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Representation theory of Drinfeld modular forms of level T

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A B S T R A C T

This paper expands upon our results for the arithmetic of Drinfeld modular forms of level T [12] by providing an interpretation from a representation theoretic point of view.
We identify $\mathrm{GL}(2, \mathbb{F}_q)$-modules that arise naturally from the theory of Drinfeld modular forms of level T with classical $\mathrm{GL}(2, \mathbb{F}_q)$-modules. Using the arithmetic of the so-called modified Eisenstein series all isomorphisms are stated explicitly.
In particular, we examine the close connection between Drinfeld modular forms of level T and the theory of symmetric powers of the tautological representation of $\mathrm{GL}(2, \mathbb{F}_q)$ described in previous work [13].

0. Introduction

Drinfeld modular forms are the function field analogue to classical elliptic modular forms. The origins of this theory go back to Drinfeld's central work [5], which has been made more accessible to a wider audience by Deligne and Husemöller [4]. For early work on function field modular forms see for example Goss [9,10] or Gekeler [6].

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However, until now Drinfeld modular forms have not been studied from a representation theoretic point of view. In the present paper we examine the interaction of these two concepts.

Group actions arise naturally from the definition of Drinfeld modular forms. We focus on the case of modular forms of level T (where $A = \mathbb{F}_q[T]$ is the base ring) for two reasons: On the one hand, the structure of the algebra of modular forms of level T is well-known. The principal results are due to Cornelissen, see [3]. I have expanded upon these in [12]. On the other hand, restriction to level T allows us to focus on actions of the group $G = \mathrm{GL}(2, \mathbb{F}_q)$; the modular representation of which has been studied extensively before, see for example [2] and [14].

The basic idea of our approach can be applied to the general case of modular forms of level N. However, both the algebra of modular forms as well as the group action will be more complicated in this case. Therefore, the generalization of the present results is non-trivial.

The aim of the present paper is to determine the G-module structure of naturally occurring modules of Drinfeld modular forms. Specifically, we provide identifications with classical G-modules such as symmetric powers of the tautological two-dimensional module V. In general our results are given explicitly in terms of distinguished bases.

The principal results are Theorem 2.5, which describes the modules of Eisenstein series, Theorem 3.2, in which we state the connection between modular forms of weight k and certain symmetric powers of V, Theorem 4.7 and Theorem 4.14, which describe the smallest non-trivial submodule and the successive quotients of the cusp filtration, respectively, and Theorem 5.3, in which we identify modules of cusp forms with twisted symmetric powers of V.

The present paper is a summary of chapters 7 through 9 of the author's dissertation [11], with some additional content from chapter 10.

Also, it is a continuation of prior articles [12] and [13]. These preliminaries are briefly summarized in section 1 of the present work. Specifically, we fix notation for the objects of interest in the Drinfeld setting and for the necessary tools from modular representation theory.

In the second section we study the action of G on the module of Eisenstein series, which is central to the results of the following sections.

In the third section we show that the module of modular forms of weight k is isomorphic to $\mathrm{Sym}^{kq}(V)$.

The fourth section deals with the cusp filtration. We study relations between filtrations of different weights and describe the G-module structure of the successive quotients using results from [12].

In the final section we apply results from [13] to identify modules of cusp forms with determinant twists of symmetric powers of V.

1. Preliminaries

In this section we provide a brief recap of the Drinfeld situation as well as the necessary concepts from modular representation theory.

On the Drinfeld side we focus on modular forms of level T. That is, we are dealing with the same situation as in [12]. Consider again the following basic notation:

1.1. Notation. Let $q = p^r$ be a prime power. Let $A = \mathbb{F}_q[T]$ be the ring of polynomials over the field with q elements and $K = \mathbb{F}_q(T)$ its field of quotients. On K we fix the normalized absolute value "$|\cdot|$" induced by the degree valuation on A. The completion of K with respect to this absolute value is $K_\infty = \mathbb{F}_q[[T^{-1}]]$, the field of formal Laurent series in T^{-1}. The algebraic closure of K_∞ is not itself complete with respect to the unique continuation of "$|\cdot|$". Its completion $\mathcal{C}_\infty = \widehat{\overline{K}}_\infty$ is, however, again algebraically closed. We call $\Omega = \mathcal{C}_\infty \setminus K_\infty$ the *Drinfeld upper half-plane.*

The *full modular group* $\Gamma(1) = \mathrm{GL}(2, A)$ acts on Ω from the left via Möbius transformation. In this paper, we focus on the principal congruence subgroup of level T

$$\Gamma(T) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a \equiv d \equiv 1 \bmod T,\, b \equiv c \equiv 0 \bmod T \right\}.$$

In particular, by a *modular form* we always mean a Drinfeld modular form of level T, that is, a rigid analytic function $f : \Omega \to \mathcal{C}_\infty$ that is holomorphic on Ω and at the cusps of $\Gamma(T)$, and satisfies

$$f|_{[\gamma]_k}(z) := (cz + d)^{-k} f(\gamma z) = f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(T) \tag{1}$$

for some number $k_0 \in \mathbb{N}$, which is called the *weight* of f.

A more detailed explanation of this concept can be found in [12, Definition 1.3], which is in turn a summary of [8]. Holomorphy at the cusps is defined in terms of a Fourier expansion at each cusp with a vanishing principal part. This allows the definition of the concept of *cusp forms*, with a cusp form of order n being defined as a modular form possessing a Fourier expansion of shape $\sum_{i \geq n} a_n \tau(z)^n$ at each cusp (here, $\tau(z)$ is a uniformizer akin to $e^{2\pi i z}$ in the classical case).

Modular forms of level T form a graded $\mathcal{C}_\infty$-algebra $M = \bigoplus_k M_k$, where M_k is the space of all modular forms of weight k. From the fact that the line bundle of modular forms of weight 1 has degree q (see [6, VII, (6.1)]) it follows that $\dim M_k = kq + 1$. The cusp forms of order n form the subspace $M_k^n \subseteq M_k$. The resulting filtration of finite length

$$M_k = M_k^0 \supseteq M_k^1 \supseteq M_k^2 \supseteq \dots$$

is called the *cusp filtration.*

A very important tool to describe explicit bases are the *Eisenstein series*. As in the classical case, they are given by certain lattice sums

$$E_\nu^{(k)}(z) := \frac{1}{T} \sum_{\substack{(a,b)\in A^2 \\ (a,b)\equiv\nu \bmod T}} \left(\frac{1}{az+b}\right)^k$$

with $\nu \in \mathbb{F}_q^2 \setminus \{(0,0)\}$. Here we use the same notation as in [12], which follows [3]. In particular, omission of k indicates Eisenstein series of weight 1. For a more in-depth background see for example [10].

Cornelissen shows that for $k \in \mathbb{N}$ the Eisenstein series

$$\begin{aligned} E_u^{(k)} &:= E_{(1,u)}^{(k)}, \quad u \in \mathbb{F}_q, \\ E_\infty^{(k)} &:= E_{(0,1)}^{(k)}, \end{aligned}$$

form a basis of Eis_k, the span of the Eisenstein series of weight k [3, IV, Proposition 1.1]. At the same time, he proves the direct sum decomposition of vector spaces $M_k = \mathrm{Eis}_k \oplus M_k^1$. In particular, by a simple dimension argument it follows that there are no non-trivial cusp forms of weight 1 in the present situation.

In addition, Cornelissen shows in [3, III, Theorem 3.4] that the algebra M is generated by modular forms of weight 1, that is, by Eisenstein series. The ideal of relations between products of Eisenstein series is described explicitly.

In [12], I have restated these relations and extended them to higher weights by using *modified Eisenstein series*. These are special linear combinations of the ordinary Eisenstein series based on work of Cornelissen (in particular the proof of his Theorem 3.4 mentioned above).

1.2. Definition. For $k \in \mathbb{N}$ put

$$\begin{aligned} \mathcal{E}_i^{(k)} &:= \sum_{u\in\mathbb{F}_q} u^i E_u^{(k)}, \qquad 0 \le i \le q-1, \\ \mathcal{E}_\infty^{(k)} &:= \sum_{u\in\mathbb{F}_q} u^k E_u^{(k)} + E_\infty^{(k)}, \end{aligned}$$

with the convention $0^0 = 1$. Again we may omit k for weight $k = 1$ and write additionally $\mathcal{E}_q := \mathcal{E}_\infty^{(1)}$.

With this notation, the ideal of relations is generated by the expressions

$$\mathcal{E}_i\mathcal{E}_j - \mathcal{E}_{i-1}\mathcal{E}_{j+1} \quad \text{for } 1 \le i \le j \le q-1 \tag{2}$$

(the corresponding restatement of Cornelissen's result is given in [12, Theorem 2.3]).

Another relation important to our calculations is

$$\mathcal{E}_i E_0^k = (-1)^k \mathcal{E}_{i-k} E_\infty^k, \quad 0 \le k \le i \le q, \tag{3}$$

see [12, Lemma 2.7].

We call a product of modified Eisenstein series of weight 1 a *standard monomial* if at most one factor $\mathcal{E}_b$ with $b \neq 0, q$ occurs. The standard monomials of weight k form a basis of M_k [12, Corollary 2.6].

To describe the structure of the cusp filtration we fix the unique decomposition

$$k = \mathfrak{k} + \widehat{\mathfrak{k}}(q+1) \quad \text{with} \quad 1 \le \mathfrak{k} \le q+1.$$

We can show that the length of the cusp filtration is $\mathfrak{m}(k) := \left\lfloor \frac{kq}{q+1} \right\rfloor$, see [12, Corollary 4.10]. Also, we have constructed a basis of M_k^1 that is compatible with this filtration in the sense that the intersection of the basis with a subspace of the filtration produces a basis of this subspace.

