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Differences of the Selberg trace formula and Selberg type zeta functions for Hilbert modular surfaces ☆

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A R T I C L E I N F O

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We present an example of the Selberg type zeta function for non-compact higher rank locally symmetric spaces. This is a generalization of Selberg's unpublished work [26] to non-compact cases. We study certain Selberg type zeta functions and Ruelle type zeta functions attached to the Hilbert modular group of a real quadratic field. We show that they have meromorphic extensions to the whole complex plane and satisfy functional equations. The method is based on considering the differences among several Selberg trace formulas with different weights for the Hilbert modular group. Besides as an application of the differences of the Selberg trace formula, we also obtain an asymptotic average of the class numbers of indefinite binary quadratic forms over the real quadratic integer ring.

1. Introduction

In this article, we consider Selberg type zeta functions attached to the Hilbert modular group of a real quadratic field. First of all, we recall the original Selberg zeta function constructed by Selberg in 1956. Let Γ be a co-finite discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ acting

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on the upper half plane $\mathbb{H}$. Take a hyperbolic element $\gamma \in \Gamma$, that is $|\mathrm{tr}(\gamma)| > 2$, then the centralizer of γ in Γ is infinite cyclic and γ is conjugate in $\mathrm{PSL}(2,\mathbb{R})$ to $\begin{pmatrix} N(\gamma)^{1/2} & 0 \\ 0 & N(\gamma)^{-1/2} \end{pmatrix}$ with $N(\gamma) > 1$. Put $\mathrm{Prim}(\Gamma)$ be the set of Γ-conjugacy classes of the primitive hyperbolic elements in Γ. The Selberg zeta function for Γ is defined by the following Euler product:

$$Z_\Gamma(s) := \prod_{p \in \mathrm{Prim}(\Gamma)} \prod_{k=0}^{\infty} \left(1 - N(p)^{-(k+s)}\right) \quad \text{for } \mathrm{Re}(s) > 1.$$

Selberg defined this zeta function and proved (cf. Selberg [24,25]):

(1) $Z_\Gamma(s)$ defined for $\mathrm{Re}(s) > 1$ extends meromorphically over the whole complex plane.
(2) $Z_\Gamma(s)$ has "non-trivial" zeros at $s = \frac{1}{2} \pm ir_n$ of order equal to the multiplicity of the eigenvalue $1/4 + r_n^2$ of the Laplacian $\Delta_0 = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ acting on $L^2(\Gamma \backslash \mathbb{H})$.
(3) $Z_\Gamma(s)$ satisfies a functional equation between s and $1 - s$.

The theory of Selberg zeta functions for locally symmetric spaces of rank one is evolved by Gangolli [5] (compact case) and Gangolli–Warner [6] (noncompact case). For higher rank cases, Deitmar [1] defined and studied "generalized Selberg zeta functions" for compact higher rank locally symmetric spaces. (See also Selberg [26].) Therefore, our concern is to define and study "Selberg type zeta functions" for *non-compact* higher rank locally symmetric spaces such as Hilbert modular surfaces.

Let us explain our main results on Selberg type zeta functions for Hilbert modular surfaces in more detail. Let $K/\mathbb{Q}$ be a real quadratic field with class number one and $\mathcal{O}_K$ be the ring of integers of K. Put D be the discriminant of K and $\varepsilon > 1$ be the fundamental unit of K. We denote the generator of $\mathrm{Gal}(K/\mathbb{Q})$ by σ and put $a' := \sigma(a)$ and $N(a) := aa'$ for $a \in K$. We also put $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathcal{O}_K)$. Let $\Gamma_K = \{(\gamma,\gamma') \mid \gamma \in \mathrm{PSL}(2,\mathcal{O}_K)\}$ be the Hilbert modular group of K. It is known that Γ_K is a co-finite (*non-cocompact*) irreducible discrete subgroup of $\mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ and Γ_K acts on the product $\mathbb{H}^2$ of two copies of the upper half plane $\mathbb{H}$ by component-wise linear fractional transformation. Γ_K have only one cusp (∞,∞), i.e. Γ_K-inequivalent parabolic fixed point. $X_K := \Gamma_K \backslash \mathbb{H}^2$ is called the Hilbert modular surface.

Let $(\gamma,\gamma') \in \Gamma_K$ be hyperbolic–elliptic, i.e., $|\mathrm{tr}(\gamma)| > 2$ and $|\mathrm{tr}(\gamma')| < 2$. Then the centralizer of hyperbolic–elliptic (γ,γ') in Γ_K is infinite cyclic.

Definition 1.1 *(Selberg type zeta function for Γ_K with the weight $(0,m)$).* For an even integer $m \geq 2$, we define

$$Z_K(s;m) := \prod_{(p,p')} \prod_{k=0}^{\infty} \left(1 - e^{i(m-2)\omega} N(p)^{-(k+s)}\right)^{-1} \quad \text{for } \mathrm{Re}(s) > 1.$$

Here, (p,p') runs through the set of primitive hyperbolic–elliptic Γ_K-conjugacy classes of Γ_K, and (p,p') is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$(p, p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \right).$$

Here, $N(p) > 1$, $\omega \in (0, \pi)$ and $\omega \notin \pi\mathbb{Q}$. The product is absolutely convergent for $\mathrm{Re}(s) > 1$.

Remark 1.2. Selberg also considered similar zeta functions for **co-compact** cases in his unpublished work [26]. Our definition of the zeta function is a bit different from his definition and we remark that Γ_K is **non-co-compact**.

Our main theorems on analytic properties of $Z_K(s; m)$ are the following.

Theorem 1.3. *(See Theorems 5.3 and 6.5.) For an even integer $m \geq 2$, $Z_K(s; m)$ a priori defined for $\mathrm{Re}(s) > 1$ has a meromorphic extension over the whole complex plane.*

Our Selberg zeta functions $Z_K(s; m)$ have also "non-trivial" zeros or poles and they have connections with the eigenvalues of two Laplacians. Let $\Delta_0^{(1)} := -y_1^2(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2})$ and $\Delta_m^{(2)} := -y_2^2(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y_2^2}) + imy_2\frac{\partial}{\partial x_2}$ be the Laplacians of weight 0 and m for $(z_1, z_2) \in \mathbb{H}^2$. Two Laplacians $\Delta_0^{(1)}$ and $\Delta_m^{(2)}$ act on $L^2_{\mathrm{dis}}(\Gamma_K \backslash \mathbb{H}^2; (0, m))$, the space of Hilbert Maass forms of weight $(0, m)$. (See Definition 2.16 for details.)

Theorem 1.4. *(See Theorem 5.3.) For an even integer $m \geq 4$,*

(1) *$Z_K(s; m)$ has zeros at*
 $s = \frac{1}{2} \pm i\rho_j(m)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_m^{(2)})$,
 $s = 1 - \frac{m}{2} + \frac{\pi i k}{\log \varepsilon}$ of order 1 for $k \in \mathbb{Z}$.
(2) *$Z_K(s; m)$ has poles at*
 $s = \frac{1}{2} \pm i\rho_j(m-2)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m-2)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_{m-2}^{(2)})$,
 $s = 2 - \frac{m}{2} + \frac{\pi i k}{\log \varepsilon}$ of order 1 for $k \in \mathbb{Z}$.
(3) *$Z_K(s; m)$ has zeros or poles (according to their orders are positive or negative) at $s = -k$ ($k \in \mathbb{N} \cup \{0\}$) of order $(2k+1)E(X_K) + 2\sum_{j=1}^{N}[k/\nu_j] - 2kN - \sum_{j=1}^{N}\beta_{k,j}(m)$.*
(4) *If $m = 4$, $Z_K(s, m)$ has additional simple zeros at $s = 0$ and $s = 1$.*

Here,

$$\mathrm{Ker}\left(\Lambda_q^{(2)}\right) = \left\{ f \in L^2_{\mathrm{dis}}\left(\Gamma_K \backslash \mathbb{H}^2; (0, q)\right) \;\middle|\; \Delta_q^{(2)} f = \frac{q}{2}\left(1 - \frac{q}{2}\right) f \right\}$$

for $q = m$ or $m - 2$, and $E(X_K)$ denotes the Euler characteristic of the Hilbert modular surface X_K and $\beta_{j,k}(m)$ are integers given in (5.6).

On the contrary to the case of $m \geq 4$, $Z_K(s;2)$ has no "non-trivial" poles.

Theorem 1.5. *(See Theorem 6.5.)*

(1) *$Z_K(s;2)$ has a double pole at $s=1$.*
(2) *$Z_K(s;2)$ has zeros at*
$s=\frac{1}{2}\pm i\rho_j(2)$ of order equal to twice the multiplicity of the eigenvalue $\frac{1}{4}+\rho_j(2)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_2^{(2)}) = \{f \in L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbb{H}^2;(0,2)) \mid \Delta_2^{(2)} f = 0\}$.
(3) *$Z_K(s;2)$ has zeros at $s=\pm\frac{k\pi i}{\log\varepsilon}$ $(k\in\mathbb{N})$ of order 2.*
(4) *$Z_K(s;2)$ has a zero at $s=0$ of order $E(X_K)$.*
(5) *$Z_K(s;2)$ has zeros or poles (according to their orders are positive or negative) at $s=-k$ $(k\in\mathbb{N})$ of order $(2k+1)E(X_K)+2\sum_{j=1}^{N}[k/\nu_j]-2kN$.*

Here, $E(X_K)$ denotes the Euler characteristic of the Hilbert modular surface X_K.

Actually $Z_K(s;m)$ has infinite "non-trivial" zeros by the following "Weyl's law".

Theorem 1.6. *(See Theorem 6.11.) For an even integer $m\geq 2$, let*

$$N_m^+(T) := \#\{j \bigm| 1/4+\rho_j(m)^2 \leq T\}$$

for $T>0$. Then we have

$$N_m^+(T) \sim (m-1)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2}T \quad (T\to\infty).$$

Our $Z_K(s;m)$ also satisfy a symmetric functional equation.

Theorem 1.7. *(See Theorems 5.4 and 6.6.) The zeta function $Z_K(s;m)$ satisfies the functional equation*

$$\hat{Z}_K(s;m) = \hat{Z}_K(1-s;m).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;m) := Z_K(s;m)Z_{\mathrm{id}}(s)Z_{\mathrm{ell}}(s;m)Z_{\mathrm{par/sct}}(s;m)Z_{\mathrm{hyp2/sct}}(s;m).$$

Each local Selberg zeta functions corresponding to each Γ_K-conjugacy classes of Γ_K are explicitly given. See Theorems 5.4 and 6.6 for details.

We also consider the Ruelle type zeta function.

Definition 1.8 *(Ruelle type zeta function for Γ_K).* For $\mathrm{Re}(s)>1$, the Ruelle type zeta function for Γ_K is defined by the following absolutely convergent Euler product:

$$R_K(s) := \prod_{(p,p')} \left(1 - N(p)^{-s}\right)^{-1}.$$

Here, (p, p') runs through the set of primitive hyperbolic–elliptic Γ_K-conjugacy classes of Γ_K, and (p, p') is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$(p,p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}\right).$$

Here, $N(p) > 1$, $\omega \in (0, \pi)$ and $\omega \notin \pi\mathbb{Q}$.

By the relation

$$R_K(s) = \frac{Z_K(s;2)}{Z_K(s+1;2)},$$

we have

Theorem 1.9. *(See Theorem 6.7.) The function $R_K(s)$ has a meromorphic continuation to the whole $\mathbb{C}$. $R_K(s)$ has a double pole at $s = 1$ and nonzero for* $\mathrm{Re}(s) \geq 1$.

As a byproduct of Theorem 1.7, we obtain a simple functional equation for $R_K(s)$ and an explicit formula of the coefficient of the leading term of $R_K(s)$ at $s = 0$.

Theorem 1.10. *(See Theorem 6.8.) The function $R_K(s)$ has the following functional equation*

$$R_K(s)R_K(-s) = (-1)^{E(X_K)}\big(2\sin(\pi s)\big)^{2E(X_K)} \prod_{j=1}^{N} \left(\frac{\sin(\pi s/\nu_j)}{\sin(\pi s)}\right)^2 \cdot \left(\frac{\zeta_\varepsilon(s-1)\zeta_\varepsilon(s+1)}{\zeta_\varepsilon(s)^2}\right)^2, \tag{1.1}$$

and the absolute value of the coefficient of the leading term of $R_K(s)$ at $s = 0$ is given by

$$\left|R_K^*(0)\right| = (2\pi)^{E(X_K)} \prod_{j=1}^{N} \nu_j^{-1} \frac{(2\varepsilon\log\varepsilon)^2}{(\varepsilon^2-1)^2}.$$

Here, $E(X_K)$ denotes the Euler characteristic of X_K, ε is the fundamental unit of K, $\zeta_\varepsilon(s) = (1-\varepsilon^{-2s})^{-1}$ and $\nu_1, \nu_2, \ldots, \nu_N$ be the orders of elliptic fixed points in X_K.

These analytic properties and functional equations of $Z_K(s;m)$ and $R_K(s)$ are obtained by using the "double differences" of the Selberg trace formula for Hilbert modular surfaces. The key point is considering the differences between two Selberg trace formulas with different weights. For compact cases, this was carried by Selberg [26] (without full proof) and Kelmer [18]. Therefore, we shall extend the Selberg trace formula for Hilbert modular group Γ_K with trivial weight (cf. Efrat [2] and Zograf [29]) to that with non-trivial weights (Theorem 2.22). There are new contribution from parabolic and "type 2 hyperbolic" conjugacy classes in the geometric side in the trace formula. So we have to analyze the orbital integrals for parabolic and "type 2 hyperbolic" conjugacy classes of Γ_K with non-trivial weights, which have not been fully investigated. We also obtain a more tractable expression of "type 2 hyperbolic" contribution of the geometric side of the trace formula than Efrat's or Zograf's results. Based on our Selberg trace formula for Γ_K with weight $(0, m)$, we can treat and obtain the differences and double differences of the Selberg trace formula (Theorems 4.1 and 4.4). Our full trace formula also will be used in [10].

As an application of "Double differences of the Selberg trace formula" (Theorem 4.4), we obtain a prime geodesic type theorem (Theorem 6.12) and a generalization of Sarnak's theorem [23] on class numbers of indefinite binary quadratic forms over $\mathbb{Z}$ to that for class numbers of indefinite binary quadratic forms over $\mathcal{O}_K$. Put $\mathcal{D}_{+-} := \{d \in \mathcal{O}_K \mid \exists b \in \mathcal{O}_K \text{ s.t. } d \equiv b^2 \pmod 4,\ d \text{ not a square in } \mathcal{O}_K,\ d > 0,\ d' < 0\}$. For each $d \in \mathcal{D}_{+-}$, let $h_K(d)$ denote the number of inequivalent primitive binary quadratic forms over $\mathcal{O}_K$ of discriminant d, and let $(x_d, y_d) \in \mathcal{O}_K \times \mathcal{O}_K$ be the fundamental solution of the Pellian equation $x^2 - dy^2 = 4$. Put $\varepsilon_K(d) := (x_d + \sqrt{d}y_d)/2$.

Theorem 1.11. *(See Theorem 6.14.) For $x \geq 2$, we have*

$$\sum_{\substack{d \in \mathcal{D}_{+-} \\ \varepsilon_K(d) \leq x}} h_K(d) = 2\,\mathrm{Li}\big(x^2\big) - 2 \sum_{1/2 < s_j(2) < 1} \mathrm{Li}\big(x^{2s_j(2)}\big) + O\big(x^{3/2}/\log x\big) \quad (x \to \infty).$$

Here, $s_j(2)(1 - s_j(2))$ are eigenvalues of the Laplacian $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_2^{(2)})$. See Theorem 1.5 for definition of $\mathrm{Ker}(\Lambda_2^{(2)})$.

We have a few comments on the status of this paper and related works. In this article, we consider three types of trace formulas for the Hilbert modular group $\Gamma_K \subset \mathrm{PSL}(2, \mathbb{R})^2$ with non-trivial integral weights:

(FTF) Theorem 2.22 (The full Selberg trace formula)
(DTF) Theorem 4.1 (Differences of the Selberg trace formula)
(DDTF) Theorem 4.4 (Double differences of the Selberg trace formula)

Comparing (DTF) with (DDTF), it seems that (DDTF) is "more close" to the trace formula for $\mathrm{PSL}(2, \mathbb{Z})$ than (DTF). In particular, the identity term, hyperbolic–elliptic terms and elliptic terms in (DDTF) and corresponding terms for $\mathrm{PSL}(2, \mathbb{Z})$ look very much alike. Hence, we may think that our $Z_K(s; m)$ is a natural generalization of the Selberg zeta function for $\mathrm{PSL}(2, \mathbb{Z})$. Actually, the Euler product of $Z_K(s; m)$ is more similar to that for $\mathrm{PSL}(2,\mathbb{Z})$ than that of "Partial Zeta function" in [26]. Hence, we study $Z_K(s;m)$ by using (DDTF).

An anonymous referee has kindly informed the author that Kelmer has obtained a similar trace formula, "Hybrid trace formula", to (DTF) for a non-uniform lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})^n$ for $n \geq 2$ and its consequences such as Weyl's law in [19] (2013). (See also Remark 0.5 in [19] for $n = 2$.) His proof is direct and does not use the full trace formula.

Our trace formulas (DTF) and (DDTF) are "simple version" of the full trace formula and they are derived from (FTF). The derivation and form of the full trace formula (FTF) with non-trivial integral weights does not exist anywhere in the literature and we believe that it is important and useful. We also believe that explicit results on $Z_K(s;m)$ and $R_K(s)$ are interesting and new. The results of this article were partially announced in [8] (2010) and [9] (2012).

2. The Selberg trace formula for Hilbert modular surfaces with non-trivial weights

2.1. Hilbert modular group of a real quadratic field

Let $K/\mathbb{Q}$ be a real quadratic field with class number one and $\mathcal{O}_K$ be the ring of integers of K. Put D be the discriminant of K and $\varepsilon > 1$ be the fundamental unit of K. We denote the generator of $\mathrm{Gal}(K/\mathbb{Q})$ by σ and put $a' := \sigma(a)$ and $N(a) := aa'$ for $a \in K$. We also put $\gamma' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}(2,\mathcal{O}_K)$.

Let G be $\mathrm{PSL}(2,\mathbb{R})^2 = (\mathrm{SL}(2,\mathbb{R})/\{\pm I\})^2$ and $\mathbb{H}^2$ be the direct product of two copies of the upper half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. The group G acts on $\mathbb{H}^2$ by

$$g.z = (g_1,g_2).(z_1,z_2) = \left(\frac{a_1z_1+b_1}{c_1z_1+d_1}, \frac{a_2z_2+b_2}{c_2z_2+d_2}\right) \in \mathbb{H}^2$$

for $g = (g_1,g_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)\right)$ and $z = (z_1,z_2) \in \mathbb{H}^2$.

A discrete subgroup $\Gamma \subset G$ is called irreducible if it is not commensurable with any direct product $\Gamma_1 \times \Gamma_2$ of two discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. We have classification of the elements of irreducible Γ.

Proposition 2.1 *(Classification of the elements). Let Γ be an irreducible discrete subgroup of G. Then any element of Γ is one of the followings.*

(1) $\gamma = (I,I)$ *is the identity*
(2) $\gamma = (\gamma_1,\gamma_2)$ *is hyperbolic* $\Leftrightarrow |\mathrm{tr}(\gamma_1)| > 2$ *and* $|\mathrm{tr}(\gamma_2)| > 2$
(3) $\gamma = (\gamma_1,\gamma_2)$ *is elliptic* $\Leftrightarrow |\mathrm{tr}(\gamma_1)| < 2$ *and* $|\mathrm{tr}(\gamma_2)| < 2$
(4) $\gamma = (\gamma_1,\gamma_2)$ *is hyperbolic–elliptic* $\Leftrightarrow |\mathrm{tr}(\gamma_1)| > 2$ *and* $|\mathrm{tr}(\gamma_2)| < 2$
(5) $\gamma = (\gamma_1,\gamma_2)$ *is elliptic–hyperbolic* $\Leftrightarrow |\mathrm{tr}(\gamma_1)| < 2$ *and* $|\mathrm{tr}(\gamma_2)| > 2$
(6) $\gamma = (\gamma_1,\gamma_2)$ *is parabolic* $\Leftrightarrow |\mathrm{tr}(\gamma_1)| = |\mathrm{tr}(\gamma_2)| = 2$

Note that there are no other types in Γ (parabolic–elliptic etc.) (cf. Shimizu [27]).

Let us consider the Hilbert modular group of the real quadratic field K with class number one,

$$\Gamma_K := \left\{ (\gamma, \gamma') = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \right) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathcal{O}_K) \right\}.$$

It is known that Γ_K is an irreducible discrete subgroup of $G = \mathrm{PSL}(2,\mathbb{R})^2$ with the only one cusp $\infty := (\infty, \infty)$, i.e. Γ_K-inequivalent parabolic fixed point. $X_K = \Gamma_K \backslash \mathbb{H}^2$ is called the Hilbert modular surface.

We can easily see that

Lemma 2.2 *(Stabilizer of the cusp $\infty = (\infty, \infty)$). The stabilizer of $\infty = (\infty, \infty)$ in Γ_K is given by*

$$\Gamma_\infty := \left\{ \left(\begin{pmatrix} u & \alpha \\ 0 & u^{-1} \end{pmatrix}, \begin{pmatrix} u' & \alpha' \\ 0 & u'^{-1} \end{pmatrix} \right) \,\middle|\, u \in \mathcal{O}_K^\times,\ \alpha \in \mathcal{O}_K \right\}.$$

Definition 2.3 *(Types of hyperbolic elements).* For a hyperbolic element γ, we define that

(1) γ is type 1 hyperbolic $\Leftrightarrow$ whose all fixed points are not fixed by parabolic elements.
(2) γ is type 2 hyperbolic $\Leftrightarrow$ not type 1 hyperbolic.

Lemma 2.4. *Any type 2 hyperbolic elements of Γ_K are conjugate to an element of*

$$\left\{ \gamma_{k,\alpha} = \begin{pmatrix} \varepsilon^k & \alpha \\ 0 & \varepsilon^{-k} \end{pmatrix} \,\middle|\, k \in \mathbb{N},\ \alpha \in \mathcal{O}_K \right\}$$

in Γ_K. The centralizer of $\gamma_{k,\alpha}$ in Γ_K is an infinite cyclic group.

Proof. See pp. 91–93 in [2]. $\square$

By the above lemma, we may take a generator of the centralizer $Z_{\Gamma_K}(\gamma_{k,\alpha})$ as $\gamma_{k_0,\beta}$ with $k_0 \in \mathbb{N}$ and $\beta \in \mathcal{O}_K$. We also write k_0 as $k_0(\gamma_{k,\alpha})$.

