



Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt



General Section

The bounds of the Mordell-Weil ranks in cyclotomic towers of function fields

Yusuke Aikawa

Department of Mathematics, Hokkaido University, Sapporo, 060-0810, Japan

ARTICLE INFO

Article history:

Received 15 March 2018

Received in revised form 11 January 2019

Accepted 12 January 2019

Available online 18 February 2019

Communicated by A. Pal

Keywords:

Elliptic curves

Mordell-Weil rank

Elliptic surfaces

Function fields

ABSTRACT

We present new examples of elliptic curves having the bounded rank in cyclotomic towers of function fields over \mathbb{C} . Our key method is to utilize the monodromy of the Gaussian hypergeometric functions for the computation of the Mordell-Weil groups.

© 2019 Elsevier Inc. All rights reserved.

1. Introduction

Let K be a field and E an elliptic curve over K . As is well known, the set of rational points $E(K)$ carries a structure of abelian group, which is called the Mordell-Weil group. The Mordell-Weil theorem states that this group is finitely generated in the arithmetic situation, i.e. K is a number field or the function field of an algebraic curve over a finite field. The rank of $E(K)$ as a finitely generated abelian group is called the Mordell-Weil rank of E .

E-mail address: yusuke@math.sci.hokudai.ac.jp.

In this paper we study the Mordell-Weil group $E(K)$ when K is the function field of an algebraic curve over \mathbb{C} . This is a finitely generated group if E/K satisfies certain condition [12]. Then we shall discuss a function field analogue of Mazur's conjecture [8], namely the rank growth of $E(K_n)$ for K_n/K a cyclotomic tower. In this situation, there are several previous results. Silverman gave a conditional upper bound for the rank $E(K')$ where K'/K a finite unramified extension in [15]. In [14], [4] the upper bounds were discussed and recently Pál [9] improved those results by using Hodge theory. Stiller [16] and Fastenberg [5], [6] constructed examples of elliptic curves over $\mathbb{C}(t)$ whose ranks are bounded for $\mathbb{C}(t^{\frac{1}{n}})/\mathbb{C}(t)$ independently on n . On the other hand, Shioda [11] and Ulmer [17] discovered elliptic curves over $\mathbb{F}_p(t)$ having arbitrary large rank.

The aim of this paper is to present new examples of elliptic curves E over $\mathbb{C}(t)$ having the bounded ranks for the cyclotomic extensions $\mathbb{C}(t^{\frac{1}{n}})/\mathbb{C}(t)$. The main theorem is stated in Theorem 3.1. We take a complex geometric method to prove the main theorem. In particular, the Gaussian hypergeometric function plays a central role. To be more precise, let $f : \mathcal{X} \rightarrow \mathbb{P}^1$ be an elliptic surface degenerating at three points $\{0, 1, \infty\}$ and $E/\mathbb{C}(t)$ be the generic fiber of \mathcal{E} . We assume that the singular fiber at ∞ is additive and others are multiplicative. Such elliptic surfaces are classified in [10]. Let $f_1 : \mathcal{X}_1 \rightarrow \mathbb{P}^1$ be the base change of f by the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^1; t \mapsto \alpha - t$ and $E_1/\mathbb{C}(t)$ the generic fiber. Then the main theorem asserts that the rank of $E_1(\mathbb{C}(t^{\frac{1}{n}}))$ is bounded independently on n . The proof goes in the following way. Firstly, let $f_n : \mathcal{X}_n \rightarrow \mathbb{P}^1$ be the elliptic surface obtained by the base change of $f_1 : \mathcal{X}_1 \rightarrow \mathbb{P}^1$ by $\mathbb{P}^1 \rightarrow \mathbb{P}^1; t \mapsto t^n$ and the minimal desingularization. Then the Mordell-Weil group $E(\mathbb{C}(t^{\frac{1}{n}}))$ is translated into the quotient $NS(\mathcal{X}_n)/T_n$ by the subgroup T_n generated by fibral divisors and the zero section, and this agree with the Hodge (1,1)-part of $M_n := H^2(\mathcal{X}_n)/T_n$. The surface \mathcal{X}_n is endowed with the automorphism σ given by $(x, y, t) \mapsto (x, y, \zeta_n t)$. One has the eigen decomposition $M_n = \bigoplus_{d|n} L_n^d$ where L_n^d are $\mathbb{Q}(\zeta_d)$ -vector spaces (see §3.2). It is not hard to show $\dim_{\mathbb{Q}(\zeta_d)} L_n^d = 2$. The crucial step is computing $\dim_{\mathbb{Q}(\zeta_d)} L_n^d \cap L^{1,1}$, that is either 0, 1 or 2. This step is a new encounter compared with the previous results by Silverman etc. In order to compute $\dim_{\mathbb{Q}(\zeta_d)} L_n^d \cap L^{1,1}$ we employ the period formula [1] Theorem 4.1 and apply the monodromy theorem of Gaussian hypergeometric functions [3].

2. Preliminaries

In this section, we will fix the setting and prepare some notation. Moreover we will explain how to translate the computation of the Mordell-Weil groups into the computation of the cohomology of the elliptic surfaces.

2.1. The setting

In this paper, we mean by an elliptic surface a surjective morphism $f : \mathcal{E} \rightarrow C$ from a smooth projective surface to a smooth projective curve with a section such that its generic fiber is an elliptic curve over the function field of C . We often denote by \mathcal{E}

an elliptic surface $f : \mathcal{X} \rightarrow C$ for simplicity. Moreover, for the singular fiber of elliptic surfaces, we use Kodaira's notation [7], Theorem 6.2.