1.3. Theorem. *Let $k \ge 2$. For each $1 \le i \le \mathfrak{m}(k) - 1$ we define a set $\mathcal{B}_k^i$ consisting of the modular forms*

$$\begin{aligned} \mathcal{F}_b^{(i,k)} &:= \mathcal{E}_0^{k-i-1} \mathcal{E}_b E_\infty^i, \quad 0 \le b \le q-1, \\ \mathcal{F}_\infty^{(i,k)} &:= (-1)^i \mathcal{E}_q^{k-i} E_0^i. \end{aligned}$$

Let further $\mathcal{B}_k^{\mathfrak{m}(k)}$ be the set with elements

$$\mathcal{F}_b^{(\mathfrak{m}(k),k)} := \mathcal{E}_0^{\hat{\mathfrak{k}}} \mathcal{E}_b E_\infty^{\mathfrak{m}(k)}, \quad 0 \le b \le q+1-\mathfrak{k}.$$

Finally, for $1 \le i \le \mathfrak{m}(k)$ we define

$$\mathcal{B}_k^{i,+} := \bigcup_{i \le j \le \mathfrak{m}(k)} \mathcal{B}_k^j \qquad \textit{(disjoint union)}.$$

Then $\mathcal{B}_k^{i,+}$ is a basis of M_k^i.

For the proof and further properties, see [12, Section 4].

1.4. Notation. To obtain a more uniform notation we write

$$\mathcal{B}_1^0 := \{\mathcal{F}_b^{(0,1)} := \mathcal{E}_b \mid 0 \le b \le q\}$$

for the basis of $M_1^{\mathfrak{m}(1)} = M_1 = \mathrm{Eis}_1$ that consists of the modified Eisenstein series of weight 1.

In the present paper, we are particularly interested in the following group action: The definition of $f|_{[\gamma]_k}$ in (1) induces a *right* group action of the full modular group on the algebra M of modular forms of level T.

The behavior of the ordinary Eisenstein series under this group action is known to be as follows:

1.5. Lemma *([3, I, (6.3)]). Let $k \in \mathbb{N}$. Let $\gamma \in \Gamma(1)$. We have*

$$E_{\nu}^{(k)}|_{[\gamma]_k}(z) = E_{\nu\gamma}^{(k)}(z)$$

for all $\nu \in \mathbb{F}_q^2 \setminus \{(0,0)\}$. Here $\nu\gamma$ denotes the usual (row-)vector matrix product.

By definition the subgroup $\Gamma(T)$ acts trivially on M, so that we can restrict ourselves to the action of the quotient group

$$G := \mathrm{GL}(2, \mathbb{F}_q) \cong \Gamma(T)\backslash\Gamma(1).$$

In fact, we focus on the corresponding *left* action induced by

$$\gamma f = f|_{[\gamma^{-1}]_k} \tag{4}$$

for $\gamma \in G$ and $f \in M_k$.

One sees easily that subspaces of shape M_k, Eis_k, and M_k^n are closed under the action of G.

In order to study the arising structures, we apply modular representation theory and consider the concept of a G-module. Here, we use the same language as in [13]. That is, we identify G-modules (modules for the group algebra $\mathcal{C}_\infty[G]$) and linear representations of G over $\mathcal{C}_\infty$.

Note that according to [13, Remark to Notation 1.3] all results can be transferred to the present situation despite the fact that we are using the base field $\mathcal{C}_\infty$ in place of $\overline{\mathbb{F}}_q$.

We describe the structure of naturally occurring G-modules by identifying them with classical modules. For this application the tautological two-dimensional module V and its symmetric powers $\mathrm{Sym}^n(V)$, $n \in \mathbb{N}_0$, are of particular interest to us.

As a vector space $V = \mathcal{C}_\infty^2$ and G acts from the left by matrix multiplication. The symmetric algebra of V admits a graduation $\mathrm{Sym}(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V)$. The natural action of G on V gives rise to G-module structures on the symmetric algebra $\mathrm{Sym}(V)$ and the symmetric powers $\mathrm{Sym}^n(V)$ of V.

If we write (X, Y) for the standard basis of V, the monomials $(X^{n-i}Y^i \mid 0 \leq i \leq n)$ form a basis of $\mathrm{Sym}^n(V)$. With regard to the following calculations, we observe that a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ acts on a monomial $X^{n-i}Y^i \in \mathrm{Sym}^n(V)$, $0 \leq i \leq n$, via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} X^{n-i}Y^i = (aX + cY)^{n-i}(bX + dY)^i. \tag{5}$$

Also, we note that the group G is generated by the matrices of shape $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where a and t pass through $\mathbb{F}_q^\times$.

Furthermore we will encounter determinant twists of G-modules, by which we mean modules of shape $M \otimes (\det)^n$ where $(\det)^n$ is the n-th power of the determinant character. That is, as a vector space $(\det)^n = \mathcal{C}_\infty$ and the action of a matrix $\gamma \in G$ is given by $\gamma x = (\det \gamma)^n x$ for $x \in \mathcal{C}_\infty$.

See [13, Section 1] for additional details.

Symmetric powers are related to the following class of modules, which also plays an important role in our applications. It has been studied extensively by Bardoe and Sin [1], albeit in a slightly different context (cf. [13, Remark to Notation 2.1]).

1.6. Notation. Let $B \leq G$ be the standard Borel subgroup of upper triangular matrices. For $1 \leq \delta \leq q-1$ define the induced G-module $N[\delta] := \mathrm{Ind}_B^G(\chi_\delta)$, where χ_δ is the character of B that acts by

$$\chi_\delta \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = d^\delta.$$

We have $\dim N[\delta] = q+1$ for all δ and write $N[\delta, n]$ for $N[\delta] \otimes (\det)^n$.

In the corresponding definition in [13] we chose the analytic implementation of the induced representation. However, for the applications in the present work it is sufficient to know that $N[\delta]$ has a basis $(f_i^{(\delta)}, f_\infty^{(\delta)} \mid 0 \leq i \leq q-1)$ (see [13, Lemma 2.4]), which satisfies the following transformation properties:

1.7. Lemma *([13, Lemma 2.5]). Let $1 \leq \delta \leq q-1$. The generators of G act on the basis $(f_i^{(\delta)}, f_\infty^{(\delta)} \mid 0 \leq i \leq q-1)$ of $N[\delta]$ by:*

$$\begin{aligned}
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) f_i^{(\delta)} &= a^{\delta - i} f_i^{(\delta)}, \quad 0 \leq i \leq q-1, \\
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) f_\infty^{(\delta)} &= f_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) f_i^{(\delta)} &= \sum_{j=0}^{i} \tbinom{i}{j} t^{i-j} f_j^{(\delta)}, \quad 0 \leq i \leq q-1, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) f_\infty^{(\delta)} &= \sum_{j=0}^{\delta-1} \tbinom{\delta}{j} t^{\delta-j} f_j^{(\delta)} + f_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_i^{(\delta)} &= f_{\delta-i}^{(\delta)}, \quad 1 \leq i \leq \delta - 1, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_i^{(\delta)} &= f_{q-1+\delta-i}^{(\delta)}, \quad \delta \leq i \leq q-1, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_0^{(\delta)} &= f_\infty^{(\delta)}, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) f_\infty^{(\delta)} &= f_0^{(\delta)}.
\end{aligned}$$

Thanks to results from [1], the structure of a module of this class is understood entirely. In particular, we can describe their submodule lattices and their composition factors. A version of their results adjusted to the present notation can be found in [13, Theorem 2.17].

One important original result from [13] is the following G-module filtration for symmetric powers.

1.8. Theorem. *Let $n \in \mathbb{N}$ with unique decomposition $n = \mathfrak{n} + \widehat{\mathfrak{n}}(q+1)$, $0 \le \mathfrak{n} \le q$. Then $\operatorname{Sym}^n(V)$ admits a filtration of G-submodules*

$$\{0\} \subsetneq L^{(\hat{\mathfrak{n}},n)} \subsetneq L^{(\hat{\mathfrak{n}}-1,n)} \subsetneq \cdots \subsetneq L^{(1,n)} \subsetneq L^{(0,n)} = \operatorname{Sym}^n(V),$$

where the inclusion maps are induced by multiplication with the eigenvector $XY^q - X^qY$ of $\operatorname{Sym}(V)$ and

$$L^{(i,n)} \cong \operatorname{Sym}^{n-i(q+1)}(V) \otimes (\det)^i \quad \text{for } 0 \le i \le \widehat{\mathfrak{n}}.$$

In particular, the submodule $L(n) := L^{(1,n)} \subseteq \operatorname{Sym}^n(V)$ is the image of

$$\operatorname{Sym}^{n-(q+1)}(V) \otimes (\det)^1$$

under multiplication with $XY^q - X^qY$.

For $0 \le i \le \widehat{\mathfrak{n}} - 1$ the successive quotients satisfy

$$L^{(i,n)}/L^{(i+1,n)} \cong N[n-2i, i].$$

Proof. The structure of the filtration is described in [13, Theorem 3.6]. For the role of the eigenvector $XY^q - X^qY$ see also Lemma 3.1 from the quoted article. □

2. The module of Eisenstein series

Since the algebra M of Drinfeld modular forms is generated by Eisenstein series, we begin with the modules Eis_k, $k \in \mathbb{N}$. Applying Lemma 1.5, we can describe explicitly the action of the generators of G on the ordinary Eisenstein series.

2.1. Lemma. *Let $k \in \mathbb{N}$. The generators of G act on the ordinary Eisenstein series as follows:*

$$\begin{aligned}
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) E_u^{(k)} &= a^k E_{ua}^{(k)}, \quad u \in \mathbb{F}_q, \\
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) E_\infty^{(k)} &= E_\infty^{(k)}, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) E_u^{(k)} &= E_{u-t}^{(k)}, \quad u \in \mathbb{F}_q, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) E_\infty^{(k)} &= E_\infty^{(k)},
\end{aligned}$$

$$\begin{aligned}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)E_u^{(k)}&=u^{-k}E_{u^{-1}}^{(k)},\quad u\in\mathbb{F}_q^\times,\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)E_0^{(k)}&=E_\infty^{(k)},\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)E_\infty^{(k)}&=E_0^{(k)}.\end{aligned}$$

Furthermore,

$$\begin{aligned}\left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)E_u^{(k)}&=a^kE_u^{(k)},\quad u\in\mathbb{F}_q,\\ \left(\begin{smallmatrix}a&0\\0&a\end{smallmatrix}\right)E_\infty^{(k)}&=a^kE_\infty^{(k)}.\end{aligned}$$

Proof. The fact that a scalar matrix acts by multiplication with a constant follows from the fact that scalar matrices act trivially on Ω. Using Lemma 1.5 we obtain for $u\in\mathbb{F}_q$

$$\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)E_u^{(k)}=E^{(k)}_{(1,u)\left(\begin{smallmatrix}a^{-1}&0\\0&1\end{smallmatrix}\right)}=E^{(k)}_{(a^{-1},u)}=E^{(k)}_{(1,ua)\left(\begin{smallmatrix}a^{-1}&0\\0&a^{-1}\end{smallmatrix}\right)}=a^kE_{ua}^{(k)}.$$

Note the additional inverse in the definition of the left action of G on M, cf. (4). The remaining formulae are obtained analogously. □

As a consequence of Definition 1.2, the corresponding result for the modified Eisenstein series is now immediate.

