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Integer solutions to decomposable form inequalities

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Abstract

This paper obtains a result on the finiteness of the number of integer solutions to decomposable form inequalities. Let k be a number field and let $F(X_1, \ldots, X_m)$ be a non-degenerate decomposable form with coefficients in k. We prove that, for every finite set of places S of k containing the archimedean places of k, for each real number $\lambda < \frac{1}{m-1}$ and for each constant $c > 0$, the inequality

$$0 < \prod_{v \in S} \|F(x_1, \ldots, x_m)\|_v \leqslant c H_S^{\lambda}(x_1, \ldots, x_m) \quad \text{in } (x_1, \ldots, x_m) \in \mathcal{O}_S^m. \tag{1}$$

has only finitely many $\mathcal{O}_S^*$-non-proportional solutions.

Keywords: Decomposable form inequality; Integer solutions; Diophantine approximations; Schmidt's subspace theorem

1. Introduction

Let k be a finitely generated (but not necessarily algebraic) extension field of $\mathbb{Q}$. Let $F(X_1, \ldots, X_m)$ be a form (homogeneous polynomial) in $m \geqslant 2$ variables with co-

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efficients in k and suppose that F is *decomposable*, i.e. it factorizes into linear factors over some finite extension of k. Let $b \in k^*$, where k^* is the set of non-zero elements of k, and consider the decomposable form equation

$$F(x_1, \dots, x_m) = b \quad \text{in } (x_1, \dots, x_m) \in R^m, \tag{1.1}$$

where R is a subring of k finitely generated over $\mathbb{Z}$. Equations of this type are of fundamental importance in the theory of Diophantine equations and have many applications in algebraic number theory. Important classes of such equations are Thue equations (when $m = 2$), norm form equations, discriminant form equations and index form equations. The Thue equations are named after A. Thue [Th] who proved, in the case $k = \mathbb{Q}, R = \mathbb{Z}, m = 2$, that if F is a binary form having at least three pairwise linearly independent linear factors in its factorization over the field of algebraic numbers, then (1.1) has only finitely many solutions. Later, Lang [L1] extended Thue's result to the general case when k is a finitely generated extension field of $\mathbb{Q}$ and R is a subring of k finitely generated over $\mathbb{Z}$. For the case $m \geqslant 2$, after the works of Schmidt, Schlickewei, Laurent and others (cf. [Sch1,Schli,LA]), Evertse and Györy [EG1] finally obtained a necessary and sufficient condition for (1.1) to have finitely many solutions, independently of the choice of b and R. In Section 3 of [EG1], Evertse and Györy gave an equivalent form of this condition in the case where F factors into a product of linear forms over k. The following is the statement of their result.

Theorem A (*Evertse and Györy*). *Let k be a finitely generated extension field of $\mathbb{Q}$. Let $F(X_1, \dots, X_m)$ be a decomposable form in $m \geqslant 2$ variables with coefficients in k. Assume that it factors into a product of linear forms over k. Denote by $\mathcal{L}$ a maximal set of linear factors of F which are pairwise linearly independent. Then the following two statements are equivalent:*

(i) *For every $b \in k^*$, the equation*

$$F(x_1, \dots, x_m) = b \quad \text{in } (x_1, \dots, x_m) \in R^m$$

has only finitely many solutions for every R, a subring of k finitely generated over $\mathbb{Z}$.

(ii) *The subspace $(\mathcal{L})$ of $k[X_1, \dots, X_m]$ generated by $\mathcal{L}$ has dimension m and that, for each proper non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, the intersection $(\mathcal{L}_1) \cap (\mathcal{L} \backslash \mathcal{L}_1)$ contains a non-zero element of $\mathcal{L}$.*

Note that the condition (ii) is independent of the choice of $\mathcal{L}$.

The purpose of this paper is to study decomposable form inequalities when k is assumed to be a number field. To state our result, we first recall some definitions.

