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Dmitry Kleinbock, Ronggang Shi, George Tomanov

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S-ADIC VERSION OF MINKOWSKI'S GEOMETRY OF NUMBERS AND MAHLER'S COMPACTNESS CRITERION

DMITRY KLEINBOCK, RONGGANG SHI, AND GEORGE TOMANOV

ABSTRACT. In this note we give a detailed proof of certain results on geometry of numbers in the S-adic case. These results are well-known to experts, so the aim here is to provide a convenient reference for the people who need to use them.

1. INTRODUCTION

The space of unimodular lattices in $\mathbb{R}^n$ $(n \geq 2)$ can be identified with the homogeneous space $X = \mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})$ via the correspondence $\mathbb{Z}^n g \leftrightarrow \mathrm{SL}_n(\mathbb{Z})g$ where $g \in \mathrm{SL}_n(\mathbb{R})$. It is proved by Mahler [11] that a subset R of X is relatively compact if and only if nonzero elements of the corresponding unimodular lattices are separated from zero. This phenomenon is called Mahler's compactness criterion [2, Chapter V]. It has been very useful in dynamical approach to number theory; we refer the readers to survey papers [4],[5] and [7] and references there for details.

Let S be a finite nonempty set of places of a global field K. We assume S contains all the archimedean places if K is a number field. For each place v of K, let K_v be the completion of K at v. Let $K_S = \prod_{v\in S} K_v$ and

$$I_S = \{a \in K : a \text{ is integral in } K_v \text{ for every place } v \notin S\}. \tag{1.1}$$

We consider K and hence I_S as subrings of K_S via natural embeddings $K \to K_v$. Then the homogeneous space $\mathrm{SL}_n(I_S)\backslash \mathrm{SL}_n(K_S)$ can be identified with a set of free discrete I_S-modules of rank n in K_S^n with fixed covolume. The connection between dynamics and number theory also spreads to the S-adic setting, where the corresponding version of Mahler's criterion plays an important role. The extension of Mahler's criterion to the S-adic case when K is a number field has already been used in several papers, and a proof for $K = \mathbb{Q}$ can be found in [9]. Moreover, a preliminary version [8] of the paper [9], published as a preprint of MPIM (Bonn), contains a proof of the S-adic Mahler's criterion for arbitrary number field K. When K is a function field with genus zero and S contains a single place of degree one, Mahler's criterion is proved in [6]. The general S-adic case is known to

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experts, but it is not easy to find a convenient reference. Here we provide a self-contained proof of an S-adic version of Mahler's criterion.

Theorem 1.1. *Let $n \geq 2$. A set $R \subset \mathrm{SL}_n(I_S)\backslash \mathrm{SL}_n(K_S)$ is relatively compact if and only if the subset*

$$\left\{\xi \in I_S^n g : \xi \neq 0,\ g \in \mathrm{SL}_n(K_S) \text{ and } \mathrm{SL}_n(I_S)g \in R\right\}$$

of K_S^n is separated from zero, i.e. this set has empty intersection with some open neighborhood of zero in K_S^n.

Our proof of Theorem 1.1 is based on an S-adic version of Minkowski's lemma of geometry of numbers. Let **vol** be the normalized Haar measure on the additive group K_S^n (see §2). For a discrete I_S-module Γ in K_S^n the covolume of Γ (denoted by $\mathbf{cov}(\Gamma)$) is the **vol** of a fundamental domain of Γ in K_S^n. Let $B_r(K_S^n)$ be the closed ball of radius r centered at zero in K_S^n with respect to the normalized norm (see §2). For each integer $1 \leq m \leq n$, the m-th minimum of a discrete I_S-module Γ is defined by

$$\lambda_m(\Gamma) = \inf\{r > 0 : \mathbf{dim}_K\big(\mathbf{span}_K(B_r(K_S^n) \cap \Gamma)\big) \geq m\}. \tag{1.2}$$

Here $\mathbf{span}_K$ is the K linear span of a set and $\mathbf{dim}_K$ is the dimension of a vector space over K. Similar notations are used when K is replaced by other rings. We remark here that if $K = \mathbb{Q}$ and S contains only the archimedean place, then we get the usual concept of successive minima of lattice points in $\mathbb{R}^n$.

For two nonnegative real numbers s and t the notation $s \asymp t$ means $C^{-1}s \leq t \leq Cs$ for some constant $C \geq 1$. Let σ and τ be the number of real and complex places of K respectively.[1] Let $\sharp S = \tau + \mathbf{card}(S)$ where **card** denotes the cardinality of a set. The S-adic version of Minkowski's theorem on successive minima (see [12, Chapter IV, §1] for the usual case) is the following theorem.

Theorem 1.2. *Let $n \geq 1$ and let $\Gamma \subset K_S^n$ be a discrete I_S-module with finite covolume. Then*

$$\big(\lambda_1(\Gamma) \dots \lambda_n(\Gamma)\big)^{\sharp S} \asymp \mathbf{cov}(\Gamma)$$

where the implied constants depend on K, S and n.

A refined version of Theorem 1.2 will be proved in Theorem 4.4 where the implied constants will be explicitly calculated. If K is a function field of genus zero and S consists of a single place of degree one, then Theorem 1.2 is established in [10]. The adelic versions of Theorem 1.2 are proved in [1] (resp. [13]) when K is a number field (resp. function field).