2.2. Lemma. *Let $k\in\mathbb{N}$. The modified Eisenstein series possess the following transformation properties under the generators of G:*

$$\begin{aligned}\left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\mathcal{E}_i^{(k)}&=a^{k-i}\mathcal{E}_i^{(k)},\quad 0\le i\le q-1,\\ \left(\begin{smallmatrix}a&0\\0&1\end{smallmatrix}\right)\mathcal{E}_\infty^{(k)}&=\mathcal{E}_\infty^{(k)},\\ \left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\mathcal{E}_i^{(k)}&=\sum_{j=0}^{i}\tbinom{i}{j}t^{i-j}\mathcal{E}_j^{(k)},\quad 0\le i\le q-1,\\ \left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\mathcal{E}_\infty^{(k)}&=\sum_{j=0}^{[k]-1}\tbinom{[k]}{j}t^{[k]-j}\mathcal{E}_j^{(k)}+\mathcal{E}_\infty^{(k)},\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{E}_i^{(k)}&=\mathcal{E}_{[k]-i}^{(k)},\quad 1\le i\le[k]-1,\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{E}_i^{(k)}&=\mathcal{E}_{q-1+[k]-i}^{(k)},\quad [k]\le i\le q-1,\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{E}_0^{(k)}&=\mathcal{E}_\infty^{(k)},\\ \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\mathcal{E}_\infty^{(k)}&=\mathcal{E}_0^{(k)}.\end{aligned}$$

Here, $[k]$ denotes the representative of k modulo $q-1$ in $\{1,\dots,q-1\}$.

The modified Eisenstein series of weight 1 are of particular interest for our applications:

2.3. Corollary. *For* $0 \le i \le q$ *we have*

$$\begin{aligned} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_i &= a^{q-i}\mathcal{E}_i, \\ \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_i &= \sum_{j=0}^{i} \tbinom{i}{j} t^{i-j}\mathcal{E}_j, \\ \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{E}_i &= \mathcal{E}_{q-i}. \end{aligned}$$

In particular, we observe

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_q = t\mathcal{E}_0 + \mathcal{E}_q.$$

Not only do we see that the structure of Eis_k depends only on $k \bmod q-1$, but we can identify the modules of Eisenstein series with a class of well-known G-modules.

Before we state this result, let us briefly recap some module theoretic properties:

2.4. Definition. Let W be a G-module.

1. We say W is *multiplicity free* if all composition factors in a composition series of W occur with multiplicity one.
2. The *socle* of W is the sum of all simple submodules of W. It is the largest semi-simple submodule of W.
3. The *head* of W is the largest semi-simple quotient of W.

2.5. Theorem. *Let* $k \in \mathbb{N}$. *The map* $\mathrm{Eis}_k \to N[k]$ *given by*

$$\begin{aligned} \mathcal{E}_i^{(k)} &\mapsto f_i^{([k])}, \quad 0 \le i \le q-1, \\ \mathcal{E}_\infty^{(k)} &\mapsto f_\infty^{([k])} \end{aligned}$$

and linear extension is a G*-isomorphism.*

1. *For* $k \not\equiv 0 \bmod q-1$ *the submodule lattice and the composition factors of* Eis_k *can be parametrized as in [13, Theorem 2.17]. In particular, the module* Eis_k *is multiplicity free and its socle and head are both simple* G*-modules.*
2. *If* k *is congruent to* 0 *modulo* $q-1$, *the module* Eis_k *is semi-simple and isomorphic to the direct sum* $\mathcal{C}_\infty \oplus \mathrm{Sym}^{q-1}(V)$ *of simple* G*-modules. The one-dimensional submodule is generated by*

$$\mathcal{E}_0^{(k)} - \mathcal{E}_{q-1}^{(k)} + \mathcal{E}_\infty^{(k)}.$$

Proof. The given map is obviously an isomorphism of vector spaces. Its G-equivariance follows immediately, if we compare the transformation properties stated in Lemma 1.7 and Lemma 2.2.

The structure of the modules Eis_k can be read off from the main results for the modules $N[\delta]$ given in [13], specifically Theorem 2.17 and Proposition 2.18 (as a modification of [1, Theorem C]). □

Remark. Frobenius reciprocity (cf. [13, Proposition 2.6]) implies that the isomorphism from Theorem 2.5 is unique up to a scalar factor in $\mathcal{C}_\infty$ for $k \not\equiv 0 \bmod q-1$.

In the second case, the isomorphism splits into a pair of two isomorphisms between the respective direct summands. Both isomorphisms in this pair are again unique up to a scalar factor.

Again, we take a closer look at the situation for weight 1 where $\mathrm{Eis}_1 = M_1$. Corollary 2.3 implies immediately the existence of an isomorphism of G-modules $\mathrm{Eis}_1 \cong \mathrm{Sym}^q(V)$ (the corresponding result for $N[1]$ is given in [13, Proposition 2.19]).

In this special case, the G-module structure can be described explicitly:

2.6. Theorem.

1. *If q is 2, the module $\mathrm{Eis}_1 \cong \mathrm{Sym}^2(V)$ admits a direct sum decomposition*

$$\mathrm{Eis}_1 = \underbrace{\langle \mathcal{E}_0, \mathcal{E}_2 \rangle}_{\cong V} \oplus \underbrace{\langle \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 \rangle}_{\cong \mathcal{C}_\infty}.$$

2. *For $q > 2$ the G-module Eis_1 is multiplicity free and uniserial (that is, its submodules are totally ordered under inclusion). Its $r+1$ many non-zero submodules are given by*

$$U_i := \left\langle \mathcal{E}_j \mid 0 \le j \le q,\, j \equiv 0 \bmod p^{r-i} \right\rangle, \quad 0 \le i \le r.$$

In particular, $\dim U_i = p^i + 1$. The unique composition series of Eis_1 is

$$\{0\} \subsetneq U_0 \subsetneq U_1 \subsetneq \ldots \subsetneq U_r = \mathrm{Eis}_1$$

with composition factors

$$U_0 \cong V$$

and

$$U_i/U_{i-1} \cong \mathfrak{S}\left(p^r - 2p^{r-i}, p^{r-i}\right) \quad \text{for } 1 \le i \le r.$$

Here, $\mathfrak{S}(m,\mu)$ is the unique simple submodule of $\mathrm{Sym}^m(V)\otimes(\det)^\mu$ *(we use the same classification of the simple G-modules as described in [13, Theorem 1.9]).*

Proof. The statement for $q=2$ follows immediately from the second case of Theorem 2.5.

Let therefore $q>2$. In order to prove that each U_i is a submodule of Eis_1 we only have to show that it is closed under the action of G. Thus, for $0\le i\le r$ consider $\mathcal{E}_j\in U_i$ with $j\equiv 0 \bmod p^{r-i}$. We observe $\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)E_j = a^{q-j}\mathcal{E}_j\in U_i$ and $\left(\begin{smallmatrix} 0 & 1\\ 1 & 0\end{smallmatrix}\right)\mathcal{E}_j=\mathcal{E}_{q-j}\in U_i$. Finally, we get

$$\left(\begin{smallmatrix} 1 & t\\ 0 & 1\end{smallmatrix}\right)\mathcal{E}_j=\sum_{l=0}^{j}\tbinom{j}{l}t^{j-l}\mathcal{E}_l.$$

Here, $\binom{j}{l}$ vanishes in characteristic p for $l\not\equiv 0 \bmod p^{r-i}$ due to the congruence satisfied by j (an immediate application of the Lucas congruence). Consequently, we have $\left(\begin{smallmatrix} 1 & t\\ 0 & 1\end{smallmatrix}\right)\mathcal{E}_j\in U_i$.

The remaining statements concerning the composition series are a direct consequence of the special structure of the parametrizing sets in [13, Theorem 2.17] for the case $\delta=1$. $\square$

The submodule lattice of $\mathrm{Eis}_1=M_1$ for $q>2$ gives rise to a second submodule filtration on M_k, $k\ge 2$, besides the cusp filtration: From Cornelissen's result [3, III, Theorem 3.4] we know that there is a surjective map $\mathrm{Sym}^k(M_1)\twoheadrightarrow M_k$. We can now study the images of $\mathrm{Sym}^k(U_i)$ in M_k under this map for each i.

In general, the relation between this filtration and the cusp filtration remains an open question. Initial results can be obtained by means of a suitable interpretation of results presented in [12]. For example, Theorem 3.6 from the cited paper implies that $\mathrm{Sym}^k(U_0)$ is a submodule of Eis_k for $k\le q$. Some further results for the case $k\le q$ can be found in [11, Section 8.2].

3. The module M_k

Next, we use the transformation properties of Eisenstein series of weight 1 to study a basis of M_k under the action of G. We choose the basis consisting of all standard monomials of weight $k\in\mathbb{N}$ as described in section 1. Note that the case $k=0$ is trivial, since $M_0=\mathcal{C}_\infty$.

3.1. Lemma. *Let $k\in\mathbb{N}$. For $0\le b\le q-1$ and $0\le i\le k-1$ the action of the generators of G on the standard monomials of weight k is given by*

$$\left(\begin{smallmatrix} a & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i=a^{(k-i)q-b}\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i,$$

$$\left(\begin{smallmatrix} 1 & t\\ 0 & 1\end{smallmatrix}\right)\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i=\sum_{j=0}^{b}\sum_{l=0}^{i}\tbinom{b}{j}\tbinom{i}{l}t^{i+b-l-j}\mathcal{E}_0^{k-1-l}\mathcal{E}_j\mathcal{E}_q^l,$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{E}_0^{k-1-i} \mathcal{E}_b \mathcal{E}_q^i = \mathcal{E}_0^i \mathcal{E}_{q-b} \mathcal{E}_\infty^{k-1-i}.$$

Further we have

$$\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_q^k = \mathcal{E}_q^k,$$

$$\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_q^k = \sum_{l=0}^{k} \binom{k}{l} t^{k-l} \mathcal{E}_0^{k-l} \mathcal{E}_q^l,$$

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{E}_q^k = \mathcal{E}_0^k.$$

Proof. Since the group action of G is compatible with the multiplicative structure on M, the above formulae are obtained by applying the transformation properties from Corollary 2.3 to the individual Eisenstein factors of each standard monomial.