Let k be a number field of degree d. Denote by $\mathbf{M}(k)$ the set of places (equivalent classes of absolute values) of k and write $\mathbf{M}_\infty(k)$ for the set of archimedean places of k. For $\upsilon \in \mathbf{M}(k)$ we choose the normalized absolute value $|\ \ |_\upsilon$ such that $|\ \ |_\upsilon = |\ \ |$ on $\mathbb{Q}$ (the standard absolute value) if υ is archimedean, whereas for υ non-archimedean

$|p|_\upsilon = p^{-1}$ if υ lies above the rational prime p. Denote by k_υ the completion of k with respect to υ and by $d_\upsilon = [k_\upsilon : \mathbb{Q}_\upsilon]$ the local degree. We put $\|\ \|_\upsilon = |\ |_\upsilon^{d_\upsilon/d}$. Let S be a finite subset of $\mathbf{M}(k)$ containing $\mathbf{M}_\infty(k)$. An element $x \in k$ is said to be a S-integer if $\|x\|_\upsilon \leqslant 1$ for each $\upsilon \in \mathbf{M}(k)\backslash S$. Denote by $\mathcal{O}_S$ the set of S-integers. The units of $\mathcal{O}_S$ are called S-units. The set of all S-units forms a multiplicative group which is denoted by $\mathcal{O}_S^*$. For $\mathbf{x} = (x_1, \dots, x_m) \in k^m$, we put $\|\mathbf{x}\|_\upsilon = \max_{1\leqslant i\leqslant m} \|x_i\|_\upsilon$ and we define the height of $\mathbf{x}$ by $H(\mathbf{x}) = \prod_{\upsilon\in\mathbf{M}(k)} \|\mathbf{x}\|_\upsilon$, and the logarithmic height of $\mathbf{x}$ by $h(\mathbf{x}) = \log H(\mathbf{x})$. By the product formula, $H(\lambda\mathbf{x}) = H(\mathbf{x})$ for all $\lambda \in k^*$. For $\mathbf{x} = (x_1, \dots, x_m) \in k^m$, we also define the S-height as $H_S(\mathbf{x}) = \prod_{\upsilon\in S} \|\mathbf{x}\|_\upsilon$. If $\mathbf{x} \in O_S^m\backslash\{\mathbf{0}\}$, then $H_S(\mathbf{x}) \geqslant 1$ and $H_S(\alpha\mathbf{x}) = H_S(\mathbf{x})$ for all $\alpha \in \mathcal{O}_S^*$.

Let k be a number field, and let $F(X_1, \dots, X_m)$ be a decomposable form in $m \geqslant 2$ variables with coefficients in k. For each finite set of places S of k containing the archimedean places of k, and for given two positive real numbers c and λ, we consider the solutions of the inequality

$$0 < \prod_{\upsilon\in S} \|F(x_1, \dots, x_m)\|_\upsilon \leqslant c H_S^\lambda(x_1, \dots, x_m) \quad \text{in } (x_1, \dots, x_m) \in \mathcal{O}_S^m. \tag{1.2}$$

If $(x_1, \dots, x_m)$ is a solution of (1.2), then so is $(\eta x_1, \dots, \eta x_m)$ for every $\eta \in \mathcal{O}_S^*$. Solution $(\eta x_1, \dots, \eta x_m)$ is said to be $\mathcal{O}_S^*$-proportional to $(x_1, \dots, x_m)$. To state our result, we need the following definition.

Definition 1.1. Let k be a number field and let $F(X_1, \dots, X_m)$ be a decomposable form in $m \geqslant 2$ variables with coefficients in k. We say that F is non-degenerate if it satisfies the following conditions: there exists a finite algebraic extension k' of k such that F factors into a product of linear forms over k' and if we denote by $\mathcal{L}$ a maximal set of linear factors of F which are pairwise linearly independent, then the subspace $(\mathcal{L})$ of $k'[X_1, \dots, X_m]$ generated by $\mathcal{L}$ over k' has dimension m and that, for each proper non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, the intersection $(\mathcal{L}_1) \cap (\mathcal{L}\backslash\mathcal{L}_1)$ contains a non-zero element of $\mathcal{L}$.

Note that the above definition is independent of the choice of $\mathcal{L}$. The main result of this paper is as follows.

Theorem 1.1. *Let k be a number field and let $F(X_1, \dots, X_m)$ be a non-degenerate decomposable form with coefficients in k. Then, for every finite set of places S of k containing the archimedean places of k, for each real number $\lambda < \frac{1}{m-1}$ and for each constant $c > 0$, the inequality*

$$0 < \prod_{\upsilon\in S} \|F(x_1, \dots, x_m)\|_\upsilon \leqslant c H_S^\lambda(x_1, \dots, x_m) \quad \text{in } (x_1, \dots, x_m) \in \mathcal{O}_S^m$$

has only finitely many $\mathcal{O}_S^*$-non-proportional solutions.