Let $f : \mathcal{X} \rightarrow \mathbb{P}_u^1$ be an elliptic surface over \mathbb{C} having multiplicative reduction at two points and additive reduction at a point. Here, we denote by \mathbb{P}_u^1 the projective line with inhomogeneous coordinate u . We may assume that \mathcal{X} degenerates at three points $\{0, 1, \infty\} \subset \mathbb{P}_u^1$ and has the singular fibers of type I_a (resp. I_b) at 0 (resp. 1) and the fiber of ∞ is additive fiber by the assumption. The possible types of singular fibers of such elliptic surfaces are classified in [10] and summarized in the following Table 2.1.

Table 2.1

Possible combinations of singular fibers in this situation.

	Multiplicative fiber 1	Multiplicative fiber 2	Additive fiber
Type.1	I_1	I_1	II^*
Type.2	I_1	I_2	III^*
Type.3	I_1	I_3	IV^*
Type.4	I_1	I_4	I_1^*
Type.5	I_1	I_1	I_4^*
Type.6	I_2	I_2	I_2^*

For $\alpha \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $n \in \mathbb{N}$, let $\tilde{f}_{\alpha,n} : \mathcal{X} \times_{\mathbb{P}_u^1} \mathbb{P}_{u_n}^1 \rightarrow \mathbb{P}_u^1$ be the base change of f by the morphism $g_{\alpha,n} : \mathbb{P}_{u_n}^1 \rightarrow \mathbb{P}_u^1$; $u \mapsto \alpha - u^n$, where $\mathbb{P}_{u_n}^1$ denotes the source of $g_{\alpha,n}$ with inhomogeneous coordinate u_n to distinguish \mathbb{P}_u^1 . We define an elliptic surface $f_{\alpha,n} : \mathcal{X}_{\alpha,n} \rightarrow \mathbb{P}_{u_n}^1$ by the following diagram;

$$\begin{array}{ccccc}
 \mathcal{X}_{\alpha,n} & \xrightarrow{i} & \mathcal{X} \times_{\mathbb{P}_u^1} \mathbb{P}_{u_n}^1 & \xrightarrow{pr} & \mathcal{X} \\
 & \searrow f_{\alpha,n} & \downarrow \tilde{f}_{\alpha,n} & \square & \downarrow f \\
 & & \mathbb{P}_{u_n}^1 & \xrightarrow{g_{\alpha,n}} & \mathbb{P}_u^1
 \end{array}$$

where i is the minimal desingularization and pr is the first projection. Then $\mathcal{X}_{\alpha,n}$ has the singular fibers at $(2n+1)$ -points. The I_a -type appears at $u_n = \zeta_n^k \sqrt[n]{\alpha}$ and the I_b -type appears at $u_n = \zeta_n^k \sqrt[n]{\alpha-1}$ for $k = 0, 1, \dots, n-1$, where we fix $\zeta_n := \exp(\frac{2\pi i}{n})$ throughout this paper. The following Tables 2.2–2.5 collect the variation of singular fibers at ∞ depending on the index n of the above base change. These follow from the computation of local monodromy matrices in [7], Theorem 9.1, Table.I or [2], Chapter V, Table 6.

For further discussion, we prepare several notations. Let $Z_{\alpha,n} := (f_{\alpha,n}^{-1}(0) + f_{\alpha,n}^{-1}(\infty) + \sum_{k=1}^{n-1} (f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha}) + f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha-1})))_{red}$ be the reduced divisor on $\mathcal{X}_{\alpha,n}$ and $U_{\alpha,n} \subset \mathcal{X}_{\alpha,n}$ the inverse image of $\mathbb{P}_u^1 \setminus \{0, 1, \alpha, \infty\}$ via $g_{\alpha,n} \circ f_{\alpha,n}$. Moreover, set $\mathcal{S}_{\alpha,n} := \mathbb{P}_{u_n}^1 \setminus \{u_n^n = 0, \alpha, \alpha-1, \infty\}$. We denote by $E_{\alpha,n}$ the generic fiber of $f_{\alpha,n}$. According to the Mordell-Weil theorem for function fields ([13], III, Theorem 6.1), the Mordell-Weil group $E_{\alpha,1}(\mathbb{C}(u_1^{\frac{1}{n}}))$ is a finitely generated abelian group. We will study the rank of the finitely generated abelian groups $E_{\alpha,1}(\mathbb{C}(u_1^{\frac{1}{n}}))$.

Table 2.2The type of $f_{\alpha,n}^{-1}(\infty)$ in $\mathcal{X}_{\alpha,n}$ of Type.1.

$n \bmod 6$	The type of fiber
1	II*
2	IV*
3	I ₀ *
4	IV
5	II
0	I ₀

Table 2.3The type of $f_{\alpha,n}^{-1}(\infty)$ in $\mathcal{X}_{\alpha,n}$ of Type.2.

$n \bmod 4$	The type of fiber
1	III*
2	I ₀ *
3	III
0	I ₀

Table 2.4The type of $f_{\alpha,n}^{-1}(\infty)$ in $\mathcal{X}_{\alpha,n}$ of Type.3.

$n \bmod 3$	The type of fiber
1	IV*
2	IV
0	I ₀

Table 2.5The type of $f_{\alpha,n}^{-1}(\infty)$ in $\mathcal{X}_{\alpha,n}$ of Type.4.5.6.

