



Contents lists available at ScienceDirect

## Journal of Number Theory

www.elsevier.com/locate/jnt



## Addison-type series representation for the Stieltjes constants

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## ARTICLE INFO

## Article history:

Received 12 December 2009

Revised 14 January 2010

Available online 9 April 2010

Communicated by David Goss

## MSC:

11M06

11Y60

11M35

## Keywords:

Stieltjes constants

Riemann zeta function

Laurent expansion

Series representation

Hurwitz zeta function

## ABSTRACT

The Stieltjes constants  $\gamma_k(a)$  appear in the coefficients in the regular part of the Laurent expansion of the Hurwitz zeta function  $\zeta(s, a)$  about its only pole at  $s = 1$ . We generalize a technique of Addison for the Euler constant  $\gamma = \gamma_0(1)$  to show its application to finding series representations for these constants. Other generalizations of representations of  $\gamma$  are given.

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## 1. Introduction and statement of results

The Riemann zeta function has but one simple pole, at  $s = 1$  in the complex plane [10,21,24]. In the Laurent series about that point,

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k}{k!} (s-1)^k, \quad (1.1)$$

$\gamma_k$  are called the Stieltjes constants [5–7,13,15,18–20,23,25,26], where  $\gamma_0 = \gamma$ , the Euler constant. These constants have many uses in analytic number theory, asymptotic analysis, and elsewhere. Among other applications, estimates for  $\gamma_n$  may be used to determine a zero-free region of the zeta

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function near the real axis in the critical strip  $0 < \operatorname{Re} s < 1$ . The Stieltjes constants can be related to sums over the zeta function complex zeros, as well as to the Li/Keiper constants (e.g., [9]).

The Hurwitz zeta function  $\zeta(s, a)$  may also be analytically continued to the whole complex plane  $\mathbb{C}/\{1\}$ . It has the Laurent expansion

$$\zeta(s, a) = \frac{1}{s-1} + \sum_{k=0}^{\infty} \frac{(-1)^k \gamma_k(a)}{k!} (s-1)^k, \quad s \neq 1. \quad (1.2)$$

Again  $\zeta(s, a)$  has a simple pole at  $s = 1$  with residue 1. By convention, one takes  $\gamma_k(1) = \gamma_k$  [17]. Earlier studies of  $\gamma_k(a)$  include [3,27], with the latter reference giving integral representations. In this paper, we are concerned with certain series representations of the constants  $\gamma_k(a)$ .

We let  $P_1(x) = B_1(x - [x]) = x - [x] - 1/2$  be the first periodized Bernoulli polynomial, and  $\{x\} = x - [x]$  be the fractional part of  $x$ . Addison [2] gave an interesting series representation for  $\gamma$  [11],

$$\gamma = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} n \sum_{m=2^{n-1}}^{2^n-1} \frac{1}{2m(m+1)(2m+1)} = 1 - \frac{1}{2} \sum_{n=1}^{\infty} n \sum_{m=2^{n-1}+1}^{2^n} \frac{1}{m(2m-1)}. \quad (1.3)$$

His approach uses an integral representation for the zeta function in terms of  $P_1$ . We note that various transformations of Addison's result are possible [4,11,22]. In this paper, we describe several generalizations of Addison's result. We indicate how a family of summation representations for  $\gamma_k$  may be obtained. Representative in this regard are the first two propositions and their proofs.

We let  $\psi = \Gamma'/\Gamma$  be the digamma function [1], and  $H_n = \sum_{p=1}^n 1/p$  the  $n$ th harmonic number. We have  $H_n = \psi(n+1) + \gamma$ , with  $\gamma = -\psi(1)$ .

**Proposition 1.** *We have*

$$(a) \quad -\gamma_1 = \frac{1}{4} \sum_{j=1}^{\infty} j \sum_{m=2^{j-1}}^{2^j-1} \left[ \frac{(j-1)}{2} \frac{\ln 2}{m(m+1)(2m+1)} - \frac{\ln m}{m} + \frac{\ln(m+1/2)}{2m+1} - \frac{\ln(m+1)}{m+1} \right], \quad (1.4)$$

and

$$(b) \quad -\gamma_1 = \frac{1}{3} \sum_{j=1}^{\infty} j \sum_{m=3^{j-1}}^{3^j-1} \left\{ \frac{(j-1) \ln 3}{2} \left[ \frac{1}{m(3m+1)} - \frac{1}{(m+1)(3m+2)} \right] \right. \\ \left. - \frac{\ln m}{m} + \frac{3 \ln(m+1/3)}{3m+1} + \frac{3 \ln(m+2/3)}{3m+2} - \frac{\ln(m+1)}{m+1} \right\}. \quad (1.5)$$

**Proposition 2.** *Let  $k \geq 2$  be an integer. Then we have*

$$-\gamma_1 = \sum_{j=1}^{\infty} j \sum_{m=k^{j-1}}^{k^j-1} \left\{ \left[ \frac{2km+k+1}{2km(m+1)} + \psi(km) - \psi(km+k) \right] \left( 1 + \frac{(j-1)}{2} \ln k \right) \right. \\ \left. - k \sum_{\ell=1}^k \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(\ell-1)}{k} \right] \left[ \frac{1 + \ln(m + \frac{\ell-1}{k})}{km + \ell - 1} - \frac{1 + \ln(m + \frac{\ell}{k})}{km + \ell} \right] \right\}. \quad (1.6)$$

Alternatively, on the right side we may write the difference

$$\psi(km) - \psi(km + k) = - \sum_{p=0}^{k-1} \frac{1}{mk + p} = H_{km-1} - H_{km+k-1}. \quad (1.7)$$

Series representations that generalize Addison's approach may also be obtained for  $\gamma_k(a)$ . As an illustration, we have