Theorem 1.1 implies in particular that, for each $b \in k^*$, the equation $F(x_1, \ldots, x_m) = b$ has finitely many solutions $(x_1, \ldots, x_m) \in \mathcal{O}_S{}^m$. Hence, it could be viewed as a quantitative extension of the result of Evertse and Győry [EG1] in the number field case. Consider a special case that $k = \mathbb{Q}$ and $S = \{\infty, p_1, \ldots, p_s\}$, where $p_1, \ldots, p_s$ are primes. Recall that in this case, $|\ |_\infty$ is the standard absolute value, while $|p_i|_{p_i} = p_i^{-1}$ for $i = 1, \ldots, s$. Further, every solution $\mathbf{x} \in \mathcal{O}_S^m$ of (1.2) is $\mathcal{O}_S^*$-proportional to a solution $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ with $\gcd(x_1, \ldots, x_m) = 1$ which is unique up to a factor -1. Moreover, for such a solution $(x_1, \ldots, x_m)$, $H_S(x_1, \ldots, x_m) = \max_{1 \leqslant i \leqslant m} |x_i|$. This leads us to the following consequence of Theorem 1.1.

Corollary 1.1. *Let $F(X_1, \ldots, X_m)$ be a decomposable form in $m \geqslant 2$ variables with coefficients in $\mathbb{Q}$. Assume that F is non-degenerate. Then, for each real number $\lambda < \frac{1}{m-1}$ and for each constant $c > 0$, the inequality*

$$0 < |F(x_1, \ldots, x_m)| \prod_{i=1}^{s} |F(x_1, \ldots, x_m)|_{p_i} \leqslant c \left(\max_{1 \leqslant i \leqslant m} |x_i| \right)^{\lambda}$$

has only finitely many solutions $(x_1, \ldots, x_m) \in \mathbb{Z}^m$ with $\gcd(x_1, \ldots, x_m) = 1$.

Important examples of non-degenerate decomposable forms are those $F(X_1, \ldots, X_m)$ such that $\deg F > 2(m-1)$ and that any m linear factors of F over $\overline{\mathbb{Q}}$ are linearly independent. In this case, K. Győry and the second author proved a stronger result (cf. [GR]) as follows.

Theorem B (*Győry and Ru***).** *Let k be a number field and let $F(X_1, \ldots, X_m)$ be a decomposable form in $m \geqslant 2$ variables with coefficients in k. Assume that* $\deg F > 2(m-1)$ *and* $\lambda < \deg F - 2(m-1)$. *Assume further that any m linear factors of F are linearly independent over* $\overline{\mathbb{Q}}$ *. Then, for each constant* $c > 0$, (1.2) *has only finitely many* $\mathcal{O}_S^*$*-non-proportional solutions.*

Note that, if we take $\deg F = 2(m-1) + 1$ in above, then the condition for λ becomes $\lambda < 1$. Thus, we conjecture that the condition $\lambda < \frac{1}{m-1}$ in Theorem 1.1 could be improved to $\lambda < 1$.

2. Generalization of Schmidt's subspace theorem

In this section, we prove a Schmidt's subspace-type theorem. In the theorem, we drop the "in general position" assumption for linear forms appearing in Schmidt's subspace theorem, only assuming that they are non-degenerate. Here, the meaning of that *a set of pairwise linearly independent linear forms is non-degenerate* is as follows.

Definition 2.1. Let k be a number field. A set $\mathcal{L}$ of finitely many pairwise linearly independent linear forms in $n+1$ variables with coefficients in k is said to be *non-degenerate* if the subspace $(\mathcal{L})$ of $k[X_0, \ldots, X_n]$ generated by $\mathcal{L}$ has dimension $n+1$

and that, for each proper non-empty subset $\mathcal{L}_1$ of $\mathcal{L}$, the intersection $(\mathcal{L}_1) \cap (\mathcal{L} \backslash \mathcal{L}_1)$ contains a non-zero element of $\mathcal{L}$.

To state our result, we first recall the following statement of Schmidt's subspace theorem, due to Vojta (see [V]).

Schmidt's Subspace Theorem. *Let k be a number field and let S be a finite set of places of k. Given linear forms $L_1, \ldots, L_q \in k[X_0, \ldots, X_n]$ in general position, i.e., any $n+1$ linear forms among them are linearly independent. Then, for every $\varepsilon > 0$,*

$$\sum_{j=1}^{q} \sum_{v \in S} \log \frac{\|\mathbf{x}\|_v \cdot \|L_j\|_v}{\|L_j(\mathbf{x})\|_v} \leqslant (n+1+\varepsilon) h(\mathbf{x}) \tag{2.1}$$

holds for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside a finite union of proper linear subspaces of $\mathbb{P}^n(k)$.