[1]If K is function field we have $\sigma = \tau = 0$.

2. Preliminaries: notations

Let K be a global field and let P be the set of places of K. Throughout this paper we fix a positive integer n and a finite nonempty set $S \subset P$ such that $S \supset P_0$ where P_0 (possibly empty) is the set of archimedean places of K.

For every $v \in P$ let K_v be the completion of K at v. The *S-adic numbers* and *integers* are defined as

$$K_S \stackrel{\text{def}}{=} \prod_{v \in S} K_v \quad \text{and} \quad I_S \stackrel{\text{def}}{=} \{x \in K : x \text{ is integral for all } v \in P \smallsetminus S\}$$

respectively. We consider K as a subring of K_S via the natural inclusions $K \to K_v$. For $v \in P$, let $|\cdot|_v$ be the *normalized absolute value* on K_v. If v is archimedean, we identify K_v with real or complex numbers where the usual absolute value is $|\cdot|_v$. If v is ultrametric then $|a|_v^{-1} = \mathbf{card}(I_v/aI_v)$ for all $a \in I_v$ where I_v is the ring of integers of K_v. For each ultrametric place $v \in P$ we fix a *uniformizer* ϖ_v (a generator of the maximal ideal of I_v) and take $q_v = |\varpi_v|_v^{-1}$. We define the *absolute value* and *content* for $x = (x_v)_{v \in S} \in K_S$ respectively by

$$|x| \stackrel{\text{def}}{=} \max_{v \in S} |x_v|_v \quad \text{and} \quad \mathbf{cont}(x) \stackrel{\text{def}}{=} \prod_{v \in S} |x_v|_v^{\varepsilon_v}$$

where $\varepsilon_v = 2$ if $K_v = \mathbb{C}$ and $\varepsilon_v = 1$ otherwise.

The additive group K_S^n can be naturally identified with $\prod_{v \in S} K_v^n$ and we write every $\xi \in K_S^n$ as $(\xi_v)_{v \in S}$ according to this identification. More precisely, if $\xi = (x_1, \ldots, x_n)$ where $x_i = (x_{i,v})_{v \in S}$, then $\xi_v = (x_{1,v}, \ldots, x_{n,v})$. Similarly, the group $\mathrm{GL}_n(K_S)$ can be naturally identified with $\prod_{v \in S} \mathrm{GL}_n(K_v)$ and we write every $g \in \mathrm{GL}_n(K_S)$ as $(g_v)_{v \in S}$ according to this identification. The group $\mathrm{GL}_n(K_v)$ (resp. $\mathrm{GL}_n(K_S)$) acts on K_v^n (resp. K_S^n) by matrix multiplication from the right. Moreover, the action of $g \in \mathrm{GL}_n(K_S)$ on $\xi \in K_S^n$ is consistent with these identifications, that is, $\xi g = (\xi_v g_v)_{v \in S}$ under previous notations.

For $v \in P$ we take $\mathbf{vol}_v$ to be the normalized Haar measure on K_v. For archimedean v, the measure $\mathbf{vol}_v$ is the Lebesgue measure. If v is ultrametric, the measure satisfies $\mathbf{vol}_v(I_v^n) = 1$. It follows directly from definition that

$$\mathbf{vol}_v(aB) = |a|_v^{n\varepsilon_v} \mathbf{vol}_v(B)$$

for every $a \in K_v$ and any measurable subset B of K_v^n. We take the normalized Haar measure $\mathbf{vol}$ on K_S^n to be the product measure. In the sequel we will abbreviate $\mathrm{d}\mathbf{vol}(\xi)$ by $\mathrm{d}\xi$ for the integration with respect to the volume measure. For a positive integer m, we use $\mathbf{vol}_v^m$ and $\mathbf{vol}^m$ to denote the normalized Haar measures on K_v^m and K_S^m respectively.

If v is archimedean we take $\|\cdot\|_v$ to be the Euclidean norm on K_v^n. If v is ultrametric we take $\|\cdot\|_v$ to be the sup norm with respect to coordinates, that is

$$\|(a_1, \dots, a_n)\|_v \stackrel{\text{def}}{=} \max_{1\le i\le n} |a_i|_v \quad \text{where} \quad a_i \in K_v.$$

We define the *norm* and *content* for $\xi \in K_S^n$ by

$$\|\xi\| \stackrel{\text{def}}{=} \max_{v\in S} \|\xi_v\|_v \quad \text{and} \quad \mathbf{cont}(\xi) \stackrel{\text{def}}{=} \prod_{v\in S} \|\xi_v\|_v^{\varepsilon_v}.$$

For $a \in I_S^*$, where I_S^* is the group of multiplicatively invertible elements of I_S, we have that $\mathbf{cont}(a) = 1$ and $\mathbf{cont}(a\xi) = \mathbf{cont}(\xi)$ where $\xi \in K_S^n$. Also for every $g \in \mathrm{GL}_n(K_S)$ we have

$$\mathrm{d}(\xi g) = \mathbf{cont}\big(\mathbf{det}(g)\big)\, \mathrm{d}\xi, \tag{2.1}$$

where $\mathbf{det}$ is the determinant of a matrix.