**Proposition 3.** Let  $\operatorname{Re} a > 0$ . We have

$$\begin{aligned} (a) \quad \gamma_0(a) &= -\ln a + \frac{1}{2a} + \frac{1}{4} \sum_{\nu=0}^{\infty} \frac{1}{4^{\nu}} \sum_{j=0}^{\infty} \frac{b}{(a+bj)(a+b+bj)(2a+b+2bj)} \Big|_{b=2^{-\nu}} \\ &= -\psi(a), \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} (b) \quad \gamma_1(a) &= \frac{\ln a}{2a} - \frac{\ln^2 a}{2} + \frac{1}{4} \sum_{\nu=0}^{\infty} 2^{-\nu} \sum_{j=0}^{\infty} \left[ \frac{\ln(a+bj)}{a+bj} + \frac{\ln(a+b+bj)}{a+b+bj} \right. \\ &\quad \left. - 4 \frac{\ln[a+b(j+1/2)]}{2a+b+2bj} \right] \Big|_{b=2^{-\nu}}. \end{aligned} \quad (1.9)$$

Part (a) is a known result [26]. However, we obtain it in a different way. From Proposition 3(b) we obtain

**Corollary 1.** We have

$$-\gamma_1\left(\frac{1}{2}\right) = 2\gamma \ln 2 + \ln^2 2 - \gamma_1. \quad (1.10)$$

From the proof of Proposition 3(a) we obtain

**Corollary 2.** An integral representation corresponding to the Addison series (1.3) is given by

$$\gamma = \frac{1}{2} + \frac{1}{2} \int_0^1 \left( \frac{1-x}{1+x} \right) \sum_{n=1}^{\infty} x^{2^n-1} dx. \quad (1.11)$$

Our next proposition generalizes Lemma 2.5 of [4]. We have

**Proposition 4.** For any integer  $n > 1$ , we have

$$(a) \quad \gamma^2 + \gamma_1 = - \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} \left[ k \ln n + \ln \left( \ln \left( \frac{1}{x} \right) \right) \right] x^{n^k-1} dx, \quad (1.12)$$

(b) and for  $\operatorname{Re} a > 0$ , we have

$$\ln a - \psi(a) = \ln a + \gamma_0(a) = \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} x^{an^k-1} dx, \quad (1.13)$$

$$\begin{aligned} & \gamma \ln a + \frac{1}{2} \ln^2 a - \gamma \psi(a) + \gamma_1(a) \\ &= - \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} \left[ k \ln n + \ln \left( \ln \left( \frac{1}{x} \right) \right) \right] x^{an^k-1} dx, \end{aligned} \quad (1.14)$$

$$\begin{aligned} & \left( \gamma^2 + \frac{\pi^2}{6} \right) \ln a + \gamma \ln^2 a + \frac{1}{3} \ln^3 a - \left( \gamma^2 + \frac{\pi^2}{6} \right) \psi(a) + 2\gamma \gamma_1(a) + \gamma_2(a) \\ &= \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} \left[ k^2 \ln^2 n + 2k \ln n \ln \left( \ln \left( \frac{1}{x} \right) \right) + \ln^2 \left( \ln \left( \frac{1}{x} \right) \right) \right] x^{an^k-1} dx. \end{aligned} \quad (1.15)$$

Our last proposition generalizes relation (2.87) of [8] for  $\gamma$ . We introduce the constants [8, Proposition 11]

$$p_{n+1} = -\frac{1}{n!} \int_0^1 (-x)_n dx = \frac{(-1)^{n+1}}{n!} \sum_{k=1}^n \frac{s(k, n)}{k+1}, \quad (1.16)$$

where  $(z)_n = \Gamma(z+n)/\Gamma(z)$  is the Pochhammer symbol and  $s(k, \ell)$  is the Stirling number of the first kind. Then we have

**Proposition 5.** Let  $\operatorname{Re} a > 0$ . Then we have

$$(a) \quad \ln a - \psi(a) = \ln a + \gamma_0(a) = \sum_{n=1}^{\infty} (n-1)! \frac{p_{n+1}}{(a)_n}, \quad (1.17)$$

and

$$(b) \quad \frac{1}{2} \ln^2 a + \gamma_1(a) = \sum_{n=1}^{\infty} p_{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\ln(k+a)}{(k+a)}. \quad (1.18)$$

From part (a) of this proposition we obtain the following, where  ${}_2F_1$  is the Gauss hypergeometric function. We have

**Corollary 3.** For  $\operatorname{Re} a > 0$  we have the representation

$$\ln a + \gamma_0(a) = \frac{1}{a} \int_0^1 \frac{{}_2F_1(1, 1; 1+a; v)}{[\ln^2(\frac{1-v}{v}) + \pi^2]} \frac{dv}{v}. \quad (1.19)$$

Recent analytic results for the Stieltjes constants may be found in [7,8], and a convenient and computationally effective asymptotic form in [16]. The work [8] includes an addition formula for the constants together with series representations. Although the Stieltjes constants have been investigated for approximately a century, many open questions concerning them remain, especially including characterizing their arithmetic properties.

We recall some relations useful in the following.