Let $L \in k[X_0, \ldots, X_n]$ be a linear form. Define, for every $\mathbf{x} \in \mathbb{P}^n(k)$ with $L(\mathbf{x}) \neq 0$,

$$m(\mathbf{x}, L) = \sum_{v \in S} \log \frac{\|\mathbf{x}\|_v \cdot \|L\|_v}{\|L(\mathbf{x})\|_v}$$

and

$$N(\mathbf{x}, L) = \sum_{v \notin S} \log \frac{\|\mathbf{x}\|_v \cdot \|L\|_v}{\|L(\mathbf{x})\|_v}.$$

Then, by the product formula,

$$m(\mathbf{x}, L) + N(\mathbf{x}, L) = h(\mathbf{x}) + O(1) \tag{2.2}$$

holds for all $\mathbf{x}$ with $L(\mathbf{x}) \neq 0$, where $O(1)$ is a constant, independent of $\mathbf{x}$. Hence we can rewrite (2.1) as

$$(q-n-1-\varepsilon) h(\mathbf{x}) \leqslant \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1). \tag{2.3}$$

We prove the following result which might be interesting in itself.

Theorem 2.1. *Let k be a number field and let S be a finite set of places of k. Let $\mathcal{L} = \{L_1, \ldots, L_q\}$ be a finite set of pairwise linearly independent linear forms in $n+1$ variables with coefficients in k. Assume that $\mathcal{L}$ is non-degenerate. Then, for $\varepsilon > 0$, we have,*

$$(1-\varepsilon) h(\mathbf{x}) \leqslant n \cdot \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.4}$$

for every $\mathbf{x} \in \mathbb{P}^n(k)$ with $L_j(\mathbf{x}) \neq 0$ for *$j = 1, \ldots, q$.*

Proof. Before proving Theorem 2.1, we first make two observations. First, we have the following height inequality:

$$h[z : x_1 : \cdots, x_m : y_1 : \cdots : y_l] \leqslant h[z : x_1 : \cdots : x_m] + h[z : y_1 : \cdots : y_l] \tag{2.5}$$

for $z, x_1, \ldots, x_m, y_1, \ldots, y_l \in k$ with $z \neq 0$. To show (2.5), we recall that, for every $[x_0 : \cdots : x_n] \in \mathbb{P}^n(k)$,

$$h([x_0 : \cdots : x_n]) = \sum_{v \in \mathbf{M}(k)} \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$

Hence

$$\begin{aligned} h([z : x_1 : \cdots : x_m : y_1 : \cdots : y_l]) &= h([1 : x_1/z : \cdots : x_m/z : y_1/z : \cdots : y_l/z]) \\ &\leqslant h([1 : x_1/z : \cdots : x_m/z]) \\ &\quad + h([1 : y_1/z : \cdots : y_l/z]) \\ &= h([z : x_1 : \cdots : x_m]) + h([z : y_1 : \cdots : y_l]). \end{aligned}$$

This proves (2.5). The second observation is that the non-degeneracy is preserved by restriction to the linear subspaces V of $\mathbb{P}^n$ such that none of the linear forms in $\mathcal{L}$ vanishes identically on V. To show this, let V be a subspace of $\mathbb{P}^n$ such that none of the linear forms in $\mathcal{L}$ vanishes identically on V. For a linear form $L \in \mathcal{L}$ denote by $L_{|V}$ the restriction of L to V. Let $\mathcal{M}$ be a maximal subset of pairwise linearly independent linear forms from $\{L_{|V} : L \in \mathcal{L}\}$. Then we claim that $\mathcal{M}$ is non-degenerate. In fact, since $\dim(\mathcal{L}) = n+1$, $\dim(\mathcal{M}) = \dim V + 1$, where $\dim V$ is the projective dimension of V. Next, let $\mathcal{M}_1$ be a non-empty proper subset of $\mathcal{M}$. Let $\mathcal{L}_1$ be the set of all linear forms $L \in \mathcal{L}$ such that $L_{|V}$ is proportional to a linear form in $\mathcal{M}_1$. Since $\mathcal{L}$ is non-degenerate, there is a linear form $L \in (\mathcal{L}_1) \cap (\mathcal{L} \backslash \mathcal{L}_1) \cap \mathcal{L}$. Taking restrictions to V we obtain a non-zero linear form in $(\mathcal{M}_1) \cap (\mathcal{M} \backslash \mathcal{M}_1) \cap \mathcal{M}$. So $\mathcal{M}$ is non-degenerate.

