^{n+1}(wq;q)_n. \tag{1.2}$$

In particular, $F(1;q) = F(q)$ is Kontsevich's strange function, (essentially) a quantum modular form, and $F(1;q^{-1}) = U(1;q)$ is a certain unimodal rank generating function, as described above in Section 1.1. In addition to stating Theorem 1.1, Theorem 1.2, and Corollary 1.3, in this section, we also offer a reinterpretation of a theorem from [8] in Corollary 1.4, in terms of $F(w;q)$, and provide a two-variable Hecke-type identity for $U(w;q)$ in Theorem 1.5.

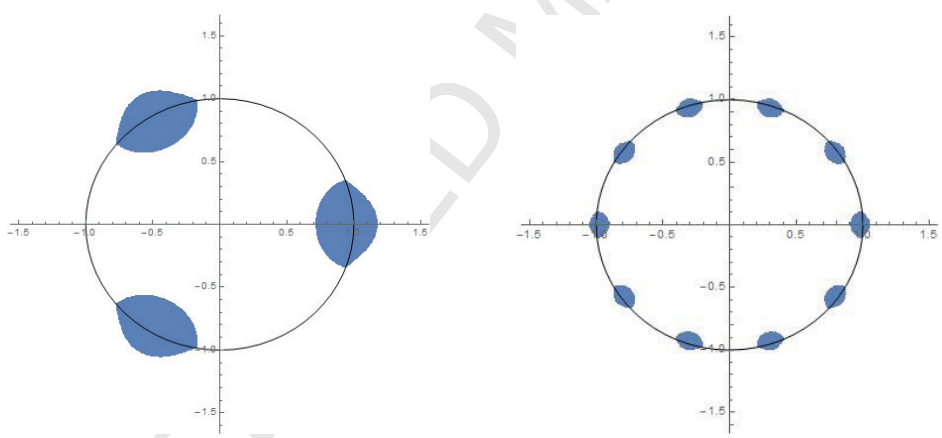
To describe our results, for $k \in \mathbb{N}$, we define the sets

$$A_k := \{w \in \mathbb{C} : |2 - w^k - w^{-k}| < 1\},$$
$$B_k := \{w \in \mathbb{C} : |w^k - w^{2k}| < 1\}.$$

Our first result generalizes the aforementioned result [6, Theorem 1.1], and shows that when q is a primitive kth root of unity, and w is any complex number in $A_k \cap B_k$, the two-variable unimodal function $U(w;q)$ is equal to our two-variable strange function $F(w;q^{-1})$.

Theorem 1.1. *If $q = \zeta_k^h$ is any primitive kth root of unity ($h \in \mathbb{Z}$, $k \in \mathbb{N}$, $\gcd(h,k) = 1$), and $w \in A_k \cap B_k$, we have that*

$$F(w;q^{-1}) = U(w;q). \tag{1.3}$$



(A) $k = 3$ (B) $k = 10$

FIGURE 1. $A_k \cap B_k$ in $\mathbb{C}$ for $k \in \{3, 10\}$

Figure 1 above illustrates the permissible values of w in Theorem 1.1 when q is a primitive 3rd or 10th root of unity. An interesting feature of Theorem 1.1 (as depicted in Figure 1) is that for a fixed root of unity q, the value w may be either inside the unit disk, outside the unit disk, or on the unit disk. When on the unit disk, we point out that w may be a complex number satisfying $|w| = 1$ that is not necessarily a root of unity. Phrasing this another way, if we write $w = re(t)$, with $r \in \mathbb{R}^+$ and $t \in \mathbb{R}$, we may either have $r > 1$, $r < 1$, or $r = 1$, in which case t is not necessarily restricted to be a rational number, but can be any real number suitably close to a fixed rational number h/k (or any translate $h/k + m$, $m \in \mathbb{Z}$). Numerical examples are given in Table 1 below.

Our next theorem in fact removes all hypotheses on the second variable w. To describe it, we define for $m \in \mathbb{N}$ the truncated functions

w	h	k	$U(w;\zeta_k^h) = F(w;\zeta_k^{-h})$
0.8	1	3	$\approx 3.92629 + 0.591093i$
$-0.9+0.3i$	2	7	$\approx -2.14078 + 1.33102i$
$1.05i$	3	8	$\approx -1.98835 - 0.431558i$
$0.68-0.7i$	5	9	$\approx 4.33875 - 5.65092i$

Table 1. Numerical values illustrating Theorem 1.1 for selected values of w both inside and outside the unit disk.

$$U_m(w;q) := \sum_{n=0}^{m-1} (wq;q)_n (w^{-1}q;q)_n q^{n+1}, \tag{1.4}$$

$$F_m(w;q) := \sum_{n=0}^{m-1} w^{n+1}(wq;q)_n. \tag{1.5}$$

In particular, $\lim_{m\to\infty} U_m(w;q) = U(w;q)$ and $\lim_{m\to\infty} F_m(w;q) = F(w;q)$. Our next theorem is a polynomial identity in the variable w, relating the truncated functions $F_k(w;q^{-1})$ and $U_k(w;q)$, when q is a fixed primitive kth root of unity.

Theorem 1.2. *If q is any primitive kth root of unity, then for all $w \in \mathbb{C}$, we have that*

$$F_k(w;q^{-1}) = w^k U_k(w;q).$$

For example, when $k = 3$ or 4, and $q = \zeta_3$ or $-i$ (respectively), Theorem 1.2 shows for any w that

$$F_3(w;\zeta_3^{-1}) = w^3 U_3(w;\zeta_3) = w\left(1 + w + \frac{1}{2}(3+i\sqrt{3})w^2 + w^3 + w^4\right),$$

$$F_4(w;i) = w^4 U_4(w;-i) = w\left(1 + w + (1-i)w^2 + (2-i)w^3 + (1-i)w^4 + w^5 + w^6\right).$$

We shall see in the course of the proofs of Theorem 1.1 and Theorem 1.2 (in Proposition 2.1) how the series $U(w;q)$ and $F(w;q)$ are related to their truncated polynomial counterparts in (1.4) and (1.5).

Next, we have the following corollary, which in a special case recovers [6, Theorem 1.1] discussed in Section 1.1.

Corollary 1.3. *Let q be any primitive kth root of unity. For all positive integers $b|k$, and all integers a, we have that*

$$F(\zeta_b^a;q^{-1}) = U(\zeta_b^a;q). \tag{1.6}$$

In particular, when $a = 0$, $b = 1$, (and q is any primitive kth root of unity) we have that

$$F(q^{-1}) = F(1;q^{-1}) = U(1;q). \tag{1.7}$$

Corollary 1.3 also gives us a new way to study the radial limit relationship between Dyson's rank mock theta function and the Andrews-Garvan crank modular form established in [8]. In particular, Dyson's rank function $R(w;q)$ and the Andrews-Garvan crank function $C(w;q)$ are defined by

$$R(w,q) := \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} N(m,n)w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n},$$

a	b	h	k	$U(\zeta_b^a;\zeta_k^h) = F(\zeta_b^a;\zeta_k^{-h})$
1	2	1	4	$-i$
2	3	5	6	$\approx -1.5 - 0.866025i$
3	7	13	14	$\approx -2.09903 + 0.820077i$
3	15	2	30	$\approx 0.582676 + 2.06846i$

TABLE 2. Numerical values illustrating Corollary 1.3 for selected a, b, and $q = \zeta_k^h$.

where $N(m,n)$ counts the number of partitions of n with rank m, and

$$C(w,q) := \sum_{n=0}^{\infty}\sum_{m\in\mathbb{Z}} M(m,n)w^m q^n = \frac{(q;q)_\infty}{(wq;q)_\infty(w^{-1}q;q)_\infty},$$

where $M(m,n)$ counts the number of partitions of n with crank m [3]. In [8, Theorem 1.2], the first author, Ono, and Rhoades address and generalize a claim of Ramanujan's, showing that asymptotically, as q approaches roots of unity radially within the unit disk, the difference between the mock modular rank generating function and (a multiple of the) modular crank generating function (with w set to be another fixed root of unity), is bounded, and specifically equals the unimodal rank generating function $U(w;q)$ evaluated at roots of unity. Applying Corollary 1.3 to [8, Theorem 1.2], we obtain the following result, which shows that "radial limits" of the difference of the mock modular partition rank and modular crank generating functions may be expressed in a new way, as special values of our two-variable Kontsevich strange function $F(w;q)$ (which we use to construct infinite families of quantum modular forms in Theorem 1.8). .

Corollary 1.4. *Let $1 \leq a < b$ and $1 \leq h < k$ be integers with $\gcd(a,b) = \gcd(h,k) = 1$ and $b|k$. If $h' \in \mathbb{Z}$ satisfies $hh' \equiv -1 \pmod{k}$ then as q approaches ζ_k^h radially within the unit disk, we have that*

$$\lim_{q\to\zeta_k^h}\left(R(\zeta_b^a;q) - \zeta_{b^2}^{-a^2h'k}C(\zeta_b^a;q)\right) = -(1-\zeta_b^a)(1-\zeta_b^{-a})F(\zeta_b^a;\zeta_k^{-h}).$$

In another direction, inspired by [6, Conjecture 1.6] on congruences associated to the coefficients of $U(-1;q)$, and [6, Theorem 1.5], which is a Hecke-type identity for $U(1;q)$, we establish a general Hecke-type identity for the two-variable unimodal rank generating function $U(w;q)$. While our work in this paper does not focus on establishing congruences related to the coefficients $u(m,n)$ of $U(w;q)$, the following identity is suggestive; establishing such congruences would be of interest.

Theorem 1.5. *We have that*

$$(1.8)\qquad U(w;q) = \frac{(wq;q)_\infty(w^{-1}q;q)_\infty}{(q;q)_\infty^2}\left(\sum_{n>0}\sum_{6n\geq|6j+1|}(-1)^{j+1}q^{2n^2-\frac{j(3j+1)}{2}}\right.$$
$$\left.+\sum_{m,n>0}\sum_{6n\geq|6j+1|}(-1)^{j+1}(w^m+w^{-m})q^{2n^2+mn-\frac{j(3j+1)}{2}}\right).$$

Remark 1.6. The result [6, Theorem 1.5] is an immediate corollary of Theorem 1.5 above when w is specialized to equal 1.

1.3. **Quantum modularity and $F(w;q)$.** In this section, we extend work of Zagier in [19] on Kontsevich's strange function $F(q) = F(1;q)$, and show that our general strange function $F(w;q)$ can be used to define infinite families of quantum modular forms. We define these forms in (1.9), (1.10), and (1.11), and establish quantum modularity in Theorem 1.8. In Proposition 1.14, we establish related asymptotic properties. To describe our results, for any positive integer b, we let $\ell_b := \mathrm{lcm}(b, 12)$. We define the set of rational numbers

$$S_b := \left\{ \frac{h}{k} \;:\; h \in \mathbb{Z}, k \in \mathbb{N}, \gcd\left(h, \frac{\ell_b k}{12}\right) = 1, b \;\middle|\; \frac{\ell_b k}{12} \right\} \subseteq \mathbb{Q},$$

and tuples of integers

$$T := \{(a, b, c) \colon b \in \mathbb{N}, a \in \mathbb{Z}, \gcd(a, b) = 1, \zeta_b^a \neq 1, c \in 2\mathbb{N}_0\},$$

where $2N_0 := \{0, 2, 4, 6, \dots\}$. Using our strange function $F(w; q)$, we define for $(a, b, c) \in T$ the functions $F^{\pm}_{a,b,c} : S_b \to \mathbb{C}$ by

$$F^{\pm}_{a,b,c}(x) := (i\ell_b)^{\frac{1}{2}} e\left(\frac{(\ell_b c + 2)^2 x}{8\ell_b}\right)(\zeta_b^{-a} e(-cx) - 1)\big(F(\zeta_b^a e(cx); e(12x/\ell_b)) \pm F(\zeta_b^{-a} e(-cx); e(12x/\ell_b))\big),$$

and the functions $\phi_{a,b,c} : S_b \to \mathbb{C}$ by

$$\phi_{a,b,c}(x) := F^{-}_{a,b,c}(x) + 2(i\ell_b)^{\frac{1}{2}} \zeta_{4b}^{-a\ell_b c}(\zeta_b^{-a} - 1) e\left(\frac{x}{2\ell_b}\right) F\left(\zeta_b^{-a}; e(12x/\ell_b)\right) \tag{1.9}$$

$$= F^{+}_{a,b,c}(x) + 2(i\ell_b)^{\frac{1}{2}} \zeta_{4b}^{-a\ell_b c} \sum_{n=1}^{\frac{\ell_b c}{2} - 1} \chi(n) \zeta_{2b}^{a(n-1)} e\left(\frac{n^2 x}{2\ell_b}\right), \tag{1.10}$$

where $\chi(n) := \left(\frac{12}{n}\right)$ is defined using the Kronecker symbol.

