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Even Unimodular Gaussian Lattices of Rank 12

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We classify even unimodular Gaussian lattices of rank 12, that is, even unimodular integral lattices of rank 12 over the ring of Gaussian integers. This is equivalent to the classification of the automorphisms τ with $\tau^2 = -1$ in the automorphism groups of all the Niemeier lattices, which are even unimodular (real) integral lattices of rank 24. There are 28 even unimodular Gaussian lattices of rank 12 up to equivalence.

Key Words: Niemeier lattice; Hermitian form; root system; automorphism group; Weyl group.

1. INTRODUCTION

Let $\mathbb{C}^n$ be an n-dimensional vector space over $\mathbb{C}$ which possesses the Hermitian form

$$(v, u) = v_1\bar{u}_1 + \cdots + v_n\bar{u}_n, \quad v = (v_1, \ldots, v_n), \quad u = (u_1, \ldots, u_n),$$

where $\bar{u}_k$ denotes the complex conjugate of u_k. A Gaussian lattice of rank n is a $\mathbb{Z}[i]$-module containing a basis of $\mathbb{C}^n$. A Gaussian lattice L is said to be integral (resp. even), if $(z, z') \in \mathbb{Z}[i]$ for all $z, z' \in L$ (resp. $\|z\|^2 \in 2\mathbb{Z}$ for all $z \in L$). Moreover L is unimodular if and only if the determinant of the Gram matrix with respect to some basis is equal to 1. Iyanaga [4] constructed an even unimodular Gaussian lattice of rank 4, and proved its uniqueness up to equivalence. The purpose of this paper is to classify the equivalence classes of even unimodular Gaussian lattices of rank ≤ 12.

An even unimodular Gaussian lattice L can be regarded as a $\mathbb{Z}$-module of rank $2n$. We denote this $\mathbb{Z}$-module by $\psi(L)$, and we regard $\psi(L)$ as a

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submodule of the Euclidean space $\mathbb{R}^{2n}$ whose symmetric bilinear form $(\cdot,\cdot)_{\mathbb{R}}$ is defined as the real part of the Hermitian form. Then $\psi(L)$ becomes an even unimodular lattice in $\mathbb{R}^{2n}$. The multiplication by i acts on $\psi(L)$ as an automorphism satisfying $i^2 = -1$. The original Hermitian form can be recovered from its real part and the action of i on $\psi(L)$ (see Lemma 2.1).

A unimodular lattice in $\mathbb{R}^{2n}$ possessing an automorphism τ with $\tau^2 = -1$ is called symplectic [1, Appendix 2]. In Section 2, we will show that all even unimodular Gaussian lattices in $\mathbb{C}^n$ are obtained from even unimodular lattices in $\mathbb{R}^{2n}$, together with an automorphism τ satisfying $\tau^2 = -1$. Moreover, isomorphisms among the Gaussian lattices are determined by the conjugacy of elements τ in the automorphism groups of lattices in $\mathbb{R}^{2n}$ satisfying $\tau^2 = -1$.

An even unimodular lattice in $\mathbb{R}^{2n}$ exists if and only if $n \equiv 0 \pmod 4$. If $n = 12$ (resp. 8, 4), then all even unimodular lattices in $\mathbb{R}^{2n}$ have been classified, and the number of the equivalence classes is 24 (resp. 2,1). An even unimodular lattice N in $\mathbb{R}^{24}$ is called a Niemeier lattice, and is characterized by its root system. The automorphism group $\mathrm{Aut}(N)$ contains the Weyl group. We will collect some properties of Weyl groups in Section 3. In Section 4, we will determine the conjugacy classes of τ ($\tau^2 = -1$) in the automorphism groups. We will need some information about $\mathrm{Aut}(N)$ and N/R, which can be found in Chapters 16 and 18 [7] in [3].

We would like to point out the similarity of the present work and that of Hashimoto–Sibner [4]. They considered the imaginary part of the Hermitian form, which becomes a non-degenerate alternating form. Such an alternating form is uniquely determined by the dimension $2n$, and thus there exists a one-to-one correspondence between the equivalence classes of even unimodular Gaussian lattices and the conjugacy classes of τ with $\tau^2 = -1$ in the symplectic group $Sp(2n,\mathbb{Z})$. This result is simple and beautiful, but it seems to be difficult to classify the conjugacy classes of τ. Our one-to-one correspondence is more complicated, but we can use the classification theorem of the even unimodular lattices of rank 24 by Niemeier [6]. Together with the knowledge of their automorphism groups, we can carry out the classification of even unimodular Gaussian lattices of rank 12. For higher dimensional cases, our approach does not work because the number of the equivalence classes becomes too large.