From the representation (cf. e.g., [24, p. 14]),

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^{\infty} x^{-(s+1)} P_1(x) dx, \quad \operatorname{Re} s > 1, \quad (1.20)$$

we obtain

$$\zeta^{(n)}(s) = \frac{(-1)^n n!}{(s-1)^{n+1}} + (-1)^n n \int_1^{\infty} \frac{P_1(x)}{x^{s+1}} \ln^{n-1} x dx - (-1)^n s \int_1^{\infty} \frac{P_1(x)}{x^{s+1}} \ln^n x dx. \quad (1.21)$$

From (1.1) we have

$$\lim_{s \rightarrow 1} \left[ \zeta^{(n)}(s) - \frac{(-1)^n n!}{(s-1)^{n+1}} \right] = (-1)^n \gamma_n. \quad (1.22)$$

Eq. (1.20) extends to

$$\zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s \int_0^{\infty} \frac{P_1(x)}{(x+a)^{s+1}} dx, \quad \operatorname{Re} s > -1. \quad (1.23)$$

We have the connection to differences of logarithmic sums

$$\gamma_j(a) - \gamma_j(b) = \sum_{n=0}^{\infty} \left[ \frac{\ln^j(n+a)}{n+a} - \frac{\ln^j(n+b)}{n+b} \right], \quad j \geq 1, \quad (1.24)$$

where  $a, b \in C/\{-1, -2, \dots\}$ .

## 2. Proof of propositions

**Proof of Proposition 1.** (a) We have from (1.20) and (1.21) the integral representation (cf. [14, p. 5]) for  $n \geq 1$

$$\gamma_n = \int_1^{\infty} P_1(x) \frac{\log^{n-1} x}{x^2} (n - \log x) dx. \quad (2.1)$$

We let  $f(x) = -P_1(x)$  and put [2]

$$g_2(x) = f(x) - \frac{1}{2} f(2x), \quad (2.2)$$

so that  $g_2$  is the rectangular function

$$\begin{aligned} g_2(x) &= 1/4, \quad 0 \leq \{x\} < 1/2, \\ &= -1/4, \quad 1/2 \leq \{x\} < 1. \end{aligned} \quad (2.3)$$

We have from (2.2)

$$\sum_{n=0}^{\infty} \frac{g_2(2^n x)}{2^n} = \sum_{n=0}^{\infty} \left[ \frac{f(2^n x)}{2^n} - \frac{f(2^{n+1} x)}{2^{n+1}} \right] = f(x). \quad (2.4)$$

Therefore, from (2.1) we obtain the representation

$$\begin{aligned} \gamma_n &= - \sum_{v=0}^{\infty} \frac{1}{2^v} \int_1^{\infty} \frac{g_2(2^v x)}{x^2} \ln^{n-1} x (n - \ln x) dx \\ &= - \sum_{v=0}^{\infty} \int_{2^v}^{\infty} \frac{g_2(y)}{y^2} \ln^{n-1}(2^{-v} y) [n - \ln(2^{-v} y)] dy, \end{aligned} \quad (2.5)$$

with the change of variable  $y = 2^v x$ . We write the integral over  $y$  as  $\int_{2^v}^{\infty} dy = \sum_{j=v}^{\infty} \int_{2^j}^{2^{j+1}} dy$  and then interchange the resulting double sum in (2.5). Therefore, we obtain

$$\gamma_n = - \sum_{j=0}^{\infty} \sum_{v=0}^j \int_{2^j}^{2^{j+1}} \frac{g_2(y)}{y^2} \ln^{n-1}(2^{-v} y) [n - \ln(2^{-v} y)] dy. \quad (2.6)$$

We now specialize to  $n = 1$ :

$$\begin{aligned} \gamma_1 &= - \sum_{j=0}^{\infty} \sum_{v=0}^j \int_{2^j}^{2^{j+1}} \frac{g_2(y)}{y^2} [1 - \ln(2^{-v} y)] dy \\ &= - \sum_{j=0}^{\infty} \sum_{v=0}^j \int_{2^j}^{2^{j+1}} \frac{g_2(y)}{y^2} [1 + v \ln 2 - \ln y] dy \\ &= - \sum_{j=0}^{\infty} (j+1) \int_{2^j}^{2^{j+1}} \frac{g_2(y)}{y^2} \left[ 1 + \frac{j}{2} \ln 2 - \ln y \right] dy. \end{aligned} \quad (2.7)$$

We then use the integral

$$\begin{aligned} \int_m^{m+1} \frac{g_2(y)}{y^2} \ln y dy &= \frac{1}{4} \left( \int_m^{m+1/2} - \int_{m+1/2}^{m+1} \right) \frac{\ln y}{y^2} dy \\ &= \frac{1}{4} \left[ \frac{1 + \ln m}{m} - \frac{4(1 + \ln(m + 1/2))}{2m + 1} + \frac{1 + \ln(m + 1)}{m + 1} \right]. \end{aligned} \quad (2.8)$$

We find

$$-\gamma_1 = \frac{1}{4} \sum_{j=1}^{\infty} j \sum_{m=2^{j-1}}^{2^j-1} \left[ \frac{1 + \frac{j-1}{2} \ln 2}{m(m+1)(2m+1)} - \frac{1 + \ln m}{m} + \frac{4(1 + \ln(m+1/2))}{2m+1} - \frac{1 + \ln(m+1)}{m+1} \right]. \quad (2.9)$$

With the use of partial fractions on the first term on the right side of this equation, we obtain the form (1.4).