Remark 1.7. We shall see in Section 3.1.2 how to deduce the equivalence of the two expressions (1.9) and (1.10) above defining $\phi_{a,b,c}$.

To state our results most generally, we use the functions $\phi_{a,b,c}$ to define further strange functions of interest. For any N-tuple of integers ($N \in \mathbb{N}$) $\boldsymbol{a} := (a_1, a_2, \dots, a_N) \in \mathbb{Z}^N$ and N-tuple of even integers $\boldsymbol{c} := (c_1, c_2, \dots, c_N) \in (2\mathbb{N}_0)^N$, and positive integer b such that $(a_j, b, c_j) \in T$ for each $1 \leq j \leq N$, we define the functions $\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}} : S_b \to \mathbb{C}$ by

$$\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}(x) := \sum_{j=1}^{N} \phi_{a_j,b,c_j}(x). \tag{1.11}$$

Note that $\Phi^{(1)}_{(a),b,(c)} = \phi_{a,b,c}$. Among the simplest examples of the functions $\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}$ (and $\phi_{a,b,c}$) are the following, the second of which is nothing but our generalized strange function $F(w; e(x))$ specialized at $w = -1$, up to a simple factor:

$$\Phi^{(1)}_{(a),b,(0)}(x) = \phi_{a,b,0}(x) = (i\ell_b)^{\frac{1}{2}} e\left(\frac{x}{2\ell_b}\right)(\zeta_b^{-a} - 1)(F(\zeta_b^a; e(12x/\ell_b)) + F(\zeta_b^{-a}; e(12x/\ell_b))),$$

$$\Phi^{(1)}_{(1),2,(0)}(x) = \phi_{1,2,0}(x) = -4(12i)^{\frac{1}{2}} e\left(\frac{x}{24}\right) F(-1; e(x)).$$

Theorem 1.8 establishes the quantum modularity of the generalized strange functions $\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}(x)$ and $\phi_{a,b,c}(x)$.

Theorem 1.8. *With hypotheses on $N, \boldsymbol{a}, b$, and $\boldsymbol{c}$ given above, for $x \in S_b$, we have that the functions $\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}$ are quantum modular forms of weight $\frac{1}{2}$. Moreover, for all $x \in S_b$ and $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(2\ell_b)$ such that $\gamma x \neq -\delta$, we have that*

$$\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}(x) - (\gamma x + \delta)^{-\frac{1}{2}} \psi_{\ell_b}^{-1}(M) \Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}(Mx) = \int_{-\frac{\delta}{\gamma}}^{i\infty} \frac{\sum_{j=1}^{N} \vartheta_{a_j,b,c_j}(u)}{\sqrt{u - x}} du. \tag{1.12}$$

In particular, when $N = 1$, $\boldsymbol{a} = (a)$, and $\boldsymbol{c} = (c)$, we have that

$$\phi_{a,b,c}(x) - (\gamma x + \delta)^{-\frac{1}{2}} \psi_{\ell_b}^{-1}(M) \phi_{a,b,c}(Mx) = \int_{-\frac{\delta}{\gamma}}^{i\infty} \frac{\vartheta_{a,b,c}(u)}{\sqrt{u-x}} du. \tag{1.13}$$

The functions $\vartheta_{a,b,c}$ are defined in (3.12), and the multiplier ψ_{ℓ_b} is defined in (3.13).

Remark 1.9. Results similar to (1.12) and (1.13) hold for odd integers c, and can be deduced easily from the results in this paper. For simplicity and ease of notation, we state Theorem 1.8 in the case of even integers c only.

Remark 1.10. Theorem 1.8 does not directly apply to the original Kontsevich strange function $F(q) = F(1; q)$ studied by Zagier, which is obtained by specializing $w = \zeta_b^a = 1$ in our generalized strange function $F(w; q)$. While $F(1; q)$ is indeed a valid specialization of our function $F(w; q)$, Zagier proved that the strange function $F(q)$ is (essentially) quantum modular of weight 3/2, which is the dual weight to our families $\phi_{\boldsymbol{a},b,\boldsymbol{c}}^{(N)}(x)$ and $\phi_{a,b,c}(x)$ of quantum modular forms of weight 1/2. For this reason, our proof of Theorem 1.8 does not directly apply to $F(q) = F(1; q)$; however, our methods here are inspired by, and are similar to, Zagier's original proof given in [19] establishing the quantum modularity of $F(q) = F(1; q)$.

An interesting feature of Theorem 1.8 is that it gives a simple closed expression in terms of either our strange function $F(w; q)$ or the unimodal rank function $U(w; q)$ for the integral appearing in the right hand side of (1.13), as we show in Corollary 1.11 below. For ease of notation, we define the following polynomials in roots of unity, where $(a, b, c) \in T$ and h and k are such that $h/k \in S_b$.

$$F_{a,b,c}(h,k) := (i\ell_b)^{\frac{1}{2}} \zeta_{8\ell_b k}^{(\ell_b c+2)^2 h} (\zeta_b^{-a}\zeta_k^{-ch} - 1) \left(F_{\mathcal{N}_{h,k}^+}\left(\zeta_b^a \zeta_k^{ch}; \zeta_{\ell_b k}^{12h}\right) + F_{\mathcal{N}_{h,k}^-}\left(\zeta_b^{-a}\zeta_k^{-ch}; \zeta_{\ell_b k}^{12h}\right)\right),$$
$$U_{a,b,c}(h,k) := 2(i\ell_b)^{\frac{1}{2}} \zeta_{8\ell_b k}^{(\ell_b c+2)^2 h} (\zeta_b^{-a}\zeta_k^{-ch} - 1) U_{\mathcal{N}_{h,k}^{\min}}\left(\zeta_b^{\pm a}\zeta_k^{\pm ch}; \zeta_{\ell_b k}^{-12h}\right),$$
$$P_{a,b,c}(h,k) := 2(i\ell_b)^{\frac{1}{2}} \zeta_{4b}^{-a\ell_b c} \sum_{n=1}^{\ell_b c/2-1} \chi(n) \zeta_{2b}^{a(n-1)} \zeta_{2\ell_b k}^{n^2 h}.$$

Recall that the truncated functions $F_m(w; q)$ and $U_m(w; q)$ appearing above are defined in (1.5) and (1.4). The numbers $\mathcal{N}_{h,k}^{\pm} = \mathcal{N}^{\pm}(a, b, c, h, k)$ are the unique integers $1 \leq \mathcal{N}_{h,k}^{\pm} \leq \frac{\ell_b k}{12}$ satisfying the congruence conditions in (3.19). The number $\mathcal{N}_{h,k}^{\min} = \mathcal{N}_{h,k}^{\min}(a, b, c, h, k) := \min\{\mathcal{N}_{h,k}^+, \mathcal{N}_{h,k}^-\}$.

Corollary 1.11 shows how the integrals appearing in Theorem 1.8 may be evaluated exactly and expressed simply in terms of ("strange" or "unimodal") polynomials, evaluated at roots of unity. The numbers $H = H_M(h, k)$ and $K = K_M(h, k)$ appearing below are defined in (3.20) and (3.21).

Corollary 1.11. *With notation and hypotheses as above, we have that*

$$\int_{-\frac{\delta}{\gamma}}^{i\infty} \frac{\vartheta_{a,b,c}(u)}{\sqrt{u-\frac{h}{k}}} du = F_{a,b,c}(h,k) + P_{a,b,c}(h,k) - \left(\gamma \tfrac{h}{k} + \delta\right)^{-\frac{1}{2}} \psi_{\ell_b}^{-1}(M) \left(F_{a,b,c}(H,K) + P_{a,b,c}(H,K)\right) \tag{1.14}$$

$$= U_{a,b,c}(h,k) + P_{a,b,c}(h,k) - \left(\gamma \tfrac{h}{k} + \delta\right)^{-\frac{1}{2}} \psi_{\ell_b}^{-1}(M) \left(U_{a,b,c}(H,K) + P_{a,b,c}(H,K)\right). \tag{1.15}$$

Remark 1.12. Corollary 1.11 follows almost immediately from Theorem 1.8 and Corollary 1.3. To be precise, the fact that the strange functions $F(w; q)$ used to define $\phi_{a,b,c}$ reduce to the truncated functions $F_{\mathcal{N}^{\pm}}$ appearing in (1.14) follows from the argument given in the proof of Theorem 1.8. The expression given in (1.15) follows using that same argument with Corollary 1.3 and Theorem 1.8.

Remark 1.13. Clearly, Corollary 1.11 gives exact values for the integrals appearing on the right hand side of (1.12) as well.

We illustrate Corollary 1.11 with the following examples.

Example 1. We take $a = 1, b = 2, c = 0, h = 1, k = 2$, so that $h/k = 1/2 \in S_2$. We choose $M = \left(\begin{smallmatrix} 1 & 0 \\ 24 & 1 \end{smallmatrix}\right) \in \Gamma(24)$, hence $H = 1$, and $K = 26$. A direct calculation using (1.14) and (1.15) reveals that

$$\begin{aligned} \phi_{1,2,0}\left(\tfrac{1}{2}\right) - 13^{-\frac{1}{2}}\phi_{1,2,0}\left(\tfrac{1}{26}\right) &= 4(12i)^{\frac{1}{2}}\zeta_{48} + 4\left(\frac{12i}{13}\right)^{\frac{1}{2}}\zeta_{624}\sum_{n=0}^{12}(-1)^{n+1}(-\zeta_{26};\zeta_{26})_n \\ &= 4(12i)^{\frac{1}{2}}\zeta_{48} + 4\left(\frac{12i}{13}\right)^{\frac{1}{2}}\zeta_{624}\sum_{n=0}^{12}\zeta_{26}^{-n-1}(-\zeta_{26}^{-1};\zeta_{26}^{-1})_n^2. \end{aligned}$$

Hence, by Corollary 1.11, the integral appearing in (1.13) may be evaluated exactly as

$$\begin{aligned} \int_{-\frac{1}{24}}^{i\infty}\frac{\vartheta_{1,2,0}(u)}{\sqrt{u-\frac{1}{2}}}du &= 4(12i)^{\frac{1}{2}}\zeta_{48} + 4\left(\frac{12i}{13}\right)^{\frac{1}{2}}\zeta_{624}\sum_{n=0}^{12}(-1)^{n+1}(-\zeta_{26};\zeta_{26})_n \\ &= 4(12i)^{\frac{1}{2}}\zeta_{48} + 4\left(\frac{12i}{13}\right)^{\frac{1}{2}}\zeta_{624}\sum_{n=0}^{12}\zeta_{26}^{-n-1}(-\zeta_{26}^{-1};\zeta_{26}^{-1})_n^2 \\ &\approx 2.61608 + 1.61783i. \end{aligned}$$