Consider for example $0 \leq b \leq q-1$ and $0 \leq i \leq k-1$. In this case we observe for generators of shape $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$

$$\begin{aligned} \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_0^{k-1-i} \mathcal{E}_b \mathcal{E}_q^i &= \left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_0\right)^{k-1-i} \left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_b\right) \left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{E}_q\right)^i \\ &= a^{(k-1-i)q} \mathcal{E}_0^{k-1-i} a^{q-b} \mathcal{E}_b \mathcal{E}_q^i \\ &= a^{(k-i)q-b} \mathcal{E}_0^{k-1-i} \mathcal{E}_b \mathcal{E}_q^i. \end{aligned}$$

The remaining cases follow analogously. $\square$

This allows for the following identification:

3.2. Theorem. *Let $k \in \mathbb{N}$. There is an isomorphism of G-modules $\Phi_k : M_k \xrightarrow{\cong} \mathrm{Sym}^{kq}(V)$ given by*

$$\begin{aligned} \mathcal{E}_0^{k-1-i} \mathcal{E}_b \mathcal{E}_q^i &\mapsto X^{kq-iq-b} Y^{iq+b}, \quad 0 \leq b \leq q-1, 0 \leq i \leq k-1, \\ \mathcal{E}_q^k &\mapsto Y^{kq} \end{aligned}$$

and linear extension.

Taken together, these maps induce an isomorphism of graded $\mathcal{C}_\infty$-algebras

$$\Phi : M = \bigoplus_{k \geq 0} M_k \xrightarrow{\cong} \bigoplus_{k \geq 0} \mathrm{Sym}^{kq}(V) \subsetneq \mathrm{Sym}(V).$$

Proof. Let us first verify the stated G-isomorphism for all $k \in \mathbb{N}$. Obviously, the map Φ_k is an isomorphism of vector spaces. In order to prove its G-equivariance we compare Lemma 3.1 to the transformation properties of monomials in $\mathrm{Sym}^{kq}(V)$ that can be obtained easily from (5).

For generators of types $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ the verification is straightforward.

For generators of type $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ consider first the image under Φ_k of a standard monomial $\mathcal{E}_0^{k-1-i} \mathcal{E}_b \mathcal{E}_q^i$ with $0 \leq b \leq q-1$ and $0 \leq i \leq k-1$. We get

$$\Phi_k\left(\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)(\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i)\right)=\sum_{j=0}^{b}\sum_{l=0}^{i}\tbinom{b}{j}\tbinom{i}{l}t^{i+b-l-j}\Phi_k(\mathcal{E}_0^{k-1-l}\mathcal{E}_j\mathcal{E}_q^l)$$
$$=\sum_{j=0}^{b}\sum_{l=0}^{i}\tbinom{b}{j}\tbinom{i}{l}t^{iq+b-lq-j}X^{kq-lq-j}Y^{lq+j}.$$

It is a well-known fact for binomial coefficients in characteristic p that, with b, j as above, we have

$$\binom{b}{j}\binom{i}{l}=\binom{iq+b}{lq+j}$$

(as a special case of the Lucas congruence).

We may extend the summation over j to all $0\le j\le q-1$, since the additional terms are zero. This allows us to write:

$$\Phi_k\left(\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)(\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i)\right)=\sum_{j=0}^{q-1}\sum_{l=0}^{i}\tbinom{iq+b}{lq+j}t^{iq+b-lq-j}X^{kq-lq-j}Y^{lq+j}$$
$$=\sum_{n=0}^{iq+b}\tbinom{iq+b}{n}t^{iq+b-n}X^{kq-n}Y^{n}$$
$$=\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)X^{kq-iq-b}Y^{iq+b}$$
$$=\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\Phi_k(\mathcal{E}_0^{k-1-i}\mathcal{E}_b\mathcal{E}_q^i).$$

Analogously, we use the Lucas congruence to show that

$$\Phi_k\left(\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\mathcal{E}_q^k\right)=\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\Phi_k\left(\mathcal{E}_q^k\right).$$

In order to obtain an isomorphism of graded algebras, we have to verify that the isomorphisms for different weights are compatible with the multiplicative structure on M. To this end, we consider two standard monomials P and Q of weights k and l, respectively. We can easily express PQ as a standard monomial of weight $k+l$ and see, after a straightforward calculation, that indeed $\Phi_k(P)\Phi_l(Q)=\Phi_{k+l}(PQ)$. $\square$

Remark.

1. For $k=1$ the map $\Phi_1 : M_1 = \mathrm{Eis}_1 \to \mathrm{Sym}^q(V)$ describes the isomorphism already mentioned in the motivation preceding Theorem 2.6.
2. We can now use results from the representation theory of symmetric powers of the tautological G-module to describe Drinfeld modular forms. Note however that for $n > q$ the structure of $\mathrm{Sym}^n(V)$ is not completely established in the literature. For example, Proposition 5.4 below provides that the multiplicities of composition factors of modules of shape M_k^n can be determined using Algorithm 4.14 from [13].

Conversely, we can transfer concepts and known results from the Drinfeld side to the representation theory of symmetric powers of V. See for example Theorem 5.5 at the end of the present article, which succinctly answers a question which remained open in the broader context of [13].

4. The quotients of the cusp filtration

Next, we consider the module M_k^1 of cusp forms. Our goal is to describe the successive quotients of the cusp filtration of M_k^1.

We have already studied M_k^1 as a vector space in [12, Section 4] and will be using the corresponding notation. In particular:

4.1. Notation. For weight $k \in \mathbb{N}$ we write

$$k = \mathfrak{k} + \widehat{\mathfrak{k}}(q+1) \quad \text{with } 1 \leq \mathfrak{k} \leq q+1$$

and put

$$\mathfrak{m}(k) = \left\lfloor \frac{kq}{q+1} \right\rfloor = \mathfrak{k} - 1 + \widehat{\mathfrak{k}}q.$$

Then $\mathfrak{m}(k)$ is the length of the cusp filtration (note that M_k^1 is non-trivial only for $k \geq 2$).

For ease of notation we simply call $M_k^{\mathfrak{m}(k)}$ the *smallest filtration module.*

We are going to give explicit descriptions of the G-module structure of the successive quotients in terms of the elements of the sets $\mathcal{B}_k^i$ (recalled in Theorem 1.3).

To begin, we consider the following eigenvector:

4.2. Lemma. *The modular form $\mathcal{E}_0 E_\infty^q \in M_{q+1}^q$ is an eigenvector with character* $(\det)^1$ *under the action of G.*

In particular, we have $M_{q+1}^q \cong (\det)^1$ as an isomorphism of G-modules.

Proof. Both statements are equivalent, since M_{q+1}^q is spanned by $\mathcal{E}_0 E_\infty^q$ according to Theorem 1.3.

The transformation properties of $\mathcal{E}_0 E_\infty^q$ under the generators of G can be obtained by means of a straightforward calculation. For the generator $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ note that $\mathcal{E}_q E_0^q = (-1)^q \mathcal{E}_0 E_\infty^q$ according to relation (3). $\square$

Remark. At this point we recall the function h, which is often encountered in the context of Drinfeld modular forms (for an example, see [7]). If we interpret h as a modular form for $\Gamma(T)$, it turns out to be a q-fold cusp form of weight $q+1$. Therefore, the above cusp form $\mathcal{E}_0 E_\infty^q$ agrees with this modular form h up to a constant factor.

Obviously, multiplication with $\mathcal{E}_0 E_\infty^q$ allows for the identification of filtration modules of different weights as long as we compensate for the additional determinant twist.

4.3. Lemma. *Let $k \in \mathbb{N}$ and $0 \le i \le \mathfrak{m}(k)$. Multiplication with $\mathcal{E}_0 E_\infty^q$ describes a G-isomorphism*

$$M_k^i \otimes (\det)^1 \xrightarrow{\cong} M_{k+q+1}^{i+q}. \qquad \square$$

In fact, this multiplication map is compatible with the constituent sets of the respective bases in the following sense:

4.4. Lemma. *Let $k \in \mathbb{N}$. Let further*

$$1 \le i \le \mathfrak{m}(k) \quad \textit{for } k \ge 2$$

or

$$i = \mathfrak{m}(k) = 0 \quad \textit{for } k = 1.$$

Then multiplication with $\mathcal{E}_0 E_\infty^q$ defines a bijective map

$$\mathcal{B}_k^i \to \mathcal{B}_{k+q+1}^{i+q}.$$

Specifically, for $i = \mathfrak{m}(k)$ we get

$$\mathcal{E}_0 E_\infty^q \mathcal{F}_b^{(\mathfrak{m}(k),k)} = \mathcal{F}_b^{(\mathfrak{m}(k+q+1),k+q+1)} \quad \textit{for } 0 \le b \le q+1-\mathfrak{k}.$$

For $1 \le i \le \mathfrak{m}(k) - 1$ we have

$$\mathcal{E}_0 E_\infty^q \mathcal{F}_b^{(i,k)} = \mathcal{F}_b^{(i+q,k+q+1)} \quad \textit{for } 0 \le b \le q-1$$

and

$$\mathcal{E}_0 E_\infty^q \mathcal{F}_\infty^{(i,k)} = \mathcal{F}_\infty^{(i+q,k+q+1)}.$$

Proof. First we consider the special case where $i = \mathfrak{m}(k)$. It is easy to see that $\mathfrak{m}(k)+q = \mathfrak{m}(k+q+1)$. Also, trivially $k+q+1$ is congruent to $\mathfrak{k}$ modulo $q+1$.