$n \bmod 2$	the Type of fiber
1	I _{nm} *
0	I _{nm}

2.2. Mordell-Weil groups and Néron-Severi groups

From now on, we denote u_1 by t and u_n by s . In the above setting, we have

$$E_{\alpha,1}(\mathbb{C}(t^{\frac{1}{n}})) \cong E_{\alpha,n}(\mathbb{C}(s)).$$

To translate the problem of elliptic curves over function fields into the problem of elliptic surfaces, we recall the relationship between Néron-Severi groups of an elliptic surface and the Mordell-Weil groups of its generic fiber. But we do not recall all of this material here; for detail, see [12]. Let $NS(\mathcal{X}_{\alpha,n})$ be the Néron-Severi group of $\mathcal{X}_{\alpha,n}$:

$$NS(\mathcal{X}_{\alpha,n}) := \{\text{divisors on } \mathcal{X}_{\alpha,n}\} / \sim.$$

Here \sim denotes algebraic equivalence. A rational point $(x(s), y(s)) \in E_{\alpha,n}(\mathbb{C}(s))$ corresponds to a section $\mathbb{P}^1 \rightarrow \mathcal{X}_{\alpha,n}; t \mapsto (x(t), y(t), t)$. This is one-to-one correspondence. By

regarding sections as divisors on $\mathcal{X}_{\alpha,n}$, rational points of $E_{\alpha,n}$ map to divisors on $\mathcal{X}_{\alpha,n}$. Let $T_{\alpha,n}$ denote the subgroup of $NS(\mathcal{X}_{\alpha,n})$ generated by fibral divisors of $f_{\alpha,n}$ and the zero section. Then the above correspondence gives an isomorphism:

$$E_{\alpha,n}(\mathbb{C}(s)) \xrightarrow{\sim} NS(\mathcal{X}_{\alpha,n})/T_{\alpha,n}.$$

Hence the Lefschetz' theorem on (1,1) classes implies an isomorphism:

$$E_{\alpha,n}(\mathbb{C}(s)) \otimes \mathbb{Q} \cong (H^2(\mathcal{X}_{\alpha,n}, \mathbb{Q}) \cap H^{1,1})/T_{\alpha,n,\mathbb{Q}}$$

where $T_{\alpha,n,\mathbb{Q}} := T_{\alpha,n} \otimes \mathbb{Q}$.

3. Computations of the cohomology

The elliptic surface $\mathcal{X}_{\alpha,n}$ has an automorphism σ given by $(x, y, t) \mapsto (x, y, \zeta_n t)$. In this section, we study the structure of the cohomology of elliptic surface $\mathcal{X}_{\alpha,n}$ as $\mathbb{Q}[\sigma]$ -module. This is the first step toward the main theorem, that is, the bound of the rank of Mordell-Weil group of elliptic curve $E_{\alpha,1}$.

3.1. The dimension of the cohomology

Set

$$\begin{aligned} M_{\alpha,n} &:= H^2(\mathcal{X}_{\alpha,n}, \mathbb{Q})/T_{\alpha,n,\mathbb{Q}} \\ &\cong W_2 H^1(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) \end{aligned}$$

where $j : \mathcal{S}_{\alpha,n} \hookrightarrow \mathbb{P}_s^1$ is the embedding. In this section, we study the structure of this module. Note that $M_{\alpha,n}$ is a \mathbb{Q} -Hodge structure on account of the inclusion $T_{\alpha,n,\mathbb{Q}} \subset H^2(\mathcal{X}_{\alpha,n}, \mathbb{Q}) \cap H^{1,1}$ endowed with multiplication by $\mathbb{Q}[\sigma]$.

Proposition 3.1. *We have*

$$\dim_{\mathbb{Q}} M_{\alpha,n} = \begin{cases} 2n-2 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is additive;} \\ 2n-3 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is multiplicative;} \\ 2n-4 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is smooth.} \end{cases}$$

Note that the terms on the right hand side are non-negative since $n \geq 1$.

Proof. We have an exact sequence

$$0 \rightarrow H^1(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) \rightarrow H^2(U_{\alpha,n}, \mathbb{Q}) \rightarrow H^2(\mathcal{X}_{\alpha,n,s}, \mathbb{Q})$$

where $\mathcal{X}_{\alpha,n,s}$ is a smooth general fiber of $f_{\alpha,n}$. By taking the graded piece of weight 2, we have an isomorphism

$$W_2 H^1(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) \cong \text{Ker}(W_2 H^2(U_{\alpha,n}, \mathbb{Q}) \rightarrow H^2(\mathcal{X}_{\alpha,n,s}, \mathbb{Q})). \quad (3.1)$$

Note that $H^2(\mathcal{X}_{\alpha,n,s}, \mathbb{Q}) \simeq \mathbb{Q}$ and the arrow in the right hand side is surjective. Hence

$$\dim_{\mathbb{Q}} M_{\alpha,n} = \dim_{\mathbb{Q}} W_2 H^2(U_{\alpha,n}, \mathbb{Q}) - 1. \quad (3.2)$$

The localization exact sequence induces the following:

$$\begin{array}{ccccccc} 0 \longrightarrow & \text{Coker}(H_{Z_{\alpha,n}}^2(\mathcal{X}_{\alpha,n}) \rightarrow H^2(\mathcal{X}_{\alpha,n})) & \longrightarrow & H^2(U_{\alpha,n}) & \longrightarrow & H_{Z_{\alpha,n}}^3(\mathcal{X}_{\alpha,n}) & \longrightarrow H^3(\mathcal{X}_{\alpha,n}) \\ & \cong \downarrow & & & & \parallel & \parallel \\ & W_2 H^2(U_{\alpha,n}) & & & & H_1(Z_{\alpha,n}) & 0. \end{array} \quad (3.3)$$

Here, all objects in the above diagram are with rational coefficient. Recall that $Z_{\alpha,n} = (f_{\alpha,n}^{-1}(0) + f_{\alpha,n}^{-1}(\infty) + \sum_{k=1}^{n-1} (f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha}) + f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha-1})))_{red}$ where $f_{\alpha,n}^{-1}(0)$ is smooth, $f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha})$ and $f_{\alpha,n}^{-1}(\zeta_n^k \sqrt[n]{\alpha-1})$ are multiplicative for $k = 1, \dots, n-1$. The fiber $f_{\alpha,n}^{-1}(\infty)$ depends on n , according to the table in §2. Thus we obtain

$$\dim_{\mathbb{Q}} H_1(Z_{\alpha,n}, \mathbb{Q}) = \begin{cases} 2n+2 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is additive;} \\ 2n+3 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is multiplicative;} \\ 2n+4 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is smooth.} \end{cases}$$