The set of vectors in K_S^n (resp. K_v^n) with norm less than or equal to r is denoted by $B_r(K_S^n)$ (resp. $B_r(K_v^n)$). It can be checked directly that

$$B_r(K_S^n) = \prod_{v\in S} B_r(K_v^n). \tag{2.2}$$

Let L be a free K_S-submodule of K_S^n with rank $m \le n$. Then

$$L = \prod_{v\in S} L_v, \tag{2.3}$$

where L_v is an m-dimensional subspace of K_v^n. There is a unique additive Haar measure $\mathbf{vol}_L$ on L (resp. $\mathbf{vol}_{L_v}$ on L_v) such that

$$\mathbf{vol}_L\big(L \cap B_1(K_S^n)\big) = \mathbf{vol}^m\big(B_1(K_S^m)\big)$$
$$(\text{resp.} \quad \mathbf{vol}_{L_v}\big(L \cap B_1(K_v^n)\big) = \mathbf{vol}_v^m\big(B_1(K_v^m)\big)\ ).$$

Moreover, the above definition, (2.2) and (2.3) imply

$$\mathbf{vol}_L = \prod_{v\in S} \mathbf{vol}_{L_v}. \tag{2.4}$$

In the case where $K = \mathbb{Q}$ and $\mathbf{card}(S) = 1$, the measure $\mathbf{vol}_L$ is the measure given by the inner product on L. Suppose $\xi = (\xi_v)_{v\in S}$ and $\xi_v \neq 0$; then the covolume of $I_S\xi$ (I_S-linear span of ξ) in $K_S\xi$ (K_S-linear span of ξ) with respect to $\mathbf{vol}_{K_S\xi}$ is equal to $\mathbf{cont}(\xi)$ multiplied by the covolume of I_S in K_S. The covolume of a discrete I_S-module Γ in K_S^n with respect to the induced measure $\mathbf{vol}_{K_S\Gamma}$ is called *relative covolume* of Γ and it is denoted by $\mathbf{cov_r}(\Gamma)$. The covolume of Γ with respect to $\mathbf{vol}$ is denoted by $\mathbf{cov}(\Gamma)$.

3. Discrete I_S-modules

Let $\Gamma \subset K_S^n$ be a discrete I_S-module. In this section, we use ideas of [9, §8] to study properties of Γ.

Lemma 3.1. *Let $\Gamma \subset K_S^n$ be a discrete I_S-module and let $\xi_1, \dots, \xi_m \in \Gamma$. The following statements are equivalent:*

(1) *$\xi_1, \dots, \xi_m$ are linearly independent over I_S;*

(2) $\xi_1, \ldots, \xi_m$ *are linearly independent over* K;
(3) $\xi_1, \ldots, \xi_m$ *are linearly independent over* K_S.

Proof. It suffices to show that (1) implies (3). We prove it by induction on m. Write $\xi_i = (\xi_{i,v})_{v\in S}$ as in §2. Suppose that ξ_1 is linearly dependent over K_S, then there exists $w \in S$ such that $\xi_{1,w} = 0$. According to the strong approximation theorem (see [3, Chapter II §15]), there is a sequence $\{c_i\}_{i\geq 1}$ of $I_S \smallsetminus \{0\}$ such that $|c_i|_v \to 0$ as $i \to \infty$ for any $v \in S \smallsetminus \{w\}$. Therefore $c_i\xi_1 \to 0$ which contradicts the assumption that Γ is discrete. This proves (3) in the case where $m = 1$.

Now suppose $m > 1$ and (1) implies (3) while m is replaced by $m-1$. By the case for $m = 1$, we know $\xi_{1,v} \neq 0$ for every $v \in S$. So there exists $g \in \mathrm{GL}_n(K_S)$ such that $\xi_1 g = (1, 0, \ldots, 0)$. The right multiplication of g on K_S^n is a K_S linear isomorphism, so we can without loss of generality assume that $\xi_1 = (1, 0, \ldots, 0)$. Let $\varphi : K_S^n \to K_S^n/K_S\xi_1 \cong K_S^{n-1}$ be the natural quotient map. Since $I_S\xi_1$ is a cocompact lattice in $K_S\xi_1$ and $\Gamma \subset K_S^n$ is discrete, the module $\varphi(\Gamma)$ is discrete and $\varphi(\xi_2), \ldots, \varphi(\xi_m)$ are linearly independent over I_S. In view of the induction hypothesis, we have $\varphi(\xi_2), \ldots, \varphi(\xi_m)$ are linear independent over K_S. Therefore $\xi_1, \ldots, \xi_m$ are linearly independent over K_S. □

Remark 3.2. The implication (2)⇒(3) holds without assuming that $\xi_1, \ldots, \xi_m$ belong to a discrete I_S-module, see [9, Lemma 7.1].

For a discrete I_S-module $\Gamma \subset K_S^n$ let $K\Gamma$ (resp. $K_S\Gamma$) be the K-linear (resp. K_S-linear) span of Γ in K_S^n. We call the dimension over K of $K\Gamma$ the *rank* of Γ. It follows from Lemma 3.1 that the rank of Γ is less than or equal to n and the equality holds if and only if Γ has finite covolume.

In the following lemma we prove a Gram-Schmidt orthogonalization process for ultrametric local fields.