Let $R_1, R_2, \dots, R_N$ be a complete system of representatives of the Γ_K-conjugacy classes of primitive elliptic elements of Γ_K. $\nu_1, \nu_2, \dots, \nu_N$ ($\nu \in \mathbb{N}$, $\nu \geq 2$) denote the orders of $R_1, R_2, \dots, R_N$. We may assume that R_j is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$R_j \sim \left(\begin{pmatrix} \cos\frac{\pi}{\nu_j} & -\sin\frac{\pi}{\nu_j} \\ \sin\frac{\pi}{\nu_j} & \cos\frac{\pi}{\nu_j} \end{pmatrix}, \begin{pmatrix} \cos\frac{t_j\pi}{\nu_j} & -\sin\frac{t_j\pi}{\nu_j} \\ \sin\frac{t_j\pi}{\nu_j} & \cos\frac{t_j\pi}{\nu_j} \end{pmatrix} \right), \quad (t_j, \nu_j) = 1.$$

For even natural number $m \geq 2$ and $l \in \{0, 1, \dots, \nu_j - 1\}$, we define $\alpha_l(m,j), \overline{\alpha_l}(m,j) \in \{0, 1, \dots, \nu_j - 1\}$ by

$$l+t_j\left(\frac{m-2}{2}\right)\equiv\alpha_l(m,j)\pmod{\nu_j}$$
$$l-t_j\left(\frac{m-2}{2}\right)\equiv\overline{\alpha_l}(m,j)\pmod{\nu_j}\tag{2.1}$$

We denote by $\mathit{\Gamma}_{\mathrm{H1}}$, $\mathit{\Gamma}_{\mathrm{E}}$, $\mathit{\Gamma}_{\mathrm{HE}}$, $\mathit{\Gamma}_{\mathrm{EH}}$ and $\mathit{\Gamma}_{\mathrm{H2}}$, type 1 hyperbolic $\mathit{\Gamma}_K$-conjugacy classes, elliptic $\mathit{\Gamma}_K$-conjugacy classes, hyperbolic–elliptic $\mathit{\Gamma}_K$-conjugacy classes, elliptic–hyperbolic $\mathit{\Gamma}_K$-conjugacy classes and type 2 hyperbolic $\mathit{\Gamma}_K$-conjugacy classes of $\mathit{\Gamma}_K$ respectively.

2.2. Preliminaries for the Selberg trace formula

Fix the weight $(m_1,m_2)\in(2\mathbb{Z})^2$. Set the automorphic factor $j_\gamma(z_j)=\frac{cz_j+d}{|cz_j+d|}$ for $\gamma\in\mathrm{PSL}(2,\mathbb{R})$ $(j=1,2)$.

Let $\Delta^{(j)}_{m_j}:=-y_j^2(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2})+im_jy_j\frac{\partial}{\partial x_j}$ $(j=1,2)$ be the Laplacians of weight m_j for the variable z_j.

Let us define the L^2-space of automorphic forms of weight (m_1,m_2) with respect to the Hilbert modular group $\mathit{\Gamma}_K$.

Definition 2.5 *(L^2-space of automorphic forms of weight (m_1,m_2)).*

$$L^2\big(\mathit{\Gamma}_K\backslash\mathbb{H}^2;(m_1,m_2)\big):=\Big\{f:\mathbb{H}^2\to\mathbb{C},C^\infty\ \Big|$$
$$(i)\ f\big((\gamma,\gamma')(z_1,z_2)\big)=j_\gamma(z_1)^{m_1}j_{\gamma'}(z_2)^{m_2}f(z_1,z_2)\quad\forall(\gamma,\gamma')\in\mathit{\Gamma}_K$$
$$(ii)\ \exists\big(\lambda^{(1)},\lambda^{(2)}\big)\in\mathbb{R}^2\quad\Delta^{(1)}_{m_1}f(z_1,z_2)=\lambda^{(1)}f(z_1,z_2),\quad\Delta^{(2)}_{m_2}f(z_1,z_2)=\lambda^{(2)}f(z_1,z_2)$$
$$(iii)\ \|f\|^2=\int\limits_{\mathit{\Gamma}_K\backslash\mathbb{H}^2}f(z)\overline{f(z)}\,d\mu(z)<\infty\Big\}.$$

Here, $d\mu(z)=\frac{dx_1dy_1}{y_1^2}\frac{dx_2dy_2}{y_2^2}$ for $z=(z_1,z_2)\in\mathbb{H}^2$.

We denote $C_c^\infty(\mathbb{R}^2)$ the space of compactly supported smooth functions on $\mathbb{R}^2$.

Take $\mathit{\Phi}\in C_c^\infty(\mathbb{R}^2)$ and introduce the point-pair invariant kernel $k(z,w)$ of weight (m_1,m_2) for $\mathit{\Phi}$ (as (6.3) on [14, p. 386]):

$$k(z,w):=\mathit{\Phi}\left[\frac{|z_1-w_1|^2}{\operatorname{Im}z_1\operatorname{Im}w_1},\frac{|z_2-w_2|^2}{\operatorname{Im}z_2\operatorname{Im}w_2}\right]H_{(m_1,m_2)}(z,w)\tag{2.2}$$

for $(z,w)=((z_1,z_2),(w_1,w_2))\in\mathbb{H}^2\times\mathbb{H}^2$. Here,

$$H_{(m_1,m_2)}(z,w):=H_{m_1}(z_1,w_1)H_{m_2}(z_2,w_2)$$

with

$$H_{m_j}(z_j,w_j):=i^{m_j}\frac{(w_j-\bar{z_j})^{m_j}}{|w_j-\bar{z_j}|^{m_j}}=i^{m_j}\frac{|z_j-\bar{w_j}|^{m_j}}{(z_j-\bar{w_j})^{m_j}}$$

for $j = 1, 2$. The reason of the last equality is that m_1, m_2 are even integers. (See [13, Definition 2.1, p. 359] and [14, (5.1), p. 349].)

Definition 2.6. For $\Phi \in C_c^\infty(\mathbb{R}^2)$, define

$$Q(w_1, w_2) := \iint\limits_{\mathbb{R}^2} \Phi\big(w_1 + v_1^2, w_2 + v_2^2\big) \prod_{j=1}^{2} \left[\frac{\sqrt{w_j + 4} + iv_j}{\sqrt{w_j + 4} - iv_j}\right]^{m_j/2} dv_1 dv_2$$
$$(w_1, w_2 \geq 0), \tag{2.3}$$

$$g(u_1, u_2) := Q\big(e^{u_1} + e^{-u_1} - 2, e^{u_2} + e^{-u_2} - 2\big), \tag{2.4}$$

$$h(r_1, r_2) := \iint\limits_{\mathbb{R}^2} g(u_1, u_2) e^{i(r_1 u_1 + r_2 u_2)} \, du_1 du_2. \tag{2.5}$$

We can easily check that $Q(w_1, w_2) \in C_c^\infty([0, \infty)^2)$, $g(u_1, u_2) \in C_c^\infty(\mathbb{R}^2)$ is an even function and $h(r_1, r_2) \in C^\infty(\mathbb{R}^2)$ is an even and rapidly decreasing function.

Proposition 2.7.

$$\Phi(x_1, x_2) = \left(-\frac{1}{\pi}\right)^2 \iint\limits_{\mathbb{R}^2} \frac{\partial^2 Q}{\partial w_1 \partial w_2}\big(x_1 + t_1^2, x_2 + t_2^2\big) \prod_{j=1}^{2} \left[\frac{\sqrt{x_j + 4 + t_j^2} - t_j}{\sqrt{x_j + 4 + t_j^2} + t_j}\right]^{m_j/2} dt_1 dt_2$$
$$(x_1, x_2 \geq 0), \tag{2.6}$$

$$g(u_1, u_2) = \left(\frac{1}{2\pi}\right)^2 \iint\limits_{\mathbb{R}^2} h(r_1, r_2) e^{-i(r_1 u_1 + r_2 u_2)} \, dr_1 dr_2. \tag{2.7}$$

Proof. See [14, p. 386], [2, Proposition 2.2] and [29, (1.1.1)]. □

2.3. Eisenstein series

Let $(m_1, m_2) \in (2\mathbb{Z})^2$, $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$, and $(s_1, s_2) \in \mathbb{C}^2$ with $\mathrm{Re}(s_1), \mathrm{Re}(s_2) \gg 0$. We define,

$$E_{(m_1, m_2)}(z, s_1, s_2) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_K} \frac{y_1^{s_1}}{|cz_1 + d|^{2s_1}} \frac{y_2^{s_2}}{|c' z_2 + d'|^{2s_2}} \frac{|cz_1 + d|^{m_1}}{(cz_1 + d)^{m_1}} \frac{|c' z_2 + d'|^{m_2}}{(c' z_2 + d')^{m_2}}.$$

Definition 2.8 *(Family of Eisenstein series).* For $(m_1, m_2) \in (2\mathbb{Z})^2$, $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$, $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $k \in \mathbb{Z}$ we define

$$E_{(m_1, m_2)}(z, s; k) := E_{(m_1, m_2)}\left(z, s + \frac{\pi i k}{2 \log \varepsilon}, s - \frac{\pi i k}{2 \log \varepsilon}\right). \tag{2.8}$$

Proposition 2.9. *For* $\mathrm{Re}(s) > 1$*, the Eisenstein series* $E_{(m_1,m_2)}(z,s;k)$ *is absolutely convergent and*

$$E_{(m_1,m_2)}(\gamma z,s;k) = j_\gamma(z_1)^{m_1} j_{\gamma'}(z_2)^{m_2} E_{(m_1,m_2)}(z,s;k)$$

for any $\gamma \in \Gamma_K$*.* $E_{(m_1,m_2)}(z,s;k)$ *is a common eigenfunction of* $\Delta^{(1)}_{m_1}$ *and* $\Delta^{(2)}_{m_2}$*.*

Proof. See pp. 38–44 in [2]. □

Proposition 2.10 *(Fourier expansion of Eisenstein series). Put*

$$L := \{ l = (l_1,l_2) \in K^2 \mid l_1\alpha + l_2\alpha' \in \mathbb{Z}\ \forall \alpha \in \mathcal{O}_K \}$$

and $\langle l,x\rangle := l_1x_1 + l_2x_2$ *for* $x = (x_1,x_2) \in \mathbb{R}^2$*. We write the Fourier coefficients as* $a_l(y,s;k)$ *for* $l \in L$*:*

$$E_{(m_1,m_2)}(z,s;k) = \sum_{l \in L} a_l(y,s;k) e^{2\pi i\langle l,x\rangle}.$$

Then the constant term $a_0(y,s;k)$ *is given by*

$$y_1^{s+\frac{\pi ik}{2\log\varepsilon}} y_2^{s-\frac{\pi ik}{2\log\varepsilon}} + \varphi_{(m_1,m_2)}(s,k) y_1^{1-s-\frac{\pi ik}{2\log\varepsilon}} y_2^{1-s+\frac{\pi ik}{2\log\varepsilon}}$$

with

$$\begin{aligned}\varphi_{(m_1,m_2)}(s,k) = {}& \frac{(-1)^{\frac{m_1+m_2}{2}}\pi}{\sqrt{D}} \frac{L(2s-1,\chi_{-k})}{L(2s,\chi_{-k})} \frac{\Gamma(s+\frac{\pi ik}{2\log\varepsilon}-\frac{1}{2})\Gamma(s+\frac{\pi ik}{2\log\varepsilon})}{\Gamma(s+\frac{\pi ik}{2\log\varepsilon}+\frac{m_1}{2})\Gamma(s+\frac{\pi ik}{2\log\varepsilon}-\frac{m_1}{2})} \\ & \times \frac{\Gamma(s-\frac{\pi ik}{2\log\varepsilon}-\frac{1}{2})\Gamma(s-\frac{\pi ik}{2\log\varepsilon})}{\Gamma(s-\frac{\pi ik}{2\log\varepsilon}+\frac{m_2}{2})\Gamma(s-\frac{\pi ik}{2\log\varepsilon}-\frac{m_2}{2})}. \end{aligned} \tag{2.9}$$

Here, $L(s,\chi_{-k})$ *is defined by* $L(s,\chi_{-k}) := \sum_{0\neq(c)\subset\mathcal{O}_K} |\frac{c}{c'}|^{-\frac{i\pi k}{\log\varepsilon}} |N(c)|^{-s}$ *for* $k \in \mathbb{Z}$*.*

For $l \neq (0,0)$*,* $a_l(y,s;k)$ *is given by*

$$\begin{aligned}& \frac{(-1)^{\frac{m_1+m_2}{2}}}{\sqrt{D}} \frac{\sigma_{1-2s,-k}(l)}{L(2s,\chi_{-k})} \frac{\pi^{2s}|l_1|^{s+\frac{\pi ik}{2\log\varepsilon}-1} |l_2|^{s-\frac{\pi ik}{2\log\varepsilon}-1}}{\Gamma(s+\frac{\pi ik}{2\log\varepsilon}+\mathrm{sgn}(l_1)\frac{m_1}{2})\Gamma(s-\frac{\pi ik}{2\log\varepsilon}+\mathrm{sgn}(l_2)\frac{m_2}{2})} \\ & \times W_{\mathrm{sgn}(l_1)\cdot\frac{m_1}{2}, s+\frac{\pi ik}{2\log\varepsilon}-\frac{1}{2}}(4\pi|l_1|y_1) W_{\mathrm{sgn}(l_2)\cdot\frac{m_2}{2}, s-\frac{\pi ik}{2\log\varepsilon}-\frac{1}{2}}(4\pi|l_2|y_2).\end{aligned}$$

Here, $W_{\kappa,\mu}(z)$ *is Whittaker's confluent hypergeometric function (see [28, Chapter 16] for definition) and* $\sigma_{1-2s,-k}(l) = \sum_{\{c\},\ \frac{l}{c}\in\mathcal{D}_k^{-1}} \frac{\chi_{-k}(c)}{|N(c)|^{2s-1}}$*, where* $\mathcal{D}_k^{-1}$ *is the inverse different of* K*. (See [2, p. 50].)*

Proof. The case of $(m_1, m_2) = (0, 0)$ is proved in [2]. For general case, we use the formulas (see [3, p. 55] and [12, 3.384 (9)]):

$$\int_{-\infty}^{\infty} \frac{dx}{|x+i|^{2s-m}(x+i)^m} = (-1)^{m/2}\pi^{1/2}\frac{\Gamma(s-\frac{1}{2})\Gamma(s)}{\Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})},$$

and

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi ipxy}}{|x+i|^{2s-m}(x+i)^m}\,dx = (-1)^{m/2}\frac{\pi^s(|p|y)^{s-1}}{\Gamma(s+\operatorname{sgn}(p)\frac{m}{2})}W_{\operatorname{sgn}(p)\cdot\frac{m}{2},s-\frac{1}{2}}\bigl(4\pi|p|y\bigr)$$

for $0 \neq p \in \mathbb{R}$. The rest of the proof is quite the same as in pp. 47–50 in [2]. □

We can prove the following theorem and proposition by the similar method in pp. 58–64 in [2].

Theorem 2.11 *(Functional equation). For any $k \in \mathbb{Z}$, $E_{(m_1,m_2)}(z, s; k)$ and $\varphi(s, k)$ can be continued meromorphically to all of $s \in \mathbb{C}$. Moreover, we have*

$$E_{(m_1,m_2)}(z, 1-s; -k) = \varphi_{(m_1,m_2)}(1-s, -k)E_{(m_1,m_2)}(z, s; k),$$

and

$$\varphi_{(m_1,m_2)}(s, k)\varphi_{(m_1,m_2)}(1-s, -k) = 1.$$

Proposition 2.12. *$E_{(m_1,m_2)}(z, s; k)$ and $\varphi_{(m_1,m_2)}(s, k)$ have no poles in $\operatorname{Re}(s) > \frac{1}{2}$, except for finitely many in $(\frac{1}{2}, 1]$ when $k = 0$.*

For $Y > 1$ and $\underline{m} = (m_1, m_2) \in (2\mathbb{Z})^2$, define

$$E^Y_{\underline{m}}(z, s; k) := \begin{cases} E_{\underline{m}}(z, s; k) - a_0(y, s; k) & \text{if } y_1y_2 \geq Y, \\ E_{\underline{m}}(z, s; k) & \text{if } y_1y_2 < Y. \end{cases} \tag{2.10}$$

For $\operatorname{Re}(s) > 1$, we note that $E^Y_{\underline{m}}(z, s; k)$ is a square-integrable function on the fundamental domain for Γ_K.

Theorem 2.13 *(Maass–Selberg relation). Let $\underline{m}, \underline{m}' \in (2\mathbb{Z})^2$ with $\underline{m} + \underline{m}' = (0, 0)$. For $(s, k) \neq (s', k')$ and $(s, k) + (s', k') \neq (1, 0)$, we have*

$$\begin{aligned}&\int_{\Gamma_K\backslash\mathbb{H}^2} E^Y_{\underline{m}}(z, s; k)E^Y_{\underline{m}'}\bigl(z, s'; k'\bigr)\,d\mu(z)\\ &\quad = 2\sqrt{D}\log\varepsilon\biggl[\delta_{k,-k'}\frac{Y^{s+s'-1} - \varphi_{\underline{m}}(s, k)\varphi_{\underline{m}'}(s', k')Y^{-s-s'+1}}{s+s'-1}\\ &\qquad + \delta_{k,k'}\frac{\varphi_{\underline{m}'}(s', k')Y^{s-s'} - \varphi_{\underline{m}}(s, k)Y^{s'-s}}{s-s'}\biggr].\end{aligned} \tag{2.11}$$

Proof. See pp. 66–69 in [2]. $\square$

2.4. Selberg trace formula for Hilbert modular surfaces

We consider a certain integral operator $\mathcal{K}_\Gamma$ acting on $L^2(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$. The kernel of this integral operator is given as follows.

Definition 2.14 *(Automorphic kernel function).* For $(z,w)\in\mathbb{H}^2\times\mathbb{H}^2$ and $\underline{m}=(m_1,m_2)\in(2\mathbb{Z})^2$, define

$$K_\Gamma(z,w):=\sum_{\gamma\in\Gamma_K}k(z,\gamma w)j_\gamma(w)=\sum_{(\gamma,\gamma')\in\Gamma_K}k\big((z_1,z_2),(\gamma w_1,\gamma' w_2)\big)\cdot\left(\frac{cw_1+d}{|cw_1+d|}\right)^{m_1}\left(\frac{c'w_2+d'}{|c'w_2+d'|}\right)^{m_2}. \tag{2.12}$$

Here, the point-pair invariant kernel $k(z,w)$ is defined in (2.2).

It is known that

Proposition 2.15. *Let $L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$ be the subspace of the discrete spectrum of $\mathcal{K}_\Gamma$ and $L^2_{\mathrm{con}}(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$ be the subspace of the continuous spectrum. Then, we have a direct sum decomposition of $\mathcal{K}_\Gamma$-invariant subspaces:*

$$L^2\big(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2)\big)=L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2)\big)\oplus L^2_{\mathrm{con}}\big(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2)\big)$$

and there is an orthonormal basis $\{\phi_j\}_{j=0}^\infty$ of $L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$.

Definition 2.16 *(Hilbert Maass forms of weight (m_1,m_2)).* Let $(m_1,m_2)\in(2\mathbb{Z})^2$. We call

$$L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2)\big)$$

the space of Hilbert Maass forms for Γ_K of weight (m_1,m_2).

To subtract continuous spectrum on $L^2_{\mathrm{con}}(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$, we introduce (see [2, p. 79] or [29, p. 1644])

Definition 2.17. For $(z,w)\in\mathbb{H}^2\times\mathbb{H}^2$ and $\underline{m}=(m_1,m_2)\in(2\mathbb{Z})^2$, define

$$H_\Gamma(z,w):=\frac{1}{8\pi\sqrt{D}\log\varepsilon}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi k}{2\log\varepsilon},r-\frac{\pi k}{2\log\varepsilon}\right)\times E_{\underline{m}}\left(z,\frac{1}{2}+ir;k\right)E_{-\underline{m}}\left(w,\frac{1}{2}-ir;-k\right)dr. \tag{2.13}$$

Let $\{\phi_j\}_{j=0}^{\infty}$ be an orthonormal basis of $L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbb{H}^2;(m_1,m_2))$ and $(\lambda_j^{(1)},\lambda_j^{(2)})\in\mathbb{R}^2$ such that

$$\Delta_{m_1}^{(1)}\phi_j=\lambda_j^{(1)}\phi_j \quad\text{and}\quad \Delta_{m_2}^{(2)}\phi_j=\lambda_j^{(2)}\phi_j.$$

Lemma 2.18. *For any j,*

$$\lambda_j^{(1)}\geq\frac{|m_1|}{2}\left(1-\frac{|m_1|}{2}\right),\qquad \lambda_j^{(2)}\geq\frac{|m_2|}{2}\left(1-\frac{|m_2|}{2}\right). \tag{2.14}$$

Proof. See [14, (6.1), p. 385]. □

Let us define the set of spectral parameters

$$\mathrm{Spec}(m_1,m_2):=\left\{\left(r_j^{(1)},r_j^{(2)}\right)\right\}_{j=0}^{\infty},$$

which is a discrete subset of

$$\left(\mathbb{R}\cup i\left[-\frac{||m_1|-1|}{2},\frac{||m_1|-1|}{2}\right]\right)\times\left(\mathbb{R}\cup i\left[-\frac{||m_2|-1|}{2},\frac{||m_2|-1|}{2}\right]\right).$$

Here, we write $\lambda_j^{(l)}=\frac{1}{4}+(r_j^{(l)})^2$ and $r_j^{(i)}$ are defined by

$$r_j^{(l)}:=\begin{cases}\sqrt{\lambda_j^{(l)}-\frac{1}{4}} & \text{if } \lambda_j^{(l)}\geq\frac{1}{4},\\ i\sqrt{\frac{1}{4}-\lambda_j^{(l)}} & \text{if } \lambda_j^{(l)}<\frac{1}{4},\end{cases} \tag{2.15}$$

for $l=1,2$.

Theorem 2.19. *$K_\Gamma(z,w)-H_\Gamma(z,w)$ is a Hilbert–Schmidt integral kernel, that is*

$$\iint_{(\Gamma_K\backslash\mathbb{H}^2)^2}\left|K_\Gamma(z,w)-H_\Gamma(z,w)\right|^2 d\mu(z)\,d\mu(w)<\infty.$$

Proof. Similar with the proof of Theorem 9.7 in [2] or p. 1644 in [29]. □

We shall assume that

$$k(z,w)=\int_{\mathbb{H}^2}k^{(1)}(z,v)k^{(2)}(v,w)\,d\mu(v)$$

where, $k^{(1)}$ and $k^{(2)}$ are defined as (2.2). Then $K_\Gamma-H_\Gamma$ defines an integral operator of trace class. So, we have

Theorem 2.20.

$$\sum_{j=0}^{\infty} h\big(r_j^{(1)}, r_j^{(2)}\big) = \int\limits_{\Gamma_K\backslash\mathbb{H}^2} \big[K_\Gamma(z,z) - H_\Gamma(z,z)\big]\, d\mu(z), \tag{2.16}$$

where the left hand side is absolutely convergent.