This means that elements of both $\mathcal{B}_k^{\mathfrak{m}(k)}$ and $\mathcal{B}_{k+q+1}^{\mathfrak{m}(k+q+1)}$ are parametrized by the same set of indices $0 \le b \le q+1-\mathfrak{k}$. For such b the stated identity $\mathcal{E}_0 E_\infty^q \mathcal{F}_b^{(\mathfrak{m}(k),k)} = \mathcal{F}_b^{(\mathfrak{m}(k+q+1),k+q+1)}$ follows immediately from the definition of the bases in Theorem 1.3 and the remark covering $k = 1$.

In the remaining cases, the equality of $\mathcal{E}_0 E_\infty^q \mathcal{F}_b^{(i,k)}$ and $\mathcal{F}_b^{(i+q,k+q+1)}$ for $0 \le b \le q-1$ is again trivial. For the remaining statement we apply relation (3) and observe

$$\begin{aligned}\mathcal{E}_0 E_\infty^q \mathcal{F}_\infty^{(i,k)} &= (-1)^q \mathcal{E}_q E_0^q \mathcal{F}_\infty^{(i,k)} \\ &= (-1)^q \mathcal{E}_q E_0^q (-1)^i \mathcal{E}_q^{k-i} E_0^i \\ &= (-1)^{i+q} \mathcal{E}_q^{k+q+1-(i+q)-1} E_0^{i+q} \\ &= \mathcal{F}_\infty^{(i+q,k+q+1)}. \qquad \square\end{aligned}$$

Remark. Note in the above lemma that $i = 0$ is only allowed for weight $k = 1$. The reason for this is that M_1 is mapped onto M_{q+2}^q, the smallest non-trivial submodule of the cusp filtration for weight $q + 2$. Due to the special shape of the basis elements of the smallest filtration module, $\mathcal{B}_1^0$ and $\mathcal{B}_{q+2}^q$ can be identified by means of multiplication with $\mathcal{E}_0 E_\infty^q$.

This is not true for $k \geq 2$, since then q is strictly less than $\mathfrak{m}(k + q + 1)$. In this case, the set $\mathcal{B}_{k+q+1}^q$ is the image of the set $\{\mathcal{E}_0^{k-1}\mathcal{E}_b \mid 0 \leq b \leq q - 1\} \cup \{\mathcal{E}_q^k\}$ under multiplication with the distinguished eigenvector. While this set complements $\mathcal{B}_k^{1,+}$ to form a basis of M_k, it is not itself a basis of Eis_k as a G-module; it is not closed under the action of G. We may apply results for Eisenstein series to cover the case $\mathcal{B}_{k+q+1}^q$ only if we focus solely on the successive quotients of the cusp filtration.

If we changed the definition of the basis elements to be compatible with the direct sum decomposition $M_k = \mathrm{Eis}_k \oplus M_k^1$, the modified basis would no longer have the desired behavior at the cusps. This behavior, however, was essential for our work in [12]. Additionally, we will see in the next section that the favorable behavior at the cusps is an expression of the close relation to symmetric powers.

The situation described so far can be visualized as follows:

$$\begin{array}{ccccccccccc} M_{k+q+1}^1 & \supseteq & \cdots & \supseteq & M_{k+q+1}^q & \supseteq & M_{k+q+1}^{q+1} & \supseteq & \cdots & \supseteq & M_{k+q+1}^{\mathfrak{m}(k+q+1)} \\ & & & & \cong \big\uparrow \cdot\mathcal{E}_0E_\infty^q & & \cong \big\uparrow \cdot\mathcal{E}_0E_\infty^q & & & & \cong \big\uparrow \cdot\mathcal{E}_0E_\infty^q \\ & & & & M_k \otimes (\det)^1 & \supseteq & M_k^1 \otimes (\det)^1 & \supseteq & \cdots & \supseteq & M_k^{\mathfrak{m}(k)} \otimes (\det)^1 \end{array}$$

and in terms of the corresponding bases:

$$\begin{array}{ccccccccccc} \mathcal{B}_{k+q+1}^1 & \cup & \cdots & \cup & \mathcal{B}_{k+q+1}^q & \cup & \mathcal{B}_{k+q+1}^{q+1} & \cup & \cdots & \cup & \mathcal{B}_{k+q+1}^{\mathfrak{m}(k+q+1)} \\ & & & & & & \big\uparrow \cdot\mathcal{E}_0E_\infty^q & & & & \big\uparrow \cdot\mathcal{E}_0E_\infty^q \\ & & & & & & \mathcal{B}_k^1 & \cup & \cdots & \cup & \mathcal{B}_k^{\mathfrak{m}(k)} \end{array}$$

Let us first examine the smallest filtration module for arbitrary weight $k \in \mathbb{N}$. The above identification allows us to reduce the question to one for smaller weights. To be precise, by applying Lemma 4.3 $\widehat{\mathfrak{k}}$-times we observe:

4.5. Lemma. *For $k \geq 2$ there is a G-isomorphism*

$$M_{\mathfrak{k}}^{\mathfrak{m}(\mathfrak{k})} \otimes (\det)^{\hat{\mathfrak{k}}} \xrightarrow{\cong} M_k^{\mathfrak{m}(k)},$$

which is given by

$$\mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} \mapsto \mathcal{F}_b^{(\mathfrak{m}(k),k)}, \quad 0 \leq b \leq q+1-\mathfrak{k},$$

and linear extension. □

It is therefore sufficient to study the action of G on the sets $\mathcal{B}_{\mathfrak{k}}^{\mathfrak{m}(\mathfrak{k})}$ with $1 \leq \mathfrak{k} \leq q+1$.

4.6. Lemma. *Let $1 \leq \mathfrak{k} \leq q+1$. The map*

$$M_{\mathfrak{k}}^{\mathfrak{m}(\mathfrak{k})} \to \mathrm{Sym}^{q+1-\mathfrak{k}}(V) \otimes (\det)^{\mathfrak{m}(\mathfrak{k})}$$

given by

$$\mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} \mapsto X^{q+1-\mathfrak{k}-b}Y^b \quad \textit{for } 0 \leq b \leq q+1-\mathfrak{k}$$

and linear extension is an isomorphism of G-modules.

Proof. Obviously, the map in question is a vector space isomorphism. As usual, in the next step we study the basis elements $\mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})}$ under the action of G.

Since $\mathfrak{m}(\mathfrak{k})$ equals $\mathfrak{k}-1$, in this case the definition reads $\mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} = \mathcal{E}_b E_\infty^{\mathfrak{m}(\mathfrak{k})}$. By means of a straightforward application of the transformation properties of ordinary and modified Eisenstein series, we obtain

$$\begin{aligned}
\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} &= a^{q-b}\mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})}, \\
\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} &= \sum_{j=0}^{b} \binom{b}{j} t^{b-j}\mathcal{F}_j^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})}, \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{F}_b^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})} &= (-1)^{\mathfrak{m}(\mathfrak{k})}\mathcal{F}_{q+1-\mathfrak{k}-b}^{(\mathfrak{m}(\mathfrak{k}),\mathfrak{k})}.
\end{aligned}$$

The last identity again makes use of relation (3).

On the other hand, the transformation properties of the monomials in the twisted module $\mathrm{Sym}^{q+1-\mathfrak{k}}(V) \otimes (\det)^{\mathfrak{m}(\mathfrak{k})}$ are well-known (they can be read off easily from equation (5) taking into account the additional determinant twist). If we compare our results, we see that the map in question is indeed G-equivariant. □

We obtain the following result for the smallest filtration module in the general case.

4.7. Theorem. *Let $k \in \mathbb{N}$. There is a G-isomorphism*

$$M_k^{\mathfrak{m}(k)} \xrightarrow{\cong} \mathrm{Sym}^{q+1-\mathfrak{k}}(V) \otimes (\det)^{\mathfrak{m}(k)}$$

given by

$$\mathcal{F}_b^{(\mathfrak{m}(k),k)} \mapsto X^{q+1-\mathfrak{k}-b}Y^b, \quad 0 \le b \le q+1-\mathfrak{k}$$

and linear extension.

Proof. This is a direct consequence of Lemma 4.5 and Lemma 4.6 (note that $(\det)^{\mathfrak{m}(\mathfrak{k})+\hat{\mathfrak{k}}} = (\det)^{\mathfrak{m}(k)}$). $\square$

In case of the remaining filtration modules, we can describe an analogue to Lemma 4.5. Note that $\mathfrak{m}(k)$ is strictly greater than 1 if and only if k is at least 3.

4.8. Lemma. *Let $k \ge 3$ and $1 \le i \le \mathfrak{m}(k)-1$. Consider the unique decomposition $i = j+lq$ with $1 \le j \le q$ and $l \in \mathbb{N}_0$. Then multiplication with $(\mathcal{E}_0 E_\infty^q)^l$ induces an isomorphism of G-modules*

$$M_{k-l(q+1)}^j \otimes (\det)^l \xrightarrow{\cong} M_k^i.$$

In particular, the elements of $\mathcal{B}_k^i$ satisfy

$$\mathcal{F}_b^{(i,k)} = (\mathcal{E}_0 E_\infty^q)^l \mathcal{F}_b^{(j,k-l(q+1))} \quad \textit{for } 0 \le b \le q-1$$

and

$$\mathcal{F}_\infty^{(i,k)} = (\mathcal{E}_0 E_\infty^q)^l \mathcal{F}_\infty^{(j,k-l(q+1))}.$$

Proof. It is sufficient to verify that the module $M_{k-l(q+1)}^j$ is well-defined. Then the claims are proven by the l-fold application of Lemma 4.3 and Lemma 4.4, respectively.

Indeed, in the present situation we observe $l \le \widehat{\mathfrak{k}}$. Therefore $k - l(q+1) > 0$ and in particular

$$j = i - lq < \mathfrak{m}(k) - lq = \mathfrak{m}(k - l(q+1)). \qquad \square$$

Remark. At first it may seem counter-intuitive to restrict ourselves to modules with a low vanishing order (that is, the largest filtration modules of each weight). However, the essential consequence of the above lemma is the resulting restriction of the basis elements that have to be studied for each weight.

Taking into account our result for sets of shape $\mathcal{B}_k^{\mathfrak{m}(k)}$, $k \ge 2$, in Theorem 4.7, knowing the transformation properties under G of the elements of the sets

$$\mathcal{B}_k^i \quad \text{for } k \ge 3 \text{ and } 1 \le i \le \min(\mathfrak{m}(k)-1, q)$$

is sufficient in order to have a complete description of the transformation properties for all basis elements for all weights.