For part (b), we proceed similarly, employing the function

$$\begin{aligned} g_3(x) &= f(x) - \frac{1}{3}f(3x) = 1/3, \quad 0 \leq \{x\} < 1/3, \\ &= 0, \quad 1/3 \leq \{x\} < 2/3, \\ &= -1/3, \quad 2/3 \leq \{x\} < 1, \end{aligned} \quad (2.10)$$

so that  $\sum_{n=0}^{\infty} \frac{g_3(3^n x)}{3^n} = f(x)$ . We use the integrals

$$\begin{aligned} \int_m^{m+1} \frac{g_3(y)}{y^2} dy &= \frac{1}{3} \left( \int_m^{m+1/3} - \int_{m+2/3}^{m+1} \right) \frac{dy}{y^2} \\ &= \frac{1}{3} \left[ \frac{1}{m(3m+1)} - \frac{1}{(m+1)(3m+2)} \right], \end{aligned} \quad (2.11a)$$

and

$$\begin{aligned} \int_m^{m+1} \frac{g_3(y)}{y^2} \ln y dy &= \frac{1}{3} \left( \int_m^{m+1/3} - \int_{m+2/3}^{m+1} \right) \frac{\ln y}{y^2} dy \\ &= \frac{1}{3} \left[ \frac{1 + \ln m}{m} - \frac{3(1 + \ln(m+1/3))}{3m+1} \right. \\ &\quad \left. - \frac{3(1 + \ln(m+2/3))}{3m+2} + \frac{1 + \ln(m+1)}{m+1} \right]. \end{aligned} \quad (2.11b)$$

Again, an application of partial fractions gives the final form (1.5).  $\square$

**Proof of Proposition 2.** Proposition 1 is further extended with the use of functions for integers  $k \geq 2$

$$g_k(x) = f(x) - \frac{1}{k}f(kx), \quad (2.12)$$

with  $\sum_{n=0}^{\infty} \frac{g_k(k^n x)}{k^n} = f(x)$ . Since  $f$  is periodic,  $g_k$  is also periodic of period 1. From the expression

$$g_k(x) = \frac{1}{2} \left( 1 - \frac{1}{k} \right) + \frac{1}{k} \{kx\} - \{x\}, \quad (2.13)$$

we obtain the values of  $g_k(x)$  on subintervals  $[\frac{j-1}{k}, \frac{j}{k}]$  for  $j = 1, 2, \dots, k$ . We have

$$g_k(x) = \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(j-1)}{k}, \quad x \in \left[ \frac{j-1}{k}, \frac{j}{k} \right]. \quad (2.14)$$

This enables the development of summation expressions for any  $\gamma_n$  as a result of integrations over a function  $g_k(x)$ . We have from (2.1)

$$\begin{aligned} \gamma_n &= - \sum_{\nu=0}^{\infty} \frac{1}{k^{\nu}} \int_1^{\infty} \frac{g_k(k^{\nu}x)}{x^2} \ln^{n-1} x (n - \ln x) dx \\ &= - \sum_{\nu=0}^{\infty} \int_{k^{\nu}}^{\infty} \frac{g_k(y)}{y^2} \ln^{n-1}(k^{-\nu}y) [n - \ln(k^{-\nu}y)] dy \\ &= - \sum_{j=0}^{\infty} \sum_{\nu=0}^j \int_{k^j}^{k^{j+1}} \frac{g_k(y)}{y^2} \ln^{n-1}(k^{-\nu}y) [n - \ln(k^{-\nu}y)] dy. \end{aligned} \quad (2.15)$$

So at  $n = 1$  we obtain

$$\begin{aligned} -\gamma_1 &= \sum_{j=0}^{\infty} \sum_{\nu=0}^j \int_{k^j}^{k^{j+1}} \frac{g_k(y)}{y^2} [1 - \ln(k^{-\nu}y)] dy \\ &= \sum_{j=0}^{\infty} (j+1) \int_{k^j}^{k^{j+1}} \frac{g_k(y)}{y^2} \left[ 1 + \frac{j}{2} \ln k - \ln y \right] dy. \end{aligned} \quad (2.16)$$

Now using the piecewise values in (2.14), we have

$$\begin{aligned} \int_m^{m+1} \frac{g_k(y)}{y^2} dy &= \sum_{\ell=1}^k \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(\ell-1)}{k} \right] \int_{m+(\ell-1)/k}^{m+\ell/k} \frac{dy}{y^2} \\ &= \sum_{\ell=1}^k \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(\ell-1)}{k} \right] \frac{k}{(km+\ell)(km+\ell-1)} \\ &= \frac{2km+k+1}{2km(m+1)} + \psi(km) - \psi(km+k). \end{aligned} \quad (2.17)$$



We also have

$$\begin{aligned} \int_m^{m+1} \frac{g_k(y)}{y^2} \ln y \, dy &= \sum_{\ell=1}^k \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(\ell-1)}{k} \right] \int_{m+(\ell-1)/k}^{m+\ell/k} \ln y \frac{dy}{y^2} \\ &= \sum_{\ell=1}^k \left[ \frac{1}{2} \left( 1 - \frac{1}{k} \right) - \frac{(\ell-1)}{k} \right] \left[ \frac{1 + \ln(m + \frac{\ell-1}{k})}{km + \ell - 1} - \frac{1 + \ln(m + \frac{\ell}{k})}{km + \ell} \right] \end{aligned} \quad (2.18)$$

combining (2.16), (2.17), and (2.18), we arrive at the proposition.  $\square$

**Proof of Proposition 3.** (a) From (1.13) and (1.2) we have

$$\lim_{s \rightarrow 1} \left[ \zeta(s, a) - \frac{a^{1-s}}{s-1} \right] = -\ln a + \frac{1}{2a} - \int_0^\infty \frac{P_1(x)}{(x+a)^2} dx, \quad (2.19)$$

so that

$$\gamma_0(a) = -\ln a + \frac{1}{2a} + \int_0^\infty \frac{f(x)}{(x+a)^2} dx. \quad (2.20)$$

Using the functions  $g_k$  of (2.12) we have

$$\begin{aligned} \int_0^\infty \frac{f(x)}{(x+a)^2} dx &= \sum_{v=0}^\infty \frac{1}{k^v} \int_0^\infty \frac{g_k(k^v x)}{(x+a)^2} dx \\ &= \sum_{v=0}^\infty \frac{1}{k^{2v}} \sum_{j=0}^\infty \int_j^{j+1} \frac{g_k(y)}{(k^{-v}y + a)^2} dy. \end{aligned} \quad (2.21)$$