Our next task is to evaluate the right hand side of (2.16) explicitly.
Hereafter, we assume that the test functions are written as follows.

Assumption 2.21. We shall assume that the test functions are products of two separate functions that each involve only one independent variable. That is

$$\begin{aligned} h(r_1, r_2) &= h_1(r_1)h_2(r_2), \qquad & g(u_1, u_2) &= g_1(u_1)g_2(u_2), \\ \Phi(x_1, x_2) &= \Phi_1(x_1)\Phi_2(x_2), \qquad & Q(w_1, w_2) &= Q_1(w_1)Q_2(w_2). \end{aligned} \tag{2.17}$$

Without loss of generality we may assume that Φ_1 and Φ_2 are real valued.

Now we can state the Selberg trace formula for our cases. We give a proof of this theorem in the next section.

Theorem 2.22 *(Selberg trace formula for $L^2(\Gamma_K\backslash\mathbb{H}^2; (0,m))$ with $m \in 2\mathbb{Z}$). Let $g(u_1, u_2)$ be an even function in $C_c^\infty(\mathbb{R}^2)$ and put $h(r_1, r_2) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u_1, u_2) \times e^{i(r_1u_1 + r_2u_2)} du_1 du_2$, so that h is even, rapidly decreasing and analytic.*

Then we have,

$$\sum_{j=0}^{\infty} h\big(r_j^{(1)}, r_j^{(2)}\big) = \mathbf{I}(h) + \mathbf{II_a}(h) + \mathbf{II_b}(h) + \mathbf{III}(h). \tag{2.18}$$

Here,

$$\begin{aligned} \mathbf{I}(h) :=\ & \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2} \iint\limits_{\mathbb{R}^2} \frac{\frac{\partial^2}{\partial u_1 \partial u_2} g(u_1, u_2)}{\sinh(u_1/2)\sinh(u_2/2)} e^{-\frac{m}{2}u_2}\, du_1 du_2 \\ & + \sum_{(\gamma,\gamma')\in\Gamma_{\mathrm{H}1}} \frac{\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) g(\log N(\gamma), \log N(\gamma'))}{(N(\gamma)^{1/2} - N(\gamma)^{-1/2})(N(\gamma')^{1/2} - N(\gamma')^{-1/2})} \\ & + \sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}} \frac{-e^{-i\theta_1 + i(m-1)\theta_2}}{16\nu_R \sin\theta_1 \sin\theta_2} \iint\limits_{\mathbb{R}^2} g(u_1, u_2) e^{-\frac{u_1}{2} + \frac{(m-1)}{2}u_2} \\ & \times \prod_{j=1}^{2} \left[\frac{e^{u_j} - e^{2i\theta_j}}{\cosh u_j - \cos 2\theta_j}\right] du_1 du_2 \end{aligned}$$

$$+ \sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} \frac{ie^{i(m-1)\omega}}{4\sin\omega}$$

$$\times \int_{-\infty}^{\infty} g\big(\log N(\gamma), u\big) e^{\frac{m-1}{2}u} \left[\frac{e^u - e^{2i\omega}}{\cosh u - \cos 2\omega}\right] du$$

$$+ \sum_{(\omega',\gamma')\in\Gamma_{\mathrm{EH}}} \frac{\log N(\gamma_0')}{N(\gamma')^{1/2} - N(\gamma')^{-1/2}} \frac{ie^{-i\omega'}}{4\sin\omega'}$$

$$\times \int_{-\infty}^{\infty} g\big(u, \log N\big(\gamma'\big)\big) e^{\frac{-1}{2}u} \left[\frac{e^u - e^{2i\omega'}}{\cosh u - \cos 2\omega'}\right] du,$$

$$\mathbf{II_a}(h) := \big[\sqrt{D}A_0 - 4\log\varepsilon(\log 2 + C_E)\big] g(0,0) + \log\varepsilon \int_0^{\infty} \big[g(u,0) + g(0,u)\big] du$$

$$- \frac{\log\varepsilon}{2\pi^2} \iint_{\mathbb{R}^2} \left[\frac{\Gamma'}{\Gamma}(1+ir_1) + \frac{\Gamma'}{\Gamma}(1+ir_2)\right] h(r_1,r_2)\, dr_1 dr_2$$

$$+ 2\log\varepsilon \int_0^{\infty} \frac{g(0,u)}{e^{u/2} - e^{-u/2}} \left[1 - \cosh\frac{m}{2}u\right] du$$

$$\mathbf{II_b}(h) := -4\log\varepsilon \sum_{k=1}^{\infty} \sum_{\gamma_{k,\alpha}\in\Gamma_{\mathrm{H2}}} \frac{k_0(\gamma_{k,\alpha}) \log(N(\alpha, \varepsilon^k - \varepsilon^{-k}))}{|N(\varepsilon^k - \varepsilon^{-k})|} g(2k\log\varepsilon, 2k\log\varepsilon)$$

$$+ 4\log\varepsilon \sum_{k=1}^{\infty} \log\big(\varepsilon^k - \varepsilon^{-k}\big) g(2k\log\varepsilon, 2k\log\varepsilon)$$

$$+ 2\log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} \big[g(u, 2k\log\varepsilon) + g(2k\log\varepsilon, u)\big] \frac{\cosh(u/2)}{\sinh(u/2) + \sinh(k\log\varepsilon)}\, du$$

$$+ 2\log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} g(2k\log\varepsilon, u) \frac{1 - \cosh(m(u/2 - k\log\varepsilon))}{\sinh(u/2 - k\log\varepsilon)}\, du,$$

and

$$\mathbf{III}(h) := \frac{1}{4\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} h\left(r + \frac{\pi k}{2\log\varepsilon}, r - \frac{\pi k}{2\log\varepsilon}\right) \frac{\varphi'_{(0,m)}}{\varphi_{(0,m)}}\left(\frac{1}{2} + ir, k\right) dr$$

$$- \frac{1}{4} h(0,0)\varphi_{(0,m)}\left(\frac{1}{2}, 0\right).$$

The series and integrals converge absolutely. Here, A_0 is the constant term of the Laurent expansion of $\zeta_K(s)$ at $s = 1$ and C_E is the Euler constant. The case of $(0,m) = (0,0)$ is proved by Zograf [29] and Efrat [2].

3. Proof of the Selberg trace formula for Hilbert modular surfaces

3.1. Orbital integrals and the fundamental domain

In this section we prove Theorem 2.22. We recall Theorem 2.20:

$$\sum_{j=0}^{\infty} h\bigl(r_j^{(1)}, r_j^{(2)}\bigr) = \int\limits_{\Gamma_K\backslash\mathbb{H}^2} \bigl[K_\Gamma(z,z) - H_\Gamma(z,z)\bigr]\, d\mu(z).$$

Formally, we have

$$\int\limits_{\Gamma_K\backslash\mathbb{H}^2} K_\Gamma(z,z)\, d\mu(z) = \sum_{\gamma\in\mathrm{Conj}(\Gamma_K)} \int\limits_{\Gamma_\gamma\backslash\mathbb{H}^2} k(z,\gamma z) j_\gamma(z)\, d\mu(z).$$

Here, we put Γ_γ be the centralizer of γ in Γ_K. To prove Theorem 2.22, we calculate the orbital integral

$$\int\limits_{\Gamma_\gamma\backslash\mathbb{H}^2} k(z,\gamma z) j_\gamma(z)\, d\mu(z)$$

explicitly for each Γ_K-conjugacy classes $[\gamma]$ of Γ_K. However, γ is parabolic or type 2 hyperbolic, the above orbital integral does not converge.

Therefore, we introduce the truncated fundamental domain for Γ_K. First we construct the fundamental domain F_∞ of the group Γ_∞. (See Lemma 2.2.) By direct calculation, we have,

Lemma 3.1 *(Fundamental domain of Γ_∞). Let D be the discriminant of the quadratic field K. We write $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{H}^2$.*

(1) *If $D \equiv 1 \pmod 4$, put*

$$F_\infty := \biggl\{(x_1+iy_1, x_2+iy_2) \biggm| 0 \le \Bigl(1 - \frac{1}{\sqrt{D}}\Bigr)x_1 + \Bigl(1 + \frac{1}{\sqrt{D}}\Bigr)x_2 < 2,$$
$$0 \le x_1 - x_2 < 2\sqrt{D},\ \varepsilon^{-2} \le y_1/y_2 < \varepsilon^2\biggr\}.$$

(2) *Otherwise, put*

$$F_\infty := \bigl\{(x_1+iy_1, x_2+iy_2) \bigm| 0 \le x_1 + x_2 < 2,\ 0 \le x_1 - x_2 < 2\sqrt{D},$$
$$\varepsilon^{-2} \le y_1/y_2 < \varepsilon^2\bigr\}.$$

Then F_∞ is a fundamental domain for the group Γ_∞ acting on $\mathbb{H}^2$.

We define the standard truncated fundamental domain for Γ_K.

Definition 3.2 *(Standard fundamental domain).* Let $Y > 1$.

(1) The fundamental domain F of Γ_K, which is contained in F_∞, is called the standard fundamental domain for Γ_K.
(2) $F^Y := \{(z_1, z_2) \in F \mid y_1 y_2 < Y\}$ is called the truncated standard fundamental domain for Γ_K.
(3) Let γ be a parabolic or type 2 hyperbolic element of Γ_K.

$$F_\gamma^Y := \bigcup_{\delta \in \Gamma_\gamma \backslash \Gamma} \delta\bigl(F^Y\bigr)$$

is called the truncated standard fundamental domain for the centralizer of γ in Γ_K.

3.2. Contribution of the identity, type 1 hyperbolic, elliptic and mixed elements

In this subsection, we compute the orbital integral

$$\int_{\Gamma_\gamma \backslash \mathbb{H}^2} k(z, \gamma z) j_\gamma(z)\, d\mu(z)$$

explicitly, when γ is the identity, an elliptic, a type 1 hyperbolic, a hyperbolic–elliptic, or an elliptic–hyperbolic element. We note that all the integrals are convergent for these elements. Let $(m_1, m_2) \in (2\mathbb{Z})^2$.

• Identity term: By definition, we have

$$\begin{aligned} I(m_1, m_2) &:= \int_{\Gamma_K \backslash \mathbb{H}^2} k(z, z)\, d\mu(z) = \int_{\Gamma_K \backslash \mathbb{H}^2} H_{(m_1, m_2)}(z, z) \Phi(0, 0)\, d\mu(z) \\ &= (-1)^{m_1 + m_2} \mathrm{vol}\bigl(\Gamma_K \backslash \mathbb{H}^2\bigr) \Phi(0, 0). \end{aligned} \tag{3.1}$$

And $\Phi(0, 0)$ is given by (see p. 396 in [14])

$$\begin{aligned} \Phi(0, 0) &= \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \frac{\partial^2 Q}{\partial w_1 \partial w_2}\bigl(t_1^2, t_2^2\bigr) \left[\frac{\sqrt{4 + t_1^2} - t_1}{\sqrt{4 + t_1^2} + t_1}\right]^{m_1/2} \left[\frac{\sqrt{4 + t_2^2} - t_2}{\sqrt{4 + t_2^2} + t_2}\right]^{m_2/2} dt_1 dt_2 \\ &= \frac{1}{4\pi^2} \iint_{\mathbb{R}^2} \frac{\frac{\partial^2}{\partial u_1 \partial u_2} g(u_1, u_2)}{(e^{u_1/2} - e^{-u_1/2})(e^{u_2/2} - e^{-u_2/2})} e^{-\frac{m_1}{2} u_1} e^{-\frac{m_2}{2} u_2}\, du_1 du_2. \end{aligned}$$

• Type 1 hyperbolic terms: For type 1 hyperbolic element $(\gamma, \gamma') \in \Gamma_K$, we denote it by γ for simplicity. It is known that the centralizer of γ in Γ_K is a free abelian group of rank two. (See Theorem 5.7 in [2, p. 26].) We can easily compute (see also [2, p. 31])

$$
\begin{aligned}
H_1(m_1, m_2; \gamma) &:= \int\limits_{\Gamma_\gamma \backslash \mathbb{H}^2} k(z, \gamma z) j_\gamma(z)\, d\mu(z) \\
&= \frac{\operatorname{vol}(\Gamma_\gamma \backslash \mathbb{H}^2) g(\log N(\gamma), \log N(\gamma'))}{(N(\gamma)^{1/2} - N(\gamma)^{-1/2})(N(\gamma')^{1/2} - N(\gamma')^{-1/2})}.
\end{aligned} \tag{3.2}
$$

• Elliptic terms: Let $R \in \Gamma_K$ be an elliptic element. We may assume that R is conjugate in G to the element

$$
R(\theta_1, \theta_2) = \left(\begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix}, \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \right).
$$

Let R_0 be a generator of the centralizer of R in Γ_K and denote the order of R_0 by ν_R. Then R_0 is conjugate in G to the element

$$
\left(\begin{pmatrix} \cos(\pi/\nu_R) & -\sin(\pi/\nu_R) \\ \sin(\pi/\nu_R) & \cos(\pi/\nu_R) \end{pmatrix}, \begin{pmatrix} \cos(t\pi/\nu_R) & -\sin(t\pi/\nu_R) \\ \sin(t\pi/\nu_R) & \cos(t\pi/\nu_R) \end{pmatrix} \right),
$$

where, t is an integer such that $(\nu_R, t) = 1$.

We write $R = R_0^k$ with $(1 \le k \le \nu_R - 1)$, and put $(\alpha_j, \beta_j) = (\cos\theta_j, \sin\theta_j)$ for $j = 1, 2$. Using the formulas at pp. 389–394 in [14] (see also p. 1647 in [29]), we have

$$
\begin{aligned}
E(m_1, m_2; R) &:= \int\limits_{\langle R_0 \rangle \backslash \mathbb{H}^2} k(z, Rz) j_R(z)\, d\mu(z) \\
&= \frac{1}{\nu_R} \int\limits_{\mathbb{H}^2} k\big((z_1, z_2), \big(r(\theta_1) z_1, r(\theta_2) z_2\big)\big) \\
&\quad \times \frac{(\beta_1 z_1 + \alpha_1)^{m_1}}{|\beta_1 z_1 + \alpha_1|^{m_1}} \frac{(\beta_2 z_2 + \alpha_2)^{m_2}}{|\beta_2 z_2 + \alpha_2|^{m_2}} \frac{dx_1 dy_1}{y_1^2} \frac{dx_2 dy_2}{y_2^2} \\
&= \frac{1}{16 \nu_R \beta_1 \beta_2} \big\{ i e^{i(m_1 - 1)\theta_1} \big\} \big\{ i e^{i(m_2 - 1)\theta_2} \big\} \\
&\quad \times \iint\limits_{\mathbb{R}^2} g(u_1, u_2) \prod_{j=1}^{2} \left[e^{\frac{(m_j - 1) u_j}{2}} \frac{e^{u_j} - e^{2i\theta_j}}{\cosh u_j - \cos 2\theta_j} \right] du_1 du_2.
\end{aligned} \tag{3.3}
$$

• Hyperbolic–elliptic terms: Let $\gamma = (\gamma, \gamma') \in G$ be a hyperbolic–elliptic element. The group Γ_γ is infinite cyclic and there exists a generator $\gamma_0 = (\gamma_0, \gamma_0')$ such that $\gamma = \gamma_0^k$ with $k \ge 1$. We may assume that (γ_0, γ_0') is conjugate in G to the element

$$
\left(\begin{pmatrix} N(\gamma_0)^{1/2} & 0 \\ 0 & N(\gamma_0)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega_0 & -\sin\omega_0 \\ \sin\omega_0 & \cos\omega_0 \end{pmatrix} \right).
$$

Here, $N(\gamma_0) > 1$, $\omega_0 \in (0, \pi)$ and $\omega_0 \notin \pi\mathbb{Q}$. Using the formulas at pp. 389–394 in [14] (see also p. 1647 in [29]), we have

$$\begin{aligned}
HE(m_1,m_2;\gamma) &:= \int\limits_{\Gamma_\gamma\backslash\mathbb{H}^2} k(z,\gamma z) j_\gamma(z)\, d\mu(z) \\
&= \int\limits_{\langle\gamma_0\rangle\backslash\mathbb{H}^2} k\big((z_1,z_2),\big(N(\gamma)z_1, r(\omega)z_2\big)\big) \\
&\quad \times \frac{(N(\gamma)^{-1/2})^{m_1}}{|N(\gamma)^{-1/2}|^{m_1}} \frac{(z_2\sin\omega+\cos\omega)^{m_2}}{|z_2\sin\omega+\cos\omega|^{m_2}} \frac{dx_1dy_1}{y_1^2} \frac{dx_2dy_2}{y_2^2} \\
&= \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} \frac{ie^{i(m_2-1)\omega}}{4\sin\omega} \\
&\quad \times \int_{-\infty}^{\infty} g\big(\log N(\gamma),u_2\big) e^{\frac{m_2-1}{2}u_2} \left[\frac{e^{u_2}-e^{2i\omega}}{\cosh u_2-\cos 2\omega}\right] du_2. \qquad (3.4)
\end{aligned}$$

• Elliptic–hyperbolic terms: Let $\gamma=(\gamma,\gamma')\in G$ be an elliptic–hyperbolic element. The group Γ_γ is infinite cyclic and there exists a generator $\gamma_0=(\gamma_0,\gamma_0')$ such that $\gamma=\gamma_0^l$ with $l\geq 1$. We may assume that (γ_0,γ_0') is conjugate in G to the element

$$\left(\begin{pmatrix}\cos\omega_0' & -\sin\omega_0' \\ \sin\omega_0' & \cos\omega_0'\end{pmatrix}, \begin{pmatrix}N(\gamma_0')^{1/2} & 0 \\ 0 & N(\gamma_0')^{-1/2}\end{pmatrix}\right).$$

Here, $N(\gamma_0')>1$, $\omega_0'\in(0,\pi)$ and $\omega_0'\notin\pi\mathbb{Q}$. Then we have

$$\begin{aligned}
EH(m_1,m_2;\gamma) &:= \int\limits_{\Gamma_\gamma\backslash\mathbb{H}^2} k(z,\gamma z) j_\gamma(z)\, d\mu(z) \\
&= \frac{\log N(\gamma_0')}{N(\gamma')^{1/2}-N(\gamma')^{-1/2}} \frac{ie^{i(m_1-1)\omega'}}{4\sin\omega'} \\
&\quad \times \int_{-\infty}^{\infty} g\big(u_1,\log N\big(\gamma'\big)\big) e^{\frac{m_1-1}{2}u_1} \left[\frac{e^{u_1}-e^{2i\omega'}}{\cosh u_1-\cos 2\omega'}\right] du_1. \qquad (3.5)
\end{aligned}$$

Putting together with the all results in this subsection, we obtain the term $\mathbf{I}(h)$ in Theorem 2.22.

3.3. Parabolic contribution

Let Γ_{P} be the set of Γ_K-conjugacy classes of parabolic elements in Γ_K. Let $(m_1,m_2)\in(2\mathbb{Z})^2$ and $Y>1$. We consider the parabolic contribution to the trace formula with the truncation parameter Y:

$$P^Y(m_1,m_2) := \sum_{\gamma\in\Gamma_{\mathrm{P}}} \int\limits_{F_\gamma^Y} k(z,\gamma z) j_\gamma(z)\, d\mu(z).$$

Here, $F_\gamma^Y = \bigcup_{\delta\in\Gamma_\gamma\backslash\Gamma}\delta(F^Y)$ and $F^Y = \{(z_1,z_2)\in F \mid \mathrm{Im}(z_1)\,\mathrm{Im}(z_2) < Y\}$ is the truncated standard fundamental domain for Γ_K, which is defined in Definition 3.2.

Then we have

Proposition 3.3. *For $m\in 2\mathbb{Z}$ and $Y>1$, we have*

$$\begin{aligned}
P^Y(0,m) &= 2\log\varepsilon\log Y g(0,0)\\
&\quad + \bigl[\sqrt{D}A_0 - 4\log\varepsilon(\log 2 + C_E)\bigr]g(0,0) + \log\varepsilon\int_0^\infty \bigl[g(u,0)+g(0,u)\bigr]\,du\\
&\quad - \frac{\log\varepsilon}{2\pi^2}\iint_{\mathbb{R}^2}\left[\frac{\Gamma'}{\Gamma}(1+ir_1)+\frac{\Gamma'}{\Gamma}(1+ir_2)\right]h(r_1,r_2)\,dr_1dr_2\\
&\quad + 2\log\varepsilon\int_0^\infty \frac{g(0,u)}{e^{u/2}-e^{-u/2}}\left[1-\cosh\frac{m}{2}u\right]du + o(1)\quad (Y\to\infty).
\end{aligned}$$

Here, A_0 is the constant term of the Laurent expansion of $\zeta_K(s)$, the Dedekind zeta function of K, at $s=1$ and $C_E := \lim_{n\to\infty}(1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n)$ is the Euler constant.

Proof. We recall that the test function Φ is written as $\Phi(x_1,x_2)=\Phi_1(x_1)\Phi_2(x_2)$ with real valued Φ_1 and Φ_2 by Assumption 2.21. By course of the same procedure at p. 1648 in [29] (note that Zograf's $2\sqrt{D}$ in [29] is $\sqrt{D}$ in our notation), we have

$$\begin{aligned}
&P^Y(m_1,m_2)\\
&= 4\sqrt{D}(A_{-1}\log Y + A_0)\,\mathrm{Re}\left\{\int_0^\infty\int_0^\infty \left[\frac{(2+iu_1)}{|2+iu_1|}\right]^{m_1}\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m_2}\Phi\bigl(u_1^2,u_2^2\bigr)\,du_1du_2\right\}\\
&\quad + 4\sqrt{D}A_{-1}\,\mathrm{Re}\left\{\int_0^\infty\int_0^\infty \log(u_1u_2)\left[\frac{(2+iu_1)}{|2+iu_1|}\right]^{m_1}\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m_2}\Phi\bigl(u_1^2,u_2^2\bigr)\,du_1du_2\right\}\\
&\quad + o(1)\quad (Y\to\infty).
\end{aligned}$$

Here, A_{-1}, A_0 are the coefficients of the Laurent expansion of $\zeta_K(s)$, the Dedekind zeta function of K, at $s=1$. In particular, $A_{-1}=\frac{2\log\varepsilon}{\sqrt{D}}$. Put

$$\begin{aligned}
&P_0(m_1,m_2)\\
&:= 4\sqrt{D}(A_{-1}\log Y + A_0)\,\mathrm{Re}\left\{\int_0^\infty\int_0^\infty \left[\frac{(2+iu_1)}{|2+iu_1|}\right]^{m_1}\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m_2}\Phi\bigl(u_1^2,u_2^2\bigr)\,du_1du_2\right\}\\
&= \sqrt{D}(A_{-1}\log Y + A_0)g(0,0).
\end{aligned}$$

Here, the last equality is derived from Definition 2.6.