Example 2. We take $a = 3, b = 4, c = 2, h = 33, k = 40$, so that $h/k = 33/40 \in S_4$. We choose $M = \left(\begin{smallmatrix} 97 & 48 \\ -192 & -95 \end{smallmatrix}\right) \in \Gamma(24)$, hence, $H = -5121$, and $K = 10136$. A direct calculation using (1.14) and (1.15) reveals that the strange functions $F_{3,4,2}(33,40)$ and $F_{3,4,2}(-5121,10136)$ satisfy

(1.16)

$$\begin{aligned} &F_{3,4,2}(33,40) \\ &\quad = (12i)^{\frac{1}{2}}\zeta_{320}^{1859}(\zeta_4^{-3}\zeta_{20}^{-33}-1)\left(\sum_{n=0}^{7}\zeta_5^{12(n+1)}(\zeta_{40}^{129};\zeta_{40}^{33})_n + \sum_{n=0}^{31}\zeta_5^{-12(n+1)}(\zeta_{40}^{-63};\zeta_{40}^{33})_n\right), \end{aligned}$$

(1.17)

$$= 2(12i)^{\frac{1}{2}}\zeta_{320}^{1859}(\zeta_4^{-3}\zeta_{20}^{-33}-1)\left(\sum_{n=0}^{7}\zeta_{40}^{-33(n+1)}(\zeta_{40}^{63},\zeta_{40}^{-33})_n(\zeta_{40}^{-129},\zeta_{40}^{-33})_n\right),$$

(1.18)

$$\begin{aligned} &F_{3,4,2}(-5121,10136) \\ &\quad = (12i)^{\frac{1}{2}}\zeta_{81088}^{-288483}(\zeta_4^{-3}\zeta_{5068}^{5121}-1)\left(\sum_{n=0}^{7599}\zeta_{1267}^{-330(n+1)}\left(\zeta_{10136}^{-7761};\zeta_{10136}^{-5121}\right)_n + \sum_{n=0}^{2535}\zeta_{1267}^{330(n+1)}\left(\zeta_{10136}^{-2481};\zeta_{10136}^{-5121}\right)_n\right), \end{aligned}$$

(1.19)

$$= 2(12i)^{\frac{1}{2}}\zeta_{81088}^{-288483}(\zeta_4^{-3}\zeta_{5068}^{5121}-1)\left(\sum_{n=0}^{2535}\zeta_{10136}^{5121(n+1)}(\zeta_{10136}^{7761};\zeta_{10136}^{5121})_n(\zeta_{10136}^{2481};\zeta_{10136}^{5121})_n\right),$$

and the polynomials $P_{3,4,2}(33,40)$ and $P_{3,4,2}(-5121,10136)$ satisfy

$$P_{3,4,2}(33,40) = -2(12i)^{\frac{1}{2}} \sum_{n=1}^{11} \chi(n)\zeta_8^{3(n-1)}\zeta_{960}^{33n^2}, \tag{1.20}$$

$$P_{3,4,2}(-5121,10136) = -2(12i)^{\frac{1}{2}} \sum_{n=1}^{11} \chi(n)\zeta_8^{3(n-1)}\zeta_{81088}^{-1707n^2}. \tag{1.21}$$

Hence, by Corollary 1.11, the integral appearing in (1.13) may be evaluated exactly, using the finite sums in (1.16) or (1.17), (1.18) or (1.19), and (1.20) and (1.21), as

$$\begin{aligned}&\int_{-\frac{95}{192}}^{i\infty} \frac{\vartheta_{3,4,2}(u)}{\sqrt{u-\frac{33}{40}}}du\\ &\quad= F_{3,4,2}(33,40) + P_{3,4,2}(33,40) - \left(\tfrac{-1267}{5}\right)^{-\frac{1}{2}} \left(F_{3,4,2}(-5121,10136) + P_{3,4,2}(-5121,10136)\right)\\ &\quad\approx -2.47333 + 0.934816i.\end{aligned}$$

In light of Theorem 1.8, we also offer an asymptotic expansion in Proposition 1.14 for our general strange function $F(w;q)$, which is a generating function for values of L-functions. This asymptotic expansion also gives rise to a new way to evaluate $F(\zeta_b^a;1)$ as a finite sum.

Proposition 1.14. *Let $B_r(z)$ denote the rth Bernoulli polynomial. For $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, $b \neq 6$, satisfying $\gcd(a,b)=1$ and $\cos(2\pi a/b) > 1/2$, as $t \to 0^+$, we have that*

$$e^{-t}(\zeta_b^{-a}-1)F(\zeta_b^a;e^{-24t}) \sim \sum_{r=0}^{\infty} L(-2r, C_{a,b})\frac{(-t)^r}{r!}, \tag{1.22}$$

where $L(-r,C_{a,b}) = L(-r,C_{a,b},b) := -\frac{(12b)^r}{r+1}\sum_{n=1}^{12b} C_{a,b}(n)B_{r+1}\left(\frac{n}{12b}\right),$ *and* $C_{a,b}(n) := \chi(n)\zeta_{2b}^{a(n-1)}.$

In particular, for such a and b, we have that

$$(\zeta_b^{-a}-1)F(\zeta_b^a;1) = -\frac{1}{12b}\sum_{n=1}^{12b} n\chi(n)\zeta_{2b}^{a(n-1)}. \tag{1.23}$$

2. q-hypergeometric series

In this section, we prove Theorem 1.1, Theorem 1.2, Corollary 1.3, and Theorem 1.5. We point out that Theorem 1.1 follows in part from Theorem 1.2.

2.1. **Proof of Theorem 1.1.** To prove Theorem 1.1, we establish the following results relating the two-variable unimodal rank function $U(w;q)$ and our two-variable strange function $F(w;q)$ to their truncated counterparts $U_k(w;q)$ and $F_k(w;q)$.

Proposition 2.1. *Let $k \in \mathbb{N}$, $h \in \mathbb{Z}$ be such that $\gcd(h,k)=1$. The following are true.*

i) *For all $w \in A_k$, we have that $U(w;\zeta_k^h) = (w^k + w^{-k} - 1)^{-1}U_k(w;\zeta_k^h)$.*
ii) *For all $w \in B_k$, we have that $F(w;\zeta_k^{-h}) = (w^{2k} + 1 - w^k)^{-1}F_k(w;\zeta_k^{-h})$.*

The proof of Theorem 1.1 makes use of Proposition 2.1 and Theorem 1.2, whose proofs we give below.

Proof of Theorem 1.1. From Theorem 1.2 and Proposition 2.1, we have for $w \in A_k \cap B_k$ that

$$F(w;\zeta_k^{-h}) = (w^{2k}+1-w^k)^{-1}F_k(w;\zeta_k^{-h}) = (w^k+w^{-k}-1)^{-1}U_k(w;\zeta_k^h) = U(w;\zeta_k^h).$$

$\square$

Proof of Proposition 2.1. If $|2-w^k-w^{-k}|<1$, then the geometric series $\sum_{l=0}^{\infty}(2-w^k-w^{-k})^l$ converges absolutely to $(w^k+w^{-k}-1)^{-1}$. Hence, we have

$$\begin{aligned}(w^k+w^{-k}-1)^{-1} &= \sum_{l=0}^{\infty}(2-w^k-w^{-k})^l = \lim_{r\to\infty}\sum_{l=0}^{r-1}(1-w^{-k})^l(1-w^k)^l \\ &= \lim_{r\to\infty}\sum_{l=0}^{r-1}(w\zeta_k^h;\zeta_k^h)_k^l(w^{-1}\zeta_k^h;\zeta_k^h)_k^l, \end{aligned} \tag{2.1}$$

where we have used the fact that $1-x^k=\prod_{m=1}^{k}(1-x\zeta_k^{hm})$. Multipliying both sides of (2.1) by $U_k(w;\zeta_k^h)$, and using the fact that $(x\zeta_k^h;\zeta_k^h)_k^l=(x\zeta_k^h;\zeta_k^h)_{lk}$ and $(x\zeta_k^h;\zeta_k^h)_{lk}(x\zeta_k^h;\zeta_k^h)_n=(x\zeta_k^h;\zeta_k^h)_{lk+n}$ for any $l,n\in\mathbb{N}_0$, we have that $(w^k+w^{-k}-1)^{-1}U_k(w;\zeta_k^h)$ equals

$$\begin{aligned}&= \lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=0}^{k-1}(w\zeta_k^h;\zeta_k^h)_{lk}(w^{-1}\zeta_k^h;\zeta_k^h)_{lk}(w\zeta_k^h;\zeta_k^h)_n(w^{-1}\zeta_k^h;\zeta_k^h)_n\zeta_k^{h(n+1)} \\ &= \lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=0}^{k-1}(w\zeta_k^h;\zeta_k^h)_{lk+n}(w^{-1}\zeta_k^h;\zeta_k^h)_{lk+n}\zeta_k^{h(lk+n+1)} \\ &= \lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=lk}^{(l+1)k-1}(w\zeta_k^h;\zeta_k^h)_n(w^{-1}\zeta_k^h;\zeta_k^h)_n\zeta_k^{h(n+1)} \\ &= \lim_{r\to\infty}\sum_{n=0}^{rk-1}(w\zeta_k^h;\zeta_k^h)_n(w^{-1}\zeta_k^h;\zeta_k^h)_n\zeta_k^{h(n+1)},\end{aligned}$$

which is by definition the function $U(w;\zeta_k^h)$. This proves part (i) of Proposition 2.1.

To prove part (ii) of Proposition 2.1, we proceed similarly. If $|w^k-w^{2k}|<1$, then the geometric series $\sum_{l=0}^{\infty}(w^k-w^{2k})^l$ converges absolutely to $(w^{2k}+1-w^k)^{-1}$. Hence, we have

$$(w^{2k}+1-w^k)^{-1} = \sum_{l=0}^{\infty}(w^k-w^{2k})^l = \lim_{r\to\infty}\sum_{l=0}^{r-1}w^{lk}(1-w^k)^l = \lim_{r\to\infty}\sum_{l=0}^{r-1}w^{lk}(w\zeta_k^{-h};\zeta_k^{-h})_k^l. \tag{2.2}$$

Multiplying both sides of (2.2) by $F_k(w;\zeta_k^{-h})$ we obtain that $(w^{2k}+1-w^k)^{-1}F_k(w;\zeta_k^{-h})$ equals

$$\begin{aligned}\lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=0}^{k-1}w^{kl+n+1}(w\zeta_k^{-h};\zeta_k^{-h})_k^l(w\zeta_k^{-h};\zeta_k^{-h})_n &= \lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=0}^{k-1}w^{kl+n+1}(w\zeta_k^{-h};\zeta_k^{-h})_{lk}(w\zeta_k^{-h};\zeta_k^{-h})_n \\ = \lim_{r\to\infty}\sum_{l=0}^{r-1}\sum_{n=lk}^{(l+1)k-1}w^{n+1}(w\zeta_k^{-h};\zeta_k^{-h})_n &= \lim_{r\to\infty}\sum_{n=0}^{rk-1}w^{n+1}(w\zeta_k^{-h};\zeta_k^{-h})_n = F(w;\zeta_k^{-h}).\end{aligned}$$

$\square$

2.2. **Proofs of Theorem 1.2 and Corollary 1.3.** To prove Theorem 1.2 (and hence Corollary 1.3), we establish the following important lemmas. Our methods here are inspired by and are similar to methods used in [6, 7]. For all $h \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $\gcd(h,k) = 1$, and $a \in \mathbb{N}_0$, we define the polynomials $C_k(x,w)$ and $v_a(x,w) = v_{a,h,k}(x,w)$ in the variables x and w by

$$C_k(x,w) = C_{h,k}(x,w) := w^{-k} \sum_{n=0}^{k-1} w^{n+1} \prod_{m=1}^{n} (x - w\zeta_k^{-mh}), \tag{2.3}$$

$$(w^k + w^{-k} - x^k) v_a(x,w) := C_k(\zeta_k^{-ah} x, w) - (wx; \zeta_k^{-h})_a (w^{-1}x; \zeta_k^{-h})_a C_k(x,w). \tag{2.4}$$

We also let $u_a(x,w) := v_a(\zeta_k^{ah} x, w)$. In what follows, for ease of notation, we may write $C(x)$ for $C_k(x,w)$, and $u_a(x)$ for $u_a(x,w)$. In Lemma 2.2, we establish a recursive relationship satisfied by the polynomial $C(x)$.