We now specialize to  $k=2$  and obtain

$$\begin{aligned} \int_0^\infty \frac{f(x)}{(x+a)^2} dx &= \sum_{v=0}^\infty \frac{1}{4^v} \sum_{j=0}^\infty \int_j^{j+1} \frac{g_2(y)}{(2^{-v}y + a)^2} dy \\ &= \frac{1}{4} \sum_{v=0}^\infty \frac{1}{4^v} \sum_{j=0}^\infty \left( \int_j^{j+1/2} - \int_{j+1/2}^{j+1} \right) \frac{dy}{(2^{-v}y + a)^2} \\ &= \frac{1}{4} \sum_{v=0}^\infty \frac{1}{4^v} \sum_{j=0}^\infty \frac{b}{(a+bj)(a+b+bj)(2a+b+2bj)} \Big|_{b=2^{-v}} \\ &= \frac{1}{4} \sum_{v=0}^\infty \left[ 2\psi\left(\frac{2a+b}{2b}\right) - \psi\left(\frac{a}{b}\right) - \psi\left(\frac{a+b}{b}\right) \right]_{b=2^{-v}}. \end{aligned} \quad (2.22)$$

We now apply the functional equation and duplication formula for the digamma function:

$$\begin{aligned}
 \int_0^{\infty} \frac{f(x)}{(x+a)^2} dx &= \frac{1}{4} \sum_{\nu=0}^{\infty} \left[ 2\psi\left(\frac{2a+b}{2b}\right) - 2\psi\left(\frac{a}{b}\right) - \frac{b}{a} \right]_{b=2^{-\nu}} \\
 &= \frac{1}{2} \sum_{\nu=0}^{\infty} \left[ \psi\left(\frac{a}{b} + \frac{1}{2}\right) - \psi\left(\frac{a}{b}\right) \right]_{b=2^{-\nu}} - \frac{1}{2a} \\
 &= \sum_{\nu=0}^{\infty} \left[ \psi\left(\frac{2a}{b}\right) - \psi\left(\frac{a}{b}\right) - \ln 2 \right]_{b=2^{-\nu}} - \frac{1}{2a}. \tag{2.23}
 \end{aligned}$$

We employ a standard integral representation for  $\psi$  [12, p. 943] to write

$$\begin{aligned}
 \sum_{\nu=0}^{\infty} \left[ \psi\left(\frac{2a}{b}\right) - \psi\left(\frac{a}{b}\right) - \ln 2 \right]_{b=2^{-\nu}} &= \lim_{T \rightarrow \infty} \sum_{\nu=0}^T \left[ \int_0^1 \frac{(t^{2a2^{\nu}-1} - t^{a2^{\nu}-1})}{t-1} dt - \ln 2 \right] \\
 &= \lim_{T \rightarrow \infty} \left[ \int_0^1 \frac{(t^{a2^{T+1}-1} - t^{a-1})}{t-1} dt - (T+1) \ln 2 \right] \\
 &= \lim_{T \rightarrow \infty} [\psi(a2^{T+1}) - \psi(a) - (T+1) \ln 2] \\
 &= -\psi(a) + \ln a, \tag{2.24}
 \end{aligned}$$

where the above interchange of summation and integration is justified by absolute convergence of the integral, and where in the last step we applied the large argument asymptotic form for  $\psi$  [1, p. 259]. Therefore, from (2.20), (2.23), and (2.24), we find (1.8).

(b) Using (1.23) and (1.2) we have

$$\begin{aligned}
 &\lim_{s \rightarrow 1} \left[ \zeta'(s, a) + a^{1-s} \left( \frac{1}{(s-1)^2} + \frac{\ln a}{s-1} \right) \right] \\
 &= \frac{\ln^2 a}{2} - \frac{\ln a}{2a} - \int_0^{\infty} \frac{P_1(x)}{(x+a)^2} [1 - \ln(x+a)] dx, \tag{2.25}
 \end{aligned}$$

giving

$$\gamma_1(a) = -\frac{\ln^2 a}{2} + \frac{\ln a}{2a} + \int_0^{\infty} \frac{P_1(x)}{(x+a)^2} [1 - \ln(x+a)] dx. \tag{2.26}$$

With  $k=2$  in (2.12) we have

$$\begin{aligned}
& - \int_0^{\infty} \frac{P_1(x)}{(x+a)^2} [1 - \ln(x+a)] dx \\
&= \sum_{v=0}^{\infty} \frac{1}{4^v} \sum_{j=0}^{\infty} \int_j^{j+1} \frac{g_2(y)}{(2^{-v}y+a)^2} [1 - \ln(2^{-v}y+a)] dy \\
&= \frac{1}{4} \sum_{v=0}^{\infty} \frac{1}{4^v} \sum_{j=0}^{\infty} \left( \int_j^{j+1/2} - \int_{j+1/2}^{j+1} \right) [1 - \ln(2^{-v}y+a)] \frac{dy}{(2^{-v}y+a)^2}. \tag{2.27}
\end{aligned}$$

We then carry out the integrations and simplify. Using (2.25) completes part (b).  $\square$