Lemma 3.3. *Let* K_v *be a ultrametric local field. For any* K_v*-linearly independent vectors* $\xi_1, \ldots, \xi_m \in K_v^n$ *there exist linearly independent vectors* $\eta_1, \ldots, \eta_m \in K_v^n$ *such that* $\eta_1, \ldots, \eta_r$ *are in the* K_v*-linear span of* $\xi_1, \ldots, \xi_r$ *for all* $r \leq m$, *and*

$$\|a_1\eta_1 + \cdots + a_m\eta_m\|_v = \max_{1\leq i\leq m} |a_i|_v \quad \text{for all} \quad a_i \in K_v. \tag{3.1}$$

Remark 3.4. In the sequel we call a basis of $L_v \overset{\text{def}}{=} \mathbf{span}_{K_v}\{\xi_1, \ldots, \xi_m\}$ which satisfies (3.1) an *orthonormal basis* of L_v. The map

$$\varphi : K_v^m \to L_v \quad \text{where} \quad \varphi_v(a_1, \ldots, a_m) = a_1\eta_1 + \cdots + a_m\eta_m$$

is an isometric embedding sending $\mathbf{vol}_v^m$ to $\mathbf{vol}_{L_v}$.

Proof. Contrary to the archimedean case, here we choose an entry with maximal absolute value for the corresponding vector. Write

$$\xi_i = (x_{i1}, \ldots, x_{in}), \quad \text{where} \quad x_{ij} \in K_v \quad \textit{and} \quad 1 \leq i \leq m.$$

First we choose $j_1 \le n$ such that $\|\xi_1\|_v = |x_{1j_1}|_v$ and set $\eta_1 = x_{1j_1}^{-1}\xi_1$. Next we take $\eta_2' = \xi_2 - x_{2j_1}\eta_1 = (y_{21}, \dots, y_{2n})$. We choose $j_2 \le n$ such that $\|\eta_2'\|_v = |y_{2j_2}|_v$ and set $\eta_2 = y_{2j_2}^{-1}\eta_2'$. In general after r steps we have r different integers $j_1 \dots, j_r$ and unit norm vectors $\eta_1, \dots, \eta_r$ such that η_i has j_i-th entry 1 and j_s-th entry zero for $s < i$. We take

$$\eta_{r+1}' = \xi_{r+1} - x_{r+1,j_1}\eta_1 - \dots x_{r+1,j_r}\eta_r = (y_1, \dots, y_n)$$

and choose j_{r+1} such that $\|\eta_{r+1}'\|_v = |y_{j_{r+1}}|_v$. We set $\eta_{r+1} = y_{j_{r+1}}^{-1}\eta_{r+1}'$. Then it has j_{r+1}-th entry 1 and j_s-th entry 0 for $s < r+1$. This induction process gives m unit norm vectors $\eta_1, \dots, \eta_m$.

For $(a_1, \dots, a_m) \in K_v^m$ let

$$k = \min\{1 \le r \le m : |a_{j_r}|_v = \max_{1 \le i \le m} |a_i|_v\}.$$

It is clear from the construction that

$$\|a_1\eta_1 + \dots + a_m\eta_m\|_v = |a_{j_k}|_v,$$

which proves (3.1). □

The next lemma appeared as [9, Corollary 8.4] for $K = \mathbb{Q}$ and as [8, Corollary 5.8] for K a number field.

Lemma 3.5. *Suppose Γ and Γ' are discrete I_S-modules in K_S^n with*

$$K\Gamma \cap K\Gamma' = \{0\}. \tag{3.2}$$

Then

$$\mathbf{cov_r}(\Gamma + \Gamma') \le \mathbf{cov_r}(\Gamma)\mathbf{cov_r}(\Gamma'). \tag{3.3}$$

Proof. Let $L = K_S\Gamma$, $L' = K_S\Gamma'$ and $L'' = L + L'$. In view of (3.2), Lemma 3.3 implies that L'' is a direct sum of L and L'. The right (resp. left) hand side of (3.3) is the covolume of $\Gamma + \Gamma'$ with relative to $\mathbf{vol}_L \times \mathbf{vol}_{L'}$ (resp. $\mathbf{vol}_{L''}$). Let $L = \prod_{v \in S} L_v$ and $L' = \prod_{v \in S} L_v'$ according to (2.3). In view of (2.4), it suffices to prove that for each $v \in S$ there is a positive Haar measure set R_v of $L_v + L_v'$ such that

$$\mathbf{vol}_{L_v + L_v'}(R_v) \le (\mathbf{vol}_{L_v} \times \mathbf{vol}_{L_v'})(R_v). \tag{3.4}$$

Let r and m be the rank of L and L'' respectively. For each $v \in S$ we choose an orthonormal basis $\xi_{1,v}, \dots, \xi_{r,v}$ of L_v and an orthonormal basis $\xi_{r+1,v}, \dots, \xi_{m,v}$ of L_v'. We will show that (3.4) holds for

$$R_v := \{a_1\xi_{1,v} + \dots + a_m\xi_{m,v} : a_i \in B_1(K_v)\}.$$

For all $v \in S$

$$(\mathbf{vol}_{L_v} \times \mathbf{vol}_{L_v'})(R_v) = \mathbf{vol}_v^1\big(B_1(K_v)\big)^m. \tag{3.5}$$

