Journal of Number Theory 136 (2014) 438–459

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Mean divisibility of multinomial coefficients

Shigeki Akiyama [1]

Institute of Mathematics, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki, 305-8571, Japan

ARTICLE INFO

Article history:
Received 15 December 2012
Received in revised form 15 August 2013
Accepted 18 October 2013
Available online 14 December 2013
Communicated by Jeffrey C. Lagarias

Keywords:
Multinomial coefficient
Divisibility
Riemann sum

ABSTRACT

Let $m_1, \dots, m_s$ be positive integers. Consider the sequence defined by multinomial coefficients:

$$a_n = \binom{(m_1 + m_2 + \cdots + m_s)n}{m_1 n, m_2 n, \dots, m_s n}.$$

Fix a positive integer $k \geqslant 2$. We show that there exists a positive integer $C(k)$ such that

$$\frac{\prod_{n=1}^{t} a_{kn}}{\prod_{n=1}^{t} a_n} \in \frac{1}{C(k)}\mathbb{Z}$$

for all positive integer t, if and only if $GCD(m_1, \dots, m_s) = 1$.
© 2013 Elsevier Inc. All rights reserved.

1. Mean divisibility

A sequence (a_n) $(n = 1, 2, \dots)$ of non-zero integers is *divisible* if $n \mid m$ implies $a_n \mid a_m$. It is *strongly divisible* if $GCD(a_n, a_m) = |a_{GCD(n,m)}|$. Such divisibility attracts number theorists for a long time and a lot of papers dealt with properties of such sequences

E-mail address: akiyama@math.tsukuba.ac.jp.
[1] The author is supported by the Japanese Society for the Promotion of Science (JSPS), Grant in aid 24540012.

0022-314X/$ – see front matter © 2013 Elsevier Inc. All rights reserved.
http://dx.doi.org/10.1016/j.jnt.2013.10.005

[17,16,4,3,9,1,11]. Primitive divisors of elliptic divisibility sequences and sequences arose in arithmetic dynamics are recently studied in detail [6,12,19]. In this paper, we introduce a weaker terminology which seems not studied before. We say that (a_n) is *almost mean k-divisible*, if there is a positive integer $C = C(k)$ such that $(\prod_{n=1}^{t} a_{kn})/(\prod_{i=1}^{t} a_n) \in \frac{1}{C}\mathbb{Z}$ for any positive integer t. In particular, (a_n) is *mean divisible* if $\prod_{n=1}^{t} a_n \mid \prod_{i=1}^{t} a_{kn}$ for any positive integer k and t. Clearly if (a_n) is divisible, then it is mean divisible. By definition, if a sequence is almost mean k-divisible for all k with the constant $C(k) = 1$, then it is mean divisible. We are interested in giving non-trivial examples of (almost) mean divisible sequences. In fact, we show that sequences defined by multinomial coefficients give such examples. Let $m_1, \dots, m_s$ be positive integers. A *multinomial sequence* is defined by

$$a_n = \binom{(m_1 + m_2 + \cdots + m_s)n}{m_1 n, m_2 n, \dots, m_s n} = \frac{((m_1 + m_2 + \cdots + m_s)n)!}{(m_1 n)!(m_2 n)! \dots (m_s n)!}.$$

Theorem 1. *If $GCD(m_1, m_2, \dots, m_s) = 1$, then the multinomial sequence is almost mean k-divisible for all k.*

The proof relies on an interesting integral inequality (Lemma 3) and its approximation by Riemann sums. Here are some illustrations:

Corollary 2.

$$\frac{\prod_{n=1}^{t} \binom{10n}{4n}}{\prod_{n=1}^{t} \binom{5n}{2n}} \in \frac{1}{11}\mathbb{Z}, \qquad \frac{\prod_{n=1}^{t} \binom{9n}{3n}}{\prod_{n=1}^{t} \binom{3n}{n}} \in \frac{1}{5}\mathbb{Z}, \qquad \frac{\prod_{n=1}^{t} \binom{28n}{4n,8n,16n}}{\prod_{n=1}^{t} \binom{7n}{n,2n,4n}} \in \mathbb{Z}$$

for any positive integer t.

Readers will see that Figs. 1, 2 and 3 in Section 6 essentially tell why these are true. The constant $C(k)$ is computed by an algorithm based on the proof of Theorem 1. However it is not so easy to identify the set of t's at which the denominator actually appears. For the first example, there are infinitely many t with denominator 11, but the denominator 5 in the second example appears only when $t = 2$. See Section 6 for details. We can also show

Theorem 3. *If $GCD(m_1, m_2, \dots, m_s) > 1$, then the multinomial sequence is not almost mean k-divisible for all k.*

Thus for a given k, a multinomial sequence is almost mean k-divisible if and only if $GCD(m_1, m_2, \dots, m_s) = 1$ holds. For e.g.,

Corollary 4. *Denominators of*

$$\frac{\prod_{n=1}^{t}\binom{8n}{4n}}{\prod_{n=1}^{t}\binom{4n}{2n}}$$

for $t = 1, 2, \ldots$ forms an infinite set.

The proof of Theorem 3 uses the Riemann sum approximation again, but we have to study more precisely the integral inequality of Lemma 3 at the place where it attains the equality. Indeed, we show that the set of primes in some arithmetic progression modulo $k\,LCM(m_1, \ldots, m_s, \sum_{i=1}^{s} m_i)$ must appear in the denominators. If (a_n) is a multinomial sequence with parameters $(m_1, \ldots, m_s)$, then so is $(a_{\ell n})$ with $(\ell m_1, \ldots, \ell m_s)$. Since divisibility of (a_n) is hereditary to $(a_{\ell n})$, we see from Theorem 3,

Corollary 5. *Multinomial sequences are not divisible.*

Therefore multinomial sequences supply non-trivial examples of almost mean k-divisible sequences. For the mean divisibility, we can prove

Theorem 6. *If $m_1, \ldots, m_s$ are pairwise coprime and each m_i divides $\sum_{i=1}^{s} m_i$, then the multinomial sequence is mean divisible.*

For examples, we have

Corollary 7.

$$\frac{\prod_{n=1}^{t}\binom{2kn}{kn}}{\prod_{n=1}^{t}\binom{2n}{n}} \in \mathbb{Z}, \qquad \frac{\prod_{n=1}^{t}\binom{3kn}{kn,kn,kn}}{\prod_{n=1}^{t}\binom{3n}{n,n,n}} \in \mathbb{Z}, \qquad \frac{\prod_{n=1}^{t}\binom{6kn}{kn,2kn,3kn}}{\prod_{n=1}^{t}\binom{6n}{n,2n,3n}} \in \mathbb{Z}$$

for any positive integer k and t.

The central binomial coefficient $\binom{2n}{n}$ is of historical importance. By using $\binom{2n}{n}$, Erdős [7] showed Bertran–Chebyshev's theorem that there is a prime in any interval $(n, 2n]$. Interesting divisibility problems on $\binom{2n}{n}$ are discussed in [14,8,15]. However the first example of Corollary 7 seems to be new.

Theorem 6 models Theorem 5.2 in [2] which proves

$$2^{-m}\frac{\prod_{j=m+1}^{2m}\binom{2j}{j}}{\prod_{j=1}^{m-1}\binom{2j}{j}} \in \mathbb{Z}, \qquad 2^{-m-1}\frac{\prod_{j=m+1}^{2m+1}\binom{2j}{j}}{\prod_{j=1}^{m}\binom{2j}{j}} \in \mathbb{Z}.$$

Indeed the first example of Corollary 7 follows from Lemma 5.2 in [2] as well.