For $j = 1, 2$, put

$$P_j(m_1, m_2) := \mathrm{Re}\left\{\int_0^\infty\int_0^\infty \log(u_j)\left[\frac{(2+iu_1)}{|2+iu_1|}\right]^{m_1}\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m_2}\Phi\big(u_1^2, u_2^2\big)\,du_1du_2\right\}.$$

We note that

$$P^Y(m_1, m_2) = P_0(m_1, m_2) + 8\log\varepsilon\big\{P_1(m_1, m_2) + P_2(m_1, m_2)\big\} + o(1).$$

We calculate the case of $(m_1, m_2) = (0, m)$.

$$\begin{aligned}
P_2(0, m) &= \mathrm{Re}\left\{\int_0^\infty\int_0^\infty \log(u_2)\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m}\Phi\big(u_1^2, u_2^2\big)\,du_1du_2\right\}\\
&= \mathrm{Re}\left\{\int_0^\infty \Phi_1\big(u_1^2\big)\,du_1\int_0^\infty \log(u_2)\left[\frac{(2+iu_2)}{|2+iu_2|}\right]^{m}\Phi_2\big(u_2^2\big)\,du_2\right\}\\
&= \frac{1}{2}g_1(0)\left\{-\frac{1}{2}(\log 2 + C_E)g_2(0) + \frac{1}{8}h_2(0) - \frac{1}{4\pi}\int_{\mathbb{R}} h_2(r_2)\frac{\Gamma'}{\Gamma}(1+ir_2)\,dr_2\right.\\
&\qquad\left. + \frac{1}{2}\int_0^\infty \frac{g_2(u_2)}{e^{u_2/2} - e^{-u_2/2}}\left[1 - \cosh\frac{m}{2}u_2\right]du_2\right\}.
\end{aligned}$$

We refer to pp. 406–411 in [14] for the last equality. Thus, we obtain

$$\begin{aligned}
P_2(0, m) &= -\frac{1}{4}(\log 2 + C_E)g(0,0) + \frac{1}{8}\int_0^\infty g(0,u)\,du - \frac{1}{16\pi^2}\iint_{\mathbb{R}^2} h(r_1, r_2)\frac{\Gamma'}{\Gamma}(1+ir_2)\,dr_2\\
&\quad + \frac{1}{4}\int_0^\infty \frac{g(0,u)}{e^{u/2} - e^{-u/2}}\left[1 - \cosh\frac{m}{2}u\right]du.
\end{aligned}$$

Similarly, we obtain

$$P_1(0, m) = -\frac{1}{4}(\log 2 + C_E)g(0,0) + \frac{1}{8}\int_0^\infty g(u,0)\,du - \frac{1}{16\pi^2}\iint_{\mathbb{R}^2} h(r_1, r_2)\frac{\Gamma'}{\Gamma}(1+ir_1)\,dr_1.$$

The proof is finished. □

3.4. Type 2 hyperbolic contribution

Let $(m_1, m_2) \in (2\mathbb{Z})^2$ and $Y > 1$. We consider the type 2 hyperbolic contribution to the trace formula with the truncation parameter Y:

$$H_2^Y(m_1, m_2) := \sum_{k=1}^{\infty} \sum_{\gamma_{k,\alpha} \in \Gamma_{\mathrm{H2}}} \int_{S^Y} k(z, \gamma_{k,\alpha} z) j_{\gamma_{k,\alpha}}(z)\, d\mu(z).$$

Here, $\gamma_{k,\alpha} = \left(\begin{smallmatrix} \varepsilon^k & \alpha \\ 0 & \varepsilon^{-k} \end{smallmatrix}\right)$ with $k \in \mathbb{N}$, $\alpha \in \mathcal{O}_K$ are representatives of type 2 hyperbolic conjugacy classes of Γ_k, given in Lemma 2.4,

$$S^Y := \left\{(z_1, z_2) \in F_{\gamma_{k,\alpha}} \;\middle|\; \mathrm{Im}(z_1)\,\mathrm{Im}(z_2) < Y,\ \mathrm{Im}\big(\tau(z_1)\big)\,\mathrm{Im}\big(\tau'(z_2)\big) < Y\right\}$$

and (τ, τ') is an element of Γ_K such that

$$(\tau, \tau')\big(\alpha/(\varepsilon^k - \varepsilon^{-k}), \alpha'/\big((\varepsilon')^k - (\varepsilon')^{-k}\big)\big) = (\infty, \infty).$$

We can show that (see [29, p. 1650])

$$\int_{F^Y_{\gamma_{k,\alpha}}} k(z, \gamma_{k,\alpha} z) j_{\gamma_{k,\alpha}}(z)\, d\mu(z) = \int_{S^Y} k(z, \gamma_{k,\alpha} z) j_{\gamma_{k,\alpha}}(z)\, d\mu(z) + o(1) \quad (Y \to \infty).$$

Put $\eta_k := \varepsilon^{2k} + \varepsilon^{-2k} - 2$ and recall that $k_0 = k_0(\gamma_{k,\alpha})$ was defined after Lemma 2.4. We can compute (see [2, pp. 91–97] or [29, (4.5.1), p. 1650])

$$\begin{aligned}
&\int_{S^Y} k(z, \gamma_{k,\alpha} z) j_{\gamma}(z)\, d\mu(z) \\
&= \frac{k_0 \cdot 2\log\varepsilon}{|N(\varepsilon^k - \varepsilon^{-k})|} \iint_{\mathbb{R}^2} \Phi\big(x_1^2 + \eta_k, x_2^2 + \eta_k\big) \\
&\quad \times \big[2\log Y - \log N(\Lambda)^2 + \log\big(x_1^2 + \eta_k\big) + \log\big(x_2^2 + \eta_k\big)\big] \\
&\quad \times \left(\frac{\sqrt{\eta_k + 4} + ix_1}{\sqrt{\eta_k + 4} - ix_1}\right)^{m_1/2} \left(\frac{\sqrt{\eta_k + 4} + ix_2}{\sqrt{\eta_k + 4} - ix_2}\right)^{m_2/2} dx_1 dx_2 \\
&= \frac{k_0 \cdot 2\log\varepsilon}{|N(\varepsilon^k - \varepsilon^{-k})|} \big\{R_0(m_1, m_2) + R_1(m_1, m_2) + R_2(m_1, m_2)\big\}.
\end{aligned}$$

Here, we put

$$\begin{aligned}
R_0(m_1, m_2) := &\iint_{\mathbb{R}^2} \Phi\big(x_1^2 + \eta_k, x_2^2 + \eta_k\big)\big[2\log Y - \log N(\Lambda)^2\big] \\
&\times \left(\frac{\sqrt{\eta_k + 4} + ix_1}{\sqrt{\eta_k + 4} - ix_1}\right)^{m_1/2} \left(\frac{\sqrt{\eta_k + 4} + ix_2}{\sqrt{\eta_k + 4} - ix_2}\right)^{m_2/2} dx_1 dx_2,
\end{aligned}$$

$$R_j(m_1,m_2) := \iint_{\mathbb{R}^2} \Phi\big(x_1^2+\eta_k, x_2^2+\eta_k\big) \log\big(x_j^2+\eta_k\big) \\ \times \left(\frac{\sqrt{\eta_k+4}+ix_1}{\sqrt{\eta_k+4}-ix_1}\right)^{m_1/2} \left(\frac{\sqrt{\eta_k+4}+ix_2}{\sqrt{\eta_k+4}-ix_2}\right)^{m_2/2} dx_1 dx_2 \quad (j=1,2),$$

and $\Lambda \in \mathcal{O}_K$ such that the ideal $(\Lambda) = (\alpha, \varepsilon^k - \varepsilon^{-k})$.

Firstly we note that

$$R_0(m_1,m_2) = \big[2\log Y - \log N(\Lambda)^2\big] Q(\eta_k,\eta_k) \\ = 2\big(2\log Y - \log N(\Lambda)\big) g(2k\log\varepsilon, 2k\log\varepsilon)$$

by using Proposition 2.7 and the formula $g(u_1,u_2) = Q(e^{u_1}+e^{-u_1}-2, e^{u_2}+e^{-u_2}-2)$.

Hereafter let us compute the case of $(m_1,m_2) = (0,m)$.

Proposition 3.4. *Let $m \in 2\mathbb{Z}$. We have*

$$R_1(0,m) = 2\log\big(\varepsilon^k - \varepsilon^{-k}\big) g(2k\log\varepsilon, 2k\log\varepsilon) \\ + \int_{2k\log\varepsilon}^{\infty} g(u, 2k\log\varepsilon) \frac{\cosh(u/2)}{\sinh(u/2)+\sinh(k\log\varepsilon)}\, du. \tag{3.6}$$

Proof. We recall that the test function Φ is written as $\Phi(x_1,x_2) = \Phi_1(x_1)\Phi_2(x_2)$ by Assumption 2.21. Therefore,

$$R_1(0,m) = \int_{\mathbb{R}} \Phi_1\big(x_1^2+\eta_k\big) \log\big(x_1^2+\eta_k\big)\, dx_1 \int_{\mathbb{R}} \Phi_2\big(x_2^2+\eta_k\big) \left(\frac{\sqrt{\eta_k+4}+ix_2}{\sqrt{\eta_k+4}-ix_2}\right)^{m/2} dx_2 \\ = I_1(\eta_k)\cdot Q_2(\eta_k) = I_1(\eta_k)\cdot g_2(2k\log\varepsilon).$$

Here, $I_1(\eta_k)$ is given by

$$-\frac{2}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} \log\big(x^2+\eta\big) Q_1'\big(x^2+\eta+t^2\big)\, dt dx \\ = -\frac{1}{\pi}\int_{\eta}^{\infty}\left(\int_{-\sqrt{y-\eta}}^{\sqrt{y-\eta}} \frac{\log(y-t^2)}{\sqrt{y-\eta-t^2}}\, dt\right) Q_1'(y)\, dy.$$

The inner integral is evaluated as

$$\int_{-\sqrt{y-\eta}}^{\sqrt{y-\eta}} \frac{\log(y-t^2)}{\sqrt{y-\eta-t^2}}\, dt = 2\int_0^{\pi/2} \log\big(y\cos^2\theta + \eta\sin^2\theta\big)\, d\theta = 2\pi\log\left(\frac{\sqrt{\eta}+\sqrt{y}}{2}\right).$$

Here, we used the formula (see [12, 4.399]):

$$\int_0^{\pi/2} \log\left(1+a\sin^2 x\right) dx = \int_0^{\pi/2} \log\left(1+a\cos^2 x\right) dx$$
$$= \pi \log\left(\frac{1+\sqrt{1+a}}{2}\right) \quad \text{for } a > -1.$$

Thus, we have

$$\begin{aligned} I_1(\eta_k) &= -2\int_\eta^\infty \left\{\log(\sqrt{y}+\sqrt{\eta}) - \log 2\right\} Q_1'(y)\,dy \\ &= 2\log 2 \int_{2k\log\varepsilon}^\infty g_1'(u)\,du - 2\int_{2k\log\varepsilon}^\infty \log\left(e^{u/2}-e^{-u/2}+\varepsilon^k-\varepsilon^{-k}\right) g_1'(u)\,du \\ &= 2\log\left(\varepsilon^k-\varepsilon^{-k}\right) g_1(2k\log\varepsilon) + \int_{2k\log\varepsilon}^\infty g_1(u)\frac{\cosh(u/2)}{\sinh(u/2)+\sinh(k\log\varepsilon)}\,du. \end{aligned}$$

The rest is clear. □

Proposition 3.5. *Let $m \in 2\mathbb{Z}$. We have*

$$\begin{aligned} R_2(0,m) = 2\log\left(\varepsilon^k-\varepsilon^{-k}\right) g(2k\log\varepsilon, 2k\log\varepsilon) + \int_{2k\log\varepsilon}^\infty \Bigg[\frac{\cosh(u/2)}{\sinh(u/2)+\sinh(k\log\varepsilon)} \\ + \frac{1-\cosh(m(u/2-k\log\varepsilon))}{\sinh(u/2-k\log\varepsilon)}\Bigg] g(2k\log\varepsilon, u)\,du. \end{aligned} \tag{3.7}$$

Proof. We recall that the test function Φ is written as $\Phi(x_1,x_2) = \Phi_1(x_1)\Phi_2(x_2)$ with real valued Φ_1 and Φ_2 by Assumption 2.21. Therefore,

$$\begin{aligned} R_2(0,m) &= \int_{\mathbb{R}} \Phi_1\left(x_1^2+\eta_k\right) dx_1 \int_{\mathbb{R}} \Phi_2\left(x_2^2+\eta_k\right) \log\left(x_2^2+\eta_k\right) \left(\frac{\sqrt{\eta_k+4}+ix_2}{\sqrt{\eta_k+4}-ix_2}\right)^{m/2} dx_2 \\ &= Q_1(\eta_k)\cdot I_2(\eta_k) = g_1(2k\log\varepsilon)\cdot I_2(\eta_k). \end{aligned}$$

Here, $I_2(\eta) = I_2(\eta_k)$ is given by

$$I_2(\eta) = -\frac{2}{\pi}\operatorname{Re}\Bigg[\int_0^\infty\int_{-\infty}^\infty Q_2'\big(x_2^2+\eta+t^2\big)\log\big(x_2^2+\eta\big)$$
$$\times\left(\frac{\sqrt{x_2^2+\eta+4+t^2}-t}{\sqrt{x_2^2+\eta+4+t^2}+t}\right)^{m/2}\left(\frac{\sqrt{\eta+4}+ix}{\sqrt{\eta+4}-ix}\right)^{m/2} dtdx_2\Bigg]$$
$$= -\frac{1}{\pi}\operatorname{Re}\Bigg[\int_\eta^\infty\Bigg(\int_{-\sqrt{y-\eta}}^{\sqrt{y-\eta}}\frac{\log(y-\xi^2)}{\sqrt{y-\eta-\xi^2}}\left(\frac{\sqrt{\eta+4}+i\sqrt{y-\eta-\xi^2}}{\sqrt{y+4}+\xi}\right)^m d\xi\Bigg)Q_2'(y)\,dy\Bigg],$$

by changing the variables $y = x_2^2+\eta+t^2$ and $\xi = t$. Next changing the variable $\xi = \sqrt{y-\xi}\sin\varphi$, we have

$$I_2(\eta) = -\frac{1}{\pi}\operatorname{Re}\Bigg[\int_\eta^\infty\int_{-\pi/2}^{\pi/2}\log\big((y-\eta)\cos^2\varphi+\eta\big)\left(\frac{\sinh w+i\cos\varphi}{\cosh w+\sin\varphi}\right)^m d\varphi\, Q_2'(y)\,dy\Bigg]$$

with

$$\cosh w = \sqrt{\frac{y+4}{y-\xi}}, \qquad \sinh w = \sqrt{\frac{\eta+4}{y-\xi}}. \tag{3.8}$$

Let us consider the integral

$$J(y) := \operatorname{Re}\Bigg[\int_{-\pi/2}^{\pi/2}\log\big((y-\eta)\cos^2\varphi+\eta\big)\left(\frac{\sinh w+i\cos\varphi}{\cosh w+\sin\varphi}\right)^m d\varphi\Bigg].$$

Then we see that

$$I_2 = -\frac{1}{\pi}\int_\eta^\infty J(y)Q_2'(y)\,dy = -\frac{1}{\pi}\big[J(y)Q_2(y)\big]_\eta^\infty + \frac{1}{\pi}\int_\eta^\infty J'(y)Q_2(y)\,dy$$
$$= \frac{1}{\pi}Q_2(\eta)\int_{-\pi/2}^{\pi/2}\log\eta\,d\varphi + \frac{1}{\pi}\int_\eta^\infty J'(y)Q_2(y)\,dy$$
$$= 2\log\big(\varepsilon^k-\varepsilon^{-k}\big)g_2(2k\log\varepsilon) + \frac{1}{\pi}\int_\eta^\infty J'(y)Q_2(y)\,dy.$$

Let us consider the function $f(z)$ defined by

$$f(z) := \left(\frac{ie^w-z}{ie^w+z}\right)^m\frac{\operatorname{Log}((\xi-\zeta)(\frac{z+z^{-1}}{2})^2+\eta)}{z}$$

to evaluate the derivative of $J(\xi)$. Here, $\mathrm{Log}(z)$ is the principal value logarithm whose imaginary part lies in $(-\pi, \pi]$.

Let $\epsilon, \delta > 0$ be two sufficiently small real numbers, and define the closed curve C in the complex plane, which is made up of two semi-circular arcs starting from $\varphi = -\frac{\pi}{2} + \delta$ to $\varphi = \frac{\pi}{2} - \delta$ of the radii 1 and ϵ, and besides they are joined along by the straight lines $\varphi = \pm(\frac{\pi}{2} - \delta)$.

Considering the counterclockwise contour integral of $f(z)$ along the curve C, by Cauchy's integral theorem, we have

$$\int_{-\pi/2}^{\pi/2} \left(\frac{\sinh w + i\cos\varphi}{\cosh w + \sin\varphi}\right)^m \log\bigl((\xi-\eta)\cos^2\varphi + \eta\bigr)\, id\varphi \tag{3.9}$$

$$+ \int_{i}^{i\epsilon} \left(\frac{ie^w - z}{ie^w + z}\right)^m \mathrm{Log}\left((\xi-\eta)\left(\frac{z+z^{-1}}{2}\right)^2 + \eta\right)\frac{dz}{z} \tag{3.10}$$

$$+ \int_{-\pi/2}^{\pi/2} \left(\frac{ie^w - \epsilon e^{i\varphi}}{ie^w + \epsilon e^{i\varphi}}\right)^m \mathrm{Log}\left((\xi-\eta)\left(\frac{\epsilon e^{i\varphi} + \epsilon^{-1}e^{-i\varphi}}{2}\right)^2 + \eta\right) id\varphi \tag{3.11}$$

$$+ \int_{-i\epsilon}^{-i} \left(\frac{ie^w - z}{ie^w + z}\right)^m \mathrm{Log}\left((\xi-\eta)\left(\frac{z+z^{-1}}{2}\right)^2 + \eta\right)\frac{dz}{z} \tag{3.12}$$

$$= 0$$

Put $\epsilon_0 := \sqrt{\frac{\xi}{\xi-\eta}} - \sqrt{\frac{\eta}{\xi-\eta}}$, then we see that ϵ_0 satisfies $(\xi-\eta)((\epsilon_0^{-1} - \epsilon_0)/2)^2 = \eta$ and (3.10) and (3.12) are written as follows.

$$(3.10) = \int_{\epsilon_0}^{\epsilon} \left(\frac{e^w - y}{e^w + y}\right)^m \left\{\log\left((\xi-\eta)\left(\frac{y^{-1}-y}{2}\right)^2 - \eta\right) - i\pi\right\}\frac{dy}{y}$$
$$+ \int_{1}^{\epsilon_0} \left(\frac{e^w - y}{e^w + y}\right)^m \left\{\log\left(\eta - (\xi-\eta)\left(\frac{y^{-1}-y}{2}\right)^2\right)\right\}\frac{dy}{y},$$

and

$$(3.12) = \int_{\epsilon_0}^{1} \left(\frac{e^w + y}{e^w - y}\right)^m \left\{\log\left(\eta - (\xi-\eta)\left(\frac{y^{-1}-y}{2}\right)^2\right)\right\}\frac{dy}{y}$$
$$+ \int_{\epsilon}^{\epsilon_0} \left(\frac{e^w + y}{e^w - y}\right)^m \left\{\log\left((\xi-\eta)\left(\frac{y^{-1}-y}{2}\right)^2 - \eta\right) + i\pi\right\}\frac{dy}{y}.$$

While, (3.11) is evaluated as

$$(3.11) = \int_{-\pi/2}^{\pi/2} \left[1 + O(\epsilon)\right]\left[\log\left(1/\epsilon^2\right) + \log\left(\frac{\xi - \eta}{4}\right) - 2i\varphi + O\left(\epsilon^2\right)\right] id\varphi.$$

Take the real part of

$$(-i) \times \left\{(3.9) + (3.10) + (3.11) + (3.12)\right\},$$

and we obtain,

$$J(\xi) + \pi \int_{\epsilon}^{\epsilon_0} \left(\frac{e^w - y}{e^w + y}\right)^m \frac{dy}{y} + \pi \int_{\epsilon}^{\epsilon_0} \left(\frac{e^w + y}{e^w - y}\right)^m \frac{dy}{y} - \pi\left\{\log\left(\frac{1}{\epsilon^2}\right) + \log\left(\frac{\xi - \eta}{4}\right)\right\}$$
$$= O\left(\epsilon \log\left(\frac{1}{\epsilon^2}\right)\right).$$

Therefore, we can rewrite the above formula as follows.

$$\begin{aligned} J(\xi) &+ \pi \int_{\epsilon}^{\epsilon_0} \left[\left(\frac{e^w - y}{e^w + y}\right)^m - 1\right] \frac{dy}{y} + \pi \int_{\epsilon}^{\epsilon_0} \left[\left(\frac{e^w + y}{e^w - y}\right)^m - 1\right] \frac{dy}{y} \\ &- \pi \log\left(\frac{\xi - \eta}{4}\right) + 2\pi \log \epsilon_0 \\ &= O\left(\epsilon \log\left(\frac{1}{\epsilon^2}\right)\right). \end{aligned}$$

Letting $\epsilon \to +0$, we have an expression for $J(\xi)$. Changing the variable $y = e^{-u}$ in the integral, we have

$$\begin{aligned} J(\xi) = &-\pi \int_{u_0}^{\infty} \left[\left(\frac{e^w - e^{-u}}{e^w + e^{-u}}\right)^m - 1\right] du - \pi \int_{u_0}^{\infty} \left[\left(\frac{e^w + e^{-u}}{e^w - e^{-u}}\right)^m - 1\right] du \\ &+ 2\pi \log\left(\frac{\sqrt{\xi} + \sqrt{\eta}}{2}\right) \end{aligned}$$

with $u_0 := \log \epsilon_0^{-1}$. Then we obtain an explicit formula for the derivative of $J(\xi)$.

$$\begin{aligned} \frac{dJ(\xi)}{d\xi} = &\frac{\pi}{\sqrt{\xi}(\sqrt{\xi} + \sqrt{\eta})} - 2\pi\left(\frac{\partial w}{\partial \xi} + \frac{\partial u_0}{\partial \xi}\right) \\ &+ \pi\left(\frac{\partial w}{\partial \xi} + \frac{\partial u_0}{\partial \xi}\right)\left[\left(\frac{e^w - e^{-u_0}}{e^w + e^{-u_0}}\right)^m + \left(\frac{e^w + e^{-u_0}}{e^w - e^{-u_0}}\right)^m\right]. \end{aligned}$$

By noting that

$$\frac{e^w - e^{-u_0}}{e^w + e^{-u_0}} = \frac{\sqrt{\xi+4} + \sqrt{\eta+4} - \sqrt{\xi} + \sqrt{\eta}}{\sqrt{\xi+4} + \sqrt{\eta+4} + \sqrt{\xi} - \sqrt{\eta}} = \frac{\sqrt{\eta+4} + \sqrt{\eta}}{\sqrt{\xi+4} + \sqrt{\xi}} = \frac{\varepsilon^k}{e^{u/2}},$$

$$\frac{\partial w}{\partial \xi} = -\frac{1}{2(\xi-\eta)} \frac{\sqrt{\eta+4}}{\sqrt{\xi+4}}, \qquad \frac{\partial u_0}{\partial \xi} = -\frac{1}{2(\xi-\eta)} \frac{\sqrt{\eta}}{\sqrt{\xi}},$$

$$\frac{\partial w}{\partial \xi} + \frac{\partial u_0}{\partial \xi} = -\frac{1}{\xi-\eta} \frac{\varepsilon^k e^{u/2} - \varepsilon^{-k} e^{-u/2}}{e^u - e^{-u}},$$

with $\xi = e^u + e^{-u} - 2$.