If v is archimedean, then it is clear from Eulidean geometry (i.e. volume of parallelotope) that

$$\mathbf{vol}_{L_v + L_v'}(R_v) \le \mathbf{vol}_v^1\big(B_1(K_v)\big)^m = (\mathbf{vol}_{L_v} \times \mathbf{vol}_{L_v'})(R_v).$$

If v is ultrametric, we let $\eta_{1,v}, \ldots, \eta_{m,v}$ be an orthonormal basis of $L_v + L'_v$. Then Remark 3.4 implies that R_v is contained in

$$R'_v := \{(a_1, \ldots, a_m) \in B_1(K_v^m) : a_1\eta_1 + \ldots + a_m\eta_m\}.$$

Using Remark 3.4 again together with (3.5), we have

$$\mathbf{vol}_{L_v+L'_v}(R_v) \leq \mathbf{vol}_{L_v+L'_v}(R'_v) = \mathbf{vol}_v^1\big(B_1(K_v)\big)^m = (\mathbf{vol}_{L_v} \times \mathbf{vol}_{L'_v})(R_v).$$

□

4. Successive minima

The aim of this section is to prove Theorem 1.2.

Lemma 4.1. *Let $\Gamma \subset K_S^n$ be a discrete I_S-module with finite covolume and let $R \subset K_S^n$ be a measurable subset. Then there exists $\xi \in K_S^n$ such that*

$$\mathbf{card}\big((\xi + R) \cap \Gamma\big) \geq \mathbf{vol}(R)/\mathbf{cov}(\Gamma). \tag{4.1}$$

Proof. Let χ_R be the characteristic function of R, and let $F \subset K_S^n$ be a fundamental domain for Γ. Then

$$\int_F \mathbf{card}\big((\xi + R) \cap \Gamma\big)\, \mathrm{d}\xi = \int_F \sum_{\gamma\in\Gamma} \chi_R(\gamma - \xi) d\xi = \mathbf{vol}(R).$$

Therefore there exists $\xi \in F$ such that (4.1) holds. □

Lemma 4.2. *Let $\Gamma \subset K_S^n$ be a discrete I_S-module with finite covolume. Let R_1 be a centrally symmetric convex subset of $K_{P_0}^n$ and let R_2 be a closed additive subgroup of $K_{S\smallsetminus P_0}^n$. Suppose $R \subset K_S^n$ is equal to $R_1 \times R_2$ with the natural identification of K_S^n with $K_{P_0}^n \times K_{S\smallsetminus P_0}^n$. If $\mathbf{vol}(R) > 2^{n(\sigma+2\tau)}\mathbf{cov}(\Gamma)$, then R contains a nonzero element of Γ.*

Proof. Let $R' = (\frac{1}{2}R_1) \times R_2$. It follows from the assumption on the $\mathbf{vol}(R)$ that $\mathbf{vol}(R') > \mathbf{cov}(\Gamma)$. According to Lemma 4.1, we can find two distinct elements $\gamma_1, \gamma_2 \in \Gamma$ and $\xi \in K_S^n$ such that $\gamma_i - \xi \in R'$ for $i = 1, 2$. Therefore the nonzero element $\gamma_1 - \gamma_2$ belongs to R. □

Recall that $\lambda_m(\Gamma)$ $(1 \leq m \leq n)$ is the m-th minimum of a discrete I_S-module Γ, see (1.2). It follows directly from the definition that there exist K-linearly independent vectors $\xi_1, \ldots, \xi_n \in \Gamma$ with

$$\|\xi_m\| = \lambda_m \text{ for all } 1 \leq m \leq n.$$

Moreover, by Lemma 3.1 these vectors are also linearly independent over K_S.

According to Lemma 4.2 for any $0 < t < \lambda_1(\Gamma)$ we have

$$\Big(\prod_{v\in S\smallsetminus P_0} q_v^{-n}\Big) t^{n\sharp S}\mathbf{vol}\big(B_1(K_S^n)\big) \leq \mathbf{vol}\big(B_t(K_S^n)\big) \leq 2^{n(\sigma+2\tau)}\mathbf{cov}(\Gamma). \tag{4.2}$$

Since $B_1(\mathbb{R}^m)$ contains $\{(x_1,\dots,x_m)\in\mathbb{R}^m : -m^{-1/2}\le x_i\le m^{-1/2}\}$, for archimedean $v\in P$ we have

$$\mathbf{vol}_v\big(B_1(K_v^n)\big)\ge 2^{n\varepsilon_v}n^{-n\varepsilon_v/2}, \tag{4.3}$$

where for complex place we use $B_1(\mathbb{C}^n)=B_1(\mathbb{R}^{2n})$. By (4.2), (4.3) and (2.2) we have

$$\lambda_1(\Gamma)^{n\sharp S}\le n^{n(\sigma+2\tau)/2}\Big(\prod_{v\in S\smallsetminus P_0} q_v^n\Big)\mathbf{cov}(\Gamma). \tag{4.4}$$

Lemma 4.3. *Let Γ be a discrete I_S-module with finite covolume. Suppose that $\xi_1,\dots,\xi_n\in\Gamma$ are K-linearly independent vectors and $\|\xi_m\|=\lambda_m(\Gamma)$ for all $1\le m\le n$. Then there exists $g\in\mathrm{GL}_n(K_S)$ such that*

$$\mathbf{cont}\big(\mathbf{det}(g)\big)=\prod_{i=1}^n\mathbf{cont}(\xi_i)^{-1}, \tag{4.5}$$

and any nonzero vector of $\Gamma'\overset{\text{def}}{=}\Gamma g$ has norm greater than or equal to one.