The main difficulty lies in the fact that the $\mathcal{C}_\infty$-span of such a set $\mathcal{B}_k^i$ is not closed under the action of G. In some cases we need elements from M_k^{i+1} to describe the resulting formulae.

First, we examine those cases in which the above problem does not occur. We can even forgo the restriction $i \le q$.

4.9. Lemma. *Let $k \ge 3$ and let $1 \le i \le \mathfrak{m}(k) - 1$. The elements of $\mathcal{B}_k^i$ satisfy the following transformation properties:*

$$\begin{aligned}
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)} &= a^{k-i-1+q-b} \mathcal{F}_b^{(i,k)}, \quad 0 \le b \le q-1, \\
\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mathcal{F}_\infty^{(i,k)} &= a^i \mathcal{F}_\infty^{(i,k)}, \\
\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)} &= \sum_{j=0}^{b} \tbinom{b}{j} t^{b-j} \mathcal{F}_j^{(i,k)}, \quad 0 \le b \le q-1, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_0^{(i,k)} &= (-1)^i \mathcal{F}_\infty^{(i,k)}, \\
\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_\infty^{(i,k)} &= (-1)^i \mathcal{F}_0^{(i,k)}.
\end{aligned}$$

Proof. As in previous statements, this follows by a straightforward application of Lemma 2.1 and Corollary 2.3. □

In contrast, the formulae in the remaining cases

$$\begin{aligned}
&\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)}, \quad 1 \le b \le q-1, \\
&\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \mathcal{F}_\infty^{(i,k)}
\end{aligned} \tag{6}$$

for $k \ge 3$ and $1 \le i \le \min(\mathfrak{m}(k) - 1, q)$ actually involve cusp forms of higher orders, that is, basis elements of $\mathcal{B}_k^{i+1,+}$.

While it is possible to give the exact result in each concrete example, it would be quite difficult to provide a closed formula due to the cumbersome expressions that occur.

For this reason, in the remainder of this section we restrict ourselves to studying the structure of the successive quotients of the filtration. Instead of exact transformation formulae for the elements of a set $\mathcal{B}_k^i$, $k \ge 3$, $1 \le i \le \max(q, \mathfrak{m}(k) - 1)$, we only consider congruences modulo M_k^{i+1}, thus determining the G-module structure of M_k^i / M_k^{i+1}.

In analogy with our approach for the smallest filtration module, we can then use Lemma 4.8 to generalize these results for quotients of order $i > q$.

As mentioned in the remark following Lemma 4.4, we can immediately solve the problem for quotients of vanishing order $i = q$ using our results for Eisenstein series. We assume $k > q + 2$ in the following statement to guarantee $q < \mathfrak{m}(k)$.

4.10. Lemma. *Let $k > q+2$. Then*

$$\begin{aligned} M_k^q &\cong M_{k-(q+1)} \otimes (\det)^1 \\ &\cong \mathrm{Eis}_{k-(q+1)} \otimes (\det)^1 \oplus M_{k-(q+1)}^1 \otimes (\det)^1 \end{aligned}$$

is an isomorphism of G-modules. In particular, we have

$$M_k^q / M_k^{q+1} \cong \mathrm{Eis}_{k-(q+1)} \otimes (\det)^1.$$

Proof. From Lemma 4.3 we get

$$M_k^q \cong M_{k-(q+1)}^0 \otimes (\det)^1$$

and

$$M_k^{q+1} \cong M_{k-(q+1)}^1 \otimes (\det)^1.$$

The remaining claims follow immediately from the direct sum decomposition of $M_{k-(q+1)}$ into Eisenstein series and cusp forms. $\square$

To complete our description of the quotients M_k^i / M_k^{i+1} for $i \leq q-1$ we will use the reduction formula developed in [12, Proposition 5.4] to obtain the desired congruences for the two types of transformations considered in (6).

In accordance with the notation in the quoted article, we write "$[\,x\,]$" for the representative of an integer x modulo $q-1$ in $\{1, \dots, q-1\}$ and define further

$$\langle x \rangle = \begin{cases} 0 & x = 0 \\ [x] & \text{else.} \end{cases}$$

4.11. Lemma. *Let $k \geq 3$ and $1 \leq i \leq \min(\mathfrak{m}(k)-1, q-1)$. For $1 \leq b \leq q-1$ we have*

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)} \equiv (-1)^i \mathcal{F}_{[k-2i-b]}^{(i,k)} \mod M_k^{i+1}.$$

Proof. Using relations (2) and (3), we may write

$$\begin{aligned} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)} &= \mathcal{E}_q^{k-i-1} \mathcal{E}_{q-b} E_0^i \\ &= (-1)^i \mathcal{E}_q^{k-i-2} \mathcal{E}_{q-b} \mathcal{E}_{q-i} E_\infty^i \\ &= (-1)^i \mathcal{E}_q^{k-i-2} \mathcal{E}_{q-b+1} \mathcal{E}_{q-i-1} E_\infty^i. \end{aligned}$$

Since $q-i-1 < q-i$, this product of Eisenstein series satisfies the conditions of [12, Proposition 5.4] and thus

$$\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mathcal{F}_b^{(i,k)} \equiv (-1)^i \mathcal{E}_0^{k-i-1} \mathcal{E}_{\langle (k-i-2)q + q - b + 1 + q - i - 1 \rangle} E_\infty^i \mod M_k^{i+1}.$$

By means of a straightforward calculation we verify that

$$\langle (k-i-2)q+q-b+1+q-i-1\rangle = [k-2i-b],$$

and the proof is complete. $\square$

4.12. Lemma. *Let $k\geq 3$ and $1\leq i\leq \min(\mathfrak{m}(k)-1,q-1)$. Then*

$$\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\mathcal{F}_\infty^{(i,k)} \equiv \mathcal{F}_\infty^{(i,k)} + \sum_{m=0}^{[k-2i]-1}\binom{[k-2i]}{m}t^{[k-2i]-m}\mathcal{F}_m^{(i,k)} \mod M_k^{i+1}.$$

Proof. According to relation (3), we may write $\mathcal{F}_\infty^{(i,k)} = \mathcal{E}_q^{k-i-1}\mathcal{E}_{q-i}E_\infty^i$. From this we obtain

$$\begin{aligned}\left(\begin{smallmatrix}1&t\\0&1\end{smallmatrix}\right)\mathcal{F}_\infty^{(i,k)} &= (t\mathcal{E}_0+\mathcal{E}_q)^{k-i-1}\sum_{j=0}^{q-i}\binom{q-i}{j}t^{q-i-j}\mathcal{E}_jE_\infty^i\\ &= \sum_{l=0}^{k-i-1}\sum_{j=0}^{q-i}\binom{k-i-1}{l}\binom{q-i}{j}t^{q+k-2i-1-l-j}\mathcal{E}_0^{k-i-1-l}\mathcal{E}_q^l\mathcal{E}_jE_\infty^i.\end{aligned}$$

We study the modular forms in the individual summands in more detail. We observe:

1. For $l=k-i-1$ and $j=q-i$ the modular form in question is equal to $\mathcal{F}_\infty^{(i,k)}$ according to our initial observation.
2. For $l=k-i-1$ and $0\leq j<q-i$ we may use [12, Proposition 5.4] to obtain
$$\mathcal{E}_q^{k-i-1}\mathcal{E}_jE_\infty^i \equiv \mathcal{E}_0^{k-i-1}\mathcal{E}_{\langle (k-i-1)q+j\rangle}E_\infty^i \mod M_k^{i+1}.$$
We can verify easily that $\langle (k-i-1)q+j\rangle = \langle k-i-1+j\rangle$.
3. For $0<l<k-i-1$ and $0\leq j\leq q-i$ we note that [12, Proposition 5.4] is again applicable (since the modular form in question contains at least one factor $\mathcal{E}_0$) and gives
$$\mathcal{E}_0^{k-i-1-l}\mathcal{E}_q^l\mathcal{E}_jE_\infty^i \equiv \mathcal{E}_0^{k-i-1}\mathcal{E}_{\langle lq+j\rangle}E_\infty^i \mod M_k^{i+1}.$$
As in the previous cases, we observe $\langle lq+j\rangle = \langle l+j\rangle$.
4. If l equals 0, we have the tautological statement
$$\mathcal{E}_0^{k-i-1}\mathcal{E}_jE_\infty^i = \mathcal{E}_0^{k-i-1}\mathcal{E}_{\langle j\rangle}E_\infty^i$$
for $0\leq j\leq q-i$.

According to the definition of the elements of $\mathcal{B}_k^i$, we get

$$\begin{aligned}\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)\mathcal{F}_\infty^{(i,k)} &\equiv \mathcal{F}_\infty^{(i,k)} + \sum_{l,j}\tbinom{k-i-1}{l}\tbinom{q-i}{j}t^{k-2i-l-j}\mathcal{E}_0^{k-i-1}\mathcal{E}_{\langle l+j\rangle}E_\infty^i \mod M_k^{i+1}\\ &\equiv \mathcal{F}_\infty^{(i,k)} + \sum_{l,j}\tbinom{k-i-1}{l}\tbinom{q-i}{j}t^{k-2i-l-j}\mathcal{F}_{\langle l+j\rangle}^{(i,k)} \mod M_k^{i+1},\end{aligned}$$

where the sum passes over all $0 \le l \le k-i-1$ and $0 \le j \le q-i$ such that

$$l+j < k-i-1+q-i = q-1+k-2i.$$

The Vandermonde identity for binomial coefficients implies

$$\sum_{\substack{l,j\\ l+j=m}}\binom{k-i-1}{l}\binom{q-i}{j} = \binom{q-1+k-2i}{m}$$

for $0 \le m \le q-1+k-2i-1$. Therefore we can replace the double sum by a single summation of shape

$$\left(\begin{smallmatrix}1 & t\\ 0 & 1\end{smallmatrix}\right)\mathcal{F}_\infty^{(i,k)} \equiv \mathcal{F}_\infty^{(i,k)} + \sum_{m=0}^{q-1+k-2i-1}\tbinom{q-1+k-2i}{m}t^{k-2i-m}\mathcal{F}_{\langle m\rangle}^{(i,k)} \mod M_k^{i+1}. \tag{7}$$

Next we reorder this sum and determine the coefficient for each element of the basis $\mathcal{B}_k^i$. Since $\langle m\rangle$ is zero if and only if $m=0$, we see immediately that the coefficients of $\mathcal{F}_\infty^{(i,k)}$ and $\mathcal{F}_0^{(i,k)}$ agree with the respective coefficients in the formula we set out to verify.