Moreover, the Leray spectral sequence gives an exact sequence

$$\begin{aligned} 0 \rightarrow H^1(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) &\rightarrow H^2(U_{\alpha,n}, \mathbb{Q}) \rightarrow H^0(\mathcal{S}_{\alpha,n}, j^* R^2(f_{\alpha,n})_* \mathbb{Q}) \\ &\cong H^2(\mathcal{X}_{\alpha,n,t}, \mathbb{Q})^{\pi_1(\mathcal{S}_{\alpha,n})}. \end{aligned} \quad (3.4)$$

Note that the last term is one dimensional and the last arrow is surjection. Employing the formula

$$\chi(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) = \chi(\mathcal{S}_{\alpha,n}, \mathbb{Q}) \times \text{rank } j^* R^1(f_{\alpha,n})_* \mathbb{Q},$$

we have

$$\dim H^1(\mathcal{S}_{\alpha,n}, j^* R^1(f_{\alpha,n})_* \mathbb{Q}) = 4n.$$

Hence by (3.4)

$$\dim H^2(U_{\alpha,n}, \mathbb{Q}) = 4n + 1$$

and so by (3.3)

$$\dim_{\mathbb{Q}} W_2 H^2(U_{\alpha,n}, \mathbb{Q}) = \begin{cases} 2n-1 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is additive;} \\ 2n-2 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is multiplicative;} \\ 2n-3 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is smooth.} \end{cases}$$

We reach a conclusion by (3.2). \square

3.2. The automorphism of the elliptic surface

Let $\sigma : \mathcal{X}_{\alpha,n} \rightarrow \mathcal{X}_{\alpha,n}$ be an automorphism given by $(x, y, s) \mapsto (x, y, \zeta_n s)$. Then, $\mathbb{Q}[\sigma]$ acts on $M_{\alpha,n}$. We will determine the structure of $M_{\alpha,n}$ as $\mathbb{Q}[\sigma]$ -module.

For a positive integer d which divides n , we set

$$L_{\alpha,n}^d := \text{Ker}(\Phi_d(\sigma) : M_{\alpha,n} \rightarrow M_{\alpha,n})$$

where $\Phi_d(X)$ is the minimal polynomial of ζ_d over \mathbb{Q} . We have a decomposition

$$M_{\alpha,n} = \bigoplus_{d|n} L_{\alpha,n}^d$$

of the Hodge structures. Then, we have

$$\begin{aligned} \text{rank} E_{\alpha,1}(\mathbb{C}(t^{\frac{1}{n}})) &= \text{rank} E_{\alpha,n}(\mathbb{C}(s)) \\ &= \dim_{\mathbb{Q}} M_{\alpha,n} \cap H^{1,1} \\ &= \sum_{d|n} \dim_{\mathbb{Q}} L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1}. \end{aligned} \quad (3.5)$$

Proposition 3.2. *Let d_{\min} be the minimal integer such that the fiber $f_{\alpha,d_{\min}}^{-1}(\infty)$ in $\mathcal{X}_{\alpha,d_{\min}}$ is smooth or multiplicative. According to the Tables 2.2–2.5, if the elliptic surface \mathcal{X} is Type.1 (resp. 2, 3, otherwise) in Table 2.1, then $d_{\min} = 6$ (resp. 4, 3, 2). Then, we have*

$$L_{\alpha,n}^d \cong \begin{cases} 0 & \text{if } d = 1 \\ \mathbb{Q}[\sigma]/(\Phi_d(\sigma)) & \text{if } d = d_{\min} \\ \mathbb{Q}[\sigma]/(\Phi_d(\sigma))^{\oplus 2} & \text{if } d \neq 1, d_{\min} \end{cases}$$

as $\mathbb{Q}[\sigma]$ -module.

Proof. We use induction on n . Write $n = ml$ where m, l are positive integers. Then we have an unramified cyclic covering

$$\pi_l : U_{\alpha,n} \rightarrow U_{\alpha,m} ; (x, y, u_n) \mapsto (x, y, u_n^l)$$

and this induces an injection

$$\pi_l^* : M_{\alpha,m} \hookrightarrow M_{\alpha,n}.$$

Via the above injection, since $U_n / \langle \sigma^m \rangle \cong U_m$, we obtain an isomorphism

$$M_{\alpha,m} \xrightarrow{\cong} M_{\alpha,n}^{\sigma^m=1} \quad (3.6)$$

where $M_{\alpha,n}^{\sigma^m=1}$ denotes the subspace of $M_{\alpha,n}$ consisting of elements on which σ^m acts trivially. Moreover, $\pi_l^*(L_{\alpha,m}^d) \subset L_{\alpha,n}^d$ for $d|m|n$. By the isomorphism (3.6), we have

$$L_{\alpha,m}^d \cong (L_{\alpha,n}^d)^{\sigma^m=1} = L_{\alpha,n}^d \text{ for } d|m|n.$$

Here the second equality follows from the fact that $\Phi_d(\sigma)$ divides $\sigma^m - 1$. We sum up the above discussion in the following diagram:

$$\begin{array}{ccc} & & M_{\alpha,n} \\ & & \cup \\ L_{\alpha,n}^d = (L_{\alpha,n}^d)^{\sigma^m=1} & \hookrightarrow & M_{\alpha,n}^{\sigma^m=1} \\ \uparrow \cong & & \uparrow \cong \pi_l^* \\ L_{\alpha,m}^d & \hookrightarrow & M_{\alpha,m} \end{array}$$