Proof. Suppose that $\xi_i=(\xi_{i,v})_{v\in S}$ where $\xi_{i,v}\in K_v^n$ (the notation here is the same as §2). By Lemma 3.1, for every $v\in S$ the vectors $\xi_{1,v},\dots,\xi_{n,v}$ are K_v-linearly independent in K_v^n. Using Gram-Schmidt orthogonalization process (see Lemma 3.3 for the ultrametric case), for each $v\in S$ we can find an orthonormal basis $\eta_{1,v},\dots,\eta_{n,v}$ such that for every $1\le m\le n$ the K_v-linear span of $\eta_{1,v},\dots,\eta_{m,v}$ is the same as that of $\xi_{1,v},\dots,\xi_{m,v}$. Let $b_i=(b_{i,v})_{v\in S}\in K_S$ $(1\le i\le n)$ such that $|b_{i,v}|_v=\|\xi_{i,v}\|_v$. It follows from the definition of content that

$$\mathbf{cont}(b_i)=\mathbf{cont}(\xi_i). \tag{4.6}$$

Since $\eta_i\overset{\text{def}}{=}(\eta_{i,v})_{v\in S}$ $(1\le i\le n)$ is a K_S-basis of K_S^n, there is a unique $g\in\mathrm{GL}_n(K_S)$ such that $\eta_i g=b_i^{-1}\eta_i$. We claim that this g satisfies the requirement of the lemma.

The equation (4.5) follows from

$$\mathbf{cont}\big(\mathbf{det}(g)\big)=\mathbf{cont}(b_1^{-1}\dots b_n^{-1})=\prod_{i=1}^n\mathbf{cont}(b_i)^{-1}=\prod_{i=1}^n\mathbf{cont}(\xi_i)^{-1},$$

where in the last equality we use (4.6). For the other conclusion suppose that $\zeta=c_1\eta_1+\dots+c_m\eta_m\in\Gamma'=\Gamma g$ where $c_i\in K_S$ and $c_m\ne0$. We have

$$\zeta g^{-1}=c_1b_1\eta_1+\dots+c_mb_m\eta_m\in\Gamma.$$

Since for every $v\in S$ the basis $\eta_{1,v},\dots,\eta_{n,v}$ is orthonormal, we have

$$\|\zeta g^{-1}\|\le\|\zeta\|\max_{1\le i\le m}|b_i|=\|\zeta\|\cdot\lambda_m(\Gamma). \tag{4.7}$$

On the other hand for any $1\le j\le m$, the K_S-linear span of $\eta_1,\dots,\eta_j$ is the same as that of $\xi_1,\dots,\xi_j$. Since $c_mb_m\ne0$, Lemma 3.1 implies that $\xi_1,\dots,\xi_{m-1},\zeta g^{-1}$ are K-linearly independent. Thus it follows from the definition of the m-th minimum of $\Gamma = \Gamma' g^{-1}$ that $\|\zeta g^{-1}\| \geq \lambda_m(\Gamma)$. This estimate together with (4.7) imply $\|\zeta\| \geq 1$, which completes the proof. □

Theorem 4.4. *Let Γ be a discrete I_S-module with finite covolume. Let $\xi_1, \ldots, \xi_n \in \Gamma$ be K linearly independent vectors with $\|\xi_m\| = \lambda_m(\Gamma)$ for all $1 \leq m \leq n$. Then we have*

$$\mathbf{cov}(I_S^n)^{-1}\mathbf{cov}(\Gamma) \leq \prod_{i=1}^{n} \mathbf{cont}(\xi_i) \leq n^{n(\sigma+2\tau)/2}\Big(\prod_{v\in S\setminus P_0} q_v^n\Big)\mathbf{cov}(\Gamma). \tag{4.8}$$

Proof. We first prove the upper bound of (4.8). Suppose that $\Gamma' = \Gamma g$, where $g \in \mathrm{GL}_n(K_S)$, satisfies the conclusion of Lemma 4.3. Then $\lambda_1(\Gamma') \geq 1$. Applying (4.4) for Γ' we have

$$1 \leq \lambda(\Gamma')^{n\sharp S} \leq n^{n(\sigma+2\tau)/2}\Big(\prod_{v\in S\smallsetminus P_0} q_v^n\Big)\mathbf{cov}(\Gamma'). \tag{4.9}$$

On the other hand by (2.1) and (4.5)

$$\mathbf{cov}(\Gamma') = \mathbf{cov}(\Gamma)\cdot\mathbf{cont}\big(\mathbf{det}(g)\big) = \mathbf{cov}(\Gamma)\cdot\prod_{i=1}^{n}\mathbf{cont}(\xi_i)^{-1}. \tag{4.10}$$

The upper bound of (4.8) follows from (4.9) and (4.10).