Lemma 2.2. *Assuming the notation and hypotheses above, we have that*

$$C(\zeta_k^{-h} x) = (wx - 1)(w^{-1}x - 1)C(x) + x(w^{-k} + w^k - x^k). \tag{2.5}$$

In Lemma 2.3, we establish a recursive relationship satisfied by the polynomial $u_a(x)$, and relate $u_k(x)$ to the polynomial $C(x)$.

Lemma 2.3. *For all $k \in \mathbb{N}$ and $a \in \mathbb{N}_0$, assuming the notation and hypotheses above, we have that*

$$u_k(x) = x^k C(x), \tag{2.6}$$

$$u_{a+1}(x) - u_a(x) = (wx\zeta_k^h; \zeta_k^h)_a (w^{-1}x\zeta_k^h; \zeta_k^h)_a \zeta_k^{h(a+1)} x. \tag{2.7}$$

Proof of Lemma 2.2. We have that $w^{k-1}(C(\zeta_k^{-h}x) - (wx-1)(w^{-1}x-1)C(x))$ equals

$$\begin{aligned}
&\sum_{n=0}^{k-1} w^n \prod_{m=1}^{n} (\zeta_k^{-h} x - w\zeta_k^{-hm}) - w^{1-k}(wx-1)(w^{-1}x-1) \sum_{n=0}^{k-1} w^n \prod_{m=1}^{n} (x - w\zeta_k^{-mh}) \\
&\quad = 1 - (wx-1)(w^{-1}x-1) + \sum_{n=1}^{k-1} w^n \zeta_k^{-nh}(x-w) \prod_{m=1}^{n-1} (x - w\zeta_k^{-mh}) \\
&\qquad\qquad - (wx-1)(w^{-1}x-1) \sum_{n=1}^{k-1} w^n \prod_{m=1}^{n} (x - w\zeta_k^{-mh}) \\
&\quad = -x(x - w - w^{-1}) + \sum_{n=1}^{k-1} w^n \prod_{m=1}^{n-1} (x - w\zeta_k^{-mh}) \Big(\zeta_k^{-nh}(x-w) - (wx-1)(w^{-1}x-1)(x - w\zeta_k^{-nh}) \Big) \\
&\quad = -x(x - w - w^{-1}) - x \sum_{n=1}^{k-1} w^n \prod_{m=1}^{n-1} (x - w\zeta_k^{-mh}) \Big((x-w)(x - w\zeta_k^{-nh}) - w^{-1}x + 1 \Big) \\
&\quad = -x \left(x - w - w^{-1} + \sum_{n=1}^{k-1} w^n \prod_{m=0}^{n} (x - w\zeta_k^{-mh}) - \sum_{n=0}^{k-2} w^n \prod_{m=0}^{n} (x - w\zeta_k^{-mh}) \right) \\
&\quad = -x \left(x - w - w^{-1} + w^{k-1} \prod_{m=0}^{k-1} (x - w\zeta_k^{-mh}) - (x-w) \right) \\
&\quad = xw^{k-1}(w^k + w^{-k} - x^k),
\end{aligned}$$

where we have used that $\prod_{m=0}^{k-1}(x - w\zeta_k^{-mh}) = x^k - w^k$. Multiplying through by w^{1-k} gives (2.5) as desired. □

Proof of Lemma 2.3. Letting $a = k$ in (2.4), we have that

$$\begin{aligned}(w^k + w^{-k} - x^k)u_k(x) &= C(x) - \prod_{m=0}^{k-1}(1 - wx\zeta_k^{-mh})(1 - w^{-1}x\zeta_k^{-mh})C(x) \\ &= C(x) - (w^k x^k - 1)(w^{-k}x^k - 1)C(x) = C(x)x^k(w^k + w^{-k} - x^k),\end{aligned}$$

where we have used that $\prod_{m=0}^{k-1}(1 - y\zeta_k^{-mh}) = 1 - y^k$. Dividing by $(w^k + w^{-k} - x^k)$ yields (2.6).

Next, from (2.4), letting $x \mapsto \zeta_k^{ah}x$, noting that $(\zeta_k^{ah}x)^k = x^k$, we obtain,

$$\begin{aligned}(w^k + w^{-k} - x^k)u_a(x) &= C(x) - (w\zeta_k^{ah}x;\zeta_k^{-h})_a(w^{-1}\zeta_k^{ah}x;\zeta_k^{-h})_a C(\zeta_k^{ah}x) \\ &= C(x) - (wx\zeta_k^{h};\zeta_k^{h})_a(w^{-1}x\zeta_k^{h};\zeta_k^{h})_a C(\zeta_k^{ah}x). \qquad (2.8)\end{aligned}$$

We let $a \mapsto a + 1$ in (2.8) and subtract from this result equation (2.8) to obtain

$$\begin{aligned}(2.9)\quad (w^k + &w^{-k} - x^k)(u_{a+1}(x) - u_a(x)) \\ &= (wx\zeta_k^h;\zeta_k^h)_a(w^{-1}x\zeta_k^h;\zeta_k^h)_a\Big(C(\zeta_k^{ah}x) - (1 - w\zeta_k^{(a+1)h}x)(1 - w^{-1}\zeta_k^{(a+1)h}x)C(\zeta_k^{(a+1)h}x)\Big).\end{aligned}$$

From Lemma 2.2, letting $x \mapsto \zeta_k^{h(a+1)}x$, we have that

$$C(\zeta_k^{ah}x) - (1 - w\zeta_k^{(a+1)h}x)(1 - w^{-1}\zeta_k^{(a+1)h}x)C(\zeta_k^{(a+1)h}x) = \zeta_k^{(a+1)h}x(w^k + w^{-k} - x^k).$$

Substituting this into equation (2.9) and dividing by $(w^k + w^{-k} - x^k)$ yields (2.7). $\square$

Proof of Theorem 1.2. Letting $x = 1$ in equation (2.7), we have

$$u_{a+1}(1) - u_a(1) = (w\zeta_k^h;\zeta_k^h)_a(w^{-1}\zeta_k^h;\zeta_k^h)_a\zeta_k^{(a+1)h}. \qquad (2.10)$$

From (2.6), (2.10) and the fact that $u_0(1) = 0$, we obtain

$$C(1) = u_k(1) = \sum_{a=0}^{k-1}(u_{a+1}(1) - u_a(1)) = \sum_{a=0}^{k-1}(w\zeta_k^h;\zeta_k^h)_a(w^{-1}\zeta_k^h;\zeta_k^h)_a\zeta_k^{(a+1)h}. \qquad (2.11)$$

Using (2.11) with (2.3), (1.4) and (1.5) proves Theorem 1.2. $\square$

Proof of Corollary 1.3. Let $h \in \mathbb{Z}$ be such that $\gcd(h, k) = 1$. Then there exists $h' \in \mathbb{Z}$ such that $hh' \equiv -1 \pmod{k}$. Further, since $b|k$, we define γ to be the integer in $\{0, 1, 2, \ldots, k-1\}$ such that $\gamma \equiv h'ak/b \pmod{k}$. Thus, we have that $1 - \zeta_b^a\zeta_k^{h\gamma} = 0$, and hence, that $(\zeta_b^a\zeta_k^h;\zeta_k^h)_n = 0$ for all integers $n \geq k$. Similarly, $(\zeta_b^a\zeta_k^{-h};\zeta_k^{-h})_n = 0$ for all integers $n \geq k$. Therefore, we have that $U(\zeta_b^a;\zeta_k^h) = U_k(\zeta_b^a;\zeta_k^h)$ and $F(\zeta_b^a;\zeta_k^{-h}) = F_k(\zeta_b^a;\zeta_k^{-h})$. Combining this with Theorem 1.2 proves Corollary 1.3. $\square$

2.2.1. *Proof of Theorem 1.5.*

Proof of Theorem 1.5. Our proof here extends the proof of [6, Theorem 1.5]. Here, we first make use of a general identity found in Ramanujan's lost notebook. Namely, from [2, Entry 12.2.2 (p.1)] we find that

$$\begin{aligned}\frac{(q)_\infty^2}{(x)_\infty(x^{-1}q)_\infty} &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1 - xq^n} = \frac{1}{1-x} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1 - xq^n} + \sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n-1)}{2}}}{1 - xq^{-n}} \\ &= \frac{1}{1-x} + (1 - x^{-1})\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}(1 + q^n)}{(1 - xq^n)(1 - x^{-1}q^n)},\end{aligned}$$

which implies

$$\frac{1}{(1-x)(1-x^{-1})}\left(\frac{(q)_\infty^2}{(xq)_\infty(x^{-1}q)_\infty}-1\right)=\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}(1+q^n)}{(1-xq^n)(1-x^{-1}q^n)}. \tag{2.12}$$

Next, we use Bailey's pair with $\beta_n = 1$, $\alpha_0 = 1$ and

$$\alpha_n = (1-q^{2n})q^{2n^2-n}\left(\sum_{j=-n}^{n-1}(-1)^{j+1}q^{-\frac{j(3j+1)}{2}}\right)+(-1)^n(1+q^n)q^{\frac{n(n-1)}{2}}$$

as provided in [6].[2] We substitute α_n into [6, equation (3.2)], which yields that $\sum_{n=1}^{\infty}(xq)_{n-1}(x^{-1}q)_{n-1}q^n$ equals

$$\begin{aligned}&\frac{1}{(1-x)(1-x^{-1})}\left(\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2}-1\right)\\ &\quad+\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2}\sum_{n=1}^{\infty}\sum_{j=-n}^{n-1}\frac{1-q^{2n}}{(1-xq^n)(1-x^{-1}q^n)}(-1)^{j+1}q^{2n^2-\frac{j(3j+1)}{2}}\\ &\quad+\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2}\sum_{n=1}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}(1+q^n)}{(1-xq^n)(1-x^{-1}q^n)}.\end{aligned} \tag{2.13}$$

Using (2.12), we see that the third term in (2.13) is -1 multiplied by the first term in (2.13). Thus, we deduce (after re-indexing the second term in (2.13)) that

$$\sum_{n=1}^{\infty}(xq)_{n-1}(x^{-1}q)_{n-1}q^n=\frac{(xq)_\infty(x^{-1}q)_\infty}{(q)_\infty^2}\sum_{n\geq 1}\sum_{6n\geq|6j+1|}\frac{1-q^{2n}}{(1-xq^n)(1-x^{-1}q^n)}(-1)^{j+1}q^{2n^2-\frac{j(3j+1)}{2}}. \tag{2.14}$$

It is not difficult to see that $(1-q^{2n})(1-xq^n)^{-1}(1-x^{-1}q^n)^{-1} = 1+\sum_{m\geq 1}(x^{-m}+x^m)q^{mn}$. Substituting this into (2.14) (with $x=w$) yields

$$\begin{aligned}U(w;q)=&\frac{(wq;q)_\infty(w^{-1}q;q)_\infty}{(q;q)_\infty^2}\\ &\times\left(\sum_{n>0}\sum_{6n\geq|6j+1|}(-1)^{j+1}q^{2n^2-\frac{j(3j+1)}{2}}+\sum_{m,n>0}\sum_{6n\geq|6j+1|}(-1)^{j+1}(w^m+w^{-m})q^{2n^2+mn-\frac{j(3j+1)}{2}}\right)\end{aligned}$$

as claimed. □

3. Quantum modular forms

In this section, we prove Theorem 1.8 and Proposition 1.14. We also establish the equivalence of the two expressions (1.9) and (1.10) defining our quantum modular forms $\phi_{a,b,c}(x)$ built from our strange function $F(w;q)$ studied in Section 2. Our methods extend and are inspired by methods used by Zagier to establish the quantum modular properties of Kontsevich's strange function $F(q)$. Equally important to our proof are q-hypergeometric properties of our strange function $F(w;q)$, which we establish in Section 3.1, and associated modular properties, which we establish in Section 3.2.