Put $l_{\alpha,n}^d := \dim_{\mathbb{Q}} L_{\alpha,n}^d$, then $\dim_{\mathbb{Q}} M_{\alpha,n} = \sum_{d|n} l_{\alpha,n}^d$. The above diagram yields $l_{\alpha,n}^d = l_{\alpha,m}^d$ for $d|m|n$. In particular, $l_{\alpha,n}^1 = l_{\alpha,1}^1 = 0$ by Proposition 3.1. And for $n = d_{\min}$, we have by induction

$$\begin{aligned} \sum_{d|d_{\min}} l_{\alpha,d_{\min}}^d &= l_{\alpha,d_{\min}}^{d_{\min}} + 2 \sum_{d|d_{\min}, d \neq 1, d_{\min}} \phi(d) \\ &= \begin{cases} 2n - 3 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is multiplicative i.e. } d_{\min} = 2; \\ 2n - 4 & \text{if } f_{\alpha,n}^{-1}(\infty) \text{ is smooth i.e. } d_{\min} = 3 \text{ or } 4 \text{ or } 6. \end{cases} \end{aligned}$$

Thus we obtain $l_{\alpha,d_{\min}}^{d_{\min}} = \phi(d_{\min})$. Here, ϕ denotes the Euler function.

For general n , we similarly have

$$\sum_{d|n, d \neq 1} l_{\alpha,n}^d = \begin{cases} l_{\alpha,n}^n + 2 \sum_{d|n, d \neq 1, n} \phi(d) & \text{if } d_{\min} \text{ does not divide } n; \\ l_{\alpha,n}^n + \phi(d_{\min}) + 2 \sum_{d|n, d \neq 1, d_{\min}, n} \phi(d) & \text{if } d_{\min} \text{ divides } n. \end{cases}$$

If d_{\min} not divides n , the fiber $f_{\alpha,n}^{-1}(\infty)$ is additive. Then, we have

$$\dim_{\mathbb{Q}} M_{\alpha,n} = l_{\alpha,n}^n + 2 \sum_{d|n, d \neq 1, n} \phi(d) = 2n - 2.$$

Hence $l_{\alpha,n}^n = 2\phi(n)$. If d_{\min} divides n and $f_{\alpha,n}^{-1}(\infty)$ is multiplicative, then $\phi(d_{\min}) = 1$ and

$$\dim_{\mathbb{Q}} M_{\alpha,n} = l_{\alpha,n}^n + \phi(2) + 2 \sum_{d|n, d \neq 1, d_{\min}, n} \phi(d) = 2n - 3,$$

and hence $l_{\alpha,n}^n = 2\phi(n)$. Finally, if d_{\min} divides n and $f_{\alpha,n}^{-1}(\infty)$ is smooth, then $\phi(d_{\min}) = 2$ and we conclude $l_{\alpha,n}^n = 2\phi(n)$ in the same way. We finish the proof. \square

3.3. The dimension of the subspace $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1}$

By the Proposition 3.2, the dimensions of the subspaces $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1}$ over $\mathbb{Q}(\zeta_d)$ are at most 2. In the following, we will determine the dimension completely under the assumption that α is a transcendental number.

Firstly, we treat the case that $d \leq d_{\min}$. This case can be computed as follows.

Proposition 3.3. *Let d_{\min} be an integer as in Proposition 3.2. Then, if $d \leq d_{\min}$, we have $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} = L_{\alpha,n}^d$.*

Proof. The assertion is equivalent to $(l_{\alpha,n}^d)^{2,0} := \dim_{\mathbb{Q}}(L_{\alpha,n}^d)^{2,0} = 0$. We denote the number of irreducible components of the fiber $f_{\alpha,n}^{-1}(s)$ by m_s . The Euler number $e(f_{\alpha,n}^{-1}(s))$ of the fiber $f_{\alpha,n}^{-1}(s)$ are given by

$$e(f_{\alpha,n}^{-1}(s)) = \begin{cases} 0 & \text{if } f_{\alpha,n}^{-1}(s) \text{ is smooth;} \\ m_s & \text{if } f_{\alpha,n}^{-1}(s) \text{ is multiplicative;} \\ m_s + 1 & \text{if } f_{\alpha,n}^{-1}(s) \text{ is additive.} \end{cases} \quad (3.7)$$

The Hodge number of (2,0)-part is given by the Euler numbers of fibers:

$$h_n^{2,0} := \dim_{\mathbb{Q}} H^{2,0}(\mathcal{X}_{\alpha,n}) = -1 + \frac{1}{12} \sum_{s \in \mathbb{P}_s^1} e(f_{\alpha,n}^{-1}(s)). \quad (3.8)$$

By the Tables 2.1–2.5 and (3.7), (3.8),

$$h_n^{2,0} = \lfloor \frac{n-1}{d_{\min}} \rfloor \quad (3.9)$$

where, for a real number r , $\lfloor r \rfloor$ denotes the maximum of integers which are smaller than or equal to r .

If $d \leq d_{\min}$, we have $h_d^{2,0} = 0$ by (3.9). Thus, since $(l_{\alpha,d}^d)^{2,0} = (l_{\alpha,n}^d)^{2,0}$, we have $(l_{\alpha,n}^d)^{2,0} = 0$. \square

Secondly, we will prove that $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} = 0$ for $d > d_{\min}$. In order to prove this, we set $S := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and consider a smooth fibration: $X \rightarrow S$ such that the fiber of $\alpha \in S$ is the elliptic surface $\mathcal{X}_{\alpha,n}$. Since $\pi_1(S, \alpha)$ -action commutes with σ -action, the monodromy action on the cohomology of $\mathcal{X}_{\alpha,n}$ induces $\pi_1(S, \alpha)$ -action on $L_{\alpha,n}^d$.

Proposition 3.4. *Suppose that α is a transcendental number. If $\dim_{\mathbb{Q}(\zeta_d)} L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} \neq 0$, then $\pi_1(S, \alpha)$ -action on $L_{\alpha,n}^d$ factors through a finite quotient. In other words,*

$$\mathrm{Im}(\pi_1(S, \alpha) \rightarrow \mathrm{Aut}(L_{\alpha,n}^d))$$

is a finite group.