Several further questions are exhibited in Section 7. The referee of this paper pointed out that, questions around divisibility of multinomial coefficients have long history and

are widely studied still now. In relation to the present article, we just quote [13,10,5,18]. Readers find many related works therein.

2. Some lemma

Let $m_1, \dots, m_s$ and k be positive integers. Put $m = \sum_i^s m_i$ and

$$f(x) = \lfloor mx \rfloor - \sum_{i=1}^{s} \lfloor m_i x \rfloor,$$

which is the function used in [13]. We clearly have

$$f(x+1) = f(x). \tag{1}$$

From $x - 1 < \lfloor x \rfloor \leqslant x$, we see

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases} \tag{2}$$

Using (1) and (2), we obtain

$$f(x) + f(1-x) = s - 1 \tag{3}$$

unless mx or $m_i x$ is an integer. Thus we have

$$\int_0^u f(x)\,dx + \int_{1-u}^1 f(x)\,dx = (s-1)u.$$

From this equality,

$$\int_0^u \big(f(kx) - f(x)\big)\,dx + \int_{1-u}^1 \big(f(kx) - f(x)\big)\,dx = 0.$$

Therefore we derive

$$\int_0^1 \big(f(kx) - f(x)\big)\,dx = 0 \tag{4}$$

and

$$\int_0^u \big(f(kx) - f(x)\big)\,dx = \int_0^{1-u} \big(f(kx) - f(x)\big)\,dx. \tag{5}$$

First we assume $s = 2$.

Lemma 1. *Let m_1, m_2 be coprime integers and define*

$$f(x) = \lfloor (m_1 + m_2)x \rfloor - \lfloor m_1 x \rfloor - \lfloor m_2 x \rfloor.$$

Then for any positive integer k and any positive real u, we have

$$\int_0^u f(kx)\,dx \geqslant \int_0^u f(x)\,dx$$

and the equality holds if and only if $u \in \bigcup_{a=0}^{\infty} [a - \frac{1}{k(m_1+m_2)}, a + \frac{1}{k(m_1+m_2)}]$.

Figs. 1 and 2 in Section 7 are the graphs of $\int_0^u (f(kx) - f(x))\,dx$ in special cases, which may help the reader.

Proof. By (4),

$$\int_0^u \big(f(kx) - f(x)\big)\,dx$$

is invariant under $u \mapsto u + 1$. We show the inequality for $u \in [0, 1]$. The function $f(x)$ is a right continuous step function with discontinuities at $\frac{1}{m}\mathbb{Z}$ and $\bigcup_{i=1}^{2} \frac{1}{m_i}\mathbb{Z}$ with $m = m_1 + m_2$. The discontinuities at $\frac{1}{m}\mathbb{Z}$ gives $+1$ jump and $\bigcup_{i=1}^{2} \frac{1}{m_i}\mathbb{Z}$ gives -1 jump. Thus it is clear that $f(x) = 0$ for $x \in [0, 1/m)$ and so $f(x) = 1$ for $[(m-1)/m, 1)$ by (3) and right continuity. Since both integrands are identically zero, we have

$$\int_0^u f(kx)\,dx = \int_0^u f(x)\,dx$$

for $u \in [0, \frac{1}{km}]$ and the same is true for $u \in [\frac{km-1}{km}, 1]$ by (5). By (1), we first describe the distribution of the discontinuities in $(0, 1)$, i.e.,

$$\left\{\frac{1}{m}, \frac{2}{m}, \ldots, \frac{m-1}{m}\right\} \cup \left\{\frac{1}{m_1}, \frac{2}{m_1}, \ldots, \frac{m_1 - 1}{m_1}, \frac{1}{m_2}, \frac{2}{m_2}, \ldots, \frac{m_2 - 1}{m_2}\right\}.$$

Recalling the idea of Farey fractions, let a/b and c/d be two non-negative rational numbers with $a, b, c, d \in \mathbb{Z}$, $(a, b) = (c, d) = 1$ and $a/b < c/d$. Then

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}. \tag{6}$$

Arrange elements of

$$B = \left\{ \frac{1}{m_1}, \frac{2}{m_1}, \dots, \frac{m_1-1}{m_1}, \frac{1}{m_2}, \frac{2}{m_2}, \dots, \frac{m_2-1}{m_2} \right\}$$

in the increasing order. If j_1/m_1 and j_2/m_2 are adjacent, then we find a fraction $(j_1+j_2)/m$ in between. Considering the cardinality of B, we notice that there exists exactly one element of B in the interval $(i/m, (i+1)/m)$ for $i = 1, 2, \dots, m-2$. For each i/m for $i = 1, 2, \dots, m-2$, there is a right adjacent discontinuity of the form either j_1/m_1 or j_2/m_2. For the convenience, we formally extend this idea over the discontinuities at $\mathbb{Z}$. There are no element of $\bigcup_{i=1}^{2} \frac{1}{m_i}\mathbb{Z}$ in $((m-1)/m, 1)$ and we associate j_1/m_1 with $j_1 = m_1$ for $(m-1)/m$ and j_2/m_2 with $j_2 = m_2$ for $m/m = 1$. In other words, we are formally treating in a way that

$$\left[\frac{m-1}{m}, 1\right) = \left[\frac{m-1}{m}, \frac{m_1}{m_1}\right) \cup \left[\frac{m}{m}, \frac{m_2}{m_2}\right)$$

though the last interval $[m/m, m_2/m_2)$ is empty. We extend this convention to all positive integers by periodicity (1). Then for each discontinuity of the form $\frac{1}{m}\mathbb{Z}$, there exists exactly one right adjacent discontinuity[2] in $\bigcup_{i=1}^{2} \frac{1}{m_i}\mathbb{Z}$. Thus we see that $f(x)$ is a step function which takes exactly two values $\{0, 1\}$, and for any positive u, the integral $\int_0^u f(x)\,dx$ is computed as a sum of the length of intervals where $f(x) = 1$. These half open intervals have left end points in $\frac{1}{m}\mathbb{Z}$ and right end points in $\{u\} \cup (\bigcup_{i=1}^{2} \frac{1}{m_i}\mathbb{Z})$.

The required inequality is equivalent to

$$k \int_0^u f(x)\,dx \leqslant \int_0^{ku} f(x)\,dx$$

and it suffices to show that

$$G(u) = \int_0^{ku} f(x)\,dx - k \int_0^u f(x)\,dx$$

is positive for $\frac{1}{mk} < u < \frac{mk-1}{mk}$ by (4) and (5). Moreover the positivity is clearly true for $\frac{1}{mk} < u \leqslant \varepsilon$ and $1-\varepsilon \leqslant u < \frac{mk-1}{mk}$ where ε is the next discontinuity of $f(kx)$ adjacent to $\frac{1}{mk}$. As $f(x) \in \{0, 1\}$ and $G'(u) = k(f(ku) - f(u))$, we see if $f(u) = 1$ then $G'(u) \leqslant 0$ and if $f(u) = 0$ then $G'(u) \geqslant 0$. The minimum of $G(u)$ is attained either at the end points of $[\varepsilon, 1-\varepsilon]$ or the point where $G'(u)$ changes its sign from non-positive to non-negative, i.e., where f has negative jump. Thus it is enough to show that $G(u) > 0$ for $u \in B$, because the minimum must be equal to

$$\min\left\{ G(\varepsilon), \min_{u \in B} G(u) \right\}.$$

[2] Here we think that the right adjacent discontinuity of $a = \frac{am}{m} \in \frac{1}{m}\mathbb{Z}$ is the discontinuity $\frac{am_2}{m_2} \in \frac{1}{m_2}\mathbb{Z}$.