For $1 \le b \le q-1$ the coefficient of $\mathcal{F}_b^{(i,k)}$ in (7) is $t^{[k-2i]-b}\lambda_b$ where

$$\lambda_b := \sum_{\substack{m=1\\ m\equiv b \bmod q-1}}^{q-1+k-2i-1}\tbinom{q-1+k-2i}{m}.$$

Put $d := [k-2i] > 0$ and write $k-2i = d + h(q-1)$ with $h \ge -1$. We get

$$\lambda_b = \sum_{j=0}^{h+1}\tbinom{d+(h+1)(q-1)}{b+j(q-1)} - \delta_{b,d}$$

with Kronecker delta. Here we have used the fact that

$$\binom{d+(h+1)(q-1)}{b+(h+1)(q-1)} = \begin{cases}1 & b=d\\ 0 & d<b\le q-1\end{cases}$$

to obtain the uniform upper bound $h+1$ for the summation index.

By means of a general result for binomial coefficients in characteristic p (see for example [11, Proposition A.16]) we obtain

$$\sum_{j=0}^{h+1} \tbinom{d+(h+1)(q-1)}{b+j(q-1)} = \tbinom{d}{b},$$

and in turn

$$\lambda_b = \begin{cases} \binom{d}{b} & 1 \leq b \leq d-1 \\ 0 & d \leq b \leq q-1. \end{cases}$$

Since $d = [k-2i]$, the proof is complete. $\square$

For $k \geq 3$ and $1 \leq i \leq \min(\mathfrak{m}(k)-1, q-1)$ we have now determined the transformation properties of the entire basis $\mathcal{B}_k^i$ of the quotient M_k^i/M_k^{i+1}.

4.13. Lemma. *Let $k \geq 3$ and $1 \leq i \leq \min(\mathfrak{m}(k)-1, q-1)$. There is a G-isomorphism*

$$M_k^i/M_k^{i+1} \xrightarrow{\cong} N[k-2i, i]$$

given by

$$\mathcal{F}_b^{(i,k)} \mapsto f_b^{([k-2i])}, \quad 0 \leq b \leq q-1,$$
$$\mathcal{F}_\infty^{(i,k)} \mapsto f_\infty^{([k-2i])}$$

and linear extension.

Proof. For the transformation properties of modules of type $N[\delta]$ see Lemma 1.7. Specifically, we consider the special case $\delta = [k-2i]$ with an additional twist by $(\det)^i$.

Then the statement follows by means of comparison with our results for the elements of the basis $\mathcal{B}_k^i$ of M_k^i/M_k^{i+1} in Lemma 4.9, Lemma 4.11 and Lemma 4.12. $\square$

Now we lift the restriction of the vanishing order i:

4.14. Theorem. *Let $k \geq 3$ and $1 \leq i \leq \mathfrak{m}(k)-1$. Then the map*

$$M_k^i/M_k^{i+1} \xrightarrow{\cong} N[k-2i, i],$$

given by

$$\mathcal{F}_b^{(i,k)} \mapsto f_b^{([k-2i])}, \quad 0 \leq b \leq q-1,$$
$$\mathcal{F}_\infty^{(i,k)} \mapsto f_\infty^{([k-2i])}$$

and linear extension is an isomorphism of G-modules.

Proof. Writing $i = j + lq$ with $1 \leq j \leq q$ we apply Lemma 4.8 to obtain

$$M_k^i \cong M_{k-l(q+1)}^j \otimes (\det)^l$$

and

$$M_k^{i+1} \cong M_{k-l(q+1)}^{j+1} \otimes (\det)^l.$$

Depending on j, we use either Lemma 4.10 or Lemma 4.13 to determine the quotient on the right hand side. In all cases we get

$$M_{k-l(q+1)}^j / M_{k-l(q+1)}^{j+1} \otimes (\det)^l \cong N[k-2(j+l), j] \otimes (\det)^l = N[k-2(j+l), j+l].$$

Since $i \equiv j + l \mod q - 1$, this concludes the proof. $\square$

Remark. The resulting formula is also valid for the quotient M_k/M_k^1, which we have so far excluded in this section. Indeed, our earlier results show that

$$M_k/M_k^1 \cong \mathrm{Eis}_k \cong N[k] = N[k - 2 \cdot 0, 0].$$

There is also a connection to our result for the smallest filtration module in Theorem 4.7, since we can easily verify that

$$k - 2\mathfrak{m}(k) \equiv q + 1 - \mathfrak{k} \mod q - 1.$$

5. The modules of cusp forms

Motivated by Theorem 3.2, we are now going to have a closer look at the connection between symmetric powers of V and the cusp filtration. Our goal is to apply results concerning the G-module structure of symmetric powers of V from [13], in particular the filtration recalled above in Theorem 1.8, to the present situation.

On the one hand, this alternative approach leads to additional results, specifically an identification of the modules M_k^i with classical G-modules. On the other hand, we obtain alternative proofs for some of the results of the previous section.

Let us first compare the respective notations.

5.1. Notation. For weight $k \in \mathbb{N}$ let the decomposition $k = \mathfrak{k} + \widehat{\mathfrak{k}}(q+1)$ and the number $\mathfrak{m}(k)$ be defined as in Notation 4.1.

Consider for $n = kq$ the unique decomposition from Theorem 1.8

$$n = \mathfrak{n} + \widehat{\mathfrak{n}}(q+1) \quad \text{with } 0 \leq \mathfrak{n} \leq q.$$

We observe

$$\mathfrak{n} = q + 1 - \mathfrak{k},$$
$$\widehat{\mathfrak{n}} = \mathfrak{m}(k).$$

In particular, for fixed k there are injective G-homomorphisms

$$\psi_i : \mathrm{Sym}^{kq-i(q+1)}(V) \otimes (\det)^i \hookrightarrow \mathrm{Sym}^{kq-(i-1)(q+1)}(V) \otimes (\det)^{i-1}$$

and

$$\Psi_i : \mathrm{Sym}^{kq-i(q+1)}(V) \otimes (\det)^i \hookrightarrow \mathrm{Sym}^{kq}(V),$$

for each $1 \leq i \leq \mathfrak{m}(k)$, which are given by multiplication with powers of $XY^q - X^qY$.

In order to further describe the interaction between modular forms and symmetric powers, we study the G-isomorphism $\Phi_k : M_k \xrightarrow{\cong} \mathrm{Sym}^{kq}(V)$ from Theorem 3.2 on the system

$$\mathcal{B}_k^{i,+} = \bigcup_{i \leq j \leq \mathfrak{m}(k)} \mathcal{B}_k^j, \quad 1 \leq i \leq \mathfrak{m}(k),$$

of bases of the filtration modules M_k^i.

5.2. Lemma. *Let $k \geq 2$. The elements of the sets $\mathcal{B}_k^i$ with $1 \leq i \leq \mathfrak{m}(k) - 1$ satisfy*

$$\Phi_k \left(\mathcal{F}_b^{(i,k)} \right) = \Psi_i \left(X^{kq-i(q+1)-b} Y^b \right), \quad 0 \leq b \leq q-1,$$
$$\Phi_k \left(\mathcal{F}_\infty^{(i,k)} \right) = \Psi_i \left(Y^{kq-i(q+1)} \right).$$

For the elements of $\mathcal{B}_k^{\mathfrak{m}(k)}$ we find

$$\begin{aligned}\Phi_k \left(\mathcal{F}_b^{(\mathfrak{m}(k),k)} \right) &= \Psi_{\mathfrak{m}(k)} \left(X^{kq-\mathfrak{m}(k)(q+1)-b} Y^b \right) \\ &= \Psi_{\mathfrak{m}(k)} \left(X^{q+1-\mathfrak{k}-b} Y^b \right), \quad 0 \leq b \leq q+1-\mathfrak{k}.\end{aligned}$$

Proof. To compute the images under Φ_k we use the fact that Φ_k is one component of an isomorphism of algebras $\bigoplus_j M_j \xrightarrow{\cong} \bigoplus_j \mathrm{Sym}^{jq}$ according to Theorem 3.2.

Therefore, we write the elements of $\mathcal{B}_k^{1,+}$ as polynomial expressions in terms of the modified Eisenstein series of weight 1. That is, we substitute $E_\infty = \mathcal{E}_q - \mathcal{E}_1$ and $E_0 = \mathcal{E}_0 - \mathcal{E}_{q-1}$. Then all we have to do is apply Φ_1 to the weight 1 Eisenstein series and compute the resulting product in $\mathrm{Sym}^{kq}(V)$.

This way, for $1 \leq i < \mathfrak{m}(k)$ and $0 \leq b \leq q-1$ we obtain

$$\begin{aligned}\Phi_k\left(\mathcal{F}_b^{(j,k)}\right) &= \Phi_k(\mathcal{E}_0^{k-i-1}\mathcal{E}_b(\mathcal{E}_q-\mathcal{E}_1)^i)\\ &= (X^{(k-i-1)q})(X^{q-b}Y^b)(Y^q-X^{q-1}Y)^i\\ &= \sum_{j=0}^{i}\tbinom{i}{j}(-1)^{i-j}X^{kq-i-b-j(q-1)}Y^{b+i+j(q-1)}\end{aligned}$$

(the same calculation holds verbatim for $i=\mathfrak{m}(k)$ and $0 \leq b \leq q+1-\mathfrak{k}$).