Proof. Throughout this proof, all of the fundamental groups are considered with fixed base point $\alpha \in S$ and we omit to write the base point. Take a model $X_{\overline{\mathbb{Q}}}$ of X over $\overline{\mathbb{Q}}$. Consider the following cartesian diagram

$$\begin{array}{ccc} X_{\overline{\mathbb{Q}}} & \longrightarrow & S_{\overline{\mathbb{Q}}} := \mathrm{Spec} \overline{\mathbb{Q}}[T, \frac{1}{T}, \frac{1}{1-T}] \\ \uparrow & & \uparrow \\ Y := \mathcal{X}_{\alpha,n} \times_{\mathbb{C}} \mathrm{Spec} \overline{\mathbb{Q}}(T) & \longrightarrow & \mathrm{Spec} \overline{\mathbb{Q}}(T) \\ \uparrow & & \uparrow \\ \mathcal{X}_{\alpha,n} & \longrightarrow & \mathrm{Spec} \mathbb{C} \end{array}$$

where the morphism $\mathrm{Spec} \mathbb{C} \rightarrow \mathrm{Spec} \overline{\mathbb{Q}}(T)$ is induced by the morphism $\overline{\mathbb{Q}}(T) \rightarrow \mathbb{C}; T \mapsto \alpha$ (here we use the assumption that α is a transcendental number). Let $Y_{\overline{\mathbb{Q}}(T)} := Y \times_{\overline{\mathbb{Q}}(T)} \mathrm{Spec} \overline{\mathbb{Q}}(T)$. Since $NS(\mathcal{X}_{\alpha,n}) \cong NS(Y_{\overline{\mathbb{Q}}(T)})$, one has the isomorphism

$$L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} \cong \left(NS(Y_{\overline{\mathbb{Q}}(T)})/T_{\alpha,n} \right) \cap \mathrm{Ker}(\Phi_d(\sigma) : M_{\alpha,n} \rightarrow M_{\alpha,n}),$$

and hence the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}(T)/\overline{\mathbb{Q}}(T))$ acts on this. Since the Néron-Severi group of $\mathcal{X}_{\alpha,n}$ is finitely generated and the action of the Galois group on each cycles factors through a finite quotient, we have

$$\mathrm{Gal}(K/\overline{\mathbb{Q}}(T)) \rightarrow \mathrm{Aut}(L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1})$$

for some finite extension K of $\overline{\mathbb{Q}}(T)$. This completes the proof in the case $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} = L_{\alpha,n}^d$.

Suppose $L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} \neq L_{\alpha,n}^d$, namely $\dim_{\mathbb{Q}(\zeta_d)}(L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1}) = 1$. Then there is the orthogonal decomposition

$$L_{\alpha,n}^d = (L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1}) \oplus (L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1})^\perp$$

with respect to polarization on $L_{\alpha,n}^d$ and $\pi_1(S')$ acts on each component, where S' is a smooth model of K . There is a \mathbb{Z} -lattice in each component induced from the \mathbb{Z} -lattice $H^2(\mathcal{X}_{\alpha,n}, \mathbb{Z})$ in $M_{\alpha,n}$, and $\pi_1(S')$ acts on it. Therefore, the image of $\pi_1(S')$ to $\text{Aut}(L_{\alpha,n}^d)$ is contained in $\mathbb{Z}^\times \times \mathbb{Z}^\times = \{\pm 1\} \times \{\pm 1\}$. In particular, it is finite. Thus a diagram

$$\begin{array}{ccc} \pi_1(S') & \longrightarrow & \text{Aut}(L_{\alpha,n}^d) \\ \downarrow & \nearrow & \\ \pi_1(S) & & \end{array}$$

concludes the desired assumption. \square

We postpone the proof of the following proposition to the next section.

Proposition 3.5. *For $d > d_{\min}$,*

$$\text{Im}(\pi_1(S, \alpha) \rightarrow \text{Aut}(L_{\alpha,n}^d))$$

is an infinite group.

By Proposition 3.3, 3.4 and 3.5, we have the main theorem of this paper, the explicit ranks of the Mordell-Weil group.

Theorem 3.1. *Suppose that α is a transcendental number. We have*

$$\begin{aligned} \text{rank} E_{\alpha,1}(\mathbb{C}(t^{\frac{1}{n}})) &= \sum_{1 < d \leq d_{\min}, d|n} \dim_{\mathbb{Q}} L_{\alpha,n}^d \cap (L_{\alpha,n}^d)^{1,1} \\ &= \begin{cases} \sum_{1 < d < d_{\min}, d|n} 2\phi(d) & \text{if } d_{\min} \text{ does not divide } n. \\ \sum_{1 < d < d_{\min}, d|n} 2\phi(d) + \phi(d_{\min}) & \text{if } d_{\min} \text{ divides } n, \end{cases} \end{aligned}$$

where ϕ denotes the Euler function.

4. Proof of Proposition 3.5

In this section, we will give the proof of Proposition 3.5. We assume that $d > d_{\min}$ throughout this section.

4.1. Eigendecomposition and the structure of eigenspaces

Recall

$$M_{\alpha,n} = \bigoplus_{d \neq 1, d|n} L_{\alpha,n}^d$$

and

$$L_{\alpha,n}^d \cong \mathbb{Q}[\sigma]/(\Phi_d(\sigma))^{\oplus 2} \quad \text{if } d \neq 1, d_{\min} \quad (4.1)$$

as $\mathbb{Q}[\sigma]$ -modules. If we write $L_{\alpha,n,\mathbb{C}}^d := L_{\alpha,n}^d \otimes \mathbb{C}$, we have

$$L_{\alpha,n,\mathbb{C}}^d = \bigoplus_{\chi} L_{\alpha,n}^d(\chi)$$

where χ runs through the set of homomorphisms from $\mathbb{Q}(\zeta_d)$ to $\overline{\mathbb{Q}}$ and $L_{\alpha,n}^d(\chi)$ are the spaces of eigenvectors of σ with $\pi_1(S, \alpha)$ -action. By (4.1), the spaces $L_{\alpha,n}^d(\chi)$ are two-dimensional over \mathbb{C} . To prove Proposition 3.5, it suffices to find an eigen component whose monodromy group is infinite.