Let Γ'' be the I_S-linear span of $\xi_1, \ldots, \xi_n$. Since Γ'' is a submodule of Γ, by Lemma 3.5 we get

$$\mathbf{cov}(\Gamma) \leq \mathbf{cov}(\Gamma'') \leq \prod_{i=1}^{n}\mathbf{cov}_{\mathbf{r}}(I_S\xi_i) = \mathbf{cov}(I_S^n)\cdot\prod_{i=1}^{n}\mathbf{cont}(\xi_i),$$

which implies lower bound of (4.8). □

To prove Theorem 1.2 we need a balance between contents and norms of vectors in K_S^n. The following lemma is a generalization of [9, Lemma 8.6] and [8, Lemma 5.9], and the proof is the same.

Lemma 4.5. *For any $\xi \in K_S^n$ with $\mathbf{cont}(\xi) \neq 0$, there exists $a \in I_S^*$ such that $\|a\xi\|^{\sharp S} \asymp \mathbf{cont}(\xi)$ where the implied constants depend on K and S.*

Proof. Suppose that $S = \{v_1, \ldots, v_m\}$ where $m = \mathbf{card}(S)$. Let $\mathbb{R}_+$ be the multiplicative group of positive real numbers. We define a map

$$\varphi : K \to \mathbb{R}_+^m \quad \text{by} \quad \varphi(a) = (|a|_{v_1}, \ldots, |a|_{v_m}).$$

Let

$$H = \{(r_1, \ldots, r_m) \in \mathbb{R}_+^m : \prod_{i=1}^{m} r_i^{\varepsilon_{v_i}} = 1\}.$$

It follows from Dirichlet's unit theorem (see [3, Chapter II §18]) that the group $\varphi(I_S^*) \subset H$ is a cocompact lattice in H. So there exists $A \geq 1$ which depends on K and S such that for any $(r_1, \ldots, r_m) \in H$ we can find $a \in I_S^*$ with

$$A^{-1} \leq r_i|a|_{v_i} \leq A. \tag{4.11}$$

Suppose $\xi = (\xi_v)_{v\in S}$. It follows from the definition that

$$(\|\xi_{v_1}\|_{v_1} \cdot \mathbf{cont}(\xi)^{-1/(m+\tau)}, \dots, \|\xi_{v_m}\|_{v_m} \cdot \mathbf{cont}(\xi)^{-1/(m+\tau)}) \in H.$$

By (4.11) one can find $a \in I_S^*$ such that for all $1 \le i \le m$

$$A^{-1} \le \|\xi_{v_i}\|_{v_i} \mathbf{cont}(\xi)^{-1/(m+\tau)} |a|_{v_i} \le A.$$

Therefore

$$A^{-m-\tau}\mathbf{cont}(\xi) \le \|a\xi\|^{m+\tau} \le A^{m+\tau}\mathbf{cont}(\xi).$$

□

Proof of Theorem 1.2. Let $\xi_1, \dots, \xi_n \in \Gamma$ be K-linearly independent vectors with $\|\xi_i\| = \lambda_i(\Gamma)$. By Theorem 4.4

$$\prod_{i=1}^n \mathbf{cont}(\xi_i) \asymp \mathbf{cov}(\Gamma), \tag{4.12}$$

where the implied constants depend on K, S and n. The definitions of content and norm imply

$$\mathbf{cont}(\xi_i) \le \lambda_i(\Gamma)^{\sharp S}. \tag{4.13}$$

According to Lemma 4.5 there exists $a_1, \dots, a_n \in I_S^*$ such that

$$\mathbf{cont}(\xi_i) \gg \|a_i\xi_i\|^{\sharp S}, \tag{4.14}$$

where the implied constant depends on K and S. Note that elements $a_1\xi_1, \dots, a_n\xi_n \in \Gamma$ are linear independent over K. So the definition of successive minima implies

$$\prod_{i=1}^n \lambda_i(\Gamma) \le \prod_{i=1}^n \|a_i\xi_i\|. \tag{4.15}$$

Therefore the conclusion of Theorem 1.2 follows from (4.12), (4.13), (4.14) and (4.15). □

5. Mahler's compactness criterion

Let $X = \mathrm{SL}_n(I_S)\backslash \mathrm{SL}_n(K_S)$. There is a one-to-one correspondence between X and

$$\{I_S^n g : g \in \mathrm{SL}_n(K_S)\}$$

via the map $\mathrm{SL}_n(I_S)g \to I_S^n g$. In this section $\mathbf{e}_1, \dots, \mathbf{e}_n$ denotes the standard basis of K_S^n, i.e. $\mathbf{e}_i$ has i-th entry 1 and other entries 0. Before proving Theorem 1.1 we need the following lemma. See [9, Corollary 8.6] for $K = \mathbb{Q}$ and [8, Corollary 5.1] for K a number field.

Lemma 5.1. *Let $M > 0$. Then there are only finitely many I_S-submodules Γ of I_S^n such that* $\mathbf{card}(I_S^n/\Gamma) \le M$.