**Proof of Corollary 1.** We put  $a = 1/2$  in (1.9) and manipulate the resulting series to write

$$\begin{aligned}
& \gamma_1(1/2) + \ln 2 + \frac{1}{2} \ln^2 2 \\
&= \frac{\ln(1/2)}{4} \sum_{v=0}^{\infty} 2^{-v+1} \sum_{j=0}^{\infty} \left[ \frac{1}{1+2^{-v+1}j} + \frac{1}{1+2^{-v+1}(j+1)} - \frac{2}{1+2^{-v+1}(j+1/2)} \right] \\
&+ \frac{1}{4} \sum_{v=0}^{\infty} 2^{-v} \sum_{j=0}^{\infty} \left[ \frac{\ln(1+bj)}{1+bj} + \frac{\ln[1+b(j+1)]}{1+b(j+1)} - 2 \frac{\ln[1+b(j+1/2)]}{1+b(j+1/2)} \right]_{b=2^{-v}} \\
&+ \frac{1}{2} \sum_{j=0}^{\infty} \left[ \frac{\ln(1+2j)}{1+2j} + \frac{\ln[1+2(j+1)]}{1+2(j+1)} - 2 \frac{\ln[1+2(j+1/2)]}{1+2(j+1/2)} \right]. \tag{2.28}
\end{aligned}$$

By the method of (2.23)–(2.24), the first line on the right side of this equation has the value  $-\ln 2(\gamma + \ln 2 - 1)$ . From Proposition 3(b) at  $a = 1$ , the second line on the right side of (2.28) is  $\gamma_1$ . The following evaluates the third line of the right side of (2.28).

**Lemma 1.** We have

$$\begin{aligned}
& \frac{1}{2} \sum_{j=0}^{\infty} \left[ \frac{\ln(1+2j)}{1+2j} + \frac{\ln[1+2(j+1)]}{1+2(j+1)} - 2 \frac{\ln[1+2(j+1/2)]}{1+2(j+1/2)} \right] \\
&= \frac{1}{2} \left[ \gamma_1 \left( \frac{1}{2} \right) - \gamma_1 \right] + \ln^2 2. \tag{2.29}
\end{aligned}$$

**Proof.** We write

$$\begin{aligned}
& \frac{1}{4} \sum_{j=0}^{\infty} \left[ \frac{\ln[2(j+1/2)]}{j+1/2} + \frac{\ln[2(j+3/2)]}{j+3/2} - 2 \frac{\ln[2(j+1)]}{j+1} \right] \\
&= \frac{\ln 2}{4} \sum_{j=0}^{\infty} \left[ \frac{1}{j+1/2} + \frac{1}{j+3/2} - \frac{2}{j+1} \right] \\
&+ \frac{1}{4} \left[ \gamma_1 \left( \frac{1}{2} \right) - \gamma_1 + \gamma_1 \left( \frac{3}{2} \right) - \gamma_1 \right], \tag{2.30}
\end{aligned}$$

where we used (1.24) at  $j = 1$ . Then the sum of (2.30) is given by

$$\begin{aligned} & \frac{\ln 2}{2}(2 \ln 2 - 1) + \frac{1}{4} \left[ \gamma_1 \left( \frac{1}{2} \right) + \gamma_1 \left( \frac{3}{2} \right) - 2\gamma_1 \right] \\ &= \frac{\ln 2}{2}(2 \ln 2 - 1) + \frac{1}{2} \left[ \gamma_1 \left( \frac{1}{2} \right) - \gamma_1 + \ln 2 \right] \\ &= \frac{1}{2} \left[ \gamma_1 \left( \frac{1}{2} \right) - \gamma_1 \right] + \ln^2 2. \end{aligned} \quad (2.31)$$

In obtaining this result we have used  $\gamma_k(a+1) = \gamma_k(a) - (\ln^k a)/a$ , that follows easily from the series form of  $\zeta(s, a)$ , at  $k = 1$ .

Then combining the terms on the left and right sides of (2.28) yields (1.10).  $\square$

**Proof of Corollary 2.** We have, by applying for instance Theorem 2.2 of [4],

$$\int_0^1 \left( \frac{1-x}{1+x} \right) x^p dx = -\frac{1}{p+1} + H_{p/2} - H_{(p-1)/2}, \quad \operatorname{Re} p > -1. \quad (2.32)$$

Then, with the interchange of the summation and integration justified by the absolute convergence of the integral, we have

$$\int_0^1 \left( \frac{1-x}{1+x} \right) \sum_{n=1}^{\infty} x^{2^n-1} dx = -1 + \sum_{n=1}^{\infty} [H_{p/2} - H_{(p-1)/2}]_{p=2^n-1}, \quad (2.33)$$

giving

$$\frac{1}{2} + \frac{1}{2} \int_0^1 \left( \frac{1-x}{1+x} \right) \sum_{n=1}^{\infty} x^{2^n-1} dx = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \psi \left( \frac{p+1}{2} \right) - \psi \left( \frac{p}{2} + 1 \right) \right]_{p=2^n-1}. \quad (2.34)$$

By the method of (2.22)–(2.24), the corollary follows.  $\square$

**Proof of Proposition 4.** Part (a) is based upon the integral

$$\int_0^1 \left( \frac{1}{\ln x} + \frac{1}{1-x} \right) \ln^s \left( \frac{1}{x} \right) dx = \Gamma(s) [s\zeta(s+1) - 1], \quad \operatorname{Re} s > -1. \quad (2.35)$$

This equation follows easily by term-by-term integration, as we have

$$\begin{aligned} \int_0^1 \left( \frac{1}{\ln x} + \frac{1}{1-x} \right) \ln^s \left( \frac{1}{x} \right) dx &= \int_0^1 \frac{\ln^s(1/x)}{\ln x} dx + \sum_{j=0}^{\infty} \int_0^1 x^j \ln^s \left( \frac{1}{x} \right) dx \\ &= -\Gamma(s) + \Gamma(s+1) \sum_{j=0}^{\infty} \frac{1}{(j+1)^{s+1}}, \end{aligned} \quad (2.36)$$

where we used the change of variable  $u = -\ln x$ . We apply Lemma 2.4 of [4] that we write in the following form. For any integer  $n > 1$  and  $\operatorname{Re} x > 0$ , we have

$$\frac{1}{\ln x} + \frac{1}{1-x} = \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n-1} (n-j)x^{(j-1)/n^k}}{n^k \sum_{j=0}^{n-1} x^{j/n^k}}. \quad (2.37)$$

We then proceed as in the proof of Lemma 2.5 of [4] to find

$$\int_0^1 \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) \ln^s \left( \frac{1}{u} \right) du = \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \ln^s \left( \frac{1}{x} \right) \sum_{k=1}^{\infty} n^{ks} x^{n^k-1} dx. \quad (2.38)$$

We perform  $\frac{\partial}{\partial s}|_{s=0}$  on (2.35) and (2.38), and (1.12) follows.