We obtain

$$\begin{aligned} \frac{dJ(\xi)}{d\xi} &= \frac{\pi}{\sqrt{\xi}(\sqrt{\xi}+\sqrt{\eta})} + \frac{2\pi}{\xi-\eta} \frac{\varepsilon^k e^{u/2} - \varepsilon^{-k} e^{-u/2}}{e^u - e^{-u}} \\ &\quad - \frac{\pi}{\xi-\eta} \frac{\varepsilon^k e^{u/2} - \varepsilon^{-k} e^{-u/2}}{e^u - e^{-u}} \left[\left(e^{u/2}\varepsilon^{-k}\right)^m + \left(e^{u/2}\varepsilon^{-k}\right)^{-m} \right]. \end{aligned}$$

Then, we have

$$\begin{aligned} & J'\left(e^u + e^{-u} - 2\right) \cdot \left(e^u - e^{-u}\right) \\ &= \frac{\pi(e^{u/2} + e^{-u/2})}{e^{u/2} - e^{-u/2} + \varepsilon^k - \varepsilon^{-k}} + \frac{\pi(\varepsilon^k e^{u/2} - \varepsilon^{-k} e^{-u/2})}{\xi-\eta} \left\{ 2 - \left(e^{u/2}\varepsilon^{-k}\right)^m - \left(e^{u/2}\varepsilon^{-k}\right)^{-m} \right\} \\ &= \frac{\pi \cosh(u/2)}{\sinh(u/2) + \sinh(k\log\varepsilon)} + \frac{\pi}{\sinh(u/2 - k\log\varepsilon)} \left\{ 1 - \cosh\left(m(u/2 - k\log\varepsilon)\right) \right\}. \end{aligned}$$

Substituting the above equality into the following, the proof is completed.

$$\begin{aligned} I_2 &= 2\log\left(\varepsilon^k - \varepsilon^{-k}\right) g_2(2k\log\varepsilon) + \frac{1}{\pi} \int_{\eta}^{\infty} J'(\xi) Q_2(\xi)\, d\xi \\ &= 2\log\left(\varepsilon^k - \varepsilon^{-k}\right) g_2(2k\log\varepsilon) + \frac{1}{\pi} \int_{2k\log\varepsilon}^{\infty} J'\left(e^u + e^{-u} - 2\right) g_2(u) \left(e^u - e^{-u}\right) du. \qquad \square \end{aligned}$$

Putting together with the results in this subsection, we obtain

Proposition 3.6. *For $m \in 2\mathbb{Z}$ and $Y > 1$, we have*

$$\begin{aligned} H_2^Y(0,m) &= 4\log\varepsilon \log Y \sum_{k=1}^{\infty} g(2k\log\varepsilon, 2k\log\varepsilon) \\ &\quad - 4\log\varepsilon \sum_{k=1}^{\infty} \sum_{\gamma_{k,\alpha} \in \Gamma_{\mathrm{H2}}} \frac{k_0(\gamma_{k,\alpha}) \log(N(\alpha, \varepsilon^k - \varepsilon^{-k}))}{|N(\varepsilon^k - \varepsilon^{-k})|} g(2k\log\varepsilon, 2k\log\varepsilon) \end{aligned}$$

$$+ 4\log\varepsilon \sum_{k=1}^{\infty} \log\bigl(\varepsilon^k - \varepsilon^{-k}\bigr) g(2k\log\varepsilon, 2k\log\varepsilon)$$

$$+ 2\log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} \bigl[g(u, 2k\log\varepsilon) + g(2k\log\varepsilon, u)\bigr]$$

$$\times \frac{\cosh(u/2)}{\sinh(u/2) + \sinh(k\log\varepsilon)}\,du$$

$$+ 2\log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} g(2k\log\varepsilon, u) \frac{1 - \cosh(m(u/2 - k\log\varepsilon))}{\sinh(u/2 - k\log\varepsilon)}\,du$$

$$+ o(1) \quad (Y \to \infty).$$

Proof. By noting the fact (see [2, Proposition 3.3, p. 97] or [29, p. 1650])

$$\sum_{\gamma \in \Gamma_{\mathrm{H}2},\, N(\gamma) = \varepsilon^{2k}} k_0(\gamma) = \bigl|N\bigl(\varepsilon^k - \varepsilon^{-k}\bigr)\bigr|,$$

the rest is clear. □

3.5. Contribution from Eisenstein series

Let $m \in 2\mathbb{Z}$ and $Y > 1$. Define the contribution from the Eisenstein series with the truncation parameter Y by

$$EI^Y(0, m) := \int_{F^Y} H_\Gamma(z, z)\,d\mu(z).$$

By using the Maass–Selberg relation (Theorem 2.13), we obtain

Proposition 3.7. *For $m \in 2\mathbb{Z}$, we have*

$$EI^Y(0, m) = 2\log\varepsilon\log Y \sum_{k\in\mathbb{Z}} g(2k\log\varepsilon, 2k\log\varepsilon)$$

$$- \frac{1}{4\pi} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} h\left(r + \frac{\pi k}{2\log\varepsilon}, r - \frac{\pi k}{2\log\varepsilon}\right) \frac{\varphi'_{(0,m)}}{\varphi_{(0,m)}}\left(\frac{1}{2} + ir, k\right) dr$$

$$+ \frac{1}{4} h(0, 0)\varphi_{(0,m)}\left(\frac{1}{2}, 0\right) + o(1) \quad (Y \to \infty).$$

Proof. By definition of the kernel function $H_\Gamma(z,z)$, we can check that

$$\begin{aligned}
&\int_{F^Y} H_\Gamma(z,z)\,d\mu(z) \\
&\quad= \frac{1}{8\pi\sqrt{D}\log\varepsilon}\int_F d\mu(z)\left[\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}} h\left(r+\frac{\pi k}{2\log\varepsilon}, r-\frac{\pi k}{2\log\varepsilon}\right)\left|E^Y_{(0,m)}\left(z,\frac{1}{2}+ir,k\right)\right|^2 dr\right] \\
&\qquad+o(1)\quad (Y\to\infty).
\end{aligned}$$

Next we use the following special case of Theorem 2.13:

$$\begin{aligned}
&\int_F \left|E^Y_{(0,m)}\left(z,\frac{1}{2}+ir,k\right)\right|^2 d\mu(z) \\
&\quad= 2\sqrt{D}\log\varepsilon\left[2\log Y-\frac{\varphi'_{(0,m)}}{\varphi_{(0,m)}}\left(\frac{1}{2}+ir,k\right)\right. \\
&\qquad\left.+\delta_{0,k}\frac{\varphi_{(0,m)}(\frac{1}{2}-ir,0)Y^{2ir}-\varphi_{(0,m)}(\frac{1}{2}+ir,0)Y^{-2ir}}{2ir}\right].
\end{aligned}$$

Finally we obtain the desired formula as in the proof of Proposition 1.1 in [2, p. 85]. □

3.6. Cancellation of the $\log Y$ terms

Let us complete the proof of Theorem 2.22. Let $m\in 2\mathbb{Z}$. By Propositions 3.3, 3.6 and 3.7, the $\log Y$ terms are canceled out and we have

$$\begin{aligned}
&\lim_{Y\to\infty}\left\{\sum_{\gamma\in\Gamma_{\mathrm{P}}\cup\Gamma_{\mathrm{H2}}}\int_{F^Y_\gamma} k(z,\gamma z)\,j_\gamma(z)\,d\mu(z)-\int_{F^Y} H_\Gamma(z,z)\,d\mu(z)\right\} \\
&\quad=\lim_{Y\to\infty}\left\{P^Y(0,m)+H_2^Y(0,m)-EI^Y(0,m)+o(1)\right\} \\
&\quad=P^Y(0,m)|_{Y=1}+H_2^Y(0,m)|_{Y=1}-EI^Y(0,m)|_{Y=1} \\
&\quad=: P(0,m)+H_2(0,m)+SC(0,m). \qquad (3.13)
\end{aligned}$$

We see that $P(0,m)$, $H_2(0,m)$ and $SC(0,m)$ are identified with $\mathbf{II}_a(h)$, $\mathbf{II}_b(h)$ and $\mathbf{III}(h)$ in Theorem 2.22 respectively. The series and integrals appearing in these terms are absolutely convergent by the assumption on the test functions h in this theorem. Thus we complete the proof.

4. Differences of the Selberg trace formula for Hilbert modular surfaces

4.1. Differences of the Selberg trace formula

Let $m \in 2\mathbb{Z}$. We introduce Maass operators $\Lambda_m^{(2)}$ and $K_m^{(2)}$, which play important roles in considering the "differences" of the Selberg trace formulas. We refer to [14, Proposition 5.13, p. 381] and [22, pp. 305–307] for basic properties of these Maass operators.

Firstly we consider the following "weight down" Maass operator

$$\Lambda_m^{(2)} := iy_2\frac{\partial}{\partial x_2} - y_2\frac{\partial}{\partial y_2} + \frac{m}{2} \colon L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m)\big) \to L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m-2)\big).$$

Recall that

$$\mathrm{Ker}\big(\Lambda_m^{(2)}\big) = \left\{f \in L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m)\big) \;\middle|\; \Delta_m^{(2)} f = \frac{m}{2}\left(1-\frac{m}{2}\right)f\right\},$$

i.e. $\lambda^{(2)} = \frac{m}{2}(1-\frac{m}{2})$-eigenspace.

When $\mathrm{Ker}(\Lambda_m^{(2)})$ is not trivial, let $\{\frac{1}{4}+\rho_j(m)^2\}_{j=0}^\infty$ be the set of eigenvalues of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_m^{(2)})$, then we have a direct sum decomposition into eigenspaces of the Laplacians

$$\mathrm{Ker}\big(\Lambda_m^{(2)}\big) = \bigoplus_{j=0}^{\infty} L^2_{\mathrm{dis}}\left(\Gamma_K\backslash\mathbb{H}^2;\left(\frac{1}{4}+\rho_j(m)^2,\frac{m}{2}\left(1-\frac{m}{2}\right)\right),(0,m)\right). \tag{4.1}$$

Secondly we consider the following "weight up" Maass operator

$$K_{m-2}^{(2)} := iy_2\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_2} + \frac{m-2}{2} \colon L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m-2)\big) \to L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m)\big).$$

Recall that

$$\mathrm{Ker}\big(K_{m-2}^{(2)}\big) = \left\{f \in L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,m-2)\big) \;\middle|\; \Delta_m^{(2)} f = \frac{m}{2}\left(1-\frac{m}{2}\right)f\right\},$$

i.e. $\lambda^{(2)} = \frac{m}{2}(1-\frac{m}{2})$-eigenspace.

When $\mathrm{Ker}(K_{m-2}^{(2)})$ is not trivial, let $\{\frac{1}{4}+\mu_j(m-2)^2\}_{j=0}^\infty$ be the set of eigenvalues of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(K_{m-2}^{(2)})$, then we have a direct sum decomposition into eigenspaces of the Laplacian

$$\mathrm{Ker}\big(K_{m-2}^{(2)}\big) = \bigoplus_{j=0}^{\infty} L^2_{\mathrm{dis}}\left(\Gamma_K\backslash\mathbb{H}^2;\left(\frac{1}{4}+\mu_j(m-2)^2,\frac{m}{2}\left(1-\frac{m}{2}\right)\right),(0,m-2)\right). \tag{4.2}$$

By considering the two kernel spaces (4.1) and (4.2), we *subtract* the Selberg trace formula for $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m-2))$ from the one associated with $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m))$. Then we obtain (we give a proof in the next subsection).

Theorem 4.1 *(Differences of STF for $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m)) - L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m-2))$). Let $m \in 2\mathbb{Z}$. We have*

$$\begin{aligned}
&\sum_{j=0}^{\infty} h_1\big(\rho_j(m)\big)h_2\left(\frac{i(m-1)}{2}\right) - \sum_{j=0}^{\infty} h_1\big(\mu_j(m-2)\big)h_2\left(\frac{i(m-1)}{2}\right)\\
&= (m-1)h_2\left(\frac{i(m-1)}{2}\right)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2}\int_{-\infty}^{\infty} r_1 h_1(r_1)\tanh(\pi r_1)\,dr_1\\
&\quad + \sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}} \frac{-e^{-i\theta_1+i(m-1)\theta_2}}{8\nu_R\sin\theta_1\sin\theta_2} h_2\left(\frac{i(m-1)}{2}\right)\\
&\quad \times \int_{\mathbb{R}} g_1(u_1)e^{-u_1/2}\left[\frac{e^{u_1}-e^{2i\theta_1}}{\cosh u_1 - \cos 2\theta_1}\right]du_1\\
&\quad + \sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} g_1\big(\log N(\gamma)\big)\frac{ie^{i(m-1)\omega}}{2\sin\omega} h_2\left(\frac{i(m-1)}{2}\right)\\
&\quad - \mathrm{sgn}(m-1)\log\varepsilon\, g_1(0)h_2\left(\frac{i(m-1)}{2}\right)\\
&\quad - 2\,\mathrm{sgn}(m-1)\log\varepsilon\, h_2\left(\frac{i(m-1)}{2}\right)\sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\varepsilon^{-k|m-1|}.
\end{aligned}$$

Here, we consider that the sum on $\rho_j(m)$ is empty if $\mathrm{Ker}(\Lambda_m^{(2)})$ *is trivial, and the sum on $\mu_j(m-2)$ is empty if* $\mathrm{Ker}(K_{m-2}^{(2)})$ *is trivial.* (*See also* (5.3).)

4.2. Proof of the differences of the Selberg trace formula

We prove Theorem 4.1 in this subsection. (Basic strategy is the same as the case of the trace formulas for $\mathrm{PSL}(2,\mathbb{R})$, see [14, pp. 481–485]).

• Spectral side:

By (4.1) and (4.2), the difference between the spectral sides of $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m))$ and $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m-2))$ is given by

$$\sum_{j=0}^{\infty} h_1\big(\rho_j(m)\big)h_2\left(\frac{i(m-1)}{2}\right) - \sum_{j=0}^{\infty} h_1\big(\mu_j(m-2)\big)h_2\left(\frac{i(m-1)}{2}\right).$$

• Identity term:
Put $\overline{I(m)} := I(0,m) - I(0,m-2)$. Here, $I(m_1,m_2)$ is defined in (3.1). Then, we have (see pp. 396–397 in [14])

$$\begin{aligned}\overline{I(m)} &= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{4\pi^2}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{g_1'(u_1)g_2'(u_2)}{(e^{u_1/2}-e^{-u_1/2})(e^{u_2/2}-e^{-u_2/2})} \\ &\quad\times \left\{e^{-mu_2/2} - e^{-(m-2)u_2/2}\right\} du_1 du_2 \\ &= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}\int\limits_{\mathbb{R}} r_1 h_1(r_1)\tanh(\pi r_1)\,dr_1 \int\limits_{\mathbb{R}} g_2(u_2)e^{-(m-1)u_2/2}\,du_2 \times \frac{(m-1)}{2} \\ &= (m-1)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2} h_2\left(\frac{i(m-1)}{2}\right)\int\limits_{\mathbb{R}} r_1 h_1(r_1)\tanh(\pi r_1)\,dr_1.\end{aligned}$$

• Elliptic terms:
Let R be an elliptic element. Put $\overline{E(m;R)} := E(0,m;R) - E(0,m-2;R)$. Here, $E(m_1,m_2;R)$ is defined in (3.3). Then, we have

$$\begin{aligned}\overline{E(m;R)} &= \frac{-e^{-i\theta_1+i(m-1)\theta_2}}{16\nu_R\sin\theta_1\sin\theta_2}\iint\limits_{\mathbb{R}^2} g(u_1,u_2)e^{\frac{-u_1}{2}}\left\{e^{\frac{(m-1)u_2}{2}} - e^{-2i\theta_2}e^{\frac{(m-3)u_2}{2}}\right\} \\ &\quad\times\prod_{j=1}^{2}\left[\frac{e^{u_j}-e^{2i\theta_j}}{\cosh u_j - \cos 2\theta_j}\right] du_1 du_2 \\ &= \frac{-e^{-i\theta_1+i(m-1)\theta_2}}{8\nu_R\sin\theta_1\sin\theta_2} h_2\left(\frac{i(m-1)}{2}\right)\int\limits_{\mathbb{R}} g_1(u_1)e^{-u/2}\left[\frac{e^{u_1}-e^{2i\theta_1}}{\cosh u_1 - \cos 2\theta_1}\right] du_1.\end{aligned}$$

• Hyperbolic–elliptic terms:
Let γ be a hyperbolic–elliptic element. Put $\overline{HE(m;\gamma)} := HE(0,m;\gamma) - HE(0,m-2;\gamma)$. Here, $HE(m_1,m_2;\gamma)$ is defined in (3.4). Then, we obtain,

$$\begin{aligned}\overline{HE(m;\gamma)} &= \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}}\frac{ie^{i(m-1)\omega}}{4\sin\omega}\int_{-\infty}^{\infty} g\big(\log N(\gamma),u\big)e^{\frac{m-1}{2}u}\left\{1-e^{-2i\omega}e^{-u}\right\} \\ &\quad\times\left[\frac{e^{u}-e^{2i\omega}}{\cosh u-\cos 2\omega}\right] du \\ &= \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} g_1\big(\log N(\gamma)\big)\frac{ie^{i(m-1)\omega}}{2\sin\omega} h_2\left(\frac{i(m-1)}{2}\right).\end{aligned}$$

• Parabolic contribution:
Put $\overline{P(m)} := P(0,m) - P(0,m-2)$. Here, $P(0,m)$ is defined in (3.13). Then we have,

$$\overline{P(m)} = -\log\varepsilon g_1(0)\left[\int_0^{\infty} g_2(u)e^{\frac{m-1}{2}u}\,du - \int_0^{\infty} g_2(u)e^{-\frac{m-1}{2}u}\,du\right]. \tag{4.3}$$

• Type 2 hyperbolic contribution:
Put $\overline{H_2(m)} := H_2(0,m) - H_2(0,m-2)$. Here, $H_2(0,m)$ is defined in (3.13). Then we have,

$$\overline{H_2(m)} = 2\log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} g(2k\log\varepsilon, u) \times \frac{\cosh((m-2)(u/2 - k\log\varepsilon)) - \cosh(m(u/2 - k\log\varepsilon))}{\sinh(u/2 - k\log\varepsilon)}\, du$$
$$= \log\varepsilon \sum_{k=1}^{\infty} \int_{2k\log\varepsilon}^{\infty} g(2k\log\varepsilon, u)\left\{e^{-(m-1)(u/2-k\log\varepsilon)} - e^{(m-1)(u/2-k\log\varepsilon)}\right\} du.$$

Therefore, we have

$$\overline{H_2(m)} = 2\log\varepsilon \sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\left[\varepsilon^{k(m-1)} \int_{2k\log\varepsilon}^{\infty} g_2(u) e^{-\frac{m-1}{2}u}\, du - \varepsilon^{-k(m-1)} \int_{2k\log\varepsilon}^{\infty} g_2(u) e^{\frac{m-1}{2}u}\, du\right]. \tag{4.4}$$

Finally, we calculate the scattering contribution to the differences of the trace formula for Hilbert modular surfaces. Put $\overline{SC(m)} := SC(0,m) - SC(0,m-2)$. Here, $SC(0,m)$ is defined in (3.13).

Proposition 4.2 *(Scattering contribution).*

$$\overline{SC(m)} = -2\,\mathrm{sgn}(m-1)\log\varepsilon \sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\left[\varepsilon^{-k|m-1|}\int_0^{\infty} g_2(u) e^{-\frac{|m-1|}{2}u}\, du + \varepsilon^{k|m-1|}\int_{2k\log\varepsilon}^{\infty} g_2(u)\, e^{-\frac{|m-1|}{2}u}\, du + \varepsilon^{-k|m-1|}\int_0^{2k\log\varepsilon} g_2(u) e^{\frac{|m-1|}{2}u}\, du\right]$$
$$- 2\,\mathrm{sgn}(m-1)\log\varepsilon\, g_1(0)\int_0^{\infty} g_2(u) e^{-\frac{|m-1|}{2}u}\, du. \tag{4.5}$$

Proof. Firstly, we can easily check that

$$\varphi_{(0,m)}(s,0) = \frac{(-1)^{m/2}}{D^{1-2s}\pi^{4s-2}} \frac{\Gamma(s)^2\Gamma(s-\frac{1}{2})^2}{\Gamma(s+\frac{m}{2})\Gamma(s-\frac{m}{2})\Gamma(\frac{1}{2}-s)^2} \frac{\zeta_K(2s-1)}{\zeta_K(1-2s)},$$

by using the functional equation of the Dedekind zeta $\zeta_K(s)$. Thus we have $\varphi_{(0,m)}(\frac{1}{2},0)=1$.