To continue, for a subset $I = \{i_1, \ldots, i_t\}$ of $\{1, \ldots, q\}$, we define

$$P_{I,\mathbf{x}} := [L_{i_1}(\mathbf{x}) : \cdots : L_{i_t}(\mathbf{x})].$$

We prove by induction on s the following claim:

Claim. *For every s with $2 \leqslant s \leqslant n+1$ there is a subset I of $\{1, \ldots, q\}$ (independent of $\mathbf{x}$) with rank$\{L_i : i \in I\} \geqslant s$ such that, for every $\varepsilon > 0$, the inequality*

$$(1-\varepsilon) h(P_{I,\mathbf{x}}) \leqslant (s-1) \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.6}$$

holds for every $\mathbf{x} \in \mathbb{P}^n(k)$ *outside some finite union of proper linear subspaces of* $\mathbb{P}^n(k)$.

To prove the Claim, we first settle the case $s = 2$. Since $\mathcal{L}$ is non-degenerate, there is a linear relation

$$\sum_{i \in I'} c_i L_i = 0, \tag{2.7}$$

where I' is a subset of $\{1, \ldots, q\}$ of cardinality $\geqslant 3$ and all $c_i \neq 0$. By shrinking I' if needed, we may assume that each proper subset of $\{L_i : i \in I'\}$ is linearly independent. Further, the set $\{L_i : i \in I'\}$ has rank at least 2. Without loss of generality, we assume that $I' = \{1, \ldots, t+1\}$. Let $I = \{1, \ldots, t\}$. Then $\{L_i : i \in I\}$ is linearly independent and it also has rank at least 2. Applying Schmidt's subspace theorem to the linear forms $\tilde{L}_1 = c_1 X_1, \ldots, \tilde{L}_t = c_t X_t$ and $\tilde{L}_{t+1} = c_1 X_1 + \cdots + c_t X_t$ in $\mathbb{P}^{t-1}(k)$, we have, for every $\varepsilon > 0$,

$$(1-\varepsilon) h(P) \leqslant \sum_{l=1}^{t+1} N(P, \tilde{L}_l) \tag{2.8}$$

for all $P \in \mathbb{P}^{t-1}(k)$ outside a finite union of proper linear subspaces $T_1, \ldots, T_M$. Since $\{L_i : i \in I\}$ is linearly independent, the points $\mathbf{x} \in \mathbb{P}^n(k)$ with $P_{I,\mathbf{x}} \in \cup_{\alpha=1}^M T_\alpha$ is contained in some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. Hence, outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$, we have

$$(1-\varepsilon) h(P_{I,\mathbf{x}}) \leqslant \sum_{l=1}^{t+1} N(P_{I,\mathbf{x}}, \tilde{L}_l). \tag{2.9}$$

For $1 \leqslant l \leqslant t$,

$$\begin{aligned} N(P_{I,\mathbf{x}}, \tilde{L}_l) &= \sum_{v \notin S} \log \frac{\|P_{I,\mathbf{x}}\|_v \cdot \|c_l\|_v}{\|c_l L_l(\mathbf{x})\|_v} \\ &\leqslant \sum_{v \notin S} \log \frac{\|\mathbf{x}\|_v \cdot \max_{1 \leqslant j \leqslant t} \|L_j\|_v}{\|L_l(\mathbf{x})\|_v} \\ &= \sum_{v \notin S} \log \frac{\|\mathbf{x}\|_v \cdot \|L_l\|_v}{\|L_l(\mathbf{x})\|_v} + \sum_{v \notin S} \log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_v}{\|L_l\|_v}. \end{aligned}$$