[2] We have corrected a minor typo from [6]. In particular, the exponent of q in the second summand is $n(n-1)/2$ instead of $n(n+1)/2$.

3.1. **Strange functions.** To state our results, we define two relatives $\mathcal{F}_1(w;q)$ and $\mathcal{F}_2(w;q)$ of our "strange" function $F(w;q)$ by

$$\mathcal{F}_1(w;q) := w\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1} w^j\right) q^n (wq;q)_{n-1},$$

$$\mathcal{F}_2(w;q) := \sum_{n=0}^{\infty} w^n \left((wq;q)_n - (wq;q)_\infty\right).$$

Essential to our proof of Theorem 1.8, which establishes the quantum modularity of our strange functions $\phi_{a,b,c}(x)$, is Proposition 3.1 below. This proposition relates the strange functions $F(w;q)$, $\mathcal{F}_1(w;q)$ and $\mathcal{F}_2(w;q)$ in parts (i) and (ii), and gives a related q-series identity in part (iii). We note that part (i) is established in the case $\zeta = 1$ and $d = 0$, and part (ii) is established in the case $w = 1$, by Zagier in [18].

Proposition 3.1. *The following are true:*

(i) *Let ζ be a fixed root of unity, and $d \in \mathbb{Z}$. Then the function $\mathcal{F}_1(\zeta\mathfrak{q}^d;\mathfrak{q})$ equals the function $\zeta^{-1}\mathfrak{q}^{-d}F(\zeta\mathfrak{q}^d;\mathfrak{q})$ as a power series in $\mathfrak{q} \mapsto \rho - q$ ($|q| < 1$), where ρ is any root of unity satisfying $\zeta\rho^m = 1$ for some $m \in \mathbb{Z}$.*

(ii) *The function $\mathcal{F}_1(w;q)$ equals the function $\mathcal{F}_2(w;q)$ as a power series in q, and also as a function of q in the unit disk.*

(iii) *We have that $(wq;q)_\infty + (1-w)\mathcal{F}_1(w;q) = \sum_{n=1}^{\infty}\chi(n)w^{\frac{n-1}{2}}q^{\frac{n^2-1}{24}}$.*

Remark 3.2. When $w = 1$, part (iii) of Proposition 3.1 reduces to the well-known identity

$$(q;q)_\infty = \sum_{n=1}^{\infty}\chi(n)q^{(n^2-1)/24}.$$

In this case, we obtain no useful information about the strange function $\mathcal{F}_1(1;q)$, and hence Kontsevich's strange function $F(1;q) = F(q)$ by part (i) of Proposition 3.1. However, the case $w = 1$ is already treated in [19], where $F(1;q) = F(q)$ is shown to be (essentially) quantum modular of weight 3/2. Here, we are concerned with establishing the quantum modular properties of $F(w;q)$ when $w \neq 1$.

3.1.1. *Proof of Proposition 3.1.* To prove Proposition 3.1, we first establish two auxiliary Lemmas, Lemma 3.3 and Lemma 3.4. To state Lemma 3.3, we define for $N \in \mathbb{N}$ the function

$$\mathcal{F}_{2,N}(w;q) := \sum_{n=0}^{N-1} w^n\left((wq;q)_n - (wq;q)_N\right).$$

Lemma 3.3. *For any $N \in \mathbb{N}$, we have that*

$$\mathcal{F}_{2,N}(w;q) = w\sum_{n=1}^{N}\left(\sum_{j=0}^{n-1} w^j\right) q^n (wq;q)_{n-1}. \tag{3.1}$$

In order to state Lemma 3.4, we must introduce some further notation. We define for $a \in \mathbb{N}$ and $b \in -\mathbb{N}$ the functions

$$A_a(q) = A_a(q,\rho,\zeta) := \zeta\sum_{r=0}^{a-1}\binom{a}{r+1}\rho^{a-r-1}(-q)^r,$$

$$B_b(q) = B_b(q,\rho,\zeta) := \zeta\sum_{r=0}^{\infty}\binom{b}{r+1}\rho^{b-r-1}(-q)^r,$$

where q,ρ, and ζ are as defined in Proposition 3.1 (i), and for $x\in\mathbb{Z}, y\in\mathbb{N}_0$, the binomial coefficients are defined by

$$\binom{x}{y} := \frac{\prod_{j=0}^{y-1}(x-j)}{y!}. \tag{3.2}$$

We also define $A_0(q)=B_0(q):=0$. With hypotheses given on ρ and ζ, for fixed $d\in\mathbb{Z}$, we may assume ρ is a kth root of unity and ζ another root of unity such that $\zeta\rho^{d+m}=1$ for some integer m satisfying $1\leq m\leq k$. Using the functions $A_a(q)$ and $B_b(q)$, we define the products $\Pi_j(N,q)=\Pi_j(N,q,\rho,\zeta,d)$ $(j\in\{1,2\})$ for integers $N\in\mathbb{N}$ and $d\in\mathbb{N}_0$ by

$$\Pi_1(N,q) := \prod_{\substack{\ell=d+1\\ \ell\equiv d+m \pmod k}}^{d+N} A_\ell(q), \qquad \Pi_2(N,q) := \prod_{\substack{\ell=d+1\\ \ell\not\equiv d+m \pmod k}}^{d+N} (1-\zeta\rho^\ell+qA_\ell(q)).$$

We also define for integers $N\geq 2$ and $d\in-\mathbb{N}$ the product $\Pi_3(N,q)=\Pi_3(N,q,\rho,\zeta,d)$ by

$$\Pi_3(N,q) := \begin{cases} \displaystyle\prod_{\substack{\ell=d+1\\ \ell\equiv d+m \pmod k}}^{d+N} B_\ell(q) \times \prod_{\substack{h=d+1\\ h\not\equiv d+m \pmod k}}^{d+N} (1-\zeta\rho^h+qB_h(q)), & N<-d, d\neq -1,\\[2ex] \displaystyle\prod_{\substack{\ell=d+1\\ \ell\equiv d+m \pmod k}}^{-1} B_\ell(q) \times \prod_{\substack{h=d+1\\ h\not\equiv d+m \pmod k}}^{-1} (1-\zeta\rho^h+qB_h(q)), & N\geq -d, d\neq -1,\end{cases}$$

and $\Pi_3(N,q):=(1-\zeta)$ if $d=-1$. Similarly, we define the product $\Pi_4(N,q)=\Pi_4(N,q,\rho,\zeta,d)$ for integers $N\geq 2$ and $d\in-\mathbb{N}$ by

$$\Pi_4(N,q) := \begin{cases} \displaystyle\prod_{\substack{\ell=0\\ \ell\equiv d+m \pmod k}}^{d+N} A_\ell(q) \times \prod_{\substack{h=0\\ h\not\equiv d+m \pmod k}}^{d+N} (1-\zeta\rho^h+qA_h(q)), & N\geq -d, d\neq -1,\\[2ex] \displaystyle\prod_{\substack{\ell=1\\ \ell\equiv d+m \pmod k}}^{N-1} A_\ell(q) \times \prod_{\substack{h=1\\ h\not\equiv d+m \pmod k}}^{N-1} (1-\zeta\rho^h+qA_h(q)), & d=-1,\end{cases}$$

and $\Pi_4(N,q):=1$ if $N<-d, d\neq -1$.

Lemma 3.4. *Let ζ,ρ, q, and d be as in the statement of Proposition 3.1 (i). The following are true.*
(i) If $N\in\mathbb{N}$, and $d\in\mathbb{N}_0$, then

$$(\zeta(\rho-q)^{d+1};\rho-q)_N = q^{\left\lfloor \frac{N-m}{k}+1\right\rfloor}\cdot\Pi_1(N,q)\cdot\Pi_2(N,q).$$

(ii) If $N\in\mathbb{N}, N\geq 2$, and $d\in-\mathbb{N}$, then

$$(\zeta(\rho-q)^{d+1};\rho-q)_N = q^{\left\lfloor \frac{N-m}{k}+1\right\rfloor}\cdot\Pi_3(N,q)\cdot\Pi_4(N,q).$$

Proof of Lemma 3.3. Lemma 3.3 is easily proved by induction on N. When $N=1$, (3.1) follows by direct calculation. Assuming (3.1) holds for some $N\in\mathbb{N}$, we have that

$$\begin{aligned}
\sum_{n=0}^{N} w^n((wq;q)_n - (wq;q)_{N+1}) &= \sum_{n=0}^{N} w^n((wq;q)_n - (wq;q)_N + (wq;q)_N - (wq;q)_{N+1}) \\
&= \sum_{n=0}^{N-1} w^n\left((wq;q)_n - (wq;q)_N\right) + w^N((wq;q)_N - (wq;q)_N) + \sum_{n=0}^{N} w^n\left((wq;q)_N - (wq;q)_{N+1}\right) \\
&= w\sum_{n=1}^{N}\left(\sum_{j=0}^{n-1} w^j\right) q^n(wq;q)_{n-1} + w\sum_{j=0}^{N} w^j q^{N+1}(wq;q)_N = w\sum_{n=1}^{N+1}\left(\sum_{j=0}^{n-1} w^j\right) q^n(wq;q)_{n-1},
\end{aligned}$$

as desired, establishing (3.1) for all $N \in \mathbb{N}$ by induction. □

Proof of Lemma 3.4. To prove part (i), for $\ell \in \mathbb{N}$, we have that $1 - \zeta(\rho - q)^\ell = 1 - \zeta\rho^\ell + qA_\ell(q)$. Thus, for $N \in \mathbb{N}$ and $d \in \mathbb{N}_0$, we have that

$$(\zeta(\rho-q)^{d+1}; \rho - q)_N = \prod_{\ell=d+1}^{d+N} (1 - \zeta\rho^\ell + qA_\ell(q)) = q^{\left\lfloor \frac{N-m}{k} + 1 \right\rfloor} \cdot \Pi_1(N,q) \cdot \Pi_2(N,q),$$

where the integer m is such that $\zeta\rho^{d+m} = 1, 1 \le m \le k$, as claimed. Here, we have used the fact that for $1 \le v \le y$, $y \in \mathbb{N}$,

$$\#\{\ell \mid 1 \le \ell \le b,\ \ell \equiv v \pmod{y}\} = \left\lfloor \frac{b-v}{y} + 1 \right\rfloor,$$

and also the fact that $\zeta\rho^\ell = 1$ for $\ell \equiv d + m \pmod{k}$.