Proof. Let $\Gamma \subset I_S^n$ be an I_S-submodule with $\mathbf{card}(I_S^n/\Gamma) \leq M$. For every $1 \leq i \leq n$ there is an ideal J_i of I_S such that

$$I_S\mathbf{e}_i \cap \Gamma = J_i\mathbf{e}_i \quad \text{and} \quad \mathbf{card}(I_S/J_i) \leq M.$$

Therefore

$$J_1 \times \cdots \times J_n \subset \Gamma \subset I_S^n.$$

Note that I_S is a Dedekind domain. It follows from the structure theory of ideals in I_S that there are only finitely many ideals J in I_S such that $\mathbf{card}(I_S/J) \leq M$. So the conclusion of the lemma holds. □

Proof of Theorem 1.1. Let $\pi : \mathrm{SL}_n(K_S) \to X$ be the natural quotient map and let

$$r = \inf\{\|\xi g\| : \xi \in I_S^n,\ \xi \neq 0,\ g \in \mathrm{SL}_n(K_S),\ \pi(g) \in R\}. \tag{5.1}$$

Suppose R is relatively compact. There exists a relatively compact subset $F \subset \mathrm{SL}_n(K_S)$ with $\pi(F) = R$. Therefore there exists $C > 0$ such that

$$\|\xi g\| \leq C\|\xi\| \quad \text{for every} \quad \xi \in K_S^n \quad \text{and} \quad g \in F. \tag{5.2}$$

The discreteness of Γ and (5.2) imply $r > 0$.

Now we assume $r > 0$ and prove that R is relatively compact. Let $\{g_i\}_{i\geq 1}$ be a sequence in $\pi^{-1}(R)$. It suffices to show that there exists $g \in \mathrm{SL}_n(K_S)$ such that $\pi(g)$ is a limit point of a subsequence of $\{\pi(g_i)\}_{i\geq 1}$. By Theorem 1.2 there exists $C \geq 1$ such that for any free I_S-module $\Gamma \in R$ one has

$$r \leq \lambda_1(\Gamma) \leq \lambda_n(\Gamma) \leq C. \tag{5.3}$$

For every $i \geq 1$ let $\xi_1^{(i)}, \ldots, \xi_n^{(i)} \in I_S^n$ be K-linearly independent vectors such that $\|\xi_j^{(i)} g_i\|$ equals to the j-th minimum of $I_S^n g_i$. By (5.3) we have

$$\|\xi_j^{(i)} g_i\| \leq C \quad \forall\ i \geq 1 \text{ and } 1 \leq j \leq n. \tag{5.4}$$

Let

$$\Gamma_i = \mathbf{span}_{I_S}\{\xi_j^{(i)} : 1 \leq j \leq n\}.$$

According to (5.3) and Theorem 1.2 there exists $M > 0$ such that

$$\mathbf{cov}(I_S^n) \leq \mathbf{cov}(\Gamma_i) = \mathbf{cov}(\Gamma_i g_i) \leq M \quad \forall\ i \geq 1. \tag{5.5}$$

By Lemma 5.1 and (5.5), the set $\{\Gamma_i : i \geq 1\}$ is finite. Therefore by possibly passing to a subsequence we may assume that there exists $h \in \mathrm{GL}_n(K)$ such that $\Gamma_i = I_S^n h$ for all $i \geq 1$. It follows that there is a sequence $\{f_i\}_{i\geq 1}$ in $\mathrm{GL}_n(I_S)$ such that $\mathbf{e}_j f_i h = \xi_j^{(i)}$ for all $i \geq 1$ and $1 \leq j \leq n$. By (5.4) there is a subsequence $\{g_{i_k}\}_{k\geq 1}$ of $\{g_i\}_{i\geq 1}$ and $g \in \mathrm{GL}_n(K_S)$ such that

$$f_{i_k} h g_{i_k} \to g \quad \text{as} \quad k \to \infty. \tag{5.6}$$

Since $\mathbf{det}$ is continuous and I_S is discrete in K_S, for k sufficiently large we have $\mathbf{det}(f_{i_k}) = \mathbf{det}(f_{i_{k+1}}) \in I_S^*$. Therefore by possibly passing to a subsequence and multiplying the first row of h by some element of I_S^*, we assume without loss of generality that $f_{i_k} \in \mathrm{SL}_n(I_S)$ for all k.

The group $h^{-1}\,\mathrm{SL}_n(I_S)h \cap \mathrm{SL}_n(I_S)$ has finite index in $h^{-1}\,\mathrm{SL}_n(I_S)h$. So by possibly passing to a subsequence we can find $f \in h^{-1}\,\mathrm{SL}_n(I_S)h$ and a sequence $\{h_k\}_{k\geq 1}$ of $\mathrm{SL}_n(I_S)$ such that

$$h^{-1} f_{i_k} h = f h_k \quad \forall\ k \geq 1. \tag{5.7}$$

By (5.6) and (5.7) we have $hfh_k g_{i_k} \to g$ as $k \to \infty$. Therefore

$$h_k g_{i_k} \to f^{-1}h^{-1}g \text{ as } k \to \infty.$$

Since $h_k \in \mathrm{SL}_d(I_S)$ we have $\pi(g_{i_k}) \to \pi(f^{-1}h^{-1}g)$ as $k \to \infty$. □

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Department of Mathematics, Brandeis University, Waltham, MA 02454-9110, USA

E-mail address: `kleinboc@brandeis.edu`

School of Mathematical Sciences, Xiamen University, Xiamen 361005, PR China

E-mail address: `ronggang@xmu.edu.cn`

Institut Girard Desargues, Université Claude Bernard-Lyon 1, 69622 Villeurbanne cedex, France

E-mail address: `Georges.Tomanov@desargues.univ-lyon1.fr`