Note that,

$$\log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_v}{\|L_l\|_v}$$

vanishes for all, but finitely many, places υ, so

$$\sum_{\upsilon \notin S} \log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_\upsilon}{\|L_l\|_\upsilon}$$

is a constant, independent of $\mathbf{x}$. Therefore, we have, for $1 \leqslant l \leqslant t$,

$$N(P_{I,\mathbf{x}}, \tilde{L}_l) = N(\mathbf{x}, L_l) + O(1), \tag{2.10}$$

where $O(1)$ is a constant, independent of $\mathbf{x}$. Further, by (2.7), $c_{t+1}L_{t+1}(\mathbf{x}) = -(c_1 L_1(\mathbf{x}) + \cdots + c_t L_t(\mathbf{x}))$. Hence,

$$\begin{aligned} N(P_{I,\mathbf{x}}, \tilde{L}_{t+1}) &= \sum_{\upsilon \notin S} \log \frac{\|P_{I,\mathbf{x}}\|_\upsilon \cdot \|\tilde{L}_{t+1}\|_\upsilon}{\|\tilde{L}_{t+1}(P_{I,\mathbf{x}})\|_\upsilon} \\ &= \sum_{\upsilon \notin S} \log \frac{\|P_{I,\mathbf{x}}\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|c_j\|_\upsilon}{\|c_1 L_1(\mathbf{x}) + \cdots + c_t L_t(\mathbf{x})\|_\upsilon} \\ &\leqslant \sum_{\upsilon \notin S} \log \frac{\|\mathbf{x}\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|L_j\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|c_j\|_\upsilon}{\|c_{t+1} L_{t+1}(\mathbf{x})\|_\upsilon} \\ &= \sum_{\upsilon \notin S} \log \frac{\|\mathbf{x}\|_\upsilon \cdot \|L_{t+1}\|_\upsilon}{\|L_{t+1}(\mathbf{x})\|_\upsilon} \\ &\quad + \sum_{\upsilon \notin S} \log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|c_j\|_\upsilon}{\|L_{t+1}\|_\upsilon \cdot \|c_{t+1}\|_\upsilon}. \end{aligned}$$

Again, since

$$\log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|c_j\|_\upsilon}{\|L_{t+1}\|_\upsilon \cdot \|c_{t+1}\|_\upsilon}$$

vanishes for all, but finitely many, places υ,

$$\sum_{\upsilon \notin S} \log \frac{\max_{1 \leqslant j \leqslant t} \|L_j\|_\upsilon \cdot \max_{1 \leqslant j \leqslant t} \|c_j\|_\upsilon}{\|L_{t+1}\|_\upsilon \cdot \|c_{t+1}\|_\upsilon}$$

is a constant, independent of $\mathbf{x}$. Hence,

$$N(P_{I,\mathbf{x}}, \tilde{L}_{t+1}) = N(\mathbf{x}, L_{t+1}) + O(1). \tag{2.11}$$

Combining (2.9)–(2.11), we have,

$$(1-\varepsilon)h(P_{I,\mathbf{x}})\leqslant \sum_{l=1}^{t+1} N(\mathbf{x}, L_l) + O(1)\leqslant \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.12}$$

for every $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. Hence the claim is proved for $s = 2$.

Now, let $2\leqslant s\leqslant n$ and assume that the claim holds for s, i.e., a subset I of $\{1, \ldots, q\}$ (independent of $\mathbf{x}$) exists with $\mathrm{rank}\{L_i : i \in I\}\geqslant s$, such that, for every $\varepsilon > 0$, (2.6) holds for every $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. If either $\mathrm{rank}\{L_i : i \in I\} > s$ or $s = n$, then the induction step is completed. So we assume that $\mathrm{rank}\{L_i : i \in I\} = s < n$. Let $\mathcal{A} = \{L \in \mathcal{L} : L \in (L_i : i \in I)\}$, where $(L_i : i \in I)$ is the subspace of $k[X_0, \ldots, X_n]$ generated by the linear forms $L_i, i \in I$. $\mathcal{A}$ is then a non-empty proper subset of $\mathcal{L}$. By the non-degeneracy of $\mathcal{L}$, there is a linear form $L_{i_0} \in (\mathcal{A}) \cap (\mathcal{L}\backslash\mathcal{A}) \cap \mathcal{L}$. Then $L_{i_0} = \sum_{i\in I} c_i L_i$ for certain $c_i \in k$, while also there is a linearly independent subset $\{L_j : j \in J\}$ of $\mathcal{L}\backslash\mathcal{A}$ such that $L_{i_0} = \sum_{j\in J} d_j L_j$ for certain $d_j \in k$ with $d_j \neq 0$. Notice that $\mathcal{A}$, J are independent of $\mathbf{x}$. Since $h(P_{I\cup\{i_0\},\mathbf{x}})\leqslant h(P_{I,\mathbf{x}})+O(1)$ and by the induction hypothesis, for every $\varepsilon > 0$,