Part (b) is based upon the integral

$$\int_0^1 \left( \frac{1}{\ln x} + \frac{1}{1-x} \right) x^{a-1} \ln^s \left( \frac{1}{x} \right) dx = \Gamma(s) [s\zeta(s+1, a) - a^{-s}],$$

$$\operatorname{Re} s > -1, \operatorname{Re} a > 0. \quad (2.39)$$

Using Lemma 2.4 of [4] we find

$$\int_0^1 \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) u^{a-1} \ln^s \left( \frac{1}{u} \right) du$$

$$= \int_0^1 \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \ln^s \left( \frac{1}{x} \right) \sum_{k=1}^{\infty} n^{ks} x^{an^k-1} dx. \quad (2.40)$$

Taking  $s = 0$  in (2.38) and (2.39) gives (1.13). Performing  $(\frac{\partial}{\partial s})^j|_{s=0}$ ,  $j = 1, 2$  on (2.39) and (2.40), respectively gives (1.14) and (1.15).  $\square$

**Proof of Proposition 5.** We employ the generating function [8, (2.78)]

$$\sum_{n=1}^{\infty} p_{n+1} z^{n-1} = \frac{1}{z} + \frac{1}{\ln(1-z)}, \quad |z| < 1, \quad (2.41)$$

so that by (2.39) we have

$$\Gamma(s) [s\zeta(s+1, a) - a^{-s}] = \sum_{n=1}^{\infty} p_{n+1} \int_0^1 x^{a-1} (1-x)^{n-1} \ln^s \left( \frac{1}{x} \right) dx. \quad (2.42)$$

For part (a), we take  $s = 0$  in this equation and evaluate the integral using the beta function. For part (b), we differentiate both sides of (2.42) with respect to  $s$  and put  $s = 0$ . We find

$$-\gamma \ln a - \frac{1}{2} \ln^2 a + \gamma \psi(a) - \gamma_1(a) = \sum_{n=1}^{\infty} p_{n+1} \int_0^1 x^{a-1} (1-x)^{n-1} \ln \left( \ln \left( \frac{1}{x} \right) \right) dx. \quad (2.43)$$

By logarithmic differentiation we obtain the integral for  $\operatorname{Re} s > -1$

$$\int_0^1 x^k \ln^s \left( \frac{1}{x} \right) \ln \left( \ln \left( \frac{1}{x} \right) \right) dx = \frac{\Gamma(s+1)}{(k+1)^{s+1}} [\psi(s+1) - \ln(k+1)], \quad (2.44)$$

giving at  $s = 0$

$$\int_0^1 x^k \ln \left( \ln \left( \frac{1}{x} \right) \right) dx = -\frac{[\gamma + \ln(k+1)]}{k+1}. \quad (2.45)$$

Then by binomial expansion we have

$$\begin{aligned} \int_0^1 x^{a-1} (1-x)^{n-1} \ln \left( \ln \left( \frac{1}{x} \right) \right) dx &= -\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{[\gamma + \ln(k+a)]}{k+a} \\ &= -\gamma \frac{(n-1)!}{(a)_n} - \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\ln(k+a)}{k+a}. \end{aligned} \quad (2.46)$$

By using (2.43) and (1.17) we find (1.18).

For Corollary 3 we apply the integral representation (2.85) of [8] for  $p_{n+1}$ , so that by (1.17)

$$\begin{aligned} \ln a + \gamma_0(a) &= \sum_{n=1}^{\infty} \frac{(n-1)!}{(a)_n} \int_0^{\infty} \frac{du}{(1+u)^n (\ln^2 u + \pi^2)} \\ &= \frac{1}{a} \int_0^{\infty} \frac{1}{(\ln^2 u + \pi^2)} \frac{1}{(1+u)^2} {}_2F_1 \left( 1, 1; 1+a; \frac{1}{1+u} \right) du. \quad \square \end{aligned} \quad (2.47)$$

**Remarks.** From (2.6) we have for  $n \geq 1$

$$-\gamma_n = \sum_{j=0}^{\infty} \sum_{v=0}^j \int_{2j}^{2^{j+1}} \frac{g_2(y)}{y^2} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-v \ln 2)^{\ell} \ln^{n-\ell-1} y [n + v \ln 2 - \ln y] dy, \quad (2.48)$$

and more generally from (2.15) for  $k \geq 2$  we have

$$-\gamma_n = \sum_{j=0}^{\infty} \sum_{v=0}^j \int_{k^j}^{k^{j+1}} \frac{g_k(y)}{y^2} \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} (-v \ln k)^\ell \ln^{n-\ell-1} y [n + v \ln k - \ln y] dy. \quad (2.49)$$

With (2.20) and (1.8) we have recovered a known integral representation for  $\ln \Gamma$  or the polygamma functions  $\psi^{(j)}$  in terms of  $P_1$  (e.g., [10, Section 6.3]). However, our route has been very different from say the use of Euler–Maclaurin summation.

The result (1.10) also follows from the relation  $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$ . The method of Corollary 1 shows an instance of the functional relation contained in (1.9).