To prove part (ii), we proceed similarly, additionally using where necessary the fact that for $\ell \in -\mathbb{N}$ and $|\beta| < |\alpha|$,

$$(\alpha + \beta)^\ell = \sum_{r=0}^{\infty} \binom{\ell}{r} \alpha^{\ell - r}\beta^r. \tag{3.3}$$

(This well-known expansion can be deduced by multiplying [15, 4.6.7] by α^a, with $z = \beta/\alpha$, for example.) We apply (3.3) in the case $\alpha = \rho$ and $\beta = -q$, and note that $|q| < |\rho| = 1$. That is, for any term $1 - \zeta(\rho - q)^\ell$ with $\ell \in \mathbb{N}$ encountered in expanding the product $(\zeta(\rho - q)^{d+1}; \rho - q)_N$, we proceed as above in the proof of (i). A direct calculation leads to the product $\Pi_4(N, q)$ under the hypotheses given on N and d in its definition. For any term $1 - \zeta(\rho - q)^\ell$ with $\ell \in -\mathbb{N}$ encountered, we use (3.3). A direct calculation leads to the product $\Pi_3(N, q)$ under the hypotheses given in its definition. Proceeding as in the proof of part (i) yields the claim in (ii). □

Proof of Proposition 3.1.
(ii): It follows by a straightforward calculation that

$$\lim_{N\to\infty} \left((wq;q)_N - (wq;q)_\infty\right) \sum_{n=0}^{N-1} w^n \tag{3.4}$$

converges absolutely to 0 for all $w \neq 1$ in the closed unit disk. For such w, using (3.4) and the definition of $\mathcal{F}_{2,N}$, we have that

$$0 = \lim_{N\to\infty} \left(-\mathcal{F}_{2,N}(w;q) + \sum_{n=0}^{N-1} w^n \left((wq;q)_n - (wq;q)_\infty\right) \right),$$

hence, $\lim_{N\to\infty} \mathcal{F}_{2,N}(w;q) = \mathcal{F}_2(w;q)$. Letting $N \to \infty$ in (3.1) of Lemma 3.3 yields part (ii) of Proposition 3.1.

(i): We use Lemma 3.4 and the definition of $F(w;q)$, and find for $d \in \mathbb{N}_0$ that

$$F(\zeta(\rho-q)^d;\rho-q) = \zeta(\rho-q)^d + \sum_{n=1}^{\infty} (\zeta(\rho-q)^d)^{n+1} q^{\left\lfloor \frac{n-m}{k}+1 \right\rfloor} \cdot \Pi_1(n,q) \cdot \Pi_2(n,q). \tag{3.5}$$

The integers k and m are fixed, so for any $j \in \mathbb{N}_0$, there are at most finitely many $n \in \mathbb{N}$ such that $\left\lfloor \frac{n-m}{k}+1 \right\rfloor = j$. If such an n exists for a given j, then we may define for $j \in \mathbb{N}_0$, the number

$$N_j = N_j(m,k) := \max\left\{ n \in \mathbb{N} \;\middle|\; \left\lfloor \frac{n-m}{k}+1 \right\rfloor = j \right\}.$$

If such an n does not exist for a given j, we define $N_j = N_j(m,k) := 0$. Thus, using (3.5) and the definitions of $\Pi_1(n,q)$ and $\Pi_2(n,q)$, we have shown that when $d \in \mathbb{N}_0$, the coefficient of q^j in $F(\zeta(\rho-q)^d;\rho-q)$ (as a series centered at $q=0$) for any $j \in \mathbb{N}_0$ is the coefficient of q^j in the polynomial $F_{N_j+1}(\zeta(\rho-q)^d;\rho-q)$ defined in (1.5). Making the change of variable $\mathfrak{q} \mapsto \rho - q$ (shifting the center to $\mathfrak{q} = \rho$), we thus have that for any $d \in \mathbb{N}_0$, $F(\zeta\mathfrak{q}^d;\mathfrak{q})$ is well defined as a power series in $\mathfrak{q} \mapsto \rho - q$ as claimed.

Similarly, when $d \in -\mathbb{N}$, using Lemma 3.4 and the definition of $F(w;q)$, we obtain

$$\begin{aligned} F(\zeta(\rho-q)^d;\rho-q) &= \zeta(\rho-q)^d + \zeta^2(\rho-q)^{2d}(1-\zeta\rho^{d+1} + qB_{d+1}(q)) \\ &\quad + \sum_{n=2}^{\infty} (\zeta(\rho-q)^d)^{n+1} q^{\left\lfloor \frac{n-m}{k}+1 \right\rfloor} \cdot \Pi_3(n,q) \cdot \Pi_4(n,q). \end{aligned}$$

For any $j \in \mathbb{N}_0$, similar to the number N_j, we define the numbers

$$M_j = M_j(m,k) := \max\left\{ n \in \mathbb{N} \;\middle|\; n \geq 2,\; \left\lfloor \frac{n-m}{k}+1 \right\rfloor = j \right\}$$

if such an n exists for a given j; if not, we define $M_j = M_j(m,k) := 1$. Arguing as above, we have that for $d \in -\mathbb{N}$, for any $j \in \mathbb{N}_0$, the coefficient of q^j in $F(\zeta(\rho-q)^d;\rho-q)$ (as a series centered at $q=0$) is the coefficient of q^j in the finite sum $F_{M_j+1}(\zeta(\rho-q)^d;\rho-q)$. While F_{M_j+1} is not a polynomial in this case due to the presence of terms $B_b(q)$, there are only finitely many $B_b(q)$ contributing to the finite sum F_{M_j+1}, and it is still the case that at most finitely many terms in any such expansion contribute to the coefficient of any q^j. Making the change of variable $\mathfrak{q} \mapsto \rho - q$ (shifting the center to $\mathfrak{q} = \rho$), we thus have that for any $d \in -\mathbb{N}$, $F(\zeta\mathfrak{q}^d;\mathfrak{q})$ is well defined as a power series in $\mathfrak{q} \mapsto \rho - q$.

Having established that $F(\zeta\mathfrak{q}^d;\mathfrak{q})$ is well-defined as a power series in $\mathfrak{q} \mapsto \rho - q$ for any $d \in \mathbb{Z}$, we let $x = \zeta(\rho-q)^d$ and $y = \rho - q$, and compute for $N \in \mathbb{N}$ that the difference between $x^{-1}F(x;y)$ and the polynomial $\mathcal{F}_{2,N}(x;y)$ is

$$x^{-1}F(x;y) - \mathcal{F}_{2,N}(x;y) = (xy;y)_N \sum_{n=0}^{N} x^n + \sum_{n=1}^{\infty} x^{N+n}(xy;y)_{N+n}.$$

From Lemma 3.4 and the definitions of the products $\Pi_j(N,q)$ ($j \in \{1,2,3,4\}$), we may deduce that $(xy;y)_N = O\left(q^{\left\lfloor \frac{N-m}{k}+1 \right\rfloor}\right)$. Hence, the power series expansion of the function $\zeta^{-1}(\rho-q)^{-d}F(\zeta(\rho-q)^d;\rho-q)$ (centered at $q=0$) agrees with that of the function $\mathcal{F}_{2,N}(\zeta(\rho-q)^d,\rho-q)$ up to (and including) the $\left\lfloor \frac{N-m}{k}+1 \right\rfloor - 1$ term in the series. As $N \to \infty$, $\left\lfloor \frac{N-m}{k}+1 \right\rfloor \to \infty$, using Proposition 3.1 (ii) established above, we have that (as series centered at $q=0$)

$$\mathcal{F}_1(\zeta(\rho-q)^d;\rho-q) = \mathcal{F}_2(\zeta(\rho-q)^d;\rho-q) = \zeta^{-1}(\rho-q)^{-d}F(\zeta(\rho-q)^d;\rho-q).$$

Making the change of variable $\mathfrak{q} \mapsto \rho - q$ (shifting the center to $\mathfrak{q} = \rho$) proves the claim.

(iii): Zagier shows in [18] that the function

$$S(w)=S_q(w):=\sum_{n=0}^{\infty}(w;q)_{n+1}w^n \tag{3.6}$$

satisfies the two identities (recall $\chi(n):=\left(\frac{12}{n}\right)$)

$$S(w)=\sum_{n=1}^{\infty}\chi(n)w^{\frac{n-1}{2}}q^{\frac{n^2-1}{24}}, \tag{3.7}$$

$$S(w)=(wq;q)_\infty+(1-w)\sum_{n=0}^{\infty}\left((wq;q)_n-(wq;q)_\infty\right)w^n. \tag{3.8}$$

Combining this with Proposition 3.1 (ii) proves (iii). □

3.1.2. *Equivalence of (1.9) and (1.10).* To establish the equivalence of the two expressions (1.9) and (1.10) defining our quantum modular forms $\phi_{a,b,c}(x)$, we first establish a recursive relationship satisfied by the function $S(w)$ defined in (3.6).

Lemma 3.5. *For $k\in\mathbb{N}$, we have*

$$S(w)=-\sum_{n=1}^{6k-1}\chi(n)w^{-\frac{n+1}{2}}q^{\frac{n^2-1}{24}}+\chi(6k-1)w^{-3k}q^{\frac{k(3k-1)}{2}}S(q^{-k}w). \tag{3.9}$$

Proof of Lemma 3.5. We proceed by induction on k. In [19], Zagier proves that

$$S(w)=1-qw^2-q^2w^3S(qw).$$

As such, we have that $S(q^{-1}w)=1-q^{-1}w^2-q^{-1}w^3S(w)$, which is equivalent to

$$S(w)=-w^{-1}+qw^{-3}-qw^{-3}S(q^{-1}w) \tag{3.10}$$

as desired in the case $k=1$. Now assume for some $k\in\mathbb{N}$ that (3.9) holds. Using (3.10) with $w\to q^{-k}w$, we have that

$$S(q^{-k}w)=-q^kw^{-1}+q^{3k+1}w^{-3}-q^{3k+1}w^{-3}S(q^{-(k+1)}w).$$

We substitute the above equation into (3.9) and notice that

$$\chi(6k-1)w^{-3k}q^{\frac{k(3k-1)}{2}}(-q^kw^{-1})=-\chi(6k+1)w^{-3k-1}q^{\frac{k(3k+1)}{2}},$$
$$\chi(6k-1)w^{-3k}q^{\frac{k(3k-1)}{2}}(q^{3k+1}w^{-3})=-\chi(6k+5)w^{-3k-3}q^{\frac{3k^2+5k+2}{2}},$$

to obtain $S(w)=-\sum_{n=1}^{6k+5}\chi(n)w^{-\frac{n+1}{2}}q^{\frac{n^2-1}{24}}+\chi(6k+5)w^{-3(k+1)}q^{\frac{(k+1)(3k+2)}{2}}S(q^{-(k+1)}w)$ as desired. □

Having established Lemma 3.5, we apply it with $k=\ell_bc/12$, $w=\zeta_b^{-a}$, and $q=e(12x/\ell_b)$, to obtain (using the notation $S(w)=S_q(w)$)

(3.11)

$$S_{e(12x/\ell_b)}\left(\zeta_b^{-a}\right)=-\sum_{n=1}^{\frac{\ell_bc}{2}-1}\chi(n)\zeta_{2b}^{a(n+1)}e\left(\frac{n^2-1}{2\ell_b}x\right)+\zeta_{4b}^{a\ell_bc}e\left(\frac{xc^2\ell_b}{8}\right)e\left(\frac{-xc}{2}\right)S_{e(12x/\ell_b)}(\zeta_b^{-a}e(-cx)).$$

Next, noting that $S_q(w)=w^{-1}(1-w)F(w;q)$, we rewrite

$$2(i\ell_b)^{\frac12}\zeta_{4b}^{-a\ell_bc}(\zeta_b^{-a}-1)e\left(\frac{x}{2\ell_b}\right)F\left(\zeta_b^{-a};e\left(12x/\ell_b\right)\right)=-2(i\ell_b)^{\frac12}\zeta_{4b}^{-a\ell_bc}\zeta_b^{-a}e\left(\frac{x}{2\ell_b}\right)S_{e(12x/\ell_b)}\left(\zeta_b^{-a}\right).$$

Applying equation (3.11), we obtain

$$2(i\ell_b)^{\frac{1}{2}}\zeta_{4b}^{-a\ell_b c}(\zeta_b^{-a}-1)e\left(\frac{x}{2\ell_b}\right)F\left(\zeta_b^{-a}, e\left(12x/\ell_b\right)\right)$$

$$= -2(i\ell_b)^{\frac{1}{2}}e\left(\frac{(\ell_b c+2)^2x}{8\ell_b}\right)\zeta_b^{-a}e(-cx)S_{e(12x/\ell_b)}\big(\zeta_b^{-a}e(-cx)\big)+2(i\ell_b)^{\frac{1}{2}}\zeta_{4b}^{-a\ell_b c}\sum_{n=1}^{\frac{\ell_b c}{2}-1}\chi(n)\zeta_{2b}^{a(n-1)}e\left(\frac{n^2x}{2\ell_b}\right)$$

$$= 2(i\ell_b)^{\frac{1}{2}}e\left(\frac{(\ell_b c+2)^2x}{8\ell_b}\right)\left(\zeta_b^{-a}e(-cx)-1\right)F\left(\zeta_b^{-a}e(-cx); e\left(12x/\ell_b\right)\right)$$

$$+ 2(i\ell_b)^{\frac{1}{2}}\zeta_{4b}^{-a\ell_b c}\sum_{n=1}^{\frac{\ell_b c}{2}-1}\chi(n)\zeta_{2b}^{a(n-1)}e\left(\frac{n^2x}{2\ell_b}\right),$$

from which we deduce the equivalence of (1.9) and (1.10).