Without loss of generality we may put $u = j_1/m_1$ with $j_1 = 1, 2, \dots, m_1 - 1$. Summing up the length of intervals where $f(x) = 1$, we have

$$G\left(\frac{j_1}{m_1}\right) = \left(\sum_{j=1}^{kj_1} \frac{j}{m_1} + \sum_{j=1}^{kj_2+\ell} \frac{j}{m_2} - \sum_{j=1}^{k(j_1+j_2)+\ell} \frac{j}{m}\right) - k\left(\sum_{j=1}^{j_1} \frac{j}{m_1} + \sum_{j=1}^{j_2} \frac{j}{m_2} - \sum_{j=1}^{j_1+j_2} \frac{j}{m}\right)$$

with $0 \leqslant j_2 < m_2$ and $0 \leqslant \ell \leqslant k-1$. The right side is

$$\frac{(j_2^2 k(k-1) + 2j_2 k\ell + \ell^2 + \ell)m_1^2 - 2j_1(j_2(k-1)+\ell)km_1m_2 + j_1^2 k(k-1)m_2^2}{2m_1m_2(m_1+m_2)}$$

whose numerator is a quadratic form of m_1 and m_2 with the discriminant

$$-4j_1^2 k\ell(k-\ell-1)$$

and the coefficient of m_2^2 is positive. Thus if $0 < \ell < k-1$ then $G(\frac{j_1}{m_1}) > 0$. For the remaining cases, we have

$$2m_1m_2(m_1+m_2)G\left(\frac{j_1}{m_1}\right) = \begin{cases} k(k-1)(j_2m_1 - j_1m_2)^2, & \ell = 0, \\ k(k-1)((j_2+1)m_1 - j_1m_2)^2, & \ell = k-1. \end{cases}$$

Since m_1 and m_2 are coprime and $j_1 = 1, \dots, m_1 - 1$, the right side cannot vanish in both cases. We have shown the lemma. □

We prepare an elementary inequality:

Lemma 2. *Let s and $b_1, b_2, \dots, b_{s+1}$ be positive integers with $GCD(b_1, b_2, \dots, b_{s+1}) = 1$ and $b = b_1 + b_2 + \cdots + b_{s+1}$. Then we have the following inequality*:

$$\prod_{i=1}^{s+1}\Big(GCD(b-b_i, b_i) \underset{\substack{j\neq i \\ 1\leqslant j\leqslant s+1}}{GCD} b_j\Big) \leqslant \prod_{i=1}^{s+1} \frac{b-b_i}{s}.$$

Proof. The left side is equal to:

$$\prod_{i=1}^{s+1}\Big(GCD(b-b_{i+1}, b_{i+1}) \underset{\substack{j\neq i \\ 1\leqslant j\leqslant s+1}}{GCD} b_j\Big)$$

with the convention $b_{s+2} = b_1$. Then we see that $GCD(b-b_{i+1}, b_{i+1})$ and $GCD_{\substack{j\neq i \\ 1\leqslant j\leqslant s+1}} b_j$ are coprime divisors of b_{i+1}. Therefore we have

$$\prod_{i=1}^{s+1}\Big(GCD(b-b_i, b_i) \underset{\substack{j\neq i \\ 1\leqslant j\leqslant s+1}}{GCD} b_j\Big) \leqslant \prod_{k=1}^{s+1} b_k.$$

On the other hand

$$\prod_{i=1}^{s+1}(b-b_i) \geqslant \prod_{i=1}^{s+1}\left(s\left(\prod_{j\neq i} b_j\right)^{1/s}\right) = s^{s+1}\prod_{k=1}^{s+1} b_k$$

which proves the inequality. □

We wish to show a generalization of Lemma 1.

Lemma 3. *Let $m_1,\dots,m_s$ be positive integers and put*

$$m = m_1 + m_2 + \cdots + m_s, \qquad g = GCD(m_1, m_2, \dots, m_s)$$

and $f(x) = \lfloor mx\rfloor - \sum_{i=1}^{s}\lfloor m_i x\rfloor$. Then for any positive integer k and any positive real u, we have

$$\int_0^u f(kx)\,dx \geqslant \int_0^u f(x)\,dx$$

and the equality holds if and only if $u \in \bigcup_{a=0}^{\infty}[\frac{a}{g} - \frac{1}{km}, \frac{a}{g} + \frac{1}{km}]$.

Proof. Lemma 1 shows the case $s = 2$ and $g = 1$. The case $s = 2$ and $g > 1$ is easily shown by applying Lemma 1 for $m' = m/g$ and $m_i' = m_i/g$. We assume that the statement is valid until $s(\geqslant 2)$ and prove the case $m = m_1 + m_2 + \cdots + m_{s+1}$ and $GCD(m_1, m_2, \dots, m_{s+1}) = 1$. The case $GCD(m_1, m_2, \dots, m_{s+1}) > 1$ follows similarly to the case $s = 2$.

By Lemma 2 with $b_i = m_i$, we may assume

$$GCD(m_1, m_2, \dots, m_s)\, GCD\left(\sum_{i=1}^{s} m_i, m_{s+1}\right) \leqslant \frac{\sum_{i=1}^{s} m_i}{s} \tag{7}$$

without loss of generality by changing indices. By the induction assumption, $f_1(x) = \lfloor(\sum_{i=1}^{s} m_i)x\rfloor - \sum_{i=1}^{s}\lfloor m_i x\rfloor$ satisfies the inequality:

$$\int_0^u f_1(kx)\,dx \geqslant \int_0^u f_1(x)\,dx \tag{8}$$

and the equality holds if and only if

$$x \in \bigcup_{a=0}^{\infty}\left[\frac{a}{g_1} - \frac{1}{k\sum_{i=1}^{s} m_i}, \frac{a}{g_1} + \frac{1}{k\sum_{i=1}^{s} m_i}\right]$$

with $g_1 = GCD(m_1, m_2, \dots, m_s)$. Again by the induction assumption, for $f_2(x) = \lfloor mx \rfloor - \lfloor (\sum_{i=1}^{s} m_i)x \rfloor - \lfloor m_{s+1}x \rfloor$ we have

$$\int_0^u f_2(kx)\,dx \geqslant \int_0^u f_2(x)\,dx \tag{9}$$

and the equality holds if and only if

$$x \in \bigcup_{b=0}^{\infty} \left[\frac{b}{g_2} - \frac{1}{km}, \frac{b}{g_2} + \frac{1}{km} \right]$$

with $g_2 = GCD(\sum_{i=1}^{s} m_i, m_{s+1})$. Since $f(x) = f_1(x) + f_2(x)$, we obtain

$$\int_0^u f(kx)\,dx \geqslant \int_0^u f(x)\,dx$$

from (8) and (9). Noting that g_1, g_2 are coprime, if either a/g_1 or b/g_2 is not an integer, then $|a/g_1 - b/g_2| \geqslant 1/(g_1 g_2)$. The inequality (7) shows that

$$\left[\frac{a}{g_1} - \frac{1}{k\sum_{i=1}^{s} m_i}, \frac{a}{g_1} + \frac{1}{k\sum_{i=1}^{s} m_i} \right] \cap \left[\frac{b}{g_2} - \frac{1}{km}, \frac{b}{g_2} + \frac{1}{km} \right] \neq \emptyset$$

if and only if $a/g_1 = b/g_2 \in \mathbb{Z}$. We have shown the lemma. □

Lemma 4. *The function $f(x)$ in Lemma 3 takes values in $\{0, 1, \dots, s-1\}$.*