Hereafter we fix χ to be the homomorphism $\chi(\zeta_d) = \zeta_d$. Moreover we fix rational numbers $\lambda_1, \lambda_2 \in \mathbb{Q}$ such that $\exp(2\pi i \lambda_j) (j = 1, 2)$ and $\lambda_1 \leq \lambda_2$ are eigenvalues of the local monodromy on $\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} R^1 f_* \mathbb{Q}$ where $f : \mathcal{X} \rightarrow \mathbb{P}^1$ as in the beginning of § 2.1. Since the local monodromy of elliptic surfaces over \mathbb{C} is completely classified, see [7] or [2], from the Table 2.1, we can list all of pairs (λ_1, λ_2) as in Table 4.1.

Table 4.1
Type of \mathcal{X} and d_{\min} , the pair (λ_1, λ_2) .

Type of \mathcal{X} in Table 2.1	d_{\min}	(λ_1, λ_2)
Type.1	6	$(\frac{1}{6}, \frac{5}{6})$
Type.2	4	$(\frac{1}{4}, \frac{3}{4})$
Type.3	3	$(\frac{1}{3}, \frac{2}{3})$
Type.4	2	$(\frac{1}{2}, \frac{1}{2})$
Type.5	2	$(\frac{1}{2}, \frac{1}{2})$
Type.6	2	$(\frac{1}{2}, \frac{1}{2})$

In order to study the structure of $L_{\alpha,n}^d(\chi)$ as $\mathbb{C}[\pi_1(S, \alpha)]$ -module, we make use of the Gaussian hypergeometric function.

Lemma 4.1. Put ${}_2F_1(x) := {}_2F_1\left(\begin{smallmatrix} \lambda_1 - \frac{1}{d}, \lambda_2 - \frac{1}{d} \\ 1 - \frac{1}{d} \end{smallmatrix}; x\right)$ the Gaussian hypergeometric functions. Let V_α to be two dimensional vector space over \mathbb{C} spanned by ${}_2F_1(\alpha)$ and ${}_2F_1(1 - \alpha)$, on which $\pi_1(S, \alpha)$ acts in the natural way. Then there is an isomorphism

$$V_\alpha \cong L_{\alpha,n}^d(\chi)^\vee$$

of $\mathbb{C}[\pi_1(S, \alpha)]$ -modules where $L_{\alpha,n}^d(\chi)^\vee$ denotes the dual space.

Proof. Recall $U_{\alpha,n} = (g_{\alpha,n} \circ f_{\alpha,n})^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ as in § 2.1. Put $H^2(U_{\alpha,n})_0 := \text{Ker}(H^2(U_{\alpha,n}) \rightarrow H^2(\mathcal{X}_{\alpha,n,s}))$ where $\mathcal{X}_{\alpha,n,s}$ is a general smooth fiber. There is a natural isomorphism

$$W_2 H^2(U_{\alpha,n})_0 \cong H^2(\mathcal{X}_{\alpha,n})/T_{\alpha,n} = M_{\alpha,n}$$

and this yields

$$\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} W_2 H^2(U_{\alpha,n})_0 \cong L_{\alpha,n}^d(\chi).$$

We employ [1] Theorem 4.1 (period formula). There are two homology cycles $\Gamma_1, \Gamma_2 \in H_2(U_{\alpha,n}, \overline{\mathbb{Q}})$ which form a basis of $\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} H_2(U_{\alpha,n}, \mathbb{Q})$, and two global rational forms $\omega_1, \omega_2 \in \Gamma(U_{\alpha,n}, \Omega^2)$ which form a basis of $\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} W_2 H_{\text{dR}}^2(U_{\alpha,n}/\mathbb{Q})_0$, and a differential operator Θ on $\overline{\mathbb{Q}}[\alpha]$ such that

$$\begin{pmatrix} \int_{\Gamma_1} \omega_1 & \int_{\Gamma_2} \omega_1 \\ \int_{\Gamma_1} \omega_2 & \int_{\Gamma_2} \omega_2 \end{pmatrix} = \begin{pmatrix} a_1 G_1(\alpha) & a_2 G_2(\alpha) \\ a_1 G'_1(\alpha) & a_2 G'_2(\alpha) \end{pmatrix}$$

for some constants $a_i \in \mathbb{C}^\times$ where we put $G_1(\alpha) := \Theta({}_2F_1(\alpha))$ and $G_2(\alpha) := \Theta({}_2F_1(1-\alpha))$. Note that the fact that the space $L_{\alpha,n}^d(\chi)$ is two-dimensional implies that the spaces $\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} H_2(U_{\alpha,n}, \mathbb{Q})$ and $\overline{\mathbb{Q}} \otimes_{\chi, \mathbb{Q}(\zeta_d)} W_2 H_{\text{dR}}^2(U_{\alpha,n}/\mathbb{Q})_0$ are two-dimensional. Moreover, the above matrix is invertible. This means that the composition

$$\mathbb{C}\Gamma_1 \oplus \mathbb{C}\Gamma_2 \longrightarrow H_2(U_{\alpha,n}, \mathbb{C}) \longrightarrow W_2 H_{\text{dR}}^2(U_{\alpha,n}/\mathbb{C})_0(\chi)^\vee = L_{\alpha,n}^d(\chi)^\vee$$