2. REAL LATTICES AND GAUSSIAN LATTICES

In this section, we will describe the connection between real lattices and Gaussian lattices. We use the same notation $(\cdot,\cdot), (\cdot,\cdot)_{\mathbb{R}}$ as in the Introduction.

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$, $\{e_1, f_1, \ldots, e_n, f_n\}$ be orthonormal bases of $\mathbb{C}^n$, $\mathbb{R}^{2n}$, respectively. Define $\psi : \mathbb{C}^n \to \mathbb{R}^{2n}$ by

$$\psi\left(\sum_{j=1}^{n} (x_j + iy_j)\varepsilon_j\right) = \sum_{j=1}^{n} (x_j e_j + y_j f_j),$$

where $x_j, y_j \in \mathbb{R}$, $j = 1, \ldots, n$. The following lemma is immediate from this definition.

LEMMA 2.1. *If $z, z' \in \mathbb{C}^n$, then*

$$(z, z') = (\psi(z), \psi(z'))_{\mathbb{R}} + i(\psi(z), \psi(iz'))_{\mathbb{R}}.$$

In particular, $\|\psi(z)\| = \|z\|$ holds for all $z \in \mathbb{C}^n$.

Let L be a Gaussian lattice in $\mathbb{C}^n$. Then $\psi(L)$ is a lattice in $\mathbb{R}^{2n}$. The following is immediate from Lemma 2.1.

LEMMA 2.2. *Let $L \subset \mathbb{C}^n$ be a Gaussian lattice. Then L is integral (resp. even) if and only if $\psi(L)$ is integral (resp. even).*

LEMMA 2.3. *Let $L \subset \mathbb{C}^n$ be a Gaussian lattice. Then L is unimodular if and only if $\psi(L)$ is unimodular.*

Proof. Let $\{z_1, \ldots, z_n\}$ be a basis of L, and let $G = A + iB$ be the Gram matrix with respect to this basis, where $A, B \in M_n(\mathbb{R})$. Then by Lemma 2.1, the Gram matrix of the lattice $\psi(L)$ with respect to the basis

$$\{\psi(z_1), \ldots, \psi(z_n), \psi(iz_1), \ldots, \psi(iz_n)\}$$

is given by

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix},$$

whose determinant is $|\det G|^2$. Therefore, L is unimodular if and only if $\psi(L)$ is unimodular. ∎

Let $O(2n,\mathbb{R})$, $U(n,\mathbb{C})$ denote the orthogonal group, the unitary group, respectively. Define $\tau_0 \in O(2n,\mathbb{R})$ by

$$\tau_0(e_1, f_1, \ldots, e_n, f_n) = (e_1, f_1, \ldots, e_n, f_n)\begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

Then $\psi^{-1}\tau_0\psi$ is the multiplication by i, and we have the following:

LEMMA 2.4. *The centralizer of* τ_0 *in* $O(2n,\mathbb{R})$ *is* $\psi U(n,\mathbb{C})\psi^{-1}$.

Let $\mathscr{L}$ be the set of Gaussian lattices in $\mathbb{C}^n$, and put

$$\bar{\mathscr{L}} = \{[L] \mid L \in \mathscr{L}\},$$

where $[L]$ denotes the set of lattices equivalent to L. Let $\mathscr{M}$ be the set of pairs (M,τ), where M is a lattice in $\mathbb{R}^{2n}$, τ is an automorphism of M satisfying $\tau^2 = -1$. We define an equivalence relation on $\mathscr{M}$ by

$$(M,\tau) \sim (M',\tau') \Leftrightarrow \exists\rho \in O(2n,\mathbb{R}) \qquad \text{such that } \rho(M) = M', \quad \rho\tau\rho^{-1} = \tau'.$$

Denote by $[M,\tau]$ the equivalence class containing the pair $(M,\tau) \in \mathscr{M}$, and put

$$\bar{\mathscr{M}} = \{[M,\tau] \mid (M,\tau) \in \mathscr{M}\}.$$

THEOREM 2.5. *There exists a bijection* Ψ *between* $\bar{\mathscr{L}}$ *and* $\bar{\mathscr{M}}$ *defined by* $\Psi : [L] \mapsto [\psi(L),\tau_0]$.