$$(1-\varepsilon)h(P_{I,\mathbf{x}})\leqslant (s-1)\sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1),$$

holds for every $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$, we have

$$(1-\varepsilon)h(P_{I\cup\{i_0\},\mathbf{x}})\leqslant (s-1)\sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.13}$$

for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. On the other hand, completely similar to (2.12) we have, for every $\varepsilon > 0$, the inequality

$$(1-\varepsilon)h(P_{J\cup\{i_0\},\mathbf{x}})\leqslant \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.14}$$

for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. Now let $\tilde{I} := \{i_0\}\cup I\cup J$. Then $\mathrm{rank}\{L_i : i \in \tilde{I}\}\geqslant \mathrm{rank}\{L_i : i \in I\}+1\geqslant s+1$ since each form L_j with $j \in J$ is linearly independent of the linear forms in $\mathcal{A}$, hence of $L_i, i \in I$. Further, by (2.5),

$$h(P_{\tilde{I},\mathbf{x}})\leqslant h(P_{I\cup\{i_0\},\mathbf{x}}) + h(P_{J\cup\{i_0\},\mathbf{x}}).$$

By combining this with (2.13) and (2.14) we obtain

$$(1-\varepsilon)h(P_{\tilde{I},\mathbf{x}}) \leqslant s \cdot \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1)$$

for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. This completes the induction step, and thus proves the Claim.

We now continue the proof of Theorem 2.1. Let I be a subset of $\{1, \ldots, q\}$ as in the Claim with $s = n+1$. Then all linear forms in $n+1$ variables $X_0, \ldots, X_n$ can be expressed as linear combinations of the linear forms $L_i (i \in I)$, so in particular the forms $X_0, \ldots, X_n$. Consequently, $h(\mathbf{x}) \leqslant h(P_{\tilde{I},\mathbf{x}}) + O(1)$ for $\mathbf{x} \in \mathbb{P}^n(k)$. Fixing $\varepsilon > 0$, by our claim we have

$$(1-\varepsilon)h(\mathbf{x}) \leqslant n \cdot \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \tag{2.15}$$

holds for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. We complete the proof of the theorem by induction on n. For $n = 0$ the theorem is clearly true. Suppose that the theorem is true for projective spaces of dimension at most $n-1$ for some $n \geqslant 1$. Consider inequality (2.4) for dimension n. We know, from (2.15) that (2.4) holds for all $\mathbf{x} \in \mathbb{P}^n(k)$ outside some finite union of proper linear subspaces of $\mathbb{P}^n(k)$. Let V be one of these exceptional subspaces. Since we only consider those points $\mathbf{x}$ with $L(\mathbf{x}) \neq 0$ for every $L \in \mathcal{L}$, we only need to consider those V such that none of the linear forms from $\mathcal{L}$ vanishes identically on V. Then, by our second observation stated in the beginning of the proof that $\mathcal{M}$ is non-degenerate, hence the induction hypothesis is applicable. So

$$(1-\varepsilon)h(\mathbf{x}) \leqslant \dim(V) \cdot \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1) \leqslant n \cdot \sum_{j=1}^{q} N(\mathbf{x}, L_j) + O(1)$$

for $\mathbf{x} \in V(k)$. By applying this to all exceptional subspaces we infer that (2.4) holds for all $\mathbf{x} \in \mathbb{P}^n(k)$. This completes the proof. $\square$

3. Proof of the main theorem

By the assumption, F is non-degenerate. So there exits a finite algebraic extension k' of k such that

$$F(X_1, \ldots, X_m) = L_1(X_1, \ldots, X_m) \cdots L_q(X_1, \ldots, X_m),$$

where $L_1, \ldots, L_q \in k'[X_1, \ldots, X_m]$ are linear forms and if we denote by $\mathcal{L}$ a maximal set of linear factors of F which are pairwise linearly independent over k', then $\mathcal{L}$ is non-degenerate. Let $S' \subset \mathbf{M}(k')$ consist of the extension of the places of S to k', then every S-integer in k is also an S'-integer in k'. Moreover, we have $H_S(x_1, \ldots, x_m) = H_{S'}(x_1, \ldots, x_m)$ and $\prod_{v\in S} \|F(x_1, \ldots, x_m)\|_v = \prod_{w\in S'} \|F(x_1, \ldots, x_m)\|_w$ for every $(x_1, \ldots, x_m) \in \mathcal{O}_S^m$. So (1.2) is preserved when we work on k'. Therefore, for simplicity, we assume that $k' = k$. By enlarging S if necessary, we may assume that the coefficients of L_j, $1 \leqslant j \leqslant q$, are in $\mathcal{O}_S$. Hence, for $(x_1, \ldots, x_m) \in \mathcal{O}_S^m$,

$$\prod_{v\in S} \left(\prod_{L\in\mathcal{L}} \|L(x_1, \ldots, x_m)\|_v \right) \leqslant \prod_{v\in S} \|F(x_1, \ldots, x_m)\|_v.$$