is bijection, and the image is spanned by two column vectors

$$\begin{pmatrix} G(\alpha) \\ G'(\alpha) \end{pmatrix}, \quad \begin{pmatrix} G(1-\alpha) \\ G'(1-\alpha) \end{pmatrix}$$

with respect to the dual basis of ω_1, ω_2 . Note that the action of $\pi_1(S, \alpha)$ on $L_{\alpha,n}^d(\chi)^\vee$ is compatible with that on $H_2(U_{\alpha,n}, \mathbb{C})$. Therefore, we have an isomorphism

$$L_{\alpha,n}^d(\chi)^\vee \cong \langle G(\alpha), G(1-\alpha) \rangle$$

of $\mathbb{C}[\pi_1(S, \alpha)]$ -modules. The right hand side is isomorphic to $V_\alpha = \langle {}_2F_1(\alpha), {}_2F_1(1-\alpha) \rangle$ as $\mathbb{C}[\pi_1(S, \alpha)]$ -module, so we are done. \square

4.2. Monodromy of the Gaussian hypergeometric functions

Let us put $D := x \frac{d}{dx}$ and consider the following differential equation, so-called the hypergeometric equation:

$$\left(D(D - \frac{1}{d}) - x(D + \lambda_1 - \frac{1}{d})(D + \lambda_2 - \frac{1}{d}) \right) u(x) = 0. \quad (4.2)$$

Recall that the vector space V_α is the two dimensional vector space over \mathbb{C} spanned by ${}_2F_1(\alpha)$ and ${}_2F_1(1-\alpha)$ where ${}_2F_1(x) = {}_2F_1\left(\begin{smallmatrix}\lambda_1-\frac{1}{d}, \lambda_2-\frac{1}{d} \\ 1-\frac{1}{d}\end{smallmatrix}; x\right)$ is the Gaussian hypergeometric function. Then V_α is the space of local solutions of the differential equation (4.2). Put

$$H_{\lambda_1, \lambda_2}^d := \text{Im}(\pi^1(S, \alpha) \rightarrow \text{Aut}(V_\alpha)).$$

Lemma 4.1 says that

$$H_{\lambda_1, \lambda_2}^d \cong \text{Im}(\pi_1(S, \alpha) \rightarrow \text{Aut}(I_{\alpha, n}^d(\chi))).$$

Therefore, the following proposition finishes the proof of Proposition 3.5.

Proposition 4.1. *For $d > d_{\min}$, $H_{\lambda_1, \lambda_2}^d$ is an infinite group.*

Proof. For $d > d_{\min}$, from the Table 4.1, we have

$$0 < \lambda_1 - \frac{1}{d} < \lambda_2 - \frac{1}{d} < 1 - \frac{1}{d} < 1.$$

According to Theorem 4.8 in [3], this inequality implies the infiniteness of the group $H_{\lambda_1, \lambda_2}^d$. \square

Acknowledgments

The author is deeply grateful to Masanori Asakura for useful discussion and his constant encouragement. Without his support, this work has not been accomplished.

References

- [1] M. Asakura, N. Otsubo, Regulators on K_1 of hypergeometric fibrations, in: Proceedings of Conference “Arithmetic L -functions and Differential Geometric Methods (Regulators IV)”, in press, arXiv:1709.04144.
- [2] W. Barth, K. Hulek, C. Peters, A. Van de Ven, Compact Complex Surfaces, second edition, Springer-Verlag, Berlin, 2004.
- [3] F. Beukers, G. Heckman, Monodromy for the hypergeometric function ${}_nF_{n-1}$, Invent. Math. 95 (2) (1989) 325–354.
- [4] J.S. Ellenberg, Selmer groups and Mordell-Weil groups of elliptic curves over towers of function fields, Compos. Math. 142 (5) (2006) 1215–1230.
- [5] L.A. Fastenberg, Mordell-Weil groups in procyclic extensions of a function field, Duke Math. J. 89 (2) (1997) 217–224.
- [6] L.A. Fastenberg, Computing Mordell-Weil ranks of cyclic covers of elliptic surfaces, Proc. Amer. Math. Soc. 129 (7) (2001) 1877–1883.
- [7] K. Kodaira, On compact complex analytic surface I, Ann. of Math. 71 (1960) 111–152, vol. II, Ann. of Math. 77 (1963) 563–626, vol. III, Ann. of Math. 78 (1963) 1–40.
- [8] B. Mazur, Rational points of abelian varieties with values in towers of number fields, Invent. Math. 18 (1972) 183–266.
- [9] A. Pál, Hodge theory and the Mordell-Weil rank of elliptic curves over extensions of function fields, J. Number Theory 137 (2014) 166–178.

- [10] U. Schmickler-Hirzebruch, Elliptische Flächen über $P_1\mathbb{C}$ mit drei Ausnahmefasern und die hypergeometrische Differentialgleichung, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie 33, Universität Münster, Mathematisches Institut, Münster, 1985, 170 pp.
- [11] T. Shioda, An explicit algorithm for computing the Picard number of certain algebraic surfaces, *Amer. J. Math.* 108 (2) (1986) 415–432.
- [12] T. Shioda, On the Mordell-Weil lattices, *Comment. Math. Univ. St. Pauli* 39 (2) (1990) 211–240.
- [13] J.H. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, vol. 151, Springer, New York, 1994.
- [14] J.H. Silverman, A bound for the Mordell-Weil rank of an elliptic surface after a cyclic base extension, *J. Algebraic Geom.* 9 (2) (2000) 301–308.
- [15] J.H. Silverman, The rank of elliptic surfaces in unramified Abelian towers, *J. Reine Angew. Math.* 577 (2004) 153–169.
- [16] P. Stiller, The Picard number of elliptic surfaces with many symmetries, *Pacific J. Math.* 128 (1) (1987) 157–189.
- [17] D. Ulmer, Elliptic curves with large rank over function fields, *Ann. of Math.* (2) 155 (1) (2002) 295–315.