Secondly, by the explicit formula for $\varphi_{(0,m)}(s,k)$, (see (2.9)) we see that

$$\varphi_{(0,m)}(s,k)\varphi_{(0,m-2)}(s,k)^{-1}=\left(s-\frac{\pi ik}{2\log\varepsilon}-\frac{m}{2}\right)\left(s-\frac{\pi ik}{2\log\varepsilon}+\frac{m}{2}-1\right)^{-1}.$$

So we have

$$\left(\frac{\varphi'_{(0,m)}}{\varphi_{(0,m)}}-\frac{\varphi'_{(0,m-2)}}{\varphi_{(0,m-2)}}\right)\left(\frac{1}{2}+ir,k\right)=-\frac{m-1}{(r-\frac{\pi k}{2\log\varepsilon})^2+(\frac{m-1}{2})^2}.$$

Therefore, we have

$$\begin{aligned}\overline{SC(m)}&=-\frac{1}{4\pi}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi k}{2\log\varepsilon},r-\frac{\pi k}{2\log\varepsilon}\right)\frac{m-1}{(r-\frac{\pi k}{2\log\varepsilon})^2+(\frac{m-1}{2})^2}\,dr\\&=-\frac{1}{4\pi}\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty}h\left(r+\frac{\pi k}{\log\varepsilon},r\right)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,dr.\end{aligned}\tag{4.6}$$

Thirdly, we use the Poisson summation formula to calculate (4.6) further. Let us determine the sequence $\{a_k\}$ such that

$$\sum_{k\in\mathbb{Z}}h_1\left(r+\frac{\pi k}{\log\varepsilon}\right)=\sum_{k\in\mathbb{Z}}a_k\exp\left(2\pi ikr\cdot\frac{\log\varepsilon}{\pi}\right).$$

Then

$$\begin{aligned}a_k&=\int_0^{\pi/\log\varepsilon}\sum_{k\in\mathbb{Z}}h_1\left(r+\frac{\pi k}{\log\varepsilon}\right)e^{-2k\log\varepsilon\cdot ir}\,dr=\frac{\log\varepsilon}{\pi}\int_{-\infty}^{\infty}h_1(r)e^{-2k\log\varepsilon\cdot ir}\,dr\\&=\frac{\log\varepsilon}{\pi}(2\pi)g_1(2k\log\varepsilon)=2\log\varepsilon g_1(2k\log\varepsilon).\end{aligned}$$

So (4.6) is written as

$$\overline{SC(m)}=-\frac{2\log\varepsilon}{4\pi}\sum_{k\in\mathbb{Z}}g_1(2k\log\varepsilon)\int_{-\infty}^{\infty}h_2(r)e^{2k\log\varepsilon\, ir}\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,dr.\tag{4.7}$$

Finally, let us evaluate the following integral

$$I_0:=\frac{1}{4\pi}\int_{-\infty}^{\infty}h_2(r)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,dr,$$

and

$$I_k := \frac{1}{4\pi}\int_{-\infty}^{\infty} h_2(r)\big(e^{2k\log\varepsilon\, ir} + e^{-2k\log\varepsilon\, ir}\big)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,dr$$

for $k \in \mathbb{Z}$ and $k > 0$. Recall that

$$h_2(r) = 2\int_0^{\infty} g_2(u)\cos(ru)\,du$$

and (see [12, 3.723 (2)])

$$\int_0^{\infty}\frac{\cos(ru)}{r^2+(\frac{m-1}{2})^2}\,dr = \frac{\pi}{|m-1|}e^{-\frac{|m-1|}{2}|u|}$$

for $m \neq 1$ (we assumed that $m \in 2\mathbb{Z}$). Then we obtain

$$I_0 = \operatorname{sgn}(m-1)\int_0^{\infty} g_2(u)e^{-\frac{|m-1|}{2}u}\,du.$$

While, we have

$$\begin{aligned} I_k &= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} g_2(u)\cdot 2\cos(ru)\cos(r\cdot 2k\log\varepsilon)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,drdu \\ &= \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} g_2(u)\cos\big(r(u+2k\log\varepsilon)\big)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,drdu \\ &\quad + \frac{1}{\pi}\int_0^{\infty}\int_0^{\infty} g_2(u)\cos\big(r(u-2k\log\varepsilon)\big)\frac{m-1}{r^2+(\frac{m-1}{2})^2}\,drdu. \end{aligned}$$

Therefore, we have

$$\begin{aligned} I_k = \operatorname{sgn}(m-1)&\int_0^{\infty} g_2(u)e^{-\frac{|m-1|}{2}(u+2k\log\varepsilon)}\,du \\ &+ \operatorname{sgn}(m-1)\int_0^{\infty} g_2(u)e^{-\frac{|m-1|}{2}|u-2k\log\varepsilon|}\,du \end{aligned}$$

$$
\begin{aligned}
&= \operatorname{sgn}(m-1)\varepsilon^{-k|m-1|} \int_0^\infty g_2(u) e^{-\frac{|m-1|}{2}u}\, du \\
&\quad + \operatorname{sgn}(m-1)\varepsilon^{k|m-1|} \int_{2k\log\varepsilon}^\infty g_2(u) e^{-\frac{|m-1|}{2}u}\, du \\
&\quad + \operatorname{sgn}(m-1)\varepsilon^{-k|m-1|} \int_0^{2k\log\varepsilon} g_2(u) e^{\frac{|m-1|}{2}u}\, du
\end{aligned}
$$

for $k \in \mathbb{N}$. We complete the proof. $\square$

We can now put together with, the parabolic contribution (4.3), the type 2 hyperbolic contribution (4.4) and the scattering contribution (4.5), then we obtain

Proposition 4.3.

$$
\begin{aligned}
&\overline{P(m)} + \overline{H_2(m)} + \overline{SC(m)} \\
&\quad = -\operatorname{sgn}(m-1)\log\varepsilon\, g_1(0) h_2\left(\frac{i(m-1)}{2}\right) \\
&\qquad - 2\operatorname{sgn}(m-1)\log\varepsilon \sum_{k=1}^\infty g_1(2k\log\varepsilon)\varepsilon^{-k|m-1|} h_2\left(\frac{i(m-1)}{2}\right).
\end{aligned} \tag{4.8}
$$

Proof. By (4.3) and (4.4), we see that

$$
\overline{P(m)} = -\operatorname{sgn}(m-1)\log\varepsilon g_1(0)\left[\int_0^\infty g_2(u) e^{\frac{|m-1|}{2}u}\, du - \int_0^\infty g_2(u) e^{-\frac{|m-1|}{2}u}\, du\right],
$$

and

$$
\begin{aligned}
\overline{H_2(m)} &= 2\operatorname{sgn}(m-1)\log\varepsilon \sum_{k=1}^\infty g_1(2k\log\varepsilon)\left[\varepsilon^{k|m-1|} \int_{2k\log\varepsilon}^\infty g_2(u) e^{-\frac{|m-1|}{2}u}\, du\right. \\
&\qquad \left. - \varepsilon^{-k|m-1|} \int_{2k\log\varepsilon}^\infty g_2(u) e^{\frac{|m-1|}{2}u}\, du\right].
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\overline{P(m)} + \overline{H_2(m)} + \overline{SC(m)} \\
&\quad = -\operatorname{sgn}(m-1)\log\varepsilon g_1(0)\left[\int_0^\infty g_2(u) e^{\frac{|m-1|}{2}u}\, du + \int_0^\infty g_2(u) e^{-\frac{|m-1|}{2}u}\, du\right]
\end{aligned}
$$

$$- 2\,\mathrm{sgn}(m-1)\log\varepsilon\sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\Bigg[\varepsilon^{-k|m-1|}\int_0^{\infty} g_2(u)e^{-\frac{|m-1|}{2}u}\,du$$

$$+\,\varepsilon^{-k|m-1|}\int_{2k\log\varepsilon}^{\infty} g_2(u)e^{\frac{|m-1|}{2}u}\,du + \varepsilon^{-k|m-1|}\int_0^{2k\log\varepsilon} g_2(u)e^{\frac{|m-1|}{2}u}\,du\Bigg].$$

The rest is clear. □

By using the above proposition, we complete the proof of Theorem 4.1.

4.3. Double differences of the Selberg trace formula

We wrote down the differences of the Selberg trace formulas for $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m))$ and $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m-2))$ in Theorem 4.1. Let us denote the above differences formulas as $L(m)-L(m-2)$. Next we assume that $h_2(\frac{i(m-1)}{2})\neq 0$ and $h_2(\frac{i(m-3)}{2})\neq 0$, then consider the "double differences":

$$\big(L(m)-L(m-2)\big)h_2\bigg(\frac{i(m-1)}{2}\bigg)^{-1} - \big(L(m-2)-L(m-4)\big)h_2\bigg(\frac{i(m-3)}{2}\bigg)^{-1}.$$

Then we have,

Theorem 4.4 *(Double differences of STF for $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,m))$). Let $m\in 2\mathbb{Z}$. We have*

$$\sum_{j=0}^{\infty} h_1\big(\rho_j(m)\big) - \sum_{j=0}^{\infty} h_1\big(\mu_j(m-2)\big) - \sum_{j=0}^{\infty} h_1\big(\rho_j(m-2)\big) + \sum_{j=0}^{\infty} h_1\big(\mu_j(m-4)\big)$$

$$= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}\int_{-\infty}^{\infty} rh_1(r)\tanh(\pi r)\,dr$$

$$- \sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}} \frac{ie^{-i\theta_1}e^{i(m-2)\theta_2}}{4\nu_R\sin\theta_1}\int_{-\infty}^{\infty} g_1(u)e^{-u/2}\bigg[\frac{e^u-e^{2i\theta_1}}{\cosh u-\cos 2\theta_1}\bigg]du$$

$$- \sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} g_1\big(\log N(\gamma)\big)e^{i(m-2)\omega}$$

$$- \log\varepsilon\, g_1(0)\big(\mathrm{sgn}(m-1)-\mathrm{sgn}(m-3)\big)$$

$$- 2\log\varepsilon\sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\big(\mathrm{sgn}(m-1)\varepsilon^{-k|m-1|}-\mathrm{sgn}(m-3)\varepsilon^{-k|m-3|}\big).$$

Here, we consider that the sum on $\rho_j(q)$ is empty if $\mathrm{Ker}(\Lambda_q^{(2)})$ is trivial $(q=m,m-2)$, and the sum on $\mu_j(q-2)$ is empty if $\mathrm{Ker}(K_{q-2}^{(2)})$ is trivial $(q=m,m-2)$.

Proof. By direct computation. □

5. Selberg type zeta functions for Hilbert modular surfaces

5.1. Selberg type zeta functions

Let $(\gamma, \gamma') \in \Gamma_K$ be hyperbolic–elliptic, i.e., $|\mathrm{tr}(\gamma)| > 2$ and $|\mathrm{tr}(\gamma')| < 2$. Then the centralizer of hyperbolic–elliptic (γ, γ') in Γ_K is infinite cyclic.

Definition 5.1 *(Selberg type zeta function for Γ_K with the weight $(0, m)$).* Let $m \geq 2$ be an even integer. The Selberg type zeta function for Γ_K with the weight $(0, m)$ is defined by the following Euler product:

$$Z_K(s; m) := \prod_{(p,p')} \prod_{k=0}^{\infty} \left(1 - e^{i(m-2)\omega_0} N(p)^{-(k+s)}\right)^{-1} \quad \text{for } \operatorname{Re}(s) > 1.$$

Here, (p, p') run through the set of primitive hyperbolic–elliptic Γ_K-conjugacy classes of Γ_K, and (p, p') is conjugate in $\mathrm{PSL}(2, \mathbb{R})^2$ to

$$(p, p') \sim \left(\begin{pmatrix} N(p)^{1/2} & 0 \\ 0 & N(p)^{-1/2} \end{pmatrix}, \begin{pmatrix} \cos\omega_0 & -\sin\omega_0 \\ \sin\omega_0 & \cos\omega_0 \end{pmatrix} \right),$$

where, $N(p) > 1$, $\omega_0 \in (0, \pi)$ and $\omega_0 \notin \pi\mathbb{Q}$.

Theorem 6.12, which we prove in the next section by using Theorem 4.4, ensures that the Euler product is absolutely convergent for $\operatorname{Re}(s) > 1$. Therefore, $Z_K(s; m)$ represents a holomorphic function on the half plane $\operatorname{Re}(s) > 1$. We remark that the exponent is -1 in the definition, which differs from the original one.

For an even integer $m \leq 2$, we see that

$$Z_K(s; m) = \overline{Z_K(\bar{s}; 4 - m)}.$$

Thus, it is sufficient to consider $Z_K(s; m)$ for an even integer $m \geq 2$ by the above relation.

We show that $Z_K(s; m)$ has a meromorphic extension to the whole complex plane by using Theorem 4.4 (double differences of the Selberg trace formula).

5.2. Test functions

Let us consider the logarithmic derivative of $Z_K(s; m)$. For $\operatorname{Re}(s) > 1$,

$$\begin{aligned}
\frac{d}{ds} \log Z_K(s; m) &= -\sum_{(p,p')} \sum_{k=0}^{\infty} \log N(p) \frac{e^{i(m-2)\omega_0} N(p)^{-(k+s)}}{1 - e^{i(m-2)\omega_0} N(p)^{-(k+s)}} \\
&= -\sum_{(p,p')} \sum_{l=1}^{\infty} \frac{\log N(p)}{1 - N(p^l)^{-1}} N\left(p^l\right)^{-s} e^{i(m-2)l\omega_0}. \qquad (5.1)
\end{aligned}$$

Usually, we introduce a certain test function $h(r_1, r_2)$ to get a meromorphic extension of the logarithmic derivative of the Selberg type zeta functions.

We can check that the Selberg trace formula (Theorem 2.22) holds for the test function $h(r_1, r_2)$ which satisfies the following condition (see [2, p. 105] or [29, p. 1651]):

(1) $h(\pm r_1, \pm r_2) = h(r_1, r_2)$,
(2) h is analytic in the domain $|\operatorname{Im}(r_1)| < \frac{1}{2} + \delta$, $|\operatorname{Im}(r_2)| < \frac{||m|-1|}{2} + \delta$ for some $\delta > 0$,
(3) $h(r_1, r_2) = O((1 + |r_1|^2 + |r_2|^2)^{-2-\delta})$ for some $\delta > 0$ in this domain,
(4) $g_2(u_2) \in C_c^\infty(\mathbb{R})$.

We remark that the last condition assures the absolute convergence of the geometric side of Theorem 2.22, in particular that of $\mathbf{II}_b(h)$.

Let us consider the following test function: Firstly, we fix real numbers $\beta_1, \beta_2 \geq 2$, $\beta_1 \neq \beta_2$. For $s \in \mathbb{C}$, $\operatorname{Re}(s) > 1$, we set

$$\begin{aligned} h_1(r) &:= \frac{(\beta_1^2 - (s - \frac{1}{2})^2)(\beta_2^2 - (s - \frac{1}{2})^2)}{(r^2 + (s - \frac{1}{2})^2)(r^2 + \beta_1^2)(r^2 + \beta_2^2)} \\ &= \frac{1}{r^2 + (s - \frac{1}{2})^2} + \frac{c_1(s)}{r^2 + \beta_1^2} + \frac{c_2(s)}{r^2 + \beta_2^2} \end{aligned} \tag{5.2}$$

with

$$c_1(s) = \frac{(s - \frac{1}{2})^2 - \beta_2^2}{\beta_2^2 - \beta_1^2}, \qquad c_2(s) = -\frac{(s - \frac{1}{2})^2 - \beta_1^2}{\beta_2^2 - \beta_1^2}.$$

(See Parnovskii [21] for this type test functions.) The Fourier transform of h_1 is given by

$$g_1(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_1(r) e^{-iru} \, dr = \frac{1}{2s - 1} e^{-(s - \frac{1}{2})|u|} + \frac{c_1(s)}{2\beta_1} e^{-\beta_1 |u|} + \frac{c_2(s)}{2\beta_2} e^{-\beta_2 |u|}.$$

Secondly, we take $g_2(u) \in C_c^\infty(\mathbb{R})$ such that its Fourier inverse transform $h_2(r)$ satisfies $h_2(\frac{i(m-1)}{2}) \neq 0$ and $h_2(\frac{i(m-3)}{2}) \neq 0$. Then we can easily check that our test function $h(r_1, r_2) := h_1(r_1) h_2(r_2)$ satisfies the above sufficient condition for Theorem 2.22.

Thirdly, let us assume that $m \geq 4$ for simplicity. We will treat the case of $m = 2$ in the next subsection. Let $m \geq 4$ be an even integer. Then we have

$$\left(\operatorname{Ker}\left(K_{m-2}^{(2)}\right), \operatorname{Ker}\left(K_{m-4}^{(2)}\right)\right) = \begin{cases} (\{0\}, \mathbb{C}) & \text{if } m = 4, \\ (\{0\}, \{0\}) & \text{if } m \geq 6. \end{cases} \tag{5.3}$$

Here, $K_{m-2}^{(2)}$, $K_{m-4}^{(2)}$ are the weight up Maass operators and their kernel are given by

$$\operatorname{Ker}\left(K_{q-2}^{(2)}\right) = \left\{ f \in L^2_{\mathrm{dis}}\left(\Gamma_K \backslash \mathbb{H}^2; (0, q-2)\right) \,\middle|\, \Delta_{q-2}^{(2)} f = \frac{q}{2}\left(1 - \frac{q}{2}\right) f \right\}$$

for $q = m, m-2$. The fact (5.3) is deduced from Lemma 2.18 and that the Hilbert modular group $\varGamma_K$ is an irreducible discrete subgroup. We also recall that $\{\frac{1}{4} + \rho_j(m)^2\}_{j=0}^{\infty}$ and $\{\frac{1}{4} + \rho_j(m-2)^2\}_{j=0}^{\infty}$ are the set of eigenvalues of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\varLambda_m^{(2)})$ and $\mathrm{Ker}(\varLambda_{m-2}^{(2)})$ respectively. Here, $\varLambda_m^{(2)}$, $\varLambda_{m-2}^{(2)}$ are the weight down Maass operators and their kernel are given by

$$\mathrm{Ker}\big(\varLambda_q^{(2)}\big) = \left\{ f \in L^2_{\mathrm{dis}}\big(\varGamma_K \backslash \mathbb{H}^2; (0,q)\big) \;\middle|\; \Delta_q^{(2)} f = \frac{q}{2}\left(1 - \frac{q}{2}\right) f \right\}$$

for $q = m, m-2$. If we set $\lambda_j(q) := \frac{1}{4} + \rho_j(q)^2$ for $q = m, m-2$, we note that

$$0 < \lambda_0(q) \leq \lambda_1(q) \leq \lambda_2(q) \leq \dots \tag{5.4}$$

since $\varGamma_K$ is irreducible and $m \geq 4$.

Finally, we consider Theorem 4.4, the double difference of the Selberg trace formula, for the above test function (5.2). Then we have

Theorem 5.2 *(Double differences of STF for the above test function h_1 and h_2). Let $m \geq 4$ be an even integer. For $\mathrm{Re}(s) > 1$, we have*

$$\begin{aligned}
&\sum_{j=0}^{\infty} \left[\frac{1}{\rho_j(m)^2 + (s-\frac{1}{2})^2} + \sum_{h=1}^{2} \frac{c_h(s)}{\rho_j(m)^2 + \beta_h^2} \right] \\
&\quad - \sum_{j=0}^{\infty} \left[\frac{1}{\rho_j(m-2)^2 + (s-\frac{1}{2})^2} + \sum_{h=1}^{2} \frac{c_h(s)}{\rho_j(m-2)^2 + \beta_h^2} \right] \\
&\quad + \delta_{m,4} \left[\frac{1}{s(s-1)} + \sum_{h=1}^{2} \frac{c_h(s)}{\beta_h^2 - \frac{1}{4}} \right] \\
&= 2\zeta_K(-1) \sum_{k=0}^{\infty} \left[\frac{1}{s+k} + \sum_{h=1}^{2} \frac{c_h(s)}{\beta_h + \frac{1}{2} + k} \right] + \frac{1}{2s-1} \frac{Z_K'(s;m)}{Z_K(s;m)} \\
&\quad + \sum_{h=1}^{2} \frac{c_h(s)}{2\beta_h} \frac{Z_K'(\frac{1}{2}+\beta_h;m)}{Z_K(\frac{1}{2}+\beta_h;m)} \\
&\quad + \frac{1}{2s-1} \sum_{j=1}^{N} \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m,j) - \overline{\alpha_l}(m,j)}{\nu_j^2} \psi\left(\frac{s+l}{\nu_j}\right) \\
&\quad + \sum_{h=1}^{2} \frac{c_h(s)}{2\beta_h} \sum_{j=1}^{N} \sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m,j) - \overline{\alpha_l}(m,j)}{\nu_j^2} \psi\left(\frac{\frac{1}{2}+\beta_h+l}{\nu_j}\right) \\
&\quad + \frac{1}{2s-1} \frac{d}{ds} \log\left\{ \frac{(1-\varepsilon^{-(2s+m-4)})}{(1-\varepsilon^{-(2s+m-2)})} \right\} + \sum_{h=1}^{2} \frac{c_h(s)}{2\beta_h} \frac{d}{d\beta_h} \log\left\{ \frac{(1-\varepsilon^{-(2\beta_h+m-3)})}{(1-\varepsilon^{-(2\beta_h+m-1)})} \right\}.
\end{aligned}$$

Here, $\psi(z) = \frac{d}{dz} \log \varGamma(z)$ is the digamma function.

Proof. We compute each terms appearing in Theorem 4.4 for the test function $h_1(r)$ given by (5.2).