Proof. The case $s = 2$ is shown in the proof of Lemma 1. Using the decomposition $f(x) = f_1(x) + f_2(x)$ in the proof of Lemma 3, the assertion is easily shown by induction on s. □

3. Proof of Theorem 1

Let p be a prime and $\nu_p(n)$ be the largest exponent e such that p^e divides n. Using the Legendre formula: $\nu_p(n!) = \sum_{e=1}^{\infty} \lfloor n/p^e \rfloor$, we see that

$$\nu_p\left(\prod_{n=1}^{t} \frac{a_{kn}}{a_n} \right) = \sum_{e=1}^{\infty} \sum_{n=1}^{t} \left(f\left(\frac{kn}{p^e} \right) - f\left(\frac{n}{p^e} \right) \right)$$

where f is defined in Lemma 3. Define

$$H(t) := \sum_{n=1}^{t} \left(f\left(\frac{kn}{p^e} \right) - f\left(\frac{n}{p^e} \right) \right).$$

It suffices to prove that $H(t) \geqslant 0$ for all t provided p^e is sufficiently large. First we assume that p and k are coprime. Observe that

$$\frac{1}{p^e}\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)$$

is a Riemann sum of the integral

$$\int_0^{t/p^e}\big(f(kx)-f(x)\big)\,dx.$$

Our strategy is to show that these approximation is enough fine and Lemma 3 gives the answer to our problem. Since k and p are coprime, we see that

$$f(x)+f(1-x)=f(kx)+f\big(k(1-x)\big) \tag{10}$$

holds for $x \in \frac{1}{p^e}\mathbb{Z}\setminus\mathbb{Z}$. Indeed $\lfloor x\rfloor+\lfloor -x\rfloor$ is equal to -1 or 0 by (2), both sides are equal to $s-1+\Delta(p^e)$ with

$$\Delta\big(p^e\big)=\delta\big(p^e \mid m\big)-\sum_{i=1}^{s}\delta\big(p^e \mid m_i\big)$$

for $x \in \frac{1}{p^e}\mathbb{Z}\setminus\frac{1}{p^{e-1}}\mathbb{Z}$ with $e \geqslant 1$. Here $\delta(\cdot)$ takes values 1 or 0 according to whether the inside statement is true or not. We claim that

$$H(t)=H\big(p^e-1-t\big). \tag{11}$$

In fact, from (10) we see

$$\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)+\sum_{n=p^e-t}^{p^e-1}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)=0$$

and thus

$$\sum_{n=1}^{p^e-1}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)=0.$$

Therefore we have

$$\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)=\sum_{n=1}^{p^e-1-t}\left(f\left(\frac{kn}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)$$

which shows the claim. Put $\langle x\rangle = x - \lfloor x\rfloor$. If $\langle t/p^e\rangle < 1/m$, then $H(t) \geqslant 0$ is clearly true because $f(x) = 0$ for $0 \leqslant x < 1/m$. By (11), $H(t) \geqslant 0$ also holds when $\langle t/p^e\rangle > 1 - 1/m$. From Lemma 3, there is a positive constant D such that

$$\int_0^u \big(f(kx) - f(x)\big)\,dx \geqslant D$$

for $1/m \leqslant u \leqslant (m-1)/m$. We have

$$\begin{aligned}&\int_0^{t/p^e} \big(f(kx) - f(x)\big)\,dx - \frac{1}{p^e}\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\\ &\quad= \sum_{n=1}^{t}\left\{\int_{(n-1)/p^e}^{n/p^e} \big(f(kx) - f(x)\big)\,dx - \frac{1}{p^e}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\right\}. \qquad (12)\end{aligned}$$

Since $f(kx) - f(x)$ is a step function, the last summand is zero if the interval $((n-1)/p^e, n/p^e]$ contains no discontinuities, and its modulus is bounded from above by $2(s-1)/p^e$ in light of Lemma 4. Letting E be the number of discontinuities of $f(kx) - f(x)$ in $(0,1)$, we have

$$\left|\int_0^{t/p^e} \big(f(kx) - f(x)\big)\,dx - \frac{1}{p^e}\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\right| < \frac{2(s-1)E}{p^e}.$$

Therefore if $p^e > \frac{2(s-1)E}{D}$ then

$$\frac{1}{p^e}\sum_{n=1}^{t}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) > D - \frac{2(s-1)E}{p^e} > 0.$$

Exceptional discussion is required when p divides k. Putting $k' = k/p^{\nu_p(k)}$, we have

$$\sum_{e=1}^{\infty}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) = \sum_{e=1}^{\infty}\left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)$$

since $f(x) = 0$ for $x \in \mathbb{Z}$. If k is a power of a prime p, then $k' = 1$ and the right side is identically zero. If not, we have to replace k by k' and apply the same discussion. Then corresponding D' and E' are computed and we see that if $K_p := \frac{2(s-1)E'}{D'} < p^e$ then

$$\frac{1}{p^e}\sum_{n=1}^{t}\left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) > 0.$$

Therefore $H(t) < 0$ happens only when

$$p^e \leqslant M := \max\left\{\frac{2(s-1)E}{D}, \max_{p|k \text{ and } k'>1} K_p\right\},$$

which proves Theorem 1. See Section 6 for the actual computation of $C(k)$.

4. Proof of Theorem 3

We follow the same notation as in the previous section. It suffices to prove that for any fixed k there are infinitely many pairs (p,t) such that

$$\sum_{n=1}^{t}\sum_{e=1}^{\infty}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) < 0.$$

Take the minimum gap κ between two adjacent discontinuities of $f(kx)$. We shall find infinitely many such prime p's which are greater than $\max\{1/\kappa, k^2m\}$ and coprime with $k\,LCM(m, m_1, m_2, \ldots, m_s)$. Since $p > km$, a rational number with denominator p cannot be a discontinuity of $f(kx)$.

From the assumption $g = GCD(m_1, m_2, \ldots, m_s) > 1$, we know

$$\int_0^u \big(f(kx) - f(x)\big)\,dx = 0$$

for $u \in [\frac{1}{g} - \frac{1}{km}, \frac{1}{g} + \frac{1}{km}]$ by Lemma 3. Applying (12) with $e = 1$ and $t = \lfloor p/g \rfloor$, since $p > k^2m$ implies $1/km > kt/p^2$ we have

$$\begin{aligned}
&\frac{1}{p}\sum_{n=1}^{t}\sum_{e=1}^{\infty}\left(f\left(\frac{kn}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\\
&\quad= \frac{1}{p}\sum_{n=1}^{t}\left(f\left(\frac{kn}{p}\right) - f\left(\frac{n}{p}\right)\right)\\
&\quad= \sum_{n=1}^{t}\left\{\int_{(n-1)/p}^{n/p}\left(f\left(\frac{kn}{p}\right) - f(kx)\right) - \left(f\left(\frac{n}{p}\right) - f(x)\right)dx\right\}. \qquad (13)
\end{aligned}$$