Thus, (1.2) gives

$$0 < \prod_{v\in S} \left(\prod_{L\in\mathcal{L}} \|L(x_1, \ldots, x_m)\|_v \right) \leqslant c H_S^{\lambda}(x_1, \ldots, x_m) \quad \text{in } (x_1, \ldots, x_m) \in \mathcal{O}_S^m. \tag{3.1}$$

Choose an $\varepsilon > 0$ such that $1-\varepsilon-(m-1)\lambda > 0$. By Theorem 2.1, for every $\mathbf{x} \in \mathbb{P}^{m-1}(k)$ with $L(\mathbf{x}) \neq 0$ for all $L \in \mathcal{L}$, we have

$$(1-\varepsilon)h(\mathbf{x}) \leqslant \sum_{L\in\mathcal{L}} (m-1) N(\mathbf{x}, L) + O(1).$$

By the product formula, this is equivalent to

$$(m-1) \sum_{L\in\mathcal{L}} m(\mathbf{x}, L) \leqslant [\#\mathcal{L}(m-1) - 1 + \varepsilon] h(\mathbf{x}) + O(1),$$

where $\#\mathcal{L}$ is the cardinality of the set $\mathcal{L}$. This gives

$$\sum_{v\in S} \sum_{L\in\mathcal{L}} \log \frac{\|\mathbf{x}\|_v \cdot \|L\|_v}{\|L(\mathbf{x})\|_v} \leqslant \left(\#\mathcal{L} - \frac{1-\varepsilon}{m-1} \right) h(\mathbf{x}) + O(1). \tag{3.2}$$

For $\mathbf{x} \in \mathcal{O}_S^m$, we have

$$h(\mathbf{x}) \leqslant \log H_S(\mathbf{x}). \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\frac{H_S^{\#\mathcal{L}}(\mathbf{x}) \cdot \prod_{v\in S} \prod_{L\in\mathcal{L}} \|L\|_v}{\prod_{v\in S} \prod_{L\in\mathcal{L}} \|L(\mathbf{x})\|_v} \leqslant C_1 H_S(\mathbf{x})^{\#\mathcal{L} - \frac{1-\varepsilon}{m-1}},$$

where $C_1 > 0$ is a constant. This implies that

$$H_S^{1-\varepsilon}(\mathbf{x}) \leqslant C_2 \left(\prod_{\upsilon \in S} \prod_{L \in \mathcal{L}} \|L(\mathbf{x})\|_\upsilon \right)^{m-1},$$

where $C_2 > 0$ is a constant. By (3.1), this implies that

$$H_S^{1-\varepsilon}(\mathbf{x}) < C_2 \cdot c^{m-1} \cdot H_S^{(m-1)\lambda}(\mathbf{x}).$$

Hence

$$H_S^{1-\varepsilon-(m-1)\lambda}(\mathbf{x}) < C_3,$$

where $C_3 > 0$ is a constant. With the choice of our ε, $1 - \varepsilon - (m-1)\lambda > 0$. Hence $H_S(\mathbf{x})$ is bounded. By the Dirichlet–Chevalley–Weil S-unit theorem, there is an S-unit u such that $\|u\mathbf{x}\|_\upsilon \leqslant D_\upsilon H_S(\mathbf{x})^{1/\#S}$ for $\upsilon \in S$, where the D_υ are constants depending only on k, S. Thus $\mathbf{x}$ is $\mathcal{O}_S^*$-proportional to $\mathbf{x}' := u \cdot \mathbf{x}$, and $\|\mathbf{x}'\|_\upsilon$ is bounded for every $\upsilon \in \mathbf{M}(k)$. This implies that there are only finitely many possibilities for $\mathbf{x}'$. Hence up to $\mathcal{O}_S^*$-proportionality, (1.2) has only finitely many solutions $\mathbf{x} \in \mathcal{O}_S^m$. This finishes the proof. $\square$

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