3.2. **Modularity.** Here, we establish a number of modular results, and use them together with our q-hypergeometric results from Section 3.1 to ultimately prove Theorem 1.8 and Proposition 1.14. Let $b \in \mathbb{N}$, and let $\ell_b := \operatorname{lcm}(b, 12)$. We let $a \in \mathbb{Z}$, $c \in 2\mathbb{Z}$, with $\gcd(a, b) = 1$, and define for $\tau \in \mathbb{H} := \{x + iy\colon x, y \in \mathbb{R}, y > 0\}$ the function

$$\vartheta_{a,b,c}(\tau) := \sum_{n\in\mathbb{Z}}\left(n+\frac{\ell_b c}{2}\right)\chi(n)\left(\zeta_b^a\right)^{\frac{n-1}{2}}q^{\frac{(n+\ell_b c/2)^2}{2\ell_b}}. \tag{3.12}$$

We show in Lemma 3.6 that $\vartheta_{a,b,c}$ is an ordinary modular form on the congruence subgroup

$$\Gamma(2\ell_b) := \left\{\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) : \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right) \pmod{2\ell_b},\ \alpha\delta-\beta\gamma = 1\right\} \subseteq \mathrm{SL}_2(\mathbb{Z})$$

with respect to the multiplier ψ_{ℓ_b} defined for matrices $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma(2\ell_b)$ by

$$\psi_{\ell_b}(M) := \left(\frac{\ell_b}{\delta}\right)\left(\frac{2\gamma}{\delta}\right)\epsilon_\delta^{-1}, \tag{3.13}$$

where as usual, $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol, and

$$\epsilon_\delta := \begin{cases}1, & \delta \equiv 1 \pmod 4,\\ i, & \delta \equiv 3 \pmod 4.\end{cases}$$

Lemma 3.6. *For any $M = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \Gamma(2\ell_b)$ and $\tau \in \mathbb{H}$, we have that*

$$\vartheta_{a,b,c}(M\tau) = \psi_{\ell_b}(M)(\gamma\tau+\delta)^{\frac{3}{2}}\vartheta_{a,b,c}(\tau).$$

Next we define a "period integral" of our modular theta function $\vartheta_{a,b,c}$, namely, the function $\vartheta^*_{a,b,c}$, which is defined on the lower-half of the complex plane $\mathbb{H}^- := \{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$. In order to remain consistent with our previous notation, we let $\tau \in \mathbb{H}$ as usual, and define $\vartheta^*_{a,b,c}$ with respect to the variable $\overline{\tau} \in \mathbb{H}^-$, the complex conjugate of τ, as follows:

$$\vartheta^*_{a,b,c}(\overline{\tau}) := \int_\tau^{i\infty}\frac{\vartheta_{a,b,c}(u)}{\sqrt{u-\overline{\tau}}}du.$$

Using the modular properties of the theta function $\vartheta_{a,b,c}(\tau)$ established in Lemma 3.6, we establish near modular properties of the integral $\vartheta^*_{a,b,c}(\overline{\tau})$ in Lemma 3.7.

Lemma 3.7. *For any $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(2\ell_b)$ and $\tau \in \mathbb{H}$, we have that*

$$\vartheta^*_{a,b,c}(M\overline{\tau}) - (\gamma\overline{\tau}+\delta)^{\frac{1}{2}}\psi_{\ell_b}(M)\vartheta^*_{a,b,c}(\overline{\tau}) = -(\gamma\overline{\tau}+\delta)^{\frac{1}{2}}\psi_{\ell_b}(M)\int_{-\frac{\delta}{\gamma}}^{i\infty}\frac{\vartheta_{a,b,c}(u)}{\sqrt{u-\overline{\tau}}}du.$$

Proof of Lemma 3.6. For any $N \in \mathbb{Z}$, we define

$$\theta_{N,b}(\tau) := \sum_{\substack{n\in\mathbb{Z} \\ n\equiv N \pmod{\ell_b}}} nq^{\frac{n^2}{2\ell_b}}.$$

By definition, the function $\theta_{N,b}(\tau)$ is equal to Shimura's theta function $\theta(\tau, N, \ell_b, \ell_b; x)$ (that is, $P(x) = x$) defined in [16]. By applying [16, Proposition 2.1], and simplifying, using the fact that $M \in \Gamma(2\ell_b)$, we find that for any $b \in \mathbb{N}$, $N \in \mathbb{Z}$, $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma(2\ell_b)$ and $\tau \in \mathbb{H}$, that

$$\theta_{N,b}(M\tau) = \psi_{\ell_b}(M)(\gamma\tau+\delta)^{\frac{3}{2}}\theta_{N,b}(\tau). \tag{3.14}$$

Next, we re-write

$$\begin{aligned}
\vartheta_{a,b,c}(\tau) &= \sum_{n\in\mathbb{Z}}\left(n+\frac{\ell_b c}{2}\right)\chi(n)\left(\zeta_b^a\right)^{\frac{n-1}{2}}q^{\frac{(n+\ell_b c/2)^2}{2\ell_b}} \\
&= \sum_{s\in\{1,5,7,11\}}\chi(s)\sum_{\substack{n\in\mathbb{Z} \\ n\equiv s \pmod{12}}}\left(n+\frac{\ell_b c}{2}\right)\left(\zeta_b^a\right)^{\frac{n-1}{2}}q^{\frac{(n+\ell_b c/2)^2}{2\ell_b}} \\
&= \sideset{}{^*}\sum_{\substack{k \pmod b \\ s\in\{1,5,7,11\}}}\chi(s)\left(\zeta_b^a\right)^{\frac{k-1}{2}}\sum_{\substack{n\equiv s+\frac{\ell_b c}{2} \pmod{12} \\ \text{and} \\ n\equiv k+\frac{\ell_b c}{2} \pmod b}} nq^{\frac{n^2}{2\ell_b}},
\end{aligned}$$

where $\sum^*$ denotes that the summation is only taken over those pairs (k, s) for which $\gcd(b, 12) \mid (k - s)$. In this case (and only this case, by the Chinese remainder theorem), there is a unique solution to the system of congruences

$$\begin{cases} n \equiv s + \frac{\ell_b c}{2} \pmod{12}, \\ n \equiv k + \frac{\ell_b c}{2} \pmod b, \end{cases}$$

modulo $\ell_b := \mathrm{lcm}(12, b)$, i.e. $n \equiv N_{(s,k,b)} \pmod{\ell_b}$ for some integer $N_{(s,k,b)}$. Thus, we may re-write the function $\vartheta_{a,b,c}(\tau)$ as

$$\sideset{}{^*}\sum_{\substack{k \pmod b \\ s\in\{1,5,7,11\}}}\chi(s)\left(\zeta_b^a\right)^{\frac{k-1}{2}}\theta_{N_{(s,k,b)},b}(\tau). \tag{3.15}$$

Lemma 3.6 now follows by applying (3.14) to (3.15). $\square$

Proof of Lemma 3.7. Lemma 3.7 follows by making straightforward changes of variables in the integral, combined with Lemma 3.6. To be precise, for $M \in \Gamma(2\ell_b)$, we have

$$\vartheta^*_{a,b,c}(M\overline{\tau}) = \int_{M\tau}^{i\infty}\frac{\vartheta_{a,b,c}(u)}{\sqrt{u-M\overline{\tau}}}\,du.$$

Letting $u = Mv$, and applying Lemma 3.6, after some simplification, we obtain

$$\vartheta^*_{a,b,c}(M\overline{\tau}) = \int_{\tau}^{-\frac{\delta}{\gamma}} \left(\frac{\vartheta_{a,b,c}(Mv)}{\sqrt{Mv - M\overline{\tau}}} \cdot \frac{dv}{(\gamma v + \delta)^2} \right) = -(\gamma\overline{\tau} + \delta)^{\frac{1}{2}} \psi_{l_b}(M) \int_{-\frac{\delta}{\gamma}}^{\tau} \frac{\vartheta_{a,b,c}(v)}{\sqrt{v - \overline{\tau}}} dv.$$

Hence, we have that

$$\vartheta^*_{a,b,c}(M\overline{\tau}) - (\gamma\overline{\tau} + \delta)^{\frac{1}{2}} \psi_{l_b}(M) \vartheta^*_{a,b,c}(\overline{\tau}) = -(\gamma\overline{\tau} + \delta)^{\frac{1}{2}} \psi_{l_b}(M) \int_{-\frac{\delta}{\gamma}}^{i\infty} \frac{\vartheta_{a,b,c}(u)}{\sqrt{u - \overline{\tau}}} du,$$

as desired. □

Having established the modularity of our theta function $\vartheta_{a,b,c}(\tau)$ and the near modularity of its period integral $\vartheta^*_{a,b,c}(\overline{\tau})$ in Lemma 3.6 and Lemma 3.7, we are nearly ready to prove Theorem 1.8. Our last needed ingredient is Lemma 3.8 below, which will allow us to study the asymptotic behavior of the period integral $\vartheta^*_{a,b,c}(\overline{\tau})$, where $\tau = x + iy \in \mathbb{H}$, as $y \to 0^+$.

Lemma 3.8. *Let $\tau = x + iy \in \mathbb{H}$ (i.e. $x, y \in \mathbb{R}$, with $y > 0$). With notation and hypotheses as above, we have that*

$$\begin{aligned}
\vartheta^*_{a,b,c}(\overline{\tau}) =& (i\ell_b)^{\frac{1}{2}} \sum_{n>0} \chi(n) \zeta_{2b}^{a(n-1)} e\left((n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((n + \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right) \\
&+ (i\ell_b)^{\frac{1}{2}} \sum_{0<n<\ell_b c/2} \chi(n) \zeta_{2b}^{-a(n+1)} e\left((-n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((-n + \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right) \\
&- (i\ell_b)^{\frac{1}{2}} \sum_{n>\ell_b c/2} \chi(n) \zeta_{2b}^{-a(n+1)} e\left((-n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((n - \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right).
\end{aligned}$$

Here, $\operatorname{erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du$ is the complementary error function.