For the moment, we tentatively think that no discontinuities of $f(kx)$ intersects, i.e.,

$$\left\{\frac{j}{km} \;\middle|\; j = 1, \ldots, km\right\} \cup \left(\bigcup_{i=1}^{s}\left\{\frac{j}{km_i} \;\middle|\; j = 1, \ldots, km_i\right\}\right)$$

are the set of 'distinct' $km + \sum_{i=1}^{s} km_i = 2km$ points, and compute the right side. Since we are choosing a large p, there is at most one discontinuity ξ in the interval

$((n-1)/p, n/p]$. If there is no discontinuity in $((n-1)/p, n/p]$, then $f(\frac{kn}{p}) - f(kx) = 0$. If such a discontinuity ξ exists, then we have

$$f\left(\frac{kn}{p}\right) - f(kx) = \begin{cases} 0, & x \in [\xi, \frac{n}{p}], \\ \pm 1, & x \in (\frac{n-1}{p}, \xi), \end{cases} \tag{14}$$

where ± 1 is $+1$ if ξ is the discontinuity of $\lfloor kmx \rfloor$ and -1 if ξ is the discontinuity of $\lfloor km_i x \rfloor$ for some i. Then we see

$$p \int_{(n-1)/p}^{n/p} \left(f\left(\frac{kn}{p}\right) - f(kx) \right) dx = \pm\big(p\xi - (n-1)\big) = \pm\langle p\xi \rangle. \tag{15}$$

A similar formula holds for $f(n/p) - f(x)$. Summing up, from (13) we have shown:

$$\begin{aligned} \sum_{n=1}^{t} \left(f\left(\frac{kn}{p}\right) - f\left(\frac{n}{p}\right) \right) = {} & \sum_{j=1}^{km/g} \left\langle \frac{pj}{km} \right\rangle - \sum_{i=1}^{s} \sum_{j=1}^{km_i/g} \left\langle \frac{pj}{km_i} \right\rangle \\ & - \left(\sum_{j=1}^{m/g} \left\langle \frac{pj}{m} \right\rangle - \sum_{i=1}^{s} \sum_{j=1}^{m_i/g} \left\langle \frac{pj}{m_i} \right\rangle \right). \end{aligned} \tag{16}$$

In reality, the discontinuities of $f(kx)$ intersect in many places. For e.g., at least j/k for $j \in \mathbb{Z}$ is a common discontinuity of $\lfloor km_i x \rfloor$ and $\lfloor km_j x \rfloor$ for $i \neq j$. However the above formula is correct without any changes. This is seen by a similar convention as in the proof of Lemma 1. For e.g., if ξ belongs to two discontinuities of $\lfloor km_i x \rfloor$ and $\lfloor km_j x \rfloor$ with $i \neq j$, then

$$f\left(\frac{kn}{p}\right) - f(kx) = -2$$

in $((n-1)/p, \xi)$ instead of (14), but we computed integrand of (15) twice in (16).

Put $L = LCM(m, m_1, \ldots, m_s)$ and $L' = LCM(m_1, \ldots, m_s)$. An importance of the formula (16) is that the value is determined by $p \pmod{kL}$. As p is coprime to kL, by Dirichlet's theorem on primes in arithmetic progression, it suffices to show that there exists a single p such that the value of (16) is negative to prove our theorem. Noting

$$\sum_{j=1}^{km/g} \frac{pj}{km} - \sum_{i=1}^{s} \sum_{j=1}^{km_i/g} \frac{pj}{km_i} - \left(\sum_{j=1}^{m/g} \frac{pj}{m} - \sum_{i=1}^{s} \sum_{j=1}^{m_i/g} \frac{pj}{m_i} \right) = 0, \tag{17}$$

it is equivalent to show that there is such a p that

$$\sum_{j=1}^{km/g}\left\lfloor\frac{pj}{km}\right\rfloor-\sum_{i=1}^{s}\sum_{j=1}^{km_i/g}\left\lfloor\frac{pj}{km_i}\right\rfloor-\left(\sum_{j=1}^{m/g}\left\lfloor\frac{pj}{m}\right\rfloor-\sum_{i=1}^{s}\sum_{j=1}^{m_i/g}\left\lfloor\frac{pj}{m_i}\right\rfloor\right) \tag{18}$$

is positive.[3] Moreover using $\lfloor x\rfloor+\lfloor -x\rfloor$ is equal to a constant -1 for non-integer x by (2), the value of (18) changes its sign by the involution $p\leftrightarrow kL-p$ for p which is coprime to kL, and the same holds for (16). Therefore our task is to show that either (16) or (18) is not zero for some p which is coprime to kL. We show this by dividing into three cases.

Case $m>L'$. One can find a p that

$$p\equiv 1 \pmod{kL'},\qquad p\not\equiv 1 \pmod{km}$$

and is coprime with kL. From (17), the right side of (16) becomes

$$\begin{aligned}&\sum_{j=1}^{km/g}\left\langle\frac{pj}{km}\right\rangle-\sum_{i=1}^{s}\sum_{j=1}^{km_i/g}\frac{j}{km_i}-\left(\sum_{j=1}^{m/g}\left\langle\frac{pj}{m}\right\rangle-\sum_{i=1}^{s}\sum_{j=1}^{m_i/g}\frac{j}{m_i}\right)\\ &=\sum_{j=1}^{km/g}\left(\left\langle\frac{pj}{km}\right\rangle-\frac{j}{km}\right)-\sum_{j=1}^{m/g}\left(\left\langle\frac{pj}{m}\right\rangle-\frac{j}{m}\right)\\ &=\sum_{\substack{j=1\\ j\not\equiv 0 \pmod k}}^{km/g}\left(\left\langle\frac{pj}{km}\right\rangle-\frac{j}{km}\right).\end{aligned} \tag{19}$$

Here we have

$$\sum_{\substack{j=1\\ j\not\equiv 0 \pmod k}}^{km/g}\left\langle\frac{pj}{km}\right\rangle=\frac{1}{km}\sum_{\substack{j=1\\ j\not\equiv 0 \pmod k}}^{km/g}\mathcal{R}(pj,km),$$

where $\mathcal{R}(a,b)$ is the minimum non-negative integer congruent to a mod b. Because the function $\langle x/km\rangle$ is increasing for $0\leqslant x<km$, the minimum is attained by

$$\frac{1}{km}\sum_{\substack{j=1\\ j\not\equiv 0 \pmod k}}^{km/g}\mathcal{R}(j,km)=\sum_{\substack{j=1\\ j\not\equiv 0 \pmod k}}^{km/g}\frac{j}{km}$$

which shows that (19) is non-negative. However, by $p\not\equiv 1 \pmod{km}$, clearly $\mathcal{R}(pj,km)$ takes values outside $\{\mathcal{R}(j,km)\mid 1\leqslant j\leqslant km/g,\ j\not\equiv 0 \pmod k\}$ and thus (19) must be positive.

[3] From this expression, we see that (16) is integer valued.

Case $m < L'$. We choose a p that

$$p \not\equiv 1 \pmod{kL'}, \qquad p \equiv 1 \pmod{km}.$$

From (17), the right side of (16) becomes

$$\begin{aligned}
&\sum_{j=1}^{km/g} \frac{j}{km} - \sum_{i=1}^{s} \sum_{j=1}^{km_i/g} \left\langle \frac{pj}{km_i} \right\rangle - \left(\sum_{j=1}^{m/g} \frac{j}{m} - \sum_{i=1}^{s} \sum_{j=1}^{m_i/g} \left\langle \frac{pj}{m_i} \right\rangle \right) \\
&= -\sum_{i=1}^{s} \sum_{j=1}^{km_i/g} \left(\left\langle \frac{pj}{km_i} \right\rangle - \frac{j}{km_i} \right) - \sum_{i=1}^{s} \sum_{j=1}^{m_i/g} \left(\left\langle \frac{pj}{m_i} \right\rangle - \frac{j}{m_i} \right) \\
&= -\sum_{i=1}^{s} \sum_{\substack{j=1 \\ j \not\equiv 0 \pmod{k}}}^{km_i/g} \left(\left\langle \frac{pj}{km_i} \right\rangle - \frac{j}{km_i} \right).
\end{aligned}$$

The inner sums are non-negative and at least one of them is positive by the same discussion as in the former case.