In Proposition 5(a), when  $a = 1$ , we have  $(1)_n = n!$ , and we recover (2.87) of [8]. Other forms of part (b) may be written by using the integral representation

$$\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{\ln(k+a)}{k+a} = - \int_0^1 \left[ \frac{(n-1)!}{(a)_n} - \frac{u^{a-1}}{a} {}_2F_1(1, 1-n; 1+a; u) \right] \frac{du}{\ln u}. \quad (2.50)$$

This expression has been obtained by inserting an integral representation for  $\ln$  on the left side, interchanging summation and integration, and making a change of variable.

We note that representations of other constants may be obtained from Lemma 2.4 of [4]. We let  $\text{li}(z) \equiv \int_0^z dt / \ln t$  be the logarithmic integral. Then we have

**Corollary 4.** For any integer  $n > 1$  we have

$$-\ln(1-y) + \text{li}(y) = \int_0^y \left( \frac{n}{1-x^n} - \frac{1}{1-x} \right) \sum_{k=1}^{\infty} x^{n^k-1} dx. \quad (2.51)$$

Our results for  $\gamma_k$  also imply representations for the coefficients  $\eta_j$  that appear in the Laurent expansion of  $\zeta'/\zeta$  about  $s = 1$  (e.g., [9]). As an example, we have  $\eta_1 = \gamma^2 + 2\gamma_1$ .

### 3. Summary

We have shown that the method of Addison may be generalized in several directions. It applies not only to the Euler constant  $\gamma$ , but to the Stieltjes constants  $\gamma_k$  for the Riemann zeta function. It further extends to  $\gamma_k(a)$  for the Hurwitz zeta function. Moreover, one may develop a set of parameterized series representations by using the step-wise functions  $g_k(x)$ . On yet a broader scale, the series developments are applicable any time the first periodic Bernoulli polynomial  $P_1$  is integrated, including in numerous occurrences of Euler–Maclaurin summation.

With Proposition 4, we have generalized representations of  $\gamma$  given by Ramanujan and Berndt and Bowman [4].

### References

- [1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, 1964.
- [2] A.W. Addison, A series representation for Euler's constant, Amer. Math. Monthly 74 (1967) 823–824.
- [3] B.C. Berndt, On the Hurwitz zeta function, Rocky Mountain J. Math. 2 (1972) 151–157.
- [4] B.C. Berndt, D.C. Bowman, Ramanujan's short unpublished manuscript on integrals and series related to Euler's constant, in: M. Théra (Ed.), in: CMS Conf. Proc., vol. 27, 2000, pp. 19–27.
- [5] W.E. Briggs, Some constants associated with the Riemann zeta-function, Michigan Math. J. 3 (1955) 117–121.
- [6] W.E. Briggs, S. Chowla, The power series coefficients of  $\zeta(s)$ , Amer. Math. Monthly 62 (1955) 323–325.

- [7] M.W. Coffey, New results on the Stieltjes constants: Asymptotic and exact evaluation, *J. Math. Anal. Appl.* 317 (2006) 603–612, arXiv:math-ph/0506061.
- [8] M.W. Coffey, Series representations for the Stieltjes constants, 2009, arXiv:0905.1111.
- [9] M.W. Coffey, New results on power series expansions of the Riemann xi function and the Li/Keiper constants, *Proc. Roy. Soc. A* 464 (2008) 711–731.
- [10] H.M. Edwards, *Riemann's Zeta Function*, Academic Press, New York, 1974.
- [11] I. Gerst, Some series for Euler's constant, *Amer. Math. Monthly* 76 (1969) 273–275.
- [12] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980.
- [13] G.H. Hardy, Note on Dr. Vacca's series for  $\gamma$ , *Quart. J. Pure Appl. Math.* 43 (1912) 215–216.
- [14] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.
- [15] J.C. Kluyver, On certain series of Mr. Hardy, *Quart. J. Pure Appl. Math.* 50 (1927) 185–192.
- [16] C. Knessl, M.W. Coffey, An effective asymptotic formula for the Stieltjes constants, *Math. Comp.* (2010), in press.
- [17] R. Kreminski, Newton–Cotes integration for approximating Stieltjes (generalized Euler) constants, *Math. Comp.* 72 (2003) 1379–1397.
- [18] Y. Matsuoka, On the power series coefficients of the Riemann zeta function, *Tokyo J. Math.* 12 (1989) 49–58.
- [19] Y. Matsuoka, Generalized Euler constants associated with the Riemann zeta function, in: J. Akiyama, et al. (Eds.), *Number Theory and Combinatorics*, World Scientific, 1985, pp. 279–295.
- [20] D. Mitrović, The signs of some constants associated with the Riemann zeta function, *Michigan Math. J.* 9 (1962) 395–397.
- [21] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Grösse, *Monatsh. Preuss. Akad. Wiss.* 671 (1859–1860).
- [22] H.F. Sandham, D.F. Barrow, Advanced problems and solutions, *Amer. Math. Monthly* 58 (1951) 116–117.
- [23] T.J. Stieltjes, *Correspondance d'Hermite et de Stieltjes*, volumes 1 and 2, Gauthier-Villars, Paris, 1905.
- [24] E.C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, second ed., Oxford University Press, Oxford, 1986.
- [25] G. Vacca, A new series for the Eulerian constant  $\gamma = 0.577 \dots$ , *Quart. J. Pure Appl. Math.* 41 (1909–1910) 363–366.
- [26] J.R. Wilton, A note on the coefficients in the expansion of  $\zeta(s, x)$  in powers of  $s - 1$ , *Quart. J. Pure Appl. Math.* 50 (1927) 329–332.
- [27] N.-Y. Zhang, K.S. Williams, Some results on the generalized Stieltjes constants, *Analysis* 14 (1994) 147–162.