Similarly, we see

$$\Phi_k\left(\mathcal{F}_\infty^{(i,k)}\right) = \sum_{j=0}^{i}\tbinom{i}{j}(-1)^{i-j}X^{iq-j(q-1)}Y^{kq-i(q+1)+i+j(q-1)}.$$

On the other hand, since Ψ_i is given by multiplication with $(XY^q-X^qY)^i$, we get immediately

$$\Psi_i\left(X^{kq-i(q+1)-b}Y^b\right) = \sum_{j=0}^{i}\tbinom{i}{j}(-1)^{i-j}X^{kq-b-i-j(q-1)}Y^{b+i+j(q-1)}$$

for $1 \leq i \leq \mathfrak{m}(k)$ and $0 \leq b \leq kq-i(q+1)$. We compare this with the images under Φ_k to complete the proof. $\square$

Remark. The lemma provides an insight into the motivation for the definition of the basis $\mathcal{B}_k^{1,+}$.

In [13, Lemma 3.3] we have obtained a monomial basis of the quotient module $\mathrm{Sym}^{kq-i(q+1)}(V)/L(kq-i(q+1))$ in a natural way. The sets $\mathcal{B}_k^i$ consist precisely of the preimages of the embeddings of these bases into $\mathrm{Sym}^{kq}(V)$ (cf. Theorem 1.8).

In the context of Drinfeld modular forms, this relation between $\mathcal{B}_k^{1,+}$ and submodules of $\mathrm{Sym}^{kq}(V)$ ensures that the basis elements are "well-behaved" at the cusps.

We use the identities described in Lemma 5.2 for $1 \leq i \leq \mathfrak{m}(k)$ to map the elements of $\mathcal{B}_k^i$ into the module $\mathrm{Sym}^{kq-i(q+1)}(V)\otimes(\det)^i$. This construction provides the following G-isomorphisms:

5.3. Theorem. *Let $k \geq 2$. For $1 \leq i \leq \mathfrak{m}(k)$ there is a G-isomorphism*

$$\phi_k^{(i)} : M_k^i \xrightarrow{\cong} \mathrm{Sym}^{kq-i(q+1)}(V)\otimes(\det)^i,$$

which satisfies

$$\Psi_i \circ \phi_k^{(i)} = \Phi_k\big|_{M_k^i}. \tag{8}$$

Proof. For $i = \mathfrak{m}(k)$ the G-isomorphism $\phi_k^{(\mathfrak{m}(k))}$ is given by Theorem 4.7. Relation (8) is satisfied according to Lemma 5.2.

Next we show that the claim for $1 \le i \le \mathfrak{m}(k) - 1$ follows from the validity of the statement for $i+1$. Assume therefore that a map $\phi_k^{(i+1)}$ exists with the desired properties. We want to use it in the construction of $\phi_k^{(i)}$.

$$\begin{tikzcd}
M_k^{i+1} \arrow[rr, "\phi_k^{(i+1)}"] \arrow[d, hook] & & \mathrm{Sym}^{kq-(i+1)(q+1)}(V) \otimes (\det)^{i+1} \arrow[d, hook, "\psi_{i+1}"] \\
M_k^{i} \arrow[rr, dotted, "\phi_k^{(i)}"] \arrow[dr, hook, "\Phi_k|_{M_k^i}"'] & & \mathrm{Sym}^{kq-i(q+1)}(V) \otimes (\det)^{i} \arrow[dl, hook', "\Psi_i"] \\
 & \mathrm{Sym}^{kq}(V) &
\end{tikzcd}$$

We proceed in two steps. On the submodule $M_k^{i+1} \subseteq M_k^i$ we define $\phi_k^{(i)}$ as $\psi_{i+1} \circ \phi_k^{(i+1)}$.

To extend this definition to the entire module M_k^i, it is sufficient to prescribe the images for the elements of $\mathcal{B}_k^i$, since $\mathcal{B}_k^i$ extends a basis of M_k^{i+1} to a basis of M_k^i.

For an arbitrary element $\mathcal{F}_*^{(i,k)} \in \mathcal{B}_k^i$ we define $\phi_k^{(i)}(\mathcal{F}_*^{(i,k)})$ to be the preimage of $\Phi_k(\mathcal{F}_*^{(i,k)})$ under Ψ_i. This is well-defined, since such a preimage in $\mathrm{Sym}^{kq-i(q+1)}(V) \otimes (\det)^i$ exists according to Lemma 5.2 and is uniquely determined due to the injectivity of Ψ_i.

It is then straightforward to verify that the resulting map satisfies relation (8). In turn, this shows that $\phi_k^{(i)}$ is injective and therefore an isomorphism of vector spaces.

Using the fact that both $\Phi_k\big|_{M_k^i}$ and Ψ_i are G-equivariant, and that Ψ_i is injective, we apply relation (8) to show that $\phi_k^{(i)}$ is G-equivariant as well and the proof is complete. $\square$

Remark.

1. We can interpret Theorem 3.2 as the extension of the above theorem to the special case $i = 0$. Therefore, Theorem 5.3 holds for $k \in \mathbb{N}$. Furthermore, the case $k = 0$ is a tautology.
2. In particular, Theorem 5.3 can be used to identify the modules M_k^i with the submodules of the filtration of $\mathrm{Sym}^{kq}(V)$ recalled in Theorem 1.8. This way, the G-isomorphism

$$M_k^i / M_k^{i+1} \cong N[k - 2i, i]$$

from Theorem 4.14 can be considered a corollary to the aforementioned theorems.

Finally, we provide two examples for the transfer of results between Drinfeld modular forms and symmetric powers of V.

First, we can use the algorithms for counting the multiplicities of composition factors of symmetric powers of V given in section 4 of [13] to determine the composition factors of modules of shape M_k^i.

We recall the necessary notation of the quoted article: Let $\mathcal{R}$ be a full set of representatives modulo $q-1$ and denote by $[x]_{\mathcal{R}}$ the representative in $\mathcal{R}$ of an integer x. We use the same classification of the simple G-modules as in [13], that is, $\mathfrak{S}(m,\mu)$ is the unique simple submodule of $\mathrm{Sym}^m(V)\otimes(\det)^\mu$ where $0\le m\le q-1$ and $\mu\in\mathcal{R}$.

5.4. Proposition. *Let $k\in\mathbb{N}$ and $0\le i\le\mathfrak{m}(k)$. For $0\le m\le q-1$ and $\mu\in\mathcal{R}$ denote by $\lambda_M(k,i;m,\mu)$ the multiplicity of the simple module $\mathfrak{S}(m,\mu)$ as a composition factor of M_k^i.*

Further, for $n\in\mathbb{N}_0$ let $\lambda(n;m,\mu)$ be the multiplicity of $\mathfrak{S}(m,\mu)$ as a composition factor of $\mathrm{Sym}^n(V)$ as determined in [13, Algorithm 4.14].

Then

$$\lambda_M(k,i;m,\mu)=\lambda(kq-i(q+1);m,[\mu-i]_{\mathcal{R}}).$$

Proof. According to Theorem 5.3 and the first remark, we have

$$M_k^i\cong\mathrm{Sym}^{kq-i(q+1)}(V)\otimes(\det)^i.$$

Taking into account the proper offset of the determinant twist, the identity for the multiplicities of composition factors is immediate. □

As an example for the transfer of knowledge in the other direction, we can use a well-known fact for Drinfeld modular forms to improve [13, Theorem 3.7] by providing a necessary and sufficient condition.

5.5. Theorem. *Let $n\ge q+1$. There exists a G-module complement to $L(n)$ in $\mathrm{Sym}^n(V)$ if and only if n is divisible by q.*

Proof. If n is not a multiple of q, we have proven in [13, Theorem 3.7] that such a G-module complement cannot exist.

Now let $n=kq$. We already know from Theorem 3.2 that $\mathrm{Sym}^{kq}(V)\cong M_k$ holds. According to the definition of $L(kq)$ in Theorem 1.8, Theorem 5.3 implies $L(kq)\cong M_k^1$. Therefore the statement follows immediately from the well-known decomposition of G-modules $M_k=\mathrm{Eis}_k\oplus M_k^1$. □

References

[1] M. Bardoe, P. Sin, The permutation modules for $GL(n+1,\mathbb{F}_q)$ acting on $\mathbb{P}^n(\mathbb{F}_q)$ and $\mathbb{F}_q^{n+1}$, J. Lond. Math. Soc. (2) 61 (1) (2000) 58–80.

[2] C. Bonnafé, Representations of $SL_2(\mathbb{F}_q)$, Springer-Verlag London, Ltd., London, 2011.

[3] G. Cornelissen, Geometric properties of modular forms over rational function fields, Ph.D. thesis, Universiteit Gent, 1997.

[4] P. Deligne, D. Husemöller, Survey of Drinfeld modules, in: Current Trends in Arithmetical Algebraic Geometry, Arcata, CA, 1985, in: Contemp. Math., vol. 67, Amer. Math. Soc., Providence, RI, 1987, pp. 25–91.

[5] V.G. Drinfeld, Elliptic modules, Mat. Sb. (N. S.) 94 (136) (1974) 594–627, p. 656; English translation: Math. USSR, Sb. 23 (1976) 561–562.

[6] E.-U. Gekeler, Drinfeld Modular Curves, Lecture Notes in Math., vol. 1231, Springer-Verlag, Berlin, 1986.

[7] E.-U. Gekeler, On the coefficients of Drinfeld modular forms, Invent. Math. 93 (3) (1988) 667–700.

[8] E.-U. Gekeler, M. Reversat, Jacobians of Drinfeld modular curves, J. Reine Angew. Math. 476 (1996) 27–93.

[9] D. Goss, Modular forms for $\mathbb{F}_r[T]$, J. Reine Angew. Math. 317 (1980) 16–39.

[10] D. Goss, π-adic Eisenstein series for function fields, Compos. Math. 41 (1) (1980) 3–38.

[11] E. Varela Roldán, Darstellungstheorie Drinfeld'scher Modulformen, Ph.D. thesis, Universität des Saarlandes, 2015.

[12] E. Varela Roldán, Arithmetic of Eisenstein series of level T for the function field modular group $\mathrm{GL}(2, \mathbb{F}_q[T])$, J. Number Theory 167 (2016) 180–201.

[13] E. Varela Roldán, Composition factors of symmetric powers of the tautological representation of $\mathrm{GL}(2, \mathbb{F}_q)$, J. Number Theory (2017), under review (JNT-D-16-00031).

[14] B. Wack, Darstellungen von $\mathrm{SL}(2, \mathbb{F}_q)$ und $\mathrm{GL}(2, \mathbb{F}_q)$ in definierender Charakteristik, Diploma thesis, Universität des Saarlandes, 1996.