Case[4] $m = L'$. We rewrite (18) into

$$\mathcal{F}(p) := \sum_{\substack{j=1 \\ j \not\equiv 0 \pmod{k}}}^{km/g} \left\lfloor \frac{pj}{km} \right\rfloor - \sum_{i=1}^{s} \sum_{\substack{j=1 \\ j \not\equiv 0 \pmod{k}}}^{km_i/g} \left\lfloor \frac{pj}{km_i} \right\rfloor.$$

It suffices to show that there are two integers p_1 and p_2 which are coprime with $km = k\,LCM(m, m_1, \dots, m_s)$ and $\mathcal{F}(p_1) \neq \mathcal{F}(p_2)$.

First we study the case that there is a prime q with $q \mid \frac{m}{g}$ and $q^2 \mid km$. Then we can take $p_1 = km/q - 1$ and $p_2 = km/q + 1$ which are coprime with km. Since

$$\left\lfloor \frac{j}{q} + \frac{j}{km} \right\rfloor - \left\lfloor \frac{j}{q} - \frac{j}{km} \right\rfloor = \begin{cases} 0, & j \not\equiv 0 \pmod{q}, \\ 1, & j \equiv 0 \pmod{q}, \end{cases}$$

we obtain

$$\mathcal{F}(p_2) - \mathcal{F}(p_1) = \sum_{\substack{1 \leqslant j \leqslant km/g \\ j \not\equiv 0 \pmod{k} \\ j \equiv 0 \pmod{q}}} 1 - \sum_{i=1}^{s} \sum_{\substack{1 \leqslant j \leqslant km_i/g \\ j \not\equiv 0 \pmod{k} \\ \frac{m}{m_i} j \equiv 0 \pmod{q}}} 1.$$

Since $m = m_1 + m_2 + \cdots + m_s$ and m/m_i are integers, the right side is expected to be negative. However a careful computation is required, because not all km_i/g are divisible by q. Since q divides either m_i/g or m/m_i, we have

[4] There are such pairs, for e.g., $1 + 2 + 3 = LCM(1, 2, 3)$, $2 + 3 + 3 + 4 = LCM(2, 3, 3, 4)$.

$$\sum_{\substack{1\leqslant j\leqslant km_i/g\\ j\not\equiv 0 \pmod k\\ \frac{m}{m_i}j\equiv 0 \pmod q}} 1 = \begin{cases} \frac{km_i}{g} - \frac{m_i}{g}, & m/m_i \equiv 0 \pmod q,\\ \frac{km_i}{gq} - \frac{km_i}{g\,LCM(k,q)}, & m/m_i \not\equiv 0 \pmod q. \end{cases}$$

Therefore

$$\begin{aligned}
&\mathcal{F}(p_2) - \mathcal{F}(p_1)\\
&\quad = \frac{km}{gq} - \frac{km}{g\,LCM(k,q)} - \sum_{\substack{1\leqslant i\leqslant s\\ \frac{m}{m_i}\not\equiv 0 \pmod q}} \left(\frac{km_i}{gq} - \frac{km_i}{g\,LCM(k,q)}\right)\\
&\qquad - \sum_{\substack{1\leqslant i\leqslant s\\ \frac{m}{m_i}\equiv 0 \pmod q}} \left(\frac{km_i}{g} - \frac{m_i}{g}\right)\\
&\quad = \sum_{\substack{1\leqslant i\leqslant s\\ \frac{m}{m_i}\equiv 0 \pmod q}} \left(\left(\frac{km_i}{gq} - \frac{km_i}{g\,LCM(k,q)}\right) - \left(\frac{km_i}{g} - \frac{m_i}{g}\right)\right)\\
&\quad = -\sum_{\substack{1\leqslant i\leqslant s\\ \frac{m}{m_i}\equiv 0 \pmod q}} \frac{m_i}{g}\frac{(k-1)((LCM(k,q)+1)(q-1)+1) - LCM(k,q) + q}{q\,LCM(k,q)}\\
&\quad \leqslant -\sum_{\substack{1\leqslant i\leqslant s\\ \frac{m}{m_i}\equiv 0 \pmod q}} \frac{(q+2)m_i}{gq\,LCM(k,q)} < 0.
\end{aligned}$$

Here we used $k \geqslant 2$ and $q \geqslant 2$. Note that the last sum is non-empty, because $q \mid (m/g)$ and $GCD((m_1/g), \ldots, (m_s/g)) = 1$ implies that q divides m/m_i for at least one i.

Second, consider the case that there is a prime divisor $q > 3$ of m/g. We may assume that either

$$(p_1, p_2) = \left(\frac{km}{q} - 1, \frac{km}{q} + 1\right) \quad \text{or} \quad \left(\frac{2km}{q} - 1, \frac{2km}{q} + 1\right)$$

satisfies $GCD(p_i, km) = 1$ for $i = 1$ and 2. In fact, if for e.g. $km/q - 1 \equiv 0 \pmod q$ and $2km/q + 1 \equiv 0 \pmod q$ hold, then q^2 divides km, which is reduced to the first case. Once we have such a pair (p_1, p_2), we can show

$$\mathcal{F}(p_2) - \mathcal{F}(p_1) < 0$$

in the same manner. So we finally consider the case that all the prime divisors of m/g is 2 and 3 and m/g is square free, which covers the remaining cases. There exists only one such case with

$$\frac{m_1}{g}+\cdots+\frac{m_s}{g}=LCM\left(\frac{m_1}{g},\dots,\frac{m_s}{g}\right),$$

that is, $s=3$ and $(m_1/g,m_2/g,m_3/g)=(1,2,3)$. So our last task is to consider the case: $(m_1,m_2,m_3)=(g,2g,3g)$. Since $m=6g$ is even we can choose:

$$(p_1,p_2)=\begin{cases}(\frac{km}{2}-1,\frac{km}{2}+1), & km\equiv 0 \pmod 4,\\ (\frac{km}{2}-2,\frac{km}{2}+2), & km\equiv 2 \pmod 4.\end{cases}$$

Then we see $\mathcal{F}(p_2)-\mathcal{F}(p_1)<0$ in the same manner.

5. Proof of Theorem 6

This proof is inspired by Theorem 5.2 and Lemma 5.2 in [2]. We use the same terminology as in Section 3. Under the assumption, $f(x)$ has no discontinuity of negative jump in $(0,1)$ by cancellation. Thus $f(x)$ is non-decreasing. We show that

$$\sum_{n=1}^{t}\left(f\left(\frac{k'n}{p^e}\right)-f\left(\frac{n}{p^e}\right)\right)\geqslant 0$$

where $k'=k/p^{\nu_p(k)}$. By periodicity of f, it suffices to show the case that $t<p^e$. Since k' and p are coprime, $k'n \bmod p^e$ $(n=1,2,\dots,p^e-1)$ are distinct. From $f(1/p^e)\leqslant f(2/p^e)\leqslant\cdots\leqslant f((p^e-1)/p^e)$, we see

$$\sum_{n=1}^{t}f\left(\frac{n}{p^e}\right)$$

is the minimum of the sum of t elements in $\{f(i/p^e)\mid i=1,2,\dots,p^e-1\}$, which finishes the proof.