Proof of Lemma 3.8. We substitute the definition of $\vartheta_{a,b,c}(u)$ into the integral defining $\vartheta^*_{a,b,c}$ and integrate term by term, making the change of variable $2\pi i(n + \frac{\ell_b c}{2})^2(u - \overline{\tau})/(2\ell_b) = -w^2$ in the integral. After some simplification, we find that

$$\vartheta^*_{a,b,c}(\overline{\tau}) = (i\ell_b)^{\frac{1}{2}} \sum_{n\in\mathbb{Z}} \chi(n) \zeta_{2b}^{a(n-1)} e\left((n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \left(\frac{2}{\sqrt{\pi}} \int_{(n+\frac{\ell_b c}{2})\sqrt{\frac{2\pi y}{\ell_b}}}^{\operatorname{sgn}(n+\ell_b c/2)\cdot\infty} e^{-w^2} dw \right). \tag{3.16}$$

Hence, splitting the sum in equation (3.16) into three sums, $n > 0$, $0 > n > -\frac{\ell_b c}{2}$ and $n < -\frac{\ell_b c}{2}$ (noting that the summands equal zero for $n = 0$ and $n = -\ell_b c/2$), and changing all indices of summation to positive, we obtain

$$\begin{aligned}
\vartheta^*_{a,b,c}(\overline{\tau}) =& (i\ell_b)^{\frac{1}{2}} \sum_{n>0} \chi(n) \zeta_{2b}^{a(n-1)} e\left((n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((n + \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right) \\
&+ (i\ell_b)^{\frac{1}{2}} \sum_{0<n<\frac{\ell_b c}{2}} \chi(n) \zeta_{2b}^{a(-n-1)} e\left((-n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((-n + \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right) \\
&- (i\ell_b)^{\frac{1}{2}} \sum_{n>\frac{\ell_b c}{2}} \chi(n) \zeta_{2b}^{a(-n-1)} e\left((-n + \ell_b c/2)^2 \frac{\overline{\tau}}{2\ell_b} \right) \operatorname{erfc}\left((n - \ell_b c/2) \sqrt{\frac{2\pi y}{\ell_b}} \right).
\end{aligned}$$

□

Proof of Theorem 1.8. Let $\tau = x + iy \in \mathbb{H}$, with $x \in S_b$ and $y > 0$. Using Proposition 3.1 and Lemma 3.8, we find after a short calculation that the asymptotic expansions of the functions $\phi_{a,b,c}(\tau)$ and $\vartheta^*_{a,b,c}(\overline{\tau})$ as power series in y coincide. To be more precise, using the fact that $\operatorname{erfc}(0) = 1$, we see from Lemma 3.8 that, formally, letting $y \to 0^+$, $\vartheta^*_{a,b,c}(x)$ equals

$$(i\ell_b)^{\frac{1}{2}} \sum_{n>0} \chi(n) \zeta_{2b}^{a(n-1)} e\left((n + \ell_b c/2)^2 \frac{x}{2\ell_b}\right) + (i\ell_b)^{\frac{1}{2}} \sum_{n>0} \chi(n) \zeta_{2b}^{a(-n-1)} e\left((n - \ell_b c/2)^2 \frac{x}{2\ell_b}\right)$$
$$- 2(i\ell_b)^{\frac{1}{2}} \sum_{n>\frac{\ell_b c}{2}} \chi(n) \zeta_{2b}^{a(-n-1)} e\left((n - \ell_b c/2)^2 \frac{x}{2\ell_b}\right). \tag{3.17}$$

On the other hand, for $x \in S_b$, a short calculation reveals that $\left((\zeta_b^{\pm a} e(\pm cx); e\left(\frac{12x}{\ell_b}\right)\right)_\infty = 0$. Using this fact together with (repeated applications of) Proposition 3.1, we find that, formally, letting $y \to 0^+$, $\phi_{a,b,c}(x)$ also equals the expression in (3.17). That is, as $y \to 0^+$, $\phi_{a,b,c}(\tau) \sim \vartheta^*_{a,b,c}(\overline{\tau})$. (This is made more formal as argued in [11] using a Mellin transform.) Thus, the function $\phi_{a,b,c}$ inherits its near modular properties on $\mathbb{Q}$ from $\vartheta^*_{a,b,c}$. Precisely, from Lemma 3.7, we have for $\tau \in \mathbb{H}$ and $M \in \Gamma(2\ell_b)$ that

$$\vartheta^*_{a,b,c}(\overline{\tau}) - (\gamma\overline{\tau} + \delta)^{-\frac{1}{2}} \psi_{\ell_b}^{-1}(M) \vartheta^*_{a,b,c}(M\tau) = \int_{-\frac{\delta}{\gamma}}^{i\infty} \frac{\vartheta_{a,b,c}(u)}{\sqrt{u - \overline{\tau}}} du. \tag{3.18}$$

The function $\vartheta_{a,b,c}(u)$ appearing in the integral in (3.18) is a cusp form. Hence, we may deduce (as argued in [11, 19], for example) that this error to modularity of $\vartheta^*_{a,b,c}(\overline{\tau})$ appearing on the right hand side of (3.18) as $y \to 0^+$ is real-analytic for $x \in S_b$ such that $\gamma x \neq -\delta$. Moreover, a short calculation reveals that the functions $\phi_{a,b,c}(x)$ and $\phi_{a,b,c}(Mx)$ exist (and are finite sums) for $x \in S_b$ satisfying $\gamma x \neq -\delta$. To be precise, with integers a, b, c and rational number $x = h/k \in S_b$ satisfying the given hypotheses, there exists some integer $\widetilde{h}$ such that $h\widetilde{h} \equiv 1 \pmod{\frac{\ell_b k}{12}}$. We define the numbers $\mathcal{N}^{\pm}_{h,k} = \mathcal{N}^{\pm}(a, b, c, h, k)$ to be the unique integer $1 \leq \mathcal{N}^{\pm}_{h,k} \leq \frac{\ell_b k}{12}$ satisfying

$$\mathcal{N}^{\pm}_{h,k} \equiv \widetilde{h}\left(\mp \frac{a\ell_b k}{12b} \mp \frac{ch\ell_b}{12}\right) \left(\operatorname{mod} \frac{\ell_b k}{12}\right). \tag{3.19}$$

We also define the numbers

$$H = H_M(h, k) := \begin{cases} \alpha h + \beta k, & \gamma h + \delta k > 0, \\ -\alpha h - \beta k, & \gamma h + \delta k < 0, \end{cases} \tag{3.20}$$

$$K = K_M(h, k) := |\gamma h + \delta k|. \tag{3.21}$$

Using this notation, we see that the strange functions used to define $\phi_{a,b,c}(\frac{h}{k})$ in (1.10) are in fact finite sums; that is, $F\left(\zeta_b^{\pm a} \zeta_k^{\pm ch}; \zeta_{\ell_b k}^{12h}\right) = F_{\mathcal{N}^{\pm}_{h,k}}\left(\zeta_b^{\pm} \zeta_k^{\pm ch}; \zeta_{\ell_b k}^{12h}\right)$. Similarly, $\phi_{a,b,c}(M\frac{h}{k}) = \phi_{a,b,c}(\frac{H}{K})$, and the strange functions used to define $\phi_{a,b,c}(\frac{H}{K})$ are finite sums. Thus, all functions on the left hand side of (1.13) exist (and are calculated explicitly as polynomials in roots of unity in Corollary 1.11). Combining all of these facts, we may conclude that $\phi_{a,b,c}(x)$ is a quantum modular form of weight 1/2, and transforms as claimed in (1.13) of Theorem 1.8. Using this result, by the hypotheses on $\boldsymbol{a}, b$ and $\boldsymbol{c}$, we also have that the function $\Phi^{(N)}_{\boldsymbol{a},b,\boldsymbol{c}}(x)$ is a quantum modular form of weight 1/2, and transforms as claimed in (1.12) of Theorem 1.8. □

3.3. **Asymptotics.** For $a \in \mathbb{Z}$, and $b \in \mathbb{N}$ satisfying $\gcd(a,b)=1$, we define for $n \in \mathbb{Z}$ the function

$$C_{a,b}(n) := \chi(n)\zeta_{2b}^{a(n-1)}.$$

For ease of notation, for the remainder of this section, we will write $C(n)$ for $C_{a,b}(n)$.

Lemma 3.9. *With hypotheses and notation as above, the following are true.*

(i) *For all $b \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have that $C(n+12b) = C(n)$.*

(ii) *For all $b \in \mathbb{N}\backslash\{6\}$, we have that $\sum_{n=1}^{12b} C(n) = 0$.*

Proof of Lemma 3.9. That (i) holds follows using the definition of $C(n)$. To prove (ii), we first recall that $\gcd(a,b)=1$. We have

$$\begin{aligned}\sum_{n=1}^{12b}\chi(n)\zeta_{2b}^{a(n-1)} &= \sum_{m=0}^{b-1}\zeta_{2b}^{a(12m+1-1)} - \zeta_{2b}^{a(12m+5-1)} - \zeta_{2b}^{a(12m+7-1)} - \zeta_{2b}^{a(12m+11-1)}\\ &= (1-\zeta_b^{2a}-\zeta_b^{3a}+\zeta_b^{5a})\sum_{m=0}^{b-1}\zeta_b^{6ma}.\end{aligned}$$

Let $A_1 = A_1(a,b) := (1-\zeta_b^{2a}-\zeta_b^{3a}+\zeta_b^{5a})$ and let $A_2 = A_2(a,b) := \sum_{m=0}^{b-1}\zeta_b^{6ma}$. For $b \in \{1,2,3\}$, it is not difficult to see that $A_1 = 0$. For $b \in \mathbb{N}\setminus\{1,2,3,6\}$, let $g = g_b := \gcd(b,6)$, let $b' := b/g$, and let $s := 6/g \in \{1,2,3,6\}$. We re-write

$$A_2 = \sum_{m=0}^{b-1}\zeta_b^{6ma} = \sum_{m=0}^{b-1} e^{2\pi i(sma/b')} = g\sum_{m=0}^{b'-1} e^{2\pi i(sma/b')}.$$

Since $\gcd(b',s)=1$, the last sum above equals $g\sum_{r=0}^{b'-1} e^{2\pi i(ra/b')} = 0$. Hence, for all $b \in \mathbb{N}\setminus\{6\}$, we have shown that $\sum_{n=1}^{12b} C(n) = 0$, as claimed. ☐

Proof of Proposition 1.14. By Proposition 3.1, with $q = e^{-24t}, t \in \mathbb{R}^+$, and $w = \zeta_b^a$, we have that

$$e^{-t}(\zeta_b^a e^{-24t}; e^{-24t})_\infty + e^{-t}(\zeta_b^{-a}-1)F(\zeta_b^a; e^{-24t}) = \sum_{n=1}^{\infty} C(n)e^{-n^2 t}. \tag{3.22}$$

Using Lemma 3.9, for all $b \in \mathbb{N}\setminus\{6\}$, by a Proposition (§3, p98) in [11], as $t \to 0^+$, we have that

$$\sum_{n=1}^{\infty} C(n)e^{-n^2t} \sim \sum_{r=0}^{\infty} L(-2r, C)\frac{(-t)^r}{r!},$$

where

$$L(-r, C) = -\frac{(12b)^r}{r+1}\sum_{n=1}^{12b} C(n)B_{r+1}\left(\frac{n}{12b}\right),$$

where $B_r(z)$ denotes the rth Bernoulli polynomial. Considering the infinite product that appears in (3.22),

$$e^{-t}(\zeta_b^a e^{-24t}; e^{-24t})_\infty = \lim_{N\to\infty} e^{-t}(\zeta_b^a e^{-24t}; e^{-24t})_N = \lim_{N\to\infty}(1-\zeta_b^a)^N = 0$$

as $t \to 0^+$, which follows from the fact that $|1-\zeta_b^a|^2 = 2-2\cos(2\pi a/b) < 1$ when $\cos(2\pi a/b) > 1/2$. To prove (1.22), we observe using a similar argument to the one given above, that $F(\zeta_b^a; 1)$ exists for a and b under the hypotheses given. Thus, letting $t \to 0$ in (1.23) shows that $(\zeta_b^{-1} - 1)F(\zeta_b^a; 1)$ equals

$$-\sum_{n=1}^{2b} C(n)\left(\frac{n}{12b} - \frac{1}{2}\right) = -\frac{1}{2b}\sum_{n=1}^{2b} nC(n),$$

where we have used that $B_1(x) := x - 1/2$, and also Lemma 3.9. □

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