• Discrete spectrum: We denote the spectral side of Theorem 4.4 by $A_{\mathrm{spec}}(s;m)$. By the fact (5.3), we see that

$$A_{\mathrm{spec}}(s;m) = \sum_{j=0}^{\infty} h_1\bigl(\rho_j(m)\bigr) - \sum_{j=0}^{\infty} h_1\bigl(\rho_j(m-2)\bigr) + \delta_{m,4} h_1(i/2).$$

• Identity term: We denote the identity contribution by $A_{\mathrm{id}}(s)$. By the proof of Proposition 4.9 in [13],

$$\begin{aligned} A_{\mathrm{id}}(s) &= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}\Biggl\{\pi\tan\biggl(\pi\biggl(s-\frac{1}{2}\biggr)\biggr) + \sum_{h=1}^{2} c_h(s)\pi\tan(\pi\beta_h) \\ &\quad + \sum_{k=0}^{\infty}\Biggl[\frac{1}{s+k} - \frac{1}{s-1-k} + \sum_{h=1}^{2}\frac{c_h(s)}{\beta_h+\frac{1}{2}+k} - \sum_{h=1}^{2}\frac{c_h(s)}{\beta_h-\frac{1}{2}-k}\Biggr]\Biggr\} \\ &= 2\zeta_K(-1)\sum_{k=0}^{\infty}\Biggl[\frac{1}{s+k} + \sum_{h=1}^{2}\frac{c_h(s)}{\beta_h+\frac{1}{2}+k}\Biggr]. \end{aligned}$$

Here, we used the partial fractional expansion of $\cot(\pi z)$, the fact $c_1(s)+c_2(s)=-1$ and the formula by Siegel (see Theorem (1.1) in [7] or Proposition 5.1 in [4]):

$$\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{4\pi^2} = 2\zeta_K(-1). \tag{5.5}$$

• Elliptic term: We denote the elliptic contribution by $A_{\mathrm{ell}}(s;m)$. This is given by

$$\begin{aligned} A_{\mathrm{ell}}(s;m) &= -\sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}}\frac{ie^{-i\theta_1}e^{i(m-2)\theta_2}}{4\nu_R\sin\theta_1}\int_{-\infty}^{\infty} g_1(u)e^{-u/2}\biggl[\frac{e^u-e^{2i\theta_1}}{\cosh u-\cos 2\theta_1}\biggr]du \\ &= -\sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}}\frac{e^{i(m-2)\theta_2}}{\nu_R}\int_{0}^{\infty}\biggl[\frac{g_1(u)\cosh(u/2)}{\cosh u-\cos 2\theta_1}\biggr]du. \end{aligned}$$

By noting (10.29) and (10.31) in [15], we have

$$\begin{aligned} A_{\mathrm{ell}}(s;m) &= \frac{1}{2s-1}\sum_{j=1}^{N}\sum_{k=1}^{\nu_j-1}\sum_{l=0}^{\nu_j-1}\frac{\exp(i(m-2)(\pi ikt_j)/\nu_j)}{\nu_j^2}\frac{\sin((2l+1)\pi k/\nu_j)}{\sin(\pi k/\nu_j)}\psi\biggl(\frac{s+l}{\nu_j}\biggr) \\ &\quad + \sum_{h=1}^{2} c_h(s)\cdot\biggl\{\text{the same for } s=\frac{1}{2}+\beta_h\biggr\}. \end{aligned}$$

Next we use the following equality

$$
\begin{aligned}
&\sum_{k=1}^{\nu_j-1} e^{i(m-2)(\pi ikt_j)/\nu_j} \frac{\sin((2l+1)\pi k/\nu_j)}{\sin(\pi k/\nu_j)} \\
&= -\frac{1}{2}\left(\sum_{k=1}^{\nu_j-1} \frac{ie^{i(2\alpha_l(m,j)+1)\pi k/\nu_j}}{\sin(\pi k/\nu_j)} - \sum_{k=1}^{\nu_j-1} \frac{ie^{-i(2\overline{\alpha_l}(m,j)+1)\pi k/\nu_j}}{\sin(\pi k/\nu_j)}\right) \\
&= -\frac{1}{2}\left\{-\left(\nu_j-1-2\alpha_l(j,m)\right) - \left(\nu_j-1-2\overline{\alpha_l}(j,m)\right)\right\} \\
&= \nu_j - 1 - \alpha_l(j,m) - \overline{\alpha_l}(j,m).
\end{aligned}
$$

Here, the integers $\alpha_l(j,m), \overline{\alpha_l}(j,m) \in \{0,1,\dots \nu_j-1\}$ are defined in (2.1). The above equality is deduced from (see [3, p. 67]).

$$
\sum_{k=1}^{\nu-1} \frac{ie^{-i(2a+1)\pi k/\nu}}{\sin(\pi k/\nu)} = \nu - 1 - 2a \quad \left(a \in \{0,1,\dots \nu_j-1\}\right).
$$

Therefore, we have

$$
\begin{aligned}
A_{\mathrm{ell}}(s;m) = {} & \frac{1}{2s-1}\sum_{j=1}^{N}\sum_{l=0}^{\nu_j-1} \frac{\nu_j-1-\alpha_l(m,j)-\overline{\alpha_l}(m,j)}{\nu_j^2}\psi\left(\frac{s+l}{\nu_j}\right) \\
& + \sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\sum_{j=1}^{N}\sum_{l=0}^{\nu_j-1} \frac{\nu_j-1-\alpha_l(m,j)-\overline{\alpha_l}(m,j)}{\nu_j^2}\psi\left(\frac{\frac{1}{2}+\beta_h+l}{\nu_j}\right).
\end{aligned}
$$

• Hyperbolic–elliptic term: We denote the hyperbolic–elliptic contribution by $A_{\text{hyp-ell}}(s)$. This is given by

$$
\begin{aligned}
A_{\text{hyp-ell}}(s;m) &= -\frac{1}{2s-1}\sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2}-N(\gamma)^{-1/2}} N(\gamma)^{-(s-1/2)} e^{i(m-2)\omega} \\
&\quad - \sum_{h=1}^{2} c_h(s)\cdot\left\{\text{the same for } s = \frac{1}{2}+\beta_h\right\} \\
&= \frac{1}{2s-1}\frac{Z_K'(s;m)}{Z_K(s;m)} + \sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\frac{Z_K'(\frac{1}{2}+\beta_h;m)}{Z_K(\frac{1}{2}+\beta_h;m)}.
\end{aligned}
$$

The last equality is derived from (5.1).

• Parabolic plus scattering term: Since $m \geq 4$, $\operatorname{sgn}(m-1) - \operatorname{sgn}(m-3)$ vanishes and this term contributes zero. So, $A_{\mathrm{par/sct}}(s;m) \equiv 0$.

• Type 2 hyperbolic plus scattering term: We denote the type 2 plus scattering contribution by $A_{\mathrm{hyp2/sct}}(s;m)$. Then,

$$A_{\mathrm{hyp2/sct}}(s;m) = -\frac{2\log\varepsilon}{2s-1}\sum_{k=1}^{\infty}\varepsilon^{-k(2s-1)}\left(\varepsilon^{-k(m-1)} - \varepsilon^{-k(m-3)}\right) - \sum_{h=1}^{2} c_h(s)\cdot\left\{\text{the same for } s = \frac{1}{2}+\beta_h\right\}.$$

Nothing that

$$\sum_{k=1}^{\infty}\varepsilon^{-k(2s-1)}\varepsilon^{-k(m-1)} = \frac{\varepsilon^{-(2s+m-2)}}{1-\varepsilon^{-(2s+m-2)}} = \frac{-1}{2\log\varepsilon}\frac{d}{ds}\log\left(1-\varepsilon^{-(2s+m-2)}\right)^{-1}$$

for $\mathrm{Re}(s) > 1 - m/2$. For $\mathrm{Re}(s) > 2 - m/2$, therefore, we have

$$A_{\mathrm{hyp2/sct}}(s;m) = \frac{1}{2s-1}\frac{d}{ds}\log\left\{\frac{(1-\varepsilon^{-(2s+m-4)})}{(1-\varepsilon^{-(2s+m-2)})}\right\} + \sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\frac{d}{d\beta_h}\log\left\{\frac{(1-\varepsilon^{-(2\beta_h+m-3)})}{(1-\varepsilon^{-(2\beta_h+m-1)})}\right\}$$

The proof is finished. □

5.3. Analytic continuation of Selberg type zeta functions

We prove

Theorem 5.3. *For an even integer $m \geq 4$, the Selberg zeta function $Z_K(s;m)$, originally defined for $\mathrm{Re}(s) > 1$, has an analytic continuation to the whole complex plane as a meromorphic function.*

(1) *$Z_K(s;m)$ has zeros at*
 $s = \frac{1}{2} \pm i\rho_j(m)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_m^{(2)})$,
 $s = 1 - \frac{m}{2} + \frac{\pi i k}{\log\varepsilon}$ of order 1 for $k \in \mathbb{Z}$.
(2) *$Z_K(s;m)$ has poles at*
 $s = \frac{1}{2} \pm i\rho_j(m-2)$ of order equal to the multiplicity of the eigenvalue $\frac{1}{4} + \rho_j(m-2)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_{m-2}^{(2)})$,
 $s = 2 - \frac{m}{2} + \frac{\pi i k}{\log\varepsilon}$ of order 1 for $k \in \mathbb{Z}$.
(3) *$Z_K(s;m)$ has zeros or poles (according to their orders are positive or negative) at $s = -k$ ($k \in \mathbb{N}\cup\{0\}$) of order $(2k+1)E(X_K) + 2\sum_{j=1}^{N}[k/\nu_j] - 2kN - \sum_{j=1}^{N}\beta_{k,j}(m)$.*
(4) *If $m = 4$, $Z_K(s,m)$ has additional simple zeros at $s = 0$ and $s = 1$.*

Here,

$$\mathrm{Ker}\big(\Lambda_q^{(2)}\big) = \left\{ f \in L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,q)\big) \;\middle|\; \Delta_q^{(2)} f = \frac{q}{2}\left(1-\frac{q}{2}\right) f \right\}$$

for $q = m$ or $m-2$, and $E(X_K)$ denotes the Euler characteristic of the Hilbert modular surface X_K and the definition of the integers $\beta_{j,k}(m)$ will be given in (5.6). When the location of two zeros or poles coincide, the orders of them are added.

Proof. To get a meromorphic extension of $Z_K(s;m)$, we show that the logarithmic derivative of $Z_K(s;m)$ has a meromorphic extension to the whole complex plane and its poles are all simple with integral residues. By Theorem 5.2, it is easy to see that $(2s-1)A_{\mathrm{spec}}(s;m)$ and $-(2s-1)A_{\mathrm{hyp2/sct}}(s;m)$ are meromorphic over the complex plane and their poles are all simple with integral residues. So, we consider the function:

$$g(s;m) := -(2s-1)\big(A_{\mathrm{id}}(s) + A_{\mathrm{ell}}(s;m)\big)$$

We see that $g(s;m)$ is also meromorphic and only have simple poles at $s = -k$ for $k \in \mathbb{N} \cup \{0\}$. By the identity

$$\frac{1}{\nu}\sum_{l=0}^{\nu-1} \psi\left(\frac{s+l}{\nu}\right) = \psi(s) - \log\nu,$$

we have

$$\begin{aligned} E_j(s) &:= \frac{1}{2s-1}\sum_{l=0}^{\nu_j-1} \frac{\nu_j - 1 - \alpha_l(m,j) - \overline{\alpha}_l(m,j)}{\nu_j^2}\psi\left(\frac{s+l}{\nu_j}\right) \\ &= \frac{1}{\nu_j}\big\{\psi(s) - \log\nu_j\big\} + \frac{1}{2s-1}\sum_{l=0}^{\nu_j-1} \frac{\nu_j - 2s - \alpha_l(m,j) - \overline{\alpha}_l(m,j)}{\nu_j^2}\psi\left(\frac{s+l}{\nu_j}\right). \end{aligned}$$

Thus, for $k \in \mathbb{N} \cup \{0\}$, (we write $k = l + \nu_j n$ with $l = 0, 1, \ldots, \nu_j - 1$)

$$-\mathrm{Res}_{s=-k}(2s-1)E_j(s) = \frac{2k+1}{\nu_j} + 2\left[\frac{k}{\nu_j}\right] + 1 - \frac{\alpha_l(m,j) + \overline{\alpha}_l(m,j) - 2l}{\nu_j},$$

with $l = k - \nu_j[k/\nu_j]$. Put

$$\beta_{k,j}(m) := \frac{\alpha_l(m,j) + \overline{\alpha}_l(m,j) - 2l}{\nu_j} \quad \text{with } l = k - \nu_j\left[\frac{k}{\nu_j}\right]. \tag{5.6}$$

We see that $\beta_{k,j}(m) \in \mathbb{Z}$ since $\alpha_l(m,j) + \overline{\alpha}_l(m,j) \equiv 2l \pmod{\nu_j}$ by (2.1).

Therefore, we have

$$\begin{aligned}\operatorname{Res}_{s=-k} g(s;m) &= (2k+1)\cdot 2\zeta_K(-1) - (2k+1)\sum_{j=1}^{N}\frac{1}{\nu_j} + \sum_{j=1}^{N}\left(2\left[\frac{k}{\nu_j}\right]+1\right) \\ &\quad - \sum_{j=1}^{N}\beta_{k,j}(m) \\ &= (2k+1)E(X_K) + 2\sum_{j=1}^{N}\left[\frac{k}{\nu_j}\right] - 2kN - \sum_{j=1}^{N}\beta_{k,j}(m).\end{aligned}$$

Here, $E(X_K)$ is the Euler characteristic of the Hilbert modular surface X_K and we used the formula (see [7, Theorem (1.2), p. 60]):

$$E(X_K) = 2\zeta_K(-1) + \sum_{j=1}^{N}\frac{\nu_j - 1}{\nu_j}.$$

Hence the residues of $g(s;m)$ are all integers. The rest of proof is clear. □

5.4. Functional equation of Selberg type zeta functions

Theorem 5.4. *Let $m \geq 4$ be an even integer. The function $Z_K(s;m)$ satisfies the following functional equation*

$$\hat{Z}_K(s;m) = \hat{Z}_K(1-s;m).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;m) := Z_K(s;m)Z_{\mathrm{id}}(s)Z_{\mathrm{ell}}(s;m)Z_{\mathrm{hyp2/sct}}(s;m)$$

with

$$\begin{aligned}Z_{\mathrm{id}}(s) &:= \big(\Gamma_2(s)\Gamma_2(s+1)\big)^{2\zeta_K(-1)} \\ Z_{\mathrm{ell}}(s;m) &:= \prod_{j=1}^{N}\prod_{l=0}^{\nu_j-1}\Gamma\left(\frac{s+l}{\nu_j}\right)^{\frac{\nu_j-1-\alpha_l(m,j)-\overline{\alpha_l}(m,j)}{\nu_j}} \\ Z_{\mathrm{hyp2/sct}}(s;m) &:= \zeta_\varepsilon\left(s+\frac{m}{2}-1\right)\zeta_\varepsilon\left(s+\frac{m}{2}-2\right)^{-1},\end{aligned}$$

where, $\Gamma_2(z)$ is the double Gamma function (for definition, we refer to [17] or [11, Definition 4.10, p. 751]), $\nu_1, \nu_2, \ldots, \nu_N$ are the orders of the elliptic fixed points in X_K and the integers $\alpha_l(m,j), \overline{\alpha_l}(m,j) \in \{0,1,\ldots,\nu_j-1\}$ was defined in (2.1), $\zeta_\varepsilon(s) := (1-\varepsilon^{-2s})^{-1}$ and ε is the fundamental unit of K.

Proof. Starting from the formula in Theorem 5.2, we compute the difference of the both sides at s and $1-s$. We see that

$$\begin{aligned}&2\zeta_K(-1)\cdot(2s-1)\pi\cot(\pi s)+\left(\frac{Z_K'(s;m)}{Z_K(s;m)}+\frac{Z_K'(1-s;m)}{Z_K(1-s;m)}\right)\\&\quad+\left(\frac{Z_{\mathrm{ell}}'(s)}{Z_{\mathrm{ell}}(s)}+\frac{Z_{\mathrm{ell}}'(1-s)}{Z_{\mathrm{ell}}(1-s)}\right)+\left(\frac{Z_{\mathrm{hyp2/sct}}'(s)}{Z_{\mathrm{hyp2/sct}}(s)}+\frac{Z_{\mathrm{hyp2/sct}}'(1-s)}{Z_{\mathrm{hyp2/sct}}(1-s)}\right)\\&=0,\end{aligned}$$

by the partial fractional expansion: $\pi\cot(\pi s)=\sum_{k=0}^{\infty}[\frac{1}{s+k}-\frac{1}{1-s+k}]$. Let $I(s):=2\zeta_K(-1)\cdot(2s-1)\pi\cot(\pi s)$. It is known that the double sine function $S_2(z):=\Gamma_2(2-z)\Gamma_2(z)^{-1}$ satisfies the differential equation (see [17, Theorem 2.15, p. 860]):

$$\frac{d}{dz}\log S_2(z)=-\pi(z-1)\cot(\pi z).$$

Therefore,

$$\begin{aligned}I(s)&=-2\zeta_K(-1)\frac{d}{ds}\log\bigl(S_2(s)S_2(s+1)\bigr)=-2\zeta_K(-1)\frac{d}{ds}\log\left(\frac{\Gamma_2(2-s)}{\Gamma_2(s)}\frac{\Gamma_2(1-s)}{\Gamma_2(s+1)}\right)\\&=\frac{Z_{\mathrm{id}}'(s)}{Z_{\mathrm{id}}(s)}+\frac{Z_{\mathrm{id}}'(1-s)}{Z_{\mathrm{id}}(1-s)}.\end{aligned}$$

Integrating and exponentiating, we obtain the desired functional equation. □

6. Ruelle type zeta functions and applications

6.1. Ruelle type zeta functions

We consider the following Ruelle type zeta function

Definition 6.1 *(Ruelle type zeta function for Γ_K).* For $\mathrm{Re}(s)>1$, the Ruelle type zeta function for Γ_K is defined by the following absolutely convergent Euler product:

$$R_K(s):=\prod_{(p,p')}\bigl(1-N(p)^{-s}\bigr)^{-1}.$$

Here, (p,p') run through the set of primitive hyperbolic–elliptic Γ_K-conjugacy classes of Γ_K, and (p,p') is conjugate in $\mathrm{PSL}(2,\mathbb{R})^2$ to

$$(p,p')\sim\left(\begin{pmatrix}N(p)^{1/2}&0\\0&N(p)^{-1/2}\end{pmatrix},\begin{pmatrix}\cos\omega&-\sin\omega\\\sin\omega&\cos\omega\end{pmatrix}\right).$$

Here, $N(p)>1$, $\omega\in(0,\pi)$ and $\omega\notin\pi\mathbb{Q}$.

We note that the following relation between the Ruelle type zeta function and the Selberg type zeta function for Γ_K.

Lemma 6.2. *For* $\mathrm{Re}(s) > 1$, *we have*

$$R_K(s) = \frac{Z_K(s;2)}{Z_K(s+1;2)}.$$

Proof. For $\mathrm{Re}(s) > 1$, we have

$$\frac{Z_K(s;2)}{Z_K(s+1;2)} = \frac{\prod_{(p,p')} \prod_{k=0}^{\infty} (1 - N(p)^{-(s+k)})^{-1}}{\prod_{(p,p')} \prod_{k=0}^{\infty} (1 - N(p)^{-(s+k+1)})^{-1}} = R_K(s). \qquad \square$$

To get a meromorphic extension of $R_K(s)$, we consider meromorphic extension of the Selberg type zeta function $Z_K(s;2)$. For this, we recall Theorem 4.4, the double differences of the trace formula for the weight $(0,2)$.

Corollary 6.3 *(Double differences of STF for $L^2(\Gamma_K\backslash\mathbb{H}^2;(0,2))$). Let $m = 2$. We have*

$$\begin{aligned}
&2\sum_{j=0}^{\infty} h_1\big(\rho_j(2)\big) - 2h_1\left(\frac{i}{2}\right) \\
&= \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2} \int_{-\infty}^{\infty} r h_1(r) \tanh(\pi r)\, dr \\
&\quad - \sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}} \frac{ie^{-i\theta_1}}{4\nu_R \sin\theta_1} \int_{-\infty}^{\infty} g_1(u)\, e^{-u/2} \left[\frac{e^u - e^{2i\theta_1}}{\cosh u - \cos 2\theta_1}\right] du \\
&\quad - \sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0) g_1(\log N(\gamma))}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} - 2\log\varepsilon g_1(0) - 4\log\varepsilon \sum_{k=1}^{\infty} g_1(2k\log\varepsilon)\varepsilon^{-k}.
\end{aligned}$$

Here, $\{1/4 + \rho_j(2)^2\}_{j=0}^{\infty}$ is the set of eigenvalues of the Laplacian $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_2^{(2)})$.

Proof. We note that $\mathrm{Ker}(\Lambda_0^{(2)}) = \mathrm{Ker}(K_0^{(2)}) = \mathbb{C}$, and other two kernel spaces appearing in Theorem 4.4 for $m = 2$ are given by

$$\begin{aligned}
\mathrm{Ker}\big(\Lambda_2^{(2)}\big) &= \big\{f \in L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,2)\big) \bigm| \Delta_2^{(2)} f = 0\big\}, \\
\mathrm{Ker}\big(K_{-2}^{(2)}\big) &= \big\{f \in L^2_{\mathrm{dis}}\big(\Gamma_K\backslash\mathbb{H}^2;(0,-2)\big) \bigm| \Delta_{-2}^{(2)} f = 0\big\}.
\end{aligned}$$

The complex conjugation map $\iota: \mathrm{Ker}(\Lambda_2^{(2)}) \to \mathrm{Ker}(K_{-2}^{(2)})$ given by $\iota(f) := \bar{f}$ is onto. Hence, we see that two sets $\{1/4+\rho_j(2)^2\}_{j=0}^{\infty}$ and $\{1/4+\mu_j(-2)^2\}_{j=0}^{\infty}$ are identical. The rest is clear by Theorem 4.4. $\square$

Theorem 6.4 *(Double differences of STF for the test function h_1 (see (5.2)) with the weight $(0,2)$).*

$$
\begin{aligned}
&2\sum_{j=0}^{\infty}\left[\frac{1}{\rho_j(2)^2+(s-\frac{1}{2})^2}+\sum_{h=1}^{2}\frac{c_h(s)}{\rho_j(2)^2+\beta_h^2}\right]-2\left[\frac{1}{s(s-1)}+\sum_{h=1}^{2}\frac{c_h(s)}{\beta_h^2-\frac{1}{4}}\right]\\
&\quad=2\zeta_K(-1)\sum_{k=0}^{\infty}\left[\frac{1}{s+k}+\sum_{h=1}^{2}\frac{c_h(s)}{\beta_h+\frac{1}{2}+k}\right]+\frac{1}{2s-1}\frac{Z_K'(s;2)}{Z_K(s;2)}\\
&\qquad+\sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\frac{Z_K'(\frac{1}{2}+\beta_h;2)}{Z_K(\frac{1}{2}+\beta_h;2)}\\
&\qquad+\frac{1}{2s-1}\sum_{j=1}^{N}\sum_{l=0}^{\nu_j-1}\frac{\nu_j-1-2l}{\nu_j^2}\psi\left(\frac{s+l}{\nu_j}\right)\\
&\qquad+\sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\sum_{j=1}^{N}\sum_{l=0}^{\nu_j-1}\frac{\nu_j-1-2l}{\nu_j^2}\psi\left(\frac{\frac{1}{2}+\beta_h+l}{\nu_j}\right)\\
&\qquad+\frac{1}{2s-1}\frac{d}{ds}\log\left(\varepsilon^{-2s}\right)+\sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\frac{d}{d\beta_h}\log\left(\varepsilon^{-(2\beta_h+1)}\right)\\
&\qquad+\frac{1}{2s-1}\frac{d}{ds}\log\left\{\frac{1}{(1-\varepsilon^{-2s})^2}\right\}+\sum_{h=1}^{2}\frac{c_h(s)}{2\beta_h}\frac{d}{d\beta_h}\log\left\{\frac{1}{(1-\varepsilon^{-(2\beta_h+1)})^2}\right\}.
\end{aligned}
$$

Proof. By Corollary 6.3 and the same computation in Theorem 5.2. □

Theorem 6.5. *The Selberg zeta function $Z_K(s;2)$, originally defined for $\mathrm{Re}(s)>1$, has an analytic continuation to the whole complex plane as a meromorphic function.*

(1) *$Z_K(s;2)$ has a double pole at $s=1$.*
(2) *$Z_K(s;2)$ has zeros at*
$s=\frac{1}{2}\pm i\rho_j(2)$ of order equal to twice the multiplicity of the eigenvalue $\frac{1}{4}+\rho_j(2)^2$ of $\Delta_0^{(1)}$ acting on $\mathrm{Ker}(\Lambda_2^{(2)})=\{f\in L^2_{\mathrm{dis}}(\Gamma_K\backslash\mathbb{H}^2;(0,2))\mid \Delta_2^{(2)}f=0\}$.
(3) *$Z_K(s;2)$ has zeros at $s=\pm\frac{k\pi i}{\log\varepsilon}$ $(k\in\mathbb{N})$ of order 2.*
(4) *$Z_K(s;2)$ has a zero at $s=0$ of order $E(X_K)$.*
(5) *$Z_K(s;2)$ has zeros or poles (according to their orders are positive or negative) at $s=-k$ $(k\in\mathbb{N})$ of order $(2k+1)E(X_K)+2\sum_{j=1}^{N}[k/\nu_j]-2kN$.*

Here, $E(X_K)$ denotes the Euler characteristic of the Hilbert modular surface X_K. When the location of two zeros or poles coincide, the orders of them are added.