6. Computation of $C(k)$

In this section, we explain the computation of the constant $C(k)$ by an algorithm based on the proof of Theorem 1. For a given k, first we compute the minimum D_q of $I(u)=\int_0^u f(k'x)-f(x)\,dx$ for $u\in[1/m,1-1/m]$ for $k'=k'(q)=k/q^{\nu_q(k)}$ for all prime divisor q of k. Denote by D_1 the minimum of $I(u)$ for $k'=k$. Since the minimum of $I(u)$ is attained at the discontinuity of the step function: $I'(u)=f(k'u)-f(u)$, using (11), it is explicitly computed as

$$\min I(u)=\min\left\{\min_{1\leqslant j\leqslant\frac{m}{2}}I\left(\frac{j}{m}\right),\min_{1\leqslant i\leqslant s}\min_{\frac{k'm_i}{m}\leqslant j\leqslant\frac{k'm_i}{2}}I\left(\frac{j}{k'm_i}\right)\right\}.$$

By Theorem 1, we know $D_q > 0$. Number of discontinuities is bounded from above[5] by $2k'm$. Then we compute for prime p's with

$$p \leqslant M := \max\left\{\frac{2km(s-1)}{D_1}, \max_{q|k \text{ and } k'>1} \frac{2k'(q)m(s-1)}{D_q}\right\},$$

the values

$$\mu(p) = \min\left\{0, \min_{t=1}^{p^h-1} \sum_{n=1}^{t} \sum_{e=1}^{h+\ell-1} \left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\right\}$$

with $h = \lfloor \log(M)/\log(p) \rfloor$ and $\ell = \lfloor \log(k'm)/\log(p) \rfloor$. By the proof of Theorem 1, if $p^e > M$ then

$$\sum_{n=1}^{t} \left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) \geqslant 0$$

for any positive integer t. Further by $\frac{k'n}{p^{h+\ell}} < \frac{1}{m}$,

$$\sum_{n=1}^{t} \left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right) = 0$$

for $e \geqslant h + \ell$ and $t < p^h$. We have

$$\mu(p) = \min\left\{0, \min_{t=1}^{\infty} \sum_{n=1}^{t} \sum_{e=1}^{\infty} \left(f\left(\frac{k'n}{p^e}\right) - f\left(\frac{n}{p^e}\right)\right)\right\}$$

and $p^{\mu(p)}$ is attained as the denominator for some t. We obtained the constant $C(k) = \prod_{p \leqslant M} p^{|\mu(p)|}$. □

We briefly demonstrate this algorithm by showing Corollary 2 and make precise the comments afterwords. For $s = 2$, $m_1 = 2$, $m_2 = 3$ and $k = 2$, the graph of the function $I(u)$ for $k' = k$ is depicted in Fig. 1. We may take $D_1 = 1/60$ and $E_1 = 8$. Checking all primes which do not exceed $2 \cdot 8 \cdot 60 = 960$, we found the only non-zero output $\mu(11) = -1$. We can confirm that

$$\sum_{n=1}^{t} \sum_{e=1}^{\infty} \big(f(2n/11^e) - f(n/11^e)\big) = \sum_{n=1}^{3} \big(f(2n/11) - f(n/11)\big) = -1$$

when $t = 3{+}11^h$ and $h \geqslant 2$, since $f(x) = 0$ for $0 \leqslant x \leqslant 1/5$. Therefore the denominator 11 in the first formula in Corollary 2 appears infinitely often. For $s = 2$, $m_1 = 1$, $m_2 = 2$

[5] It is better to take the exact value to make faster the computation, as we do below in examples.

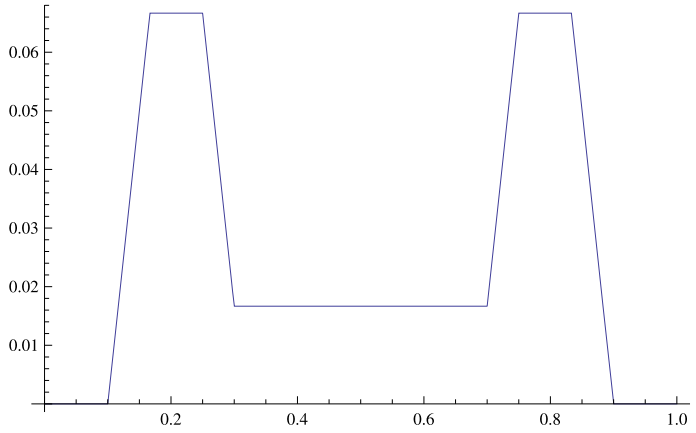


Fig. 1. $m_1 = 2,\ m_2 = 3,\ k = 2$.

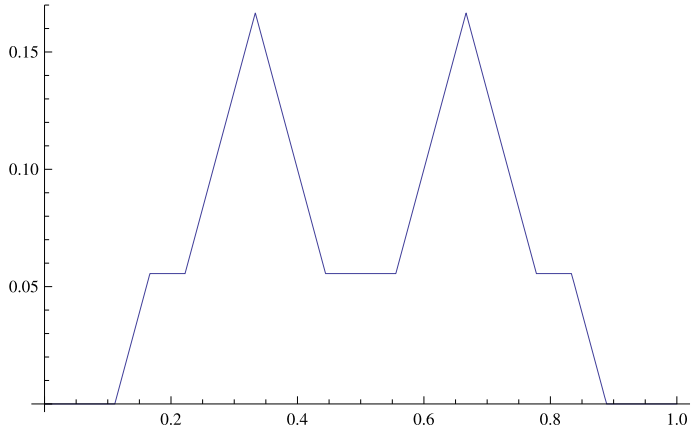


Fig. 2. $m_1 = 1,\ m_2 = 2,\ k = 3$.

and $k = 3$, the graph of $I(u)$ is depicted in Fig. 2. We have $D_1 = 1/18$, $E_1 = 10$ and the only non-zero μ-value is $\mu(5) = -1$. In this case, one can also confirm that

$$\sum_{n=1}^{2+5^h} \sum_{e=1}^{h} \bigl(f\bigl(3n/5^e\bigr) - f\bigl(n/5^e\bigr)\bigr) = -1.$$

However we have

$$\sum_{n=1}^{t} \sum_{e=1}^{h+1} \bigl(f\bigl(3n/5^e\bigr) - f\bigl(n/5^e\bigr)\bigr) \geqslant 0$$

for $2 < t < 5^h - 1$ and any positive integer h. In other words, the function $\sum_{n=1}^{t} \sum_{e=1}^{h} (f(3n/5^e) - f(n/5^e))$ has a period 5^h and attains -1 infinitely often as above, but such negative values are erased by the next period of length 5^{h+1}, except

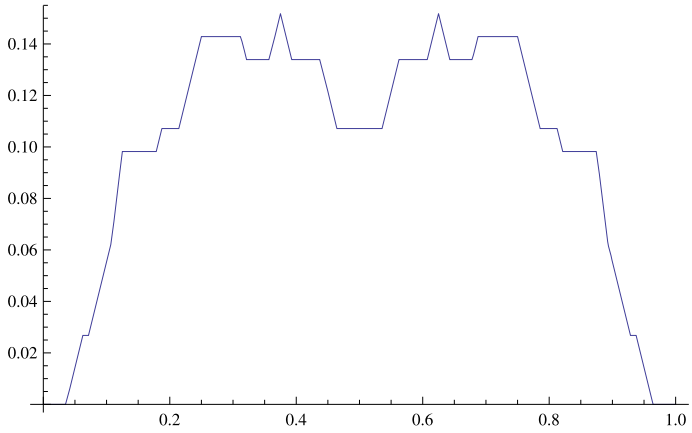


Fig. 3. $m_1 = 1$, $m_2 = 2$, $m_3 = 4$, $k = 4$.

when $t = 2$. The denominator 5 in the second formula of Corollary 2 appears only when $t = 2$.

For $s = 3$, $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ and $k = 4$, the graph of is depicted in Fig. 3. We have $D_1 = 11/112$, $E_1 = 32$ and $\mu(p) = 0$ for all prime p.

In this manner, a prime divisor p of $C(k)$ actually appears in the denominator only when the p-adic expansion of t has a special form, and not easy to describe the set of such t's.

7. Questions

We wish to list several open problems. A sequence (a_n) may be called *almost mean divisible* if there is a positive integer C such that

$$\left(\prod_{n=1}^{t} a_{kn}\right) / \left(\prod_{i=1}^{t} a_n\right) \in \frac{1}{C}\mathbb{Z}$$

for any positive integers k and t. In other words, (a_n) is almost mean divisible, if it is almost mean k-divisible with a uniform constant C independent of the choice of k.

- Is there an almost mean divisible multinomial sequence, which is not mean divisible?
- Is there a mean divisible multinomial sequence which does not satisfy the condition of Theorem 6?
- Is there any other (almost) mean divisible sequence of number theoretical interests?

I expect the answer for the first question is negative, because the bound M increases as k becomes large, in the proof of Theorem 1. For the second, there may exist such multinomial sequences for $s > 2$. For $s = 3$ and $(m_1, m_2, m_3) = (m, 1, 1)$, I checked by the algorithm in Section 6 to obtain

Corollary 8. *If* $3 \leqslant m \leqslant 10$ *and* $k \leqslant 10$, *then*

$$\prod_{n=1}^{t} \frac{\binom{(m+2)kn}{kmn,kn,kn}}{\binom{(m+2)n}{mn,n,n}} \in \mathbb{Z}$$

holds[6] *for all positive integer* t.

We do not know if this is true for all k for some $m \geqslant 3$. This sequence is factored into two:

$$\binom{(m+2)n}{mn,n,n} = \binom{(m+2)n}{2n}\binom{2n}{n}.$$

The former sequence $\left(\binom{(m+2)n}{2n}\right)$ is almost mean k-divisible for all k and an odd m by Theorem 1 and $\left(\binom{2n}{n}\right)$ is mean divisible by Theorem 6. So the denominators generated by the first sequence might be canceled by the numerators from the later one. It is an interesting problem to characterize all mean divisible multinomial sequences.

As for the third question, we can construct a different type of non-divisible almost mean k-divisible sequences. Fix an integer $\ell > 1$ and let α and β be conjugate quadratic integers so that α/β is not a root of unity. Define the ℓ-th homogeneous cyclotomic polynomial:

$$\Phi_\ell(x,y) = \prod_{\substack{0<m<\ell \\ GCD(m,\ell)=1}} (x - \zeta^m y)$$

where ζ is the primitive ℓ-th root of unity. Put

$$c_n = \Phi_\ell(\alpha^n, \beta^n).$$

Then (c_n) is a non-zero integer sequence and for any integer k coprime to ℓ, we have $c_n \mid c_{kn}$. This implies that (c_n) is almost mean k-divisible for $GCD(k,\ell) = 1$. For example, taking $\ell = 2$,

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

gives the Lucas sequence, which is non-divisible but almost mean k-divisible for all odd integer k. However, it may not be a significant construction because they already have divisibility $c_n \mid c_{kn}$ not for all but for some k.

[6] By Theorem 6, this is valid for all positive integer k when $m = 1$ and 2.

Acknowledgments

The author would like to express his gratitude to Attila Pethő for stimulating discussion and relevant references. He is also deeply grateful to Andrew Granville and the anonymous referee who supplied him good references and advices on readability and further directions.

References

[1] S. Akiyama, A criterion to estimate the least common multiple of sequences and asymptotic formulas for $\zeta(3)$ arising from recurrence relation of an elliptic function, Jpn. J. Math. (N. S.) 22 (1) (1996) 129–146.
[2] S. Akiyama, A. Pethő, On the distribution of polynomials with bounded roots, I. Polynomials with real coefficients, J. Math. Soc. Japan (2014), in press.
[3] J.-P. Bézivin, A. Pethő, A.J. van der Poorten, A full characterization of divisibility sequences, Amer. J. Math. 112 (6) (1990) 985–1001.
[4] Y. Bilu, G. Hanrot, P.M. Voutier, Existence of primitive divisors of Lucas and Lehmer numbers, J. Reine Angew. Math. 539 (2001) 75–122, with an appendix by M. Mignotte.
[5] J.W. Bober, Factorial ratios, hypergeometric series, and a family of step functions, J. Lond. Math. Soc. 79 (2) (2009) 422–444.
[6] M. Einsiedler, G. Everest, T. Ward, Primes in elliptic divisibility sequences, LMS J. Comput. Math. 4 (2001) 1–13.
[7] P. Erdős, Beweis eines Satzes von Tschebyschef, Acta Litt. Sci. Szeged 5 (1932) 194–198.
[8] P. Erdős, On some divisibility properties of $\binom{2n}{n}$, Canad. Math. Bull. 7 (1964) 513–518.
[9] A. Flatters, Primitive divisors of some Lehmer–Pierce sequences, J. Number Theory 129 (1) (2009) 209–219.
[10] A. Granville, Arithmetic properties of binomial coefficients I: Binomial coefficients modulo prime powers, Canad. Math. Soc. Conf. Proc. 20 (1997) 253–275.
[11] P. Ingram, V. Mahé, J.H. Silverman, K.E. Stange, M. Streng, Algebraic divisibility sequences over function fields, J. Aust. Math. Soc. 92 (1) (2012) 99–126.
[12] P. Ingram, J.H. Silverman, Primitive divisors in arithmetic dynamics, Math. Proc. Cambridge Philos. Soc. 146 (2) (2009) 289–302.
[13] Edm. Landau, Sur les conditions de divisibilité d'un produit de factorielles par un autre, Nouvel. Ann. Math. J. Cand. Éc. Polyt. Norm. Sér. (3) 19 (1900) 344–362.
[14] L. Moser, Notes on number theory. V. Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$, Canad. Math. Bull. 6 (1963) 167–169.
[15] Y. Moshe, On a problem of Erdös regarding binomial coefficients, Acta Arith. 121 (2) (2006) 193–197.
[16] A. Schinzel, Second order strong divisibility sequences in an algebraic number field, Arch. Math. (Brno) 23 (3) (1987) 181–186.
[17] C.L. Stewart, On divisors of Fermat, Fibonacci, Lucas, and Lehmer numbers, Proc. Lond. Math. Soc. (3) 35 (3) (1977) 425–447.
[18] Z.-W. Sun, On divisibility of binomial coefficients, J. Aust. Math. Soc. 93 (1–2) (2012) 189–201.
[19] P. Voutier, M. Yabuta, Primitive divisors of certain elliptic divisibility sequences, Acta Arith. 151 (2) (2012) 165–190.