Proof. By Theorem 6.4 and the same proof of Theorem 5.3. □

Theorem 6.6. *The Selberg type zeta function $Z_K(s;m)$ satisfies the following functional equation*

$$\hat{Z}_K(s;2) = \hat{Z}_K(1-s;2).$$

Here the completed zeta function $\hat{Z}_K(s,m)$ is given by

$$\hat{Z}_K(s;2) := Z_K(s;2) Z_{\mathrm{id}}(s) Z_{\mathrm{ell}}(s;2) Z_{\mathrm{par/sct}}(s;2) Z_{\mathrm{hyp2/sct}}(s;2)$$

with

$$Z_{\mathrm{id}}(s) := \big(\Gamma_2(s)\Gamma_2(s+1)\big)^{2\zeta_K(-1)}, \qquad Z_{\mathrm{ell}}(s;2) := \prod_{j=1}^{N} \prod_{l=0}^{\nu_j-1} \Gamma\left(\frac{s+l}{\nu_j}\right)^{\frac{\nu_j-1-2l}{\nu_j}},$$

$$Z_{\mathrm{par/sct}}(s;2) := \varepsilon^{-2s}, \qquad Z_{\mathrm{hyp2/sct}}(s;2) := \zeta_\varepsilon(s)^2 = \big(1-\varepsilon^{-2s}\big)^{-2}.$$

Proof. By using Theorem 6.4, the proof is the same as in Theorem 5.3. □

Theorem 6.7. *The function $R_K(s)$ has a meromorphic continuation to the whole $\mathbb{C}$. $R_K(s)$ has double pole at $s=1$ and nonzero for* $\mathrm{Re}(s) \geq 1$.

Proof. By Theorem 6.5 and Lemma 6.2. □

Theorem 6.8. *The function $R_K(s)$ has the following functional equation*

$$R_K(s)R_K(-s) = (-1)^{E(X_K)} \big(2\sin(\pi s)\big)^{2E(X_K)} \prod_{j=1}^{N} \left(\frac{\sin(\pi s/\nu_j)}{\sin(\pi s)}\right)^2 \cdot \left(\frac{\zeta_\varepsilon(s-1)\zeta_\varepsilon(s+1)}{\zeta_\varepsilon(s)^2}\right)^2, \tag{6.1}$$

where, $\zeta_\varepsilon(s) = (1-\varepsilon^{-2s})^{-1}$, N is the number of elliptic fixed points in X_K.

Proof. By Theorem 6.6, we have

$$\begin{aligned} R_K(s)R_K(-s) &= \frac{Z_K(s;2)}{Z_K(s+1;2)} \frac{Z_K(-s;2)}{Z_K(-s+1;2)} = \frac{Z_K(s;2)}{Z_K(1-s;2)} \frac{Z_K(-s;2)}{Z_K(1+s;2)} \\ &= B_K(s)C_K(s). \end{aligned}$$

The functions $B_K(s)$ and $C_K(s)$ are given as follows.

$$B_K(s) := \frac{Z_{\mathrm{id}}(1+s)Z_{\mathrm{id}}(1-s)}{Z_{\mathrm{id}}(s)Z_{\mathrm{id}}(-s)} \frac{Z_{\mathrm{ell}}(1+s;2)Z_{\mathrm{ell}}(1-s;2)}{Z_{\mathrm{ell}}(s;2)Z_{\mathrm{ell}}(-s;2)},$$

$$C_K(s) := \frac{Z_{\mathrm{par/sct}}(1+s;2)Z_{\mathrm{par/sct}}(1-s;2)}{Z_{\mathrm{par/sct}}(s;2)Z_{\mathrm{par/sct}}(-s;2)} \frac{Z_{\mathrm{hyp2/sct}}(1+s;2)Z_{\mathrm{hyp2/sct}}(1-s;2)}{Z_{\mathrm{hyp2/sct}}(s;2)Z_{\mathrm{hyp2/sct}}(-s;2)}.$$

We can easily check that

$$C_K(s) = \left(\frac{\zeta_\varepsilon(s-1)\zeta_\varepsilon(s+1)}{\zeta_\varepsilon(s)^2}\right)^2.$$

Let us compute $B_K(s)$. Put

$$\Xi(s) := \frac{\Gamma_2(s+1)\Gamma_2(s+2)\Gamma_2(1-s)\Gamma_2(2-s)}{\Gamma_2(s)\Gamma_2(s+1)\Gamma_2(-s)\Gamma_2(1-s)}, \qquad G_\nu(s) := \prod_{j=0}^{\nu-1}\Gamma\left(\frac{s+l}{\nu}\right)^{\frac{\nu-1-2l}{\nu}}.$$

Then we see that

$$B_K(s) = \Xi(s)^{E(X_K)}\prod_{j=1}^{N}\left[\frac{G_{\nu_j}(1+s)G_{\nu_j}(1-s)}{G_{\nu_j}(s)G_{\nu_j}(-s)}\Xi(s)^{-\frac{\nu_j-1}{\nu_j}}\right]. \tag{6.2}$$

By using $\Gamma_2(s+1)/\Gamma_2(s) = \sqrt{2\pi}\Gamma(s)^{-1}$ (see [17] or [11, Proposition 4.11]), we have

$$\Xi(s) = (2\pi)^2\big(\Gamma(s)\Gamma(s+1)\Gamma(-s)\Gamma(-s+1)\big)^{-1} = -4\sin^2(\pi s). \tag{6.3}$$

By using the multiplication formula for the Gamma function (see [12, 8.335]), we have

$$\begin{aligned} G_\nu(1+s)G_\nu(s)^{-1} &= \Gamma\left(\frac{s}{\nu}\right)^{-\frac{\nu-1}{\nu}}\left[\prod_{l=1}^{\nu-1}\Gamma\left(\frac{s+l}{\nu}\right)^{\frac{2}{\nu}}\right]\Gamma\left(\frac{s+\nu}{\nu}\right)^{\frac{1-\nu}{\nu}} \\ &= \Gamma\left(\frac{s}{\nu}\right)^{\frac{1-\nu}{\nu}}\Gamma\left(1+\frac{s}{\nu}\right)^{\frac{1-\nu}{\nu}}\left[\Gamma\left(\frac{s}{\nu}\right)^{-1}(2\pi)^{\frac{\nu-1}{2}}\nu^{1/2-s}\Gamma(s)\right]^{\frac{2}{\nu}}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\frac{G_\nu(1+s)}{G_\nu(s)}\frac{G_\nu(1-s)}{G_\nu(-s)}\Xi(s)^{-\frac{\nu-1}{\nu}} \\ &= \Gamma\left(\frac{s}{\nu}\right)^{\frac{-1-\nu}{\nu}}\Gamma\left(1+\frac{s}{\nu}\right)^{\frac{1-\nu}{\nu}}\Gamma\left(\frac{-s}{\nu}\right)^{\frac{-1-\nu}{\nu}}\Gamma\left(1+\frac{-s}{\nu}\right)^{\frac{1-\nu}{\nu}}\big[\nu\Gamma(s)\Gamma(-s)\big]^{\frac{2}{\nu}} \\ &\quad\times\big(\Gamma(s)\Gamma(s+1)\Gamma(-s)\Gamma(-s+1)\big)^{\frac{\nu-1}{\nu}} \\ &= \left[\frac{\Gamma(s)\Gamma(1-s)}{\Gamma(s/\nu)\Gamma(1-\nu/s)}\right]^2 = \left(\frac{\sin(\pi s/\nu)}{\sin(\pi s)}\right)^2. \end{aligned} \tag{6.4}$$

Substituting (6.3) and (6.4) into (6.2), we complete the proof. □

We can obtain an explicit formula of the leading term of $R_K(s)$ at $s = 0$. Let n_0 denote an integer such that $\lim_{s\to 0} s^{-n_0}R_K(s)$ is a nonzero finite value and

$$R_K^*(0) := \lim_{s\to 0} s^{-n_0}R_K(s).$$

Theorem 6.9. *The following equalities hold.*

$$n_0 = E(X_K) + 2$$

and

$$\left|R_K^*(0)\right| = (2\pi)^{E(X_K)} \prod_{j=1}^{N} \nu_j^{-1} \frac{(2\varepsilon \log \varepsilon)^2}{(\varepsilon^2 - 1)^2}.$$

Proof. By Theorem 6.8, we can compute

$$\lim_{s\to 0} \frac{R_K(s)R_K(-s)}{s^{2(E(X_K)+2)}} = (-1)^{E(X_K)}(2\pi)^{2E(X_K)} \prod_{j=1}^{N} \nu_j^{-2} \left(\frac{(2\log \varepsilon)^2}{(1-\varepsilon^2)(1-\varepsilon^{-2})} \right)^2.$$

The rest is clear. □

6.2. Weyl's law

As an application of the double difference of the trace formula for Γ_K with the weight $(0, 2)$ (Corollary 6.3), we have the following "Weyl's law".

Proposition 6.10 *(Weyl's law I). Let $T > 0$. We consider the following counting function:*

$$N(T) := \#\left\{ j \;\middle|\; 1/4 + \rho_j(2)^2 \le T \right\}.$$

Then we have

$$N(T) \sim \frac{\mathrm{vol}(\Gamma_K \backslash \mathbb{H}^2)}{16\pi^2} T \quad (T \to \infty). \tag{6.5}$$

Proof. For any $\beta > 0$, the test function $h_1(r) = e^{-\beta r^2}$ is admissible in Corollary 6.3. The Fourier transform is

$$g_1(u) = \frac{e^{-u^2/(4\beta)}}{\sqrt{4\pi\beta}},$$

so we have

$$2\sum_{j=0}^{\infty} e^{-\beta(1/4+\rho_j(2)^2)} - 2$$
$$= \frac{\mathrm{vol}(\Gamma_K \backslash \mathbb{H}^2)}{8\pi^2} \int_{-\infty}^{\infty} e^{-\beta(1/4+r^2)} r \tanh(\pi r)\, dr$$

$$- \frac{e^{-\beta/4}}{\sqrt{4\pi\beta}} \sum_{R(\theta_1,\theta_2)\in\Gamma_{\mathrm{E}}} \frac{ie^{-i\theta_1}}{4\nu_R \sin\theta_1} \int_{-\infty}^{\infty} e^{-u^2/(4\beta)} e^{-u/2} \left[\frac{e^u - e^{2i\theta_1}}{\cosh u - \cos 2\theta_1}\right] du$$
$$- \frac{e^{-\beta/4}}{\sqrt{4\pi\beta}} \sum_{(\gamma,\omega)\in\Gamma_{\mathrm{HE}}} \frac{\log N(\gamma_0)}{N(\gamma)^{1/2} - N(\gamma)^{-1/2}} e^{-(\log N(\gamma))^2/(4\beta)}$$
$$- \frac{e^{-\beta/4}}{\sqrt{4\pi\beta}} \left\{2\log\varepsilon + 4\log\varepsilon \sum_{k=1}^{\infty} e^{-(2k\log\varepsilon)^2/(4\beta)} \varepsilon^{-k}\right\}.$$

Since $\tanh(\pi r) = 1 + O(e^{-2\pi|r|})$ for any $r \in \mathbb{R}$, we obtain

$$2\sum_{j=0}^{\infty} e^{-\beta(1/4+\rho_j(2)^2)} = \frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2\beta} - \frac{2\log\varepsilon}{\sqrt{4\pi\beta}} + O(1) \quad (\beta \to +0).$$

By a classical Tauberian theorem, we complete the proof. □

We remark that the above proposition is enough to prove "a prime geodesic type theorem" for the set of primitive hyperbolic–elliptic conjugacy classes of Γ_K in the next subsection.

Besides we can prove a more general Weyl's law by using Theorem 4.1, the differences (not double differences) of the trace formula for Γ_K with the weight $(0, m)$.

Theorem 6.11 *(Weyl's law II). Let $m \in 2\mathbb{Z}$ and $T > 0$. We consider the following two counting functions:*

$$N_m^+(T) := \#\{j \mid 1/4 + \rho_j(m)^2 \le T\},$$
$$N_m^-(T) := \#\{j \mid 1/4 + \mu_j(m-2)^2 \le T\}.$$

Then we have

$$\text{If } m \ge 2, \text{ then} \quad N_m^+(T) \sim (m-1)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2} T \quad (T \to \infty), \tag{6.6}$$
$$\text{If } m \le 0, \text{ then} \quad N_m^-(T) \sim (1-m)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2} T \quad (T \to \infty). \tag{6.7}$$

Proof. We take the test function $h_1(r) = e^{-\beta r^2}$ $(\beta > 0)$ in Theorem 4.1. Then we obtain (by the same computation as in the proof of Proposition 6.10)

$$\sum_{j=0}^{\infty} e^{-\beta(1/4+\rho_j(m)^2)} - \sum_{j=0}^{\infty} e^{-\beta(1/4+\mu_j(m-2)^2)}$$
$$= (m-1)\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{16\pi^2\beta} - \mathrm{sgn}(m-1)\frac{\log\varepsilon}{\sqrt{4\pi\beta}} + O(1) \quad (\beta \to +0).$$

The rest is clear. □

6.3. Prime geodesic theorem

We can show the following asymptotic formulas for counting functions of $\mathrm{P}\Gamma_{\mathrm{HE}}$, the set of primitive hyperbolic–elliptic Γ_K-conjugacy classes of Γ_K, by Corollary 6.3 and Proposition 6.10.

Theorem 6.12 *(Prime geodesic theorem). For $X \geq 1$,*

$$\sum_{\substack{(p,p')\in \mathrm{P}\Gamma_{\mathrm{HE}} \\ N(p)\leq X}} \log N(p) = 2X - 2\sum_{1/2<s_j(2)<1} \frac{X^{s_j(2)}}{s_j(2)} + O\bigl(X^{3/4}\bigr), \tag{6.8}$$

$$\sum_{\substack{(p,p')\in \mathrm{P}\Gamma_{\mathrm{HE}} \\ N(p)\leq X}} 1 = 2\,\mathrm{Li}(X) - 2\sum_{1/2<s_j(2)<1} \mathrm{Li}\bigl(X^{s_j(2)}\bigr) + O\bigl(X^{3/4}/\log X\bigr), \tag{6.9}$$

where, $s_j(2) := 1/2 - i\rho_j(2)$ and $\mathrm{Li}(x) := \int_2^x 1/\log t\,dt$. (The condition $1/2 < s_j(2) < 1$ implies that $\rho_j(2)^2$ is negative. See (2.15).)

Proof. We follow the same procedure as in Iwaniec [15]. Let us begin with the same test function on [15, p. 155] or [16, p. 401], given by

$$g_1(u) = 2\cosh(u/2)\,q(u),$$

where, $q(u)$ is even, smooth, supported on $|u| \leq \log(X+Y)$, and such that $0 \leq q(u) \leq 1$ and $q(u) = 1$ if $|x| \leq \log X$. The parameters $X \geq Y \geq 1$ will be chosen later. For $s = 1/2 + ir$ in the segment $1/2 < s \leq 1$ we have

$$h_1(r) = \int_{-\infty}^{\infty} \bigl(e^{su} + e^{(1-s)u}\bigr) q(u)\,du = s^{-1}X^s + O\bigl(Y + X^{1/2}\bigr),$$

and for s on the line $\mathrm{Re}(s) = 1/2$ we get by partial integration that

$$h_1(r) \ll |s|^{-1}X^{1/2}\min\bigl\{1, |s|^{-2}T^2\bigr\},$$

where $T = XY^{-1}$. By using Proposition 6.10, the spectral side of Corollary 6.3 becomes (by the same method as in [20, pp. 305–307])

$$2\sum_{j=0}^{\infty} h_1\bigl(\rho_j(2)\bigr) - 2h_1(i/2) = -2X + 2\sum_{1/2<s_j(2)<1} \frac{X^{s_j(2)}}{s_j(2)} + O\bigl(Y + X^{1/2}T\bigr).$$

On the geometric side, the identity term contributes

$$\frac{\mathrm{vol}(\Gamma_K\backslash\mathbb{H}^2)}{8\pi^2}\int_{-\infty}^{\infty} rh_1(r)\tanh(\pi r)\,dr \ll X^{1/2}T$$

and the elliptic term, the parabolic plus scattering and the type 2 hyperbolic plus scattering terms contribute no more than the above bound. Gathering these estimates, we arrive at

$$-\sum_{(p,p')\in \mathrm{P}\Gamma_{\mathrm{HE}}} q\bigl(\log N(p)\bigr) = -2X + 2\sum_{1/2<s_j(2)<1}\frac{X^{s_j(2)}}{s_j(2)} + O\bigl(Y + X^{1/2}T\bigr). \quad (6.10)$$

Subtracting (6.10) from that for $X+Y$ in place of X, we deduce that

$$\sum_{X<N(p)<X+Y}\log N(p) \ll Y + X^{1/2}T.$$

Hence, we can drop the excess over $N(p)\le X$ within the error term in (6.10). As usual we choose $Y = X^{3/4}$ to minimize the error term. We completes the proof. □

6.4. Binary quadratic forms over the ring of real quadratic integers

We denote by $\mathcal{D}$ the set of discriminants of integral binary quadratic forms, that is,

$$\mathcal{D} := \bigl\{d\in\mathbb{Z}\bigm| d\equiv 0,1 \pmod 4,\ d \text{ not a square},\ d>0\bigr\}.$$

For each $d\in\mathcal{D}$, let $h(d)$ denote the number of inequivalent primitive binary quadratic forms of discriminant d, and let (x_d, y_d) be the fundamental solution of the Pellian equation $x^2-dy^2=4$ over $\mathbb{Z}$. Put

$$\varepsilon_d := \frac{x_d+\sqrt{d}y_d}{2}.$$

By using the prime geodesic theorem for $\mathrm{PSL}(2,\mathbb{Z})$, Sarnak [23] deduced the following theorem on the average behavior of $h(d)$.

Theorem 6.13. *(See Sarnak [23, Theorem 2.1].) For $x\ge 2$, we have*

$$\sum_{\substack{d\in\mathcal{D}\\ \varepsilon_d\le x}} h(d)\log\varepsilon_d = \frac{x^2}{2} + O\bigl(x^{3/2}(\log x)^3\bigr)\quad (x\to\infty).$$

$$\sum_{\substack{d\in\mathcal{D}\\ \varepsilon_d\le x}} h(d) = \mathrm{Li}\bigl(x^2\bigr) + O\bigl(x^{3/2}(\log x)^2\bigr)\quad (x\to\infty).$$

Here, $\mathrm{Li}(x)=\int_2^x 1/\log t\,dt$*.*

Let us consider a generalization of Theorem 6.13 to that for class numbers of indefinite binary quadratic forms over the real quadratic integer ring $\mathcal{O}_K$. Put

$$\mathcal{D}_{+-} := \left\{d \in \mathcal{O}_K \;\middle|\; \exists b \in \mathcal{O}_K \text{ s.t. } d \equiv b^2 \pmod 4,\ d \text{ not a square in } \mathcal{O}_K,\ d > 0,\ d' < 0\right\}.$$

For each $d \in \mathcal{D}_{+-}$, let $h_K(d)$ denote the number of inequivalent primitive binary quadratic forms of discriminant d over $\mathcal{O}_K$, and let $(x_d, y_d) \in \mathcal{O}_K \times \mathcal{O}_K$ be the fundamental solution of the Pellian equation $x^2 - dy^2 = 4$. Put

$$\varepsilon_K(d) := \frac{x_d + \sqrt{d} y_d}{2}.$$

By using Theorem 6.12, we can deduce the following theorem on the average behavior of $h_K(d)$.

Theorem 6.14. *For $x \geq 2$, we have*

$$\sum_{\substack{d \in \mathcal{D}_{+-} \\ \varepsilon_K(d) \leq x}} h_K(d) \log \varepsilon_K(d) = x^2 - \sum_{1/2 < s_j(2) < 1} \frac{X^{2s_j(2)}}{s_j(2)} + O\left(x^{3/2}\right) \quad (x \to \infty), \tag{6.11}$$

$$\sum_{\substack{d \in \mathcal{D}_{+-} \\ \varepsilon_K(d) \leq x}} h_K(d) = 2\,\mathrm{Li}\left(x^2\right) - 2 \sum_{1/2 < s_j(2) < 1} \mathrm{Li}\left(x^{2s_j(2)}\right) + O\left(x^{3/2}/\log x\right) \quad (x \to \infty). \tag{6.12}$$

Proof. We recall the assumption that the class number of K is one, so that $\mathcal{O}_K$ is a PID. Let $Q(x, y) = ax^2 + bxy + cy^2$ be a primitive indefinite quadratic forms of discriminant $d \in \mathcal{D}_{+-}$ over $\mathcal{O}_K$, i.e. $a, b, c \in \mathcal{O}_K$, the ideal $(a, b, c) = \mathcal{O}_K$ and $d = b^2 - 4ac > 0$, $d' < 0$.

The equation $Q(\theta, 1) = 0$ has two real roots, $\theta_1 = (-b + \sqrt{d})/2a$ and $\theta_2 = (-b - \sqrt{d})/2a$. By linear change of variable, $\mathrm{SL}(2, \mathcal{O}_K)$ acts on such forms and the number of equivalence classes is $h_K(d)$. The stabilizer of Q under the action of $\mathrm{SL}(2, \mathcal{O}_K)$ is equal to the stabilizer of θ_1 or θ_2 and become a free abelian group of rank one. And a generator of this group is given by

$$g(Q) = \begin{pmatrix} (t_0 - bu_0)/2 & -cu_0 \\ au_0 & (t_0 + bu_0)/2 \end{pmatrix}$$

where, (t_0, u_0) is the fundamental solution to the Pellian equation $t^2 - du^2 = 4$ over $\mathcal{O}_K$. Moreover the norm of $g(Q)$ is $\varepsilon_K(d)^2$ with $\varepsilon_K(d) = (t_0 + u_0\sqrt{d})/2$.

The map $Q \mapsto (g, g')$ sends primitive $\mathcal{O}_K$-integral quadratic forms to hyperbolic–elliptic conjugacy classes of Γ_K. It is known that it induces a bijection between classes of forms and primitive hyperbolic–elliptic conjugacy classes of Γ_K. (We refer to Efrat [2] for details.) Hence, we obtain the desired formula from Theorem 6.12. $\square$

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