Proof. If $L \in \mathscr{L}$, then $\psi(L)$ is a lattice in $\mathbb{R}^{2n}$ and $\tau_0\psi(L) = \psi(\psi^{-1}\tau_0\psi(L)) = \psi(iL) = \psi(L)$, so τ_0 is an automorphism of $\psi(L)$. Thus $(\psi(L),\tau_0) \in \mathscr{M}$. If $L, L' \in \mathscr{L}$ are isomorphic, then there exists an element $\sigma \in U(n,\mathbb{C})$ such that $\sigma(L) = L'$. Putting $\rho = \psi\sigma\psi^{-1} \in O(2n,\mathbb{R})$, we see that ρ commutes with τ_0 by Lemma 2.4, and $\rho(\psi(L)) = \psi(L')$. Thus $[\psi(L),\tau_0] = [\psi(L'),\tau_0]$. This establishes the well-definedness of the mapping Ψ.

Next we show that Ψ is injective. Suppose $L, L' \in \mathscr{L}$ and $[\psi(L),\tau_0] = [\psi(L'),\tau_0]$. Then there exists an element $\rho \in O(2n,\mathbb{R})$ such that $\rho(\psi(L)) = \psi(L')$ and $\rho\tau_0\rho^{-1} = \tau_0$. Putting $\sigma = \psi^{-1}\rho\psi$, Lemma 2.4 implies $\sigma \in U(n,\mathbb{C})$, and we have $\sigma(L) = L'$. Thus $[L] = [L']$.

Finally, we show that Ψ is surjective. Suppose $(M, \tau) \in \mathscr{M}$. Since $\tau^2 = -1$, there exists a basis $\{e'_1, f'_1, \ldots, e'_n, f'_n\}$ of $\mathbb{R}^{2n}$ such that

$$\mathbb{R}^{2n} = \langle e'_1, f'_1 \rangle \perp \cdots \perp \langle e'_n, f'_n \rangle$$

and

$$\tau(e'_1, f'_1, \ldots, e'_n, f'_n) = (e'_1, f'_1, \ldots, e'_n, f'_n) \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

Observe $\|f'_j\| = \|\tau(e'_j)\| = \|e'_j\|$ and

$$\begin{aligned} (e'_j, f'_j)_{\mathbb{R}} &= \tfrac{1}{2}(e'_j, f'_j)_{\mathbb{R}} - \tfrac{1}{2}(f'_j, -e'_j)_{\mathbb{R}} \\ &= \tfrac{1}{2}(e'_j, f'_j)_{\mathbb{R}} - \tfrac{1}{2}(\tau(e'_j), \tau(f'_j))_{\mathbb{R}} \\ &= 0 \end{aligned}$$

for $j = 1, \ldots, n$. Thus, we may assume without loss of generality that $\{e'_1, f'_1, \ldots, e'_n, f'_n\}$ is an orthonormal basis. Define $\phi \in O(2n, \mathbb{R})$ by $\phi(e'_j) = e_j$, $\phi(f'_j) = f_j$, $j = 1, \ldots, n$. Then we have $\phi\tau\phi^{-1} = \tau_0$, so $[M, \tau] = [\phi(M), \tau_0]$. Therefore $\Psi([\psi^{-1}(\phi(M))]) = [M, \tau]$. ∎

Theorem 2.5 shows that all (even, integral, unimodular) Gaussian lattices in $\mathbb{C}^n$ are obtained from (even, integral, unimodular) lattices in $\mathbb{R}^{2n}$, and isomorphisms among the Gaussian lattices are determined by the conjugacy of elements τ in the automorphism groups of lattices in $\mathbb{R}^{2n}$ satisfying $\tau^2 = -1$.

3. ROOT LATTICES

In this section, we consider the irreducible root lattice $L = L(X)$ of type X ($X = A_n (n \geq 1)$, $D_n (n \geq 4)$, $E_n (n = 6, 7, 8)$) and its Weyl group $W(X)$. We fix the presentation of these lattices as follows:

$$L(A_n) = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{Z}^{n+1} \mid \Sigma x_i = 0\},$$

$$L(D_n) = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \mid \Sigma x_i \equiv 0 \bmod 2\},$$

$$L(E_8) = \langle\ L(D_8),\ (1/2)(1,1,1,1,1,1,1,1)\ \rangle,$$

$$L(E_7) = \{(x_1, x_2, \ldots, x_8) \in L(E_8) \mid x_7 = x_8\},$$

$$L(E_6) = \{(x_1, x_2, \ldots, x_8) \in E_8 \mid x_6 = x_7 = x_8\}.$$

We denote by L^* the dual lattice of L. Then the quotient L^*/L has the following structure:

$$L^*/L \cong \begin{cases} \mathbb{Z}_{n+1} & \text{if } X = A_n, \\ \mathbb{Z}_4 & \text{if } X = D_{2k+1}, \\ \mathbb{F}_4 & \text{if } X = D_{2k}, \\ \mathbb{Z}_k & \text{if } X = E_{9-k}\ (k = 1,2,3), \end{cases}$$

where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ $(\omega^2 + \omega + 1 = 0)$. If $X = D_{2k}$, the above isomorphism is given as follows:

$$1 \leftrightarrow (1,0,\ldots,0), \quad \omega \leftrightarrow \tfrac{1}{2}(1,1,\ldots,1), \qquad \omega^2 \leftrightarrow \tfrac{1}{2}(-1,1,\ldots,1).$$

In other cases, the generator of L^*/L is as follows:

$$\begin{aligned} \tfrac{1}{n+1}(-n,1,\ldots,1) \quad & \text{if } X = A_n, \\ \tfrac{1}{2}(1,1,\ldots,1) \quad & \text{if } X = D_{2k+1}, \\ \tfrac{1}{3}(0,0,0,0,0,1,1,1) \quad & \text{if } X = E_6, \\ \tfrac{1}{2}(0,0,0,0,0,0,1,1) \quad & \text{if } X = E_7. \end{aligned}$$

The lattice vectors of squared length 2 are called the roots. The Weyl group $W(X)$ is the group generated by the reflections with respect to the roots. It is easily verified that $W(X)$ acts trivially on L^*/L. The (full) automorphism group of $L(X)$ is generated by $W(X)$ and the graph automorphisms. The (outer) graph automorphisms γ are constructed as the symmetries of the Dynkin diagram of X.

The root lattice $L(D_n)$ has an outer graph automorphism of order 2, and furthermore the root lattice $L(D_4)$ has those of order 3. These are represented by the following:

$$\text{(order 2)}\begin{pmatrix} -1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & & 0 & 1 \end{pmatrix}, \qquad \text{(order 3)}\ \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

The actions of graph automorphisms γ on L^*/L are described as follows:

the field automorphism on $\mathbb{F}_4$	$(X = D_{2k},\ \lvert\gamma\rvert = 2)$,
the multiplication by -1	$(X = D_{2k+1},\ \lvert\gamma\rvert = 2)$,
the multiplication by ω	$(X = D_4,\ \lvert\gamma\rvert = 3)$.

The root lattice of type A_n or E_6 also has outer graph automorphisms of order 2, which is obtained as the product of -1 and some element of the Weyl group.

LEMMA 3.1. *Let $W = W(X)$ be the Weyl group of type X, and set $W^* = W^*(X) = \mathrm{Aut}(L(X))$, which is generated by W and the graph automorphisms of X.*

(1) *If X is one of A_k, D_{2k+1}, E_6, E_7, then W^* contains no elements τ* with $\tau^2 = -1$.

(2) *If $X = E_8$ or $X = D_{2k}$, then W^* contains an element τ with $\tau^2 = -1$, which is contained in W if $X = E_8$ or D_{2k} with k even, and is contained in $W^* \backslash W$ if $X = D_{2k}$ with k odd. Moreover τ is uniquely determined up to the conjugation by an element of W.*

Proof.

(1) If X is one of A_k, D_{2k+1}, E_6, E_7, then $\langle -1 \rangle$ is a factor of direct product in W^*. Hence the statement holds.

(2) Suppose $X = D_{2k}$. Then W^* is the semi-direct product of the symmetric group (the group of permutation matrices) of degree $2k$ and the group consisting of the diagonal matrices with all entries ± 1.

Let τ be an element of W^* satisfying $\tau^2 = -1$. Then it is easily verified that τ is conjugate to

$$\hat{\tau} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus k}$$

in W. Set $\alpha = (1\ 2)\cdots(2k-1\ 2k) \in W$. Then the product $\alpha\hat{\tau}$ is a diagonal matrix such that the number of -1's in the diagonal entries is k. Thus $\tau \in W$ if and only if k is even. This proves (2) for the case $X = D_{2k}$.

Since $W^*(E_8) = W(E_8)$ contains $W(D_8)$ with odd index, the statement for E_8 is deduced from that for D_8. ∎

If $\tilde{L}$ is an integral lattice containing some direct sum of root lattices $L_1 \oplus \cdots \oplus L_m$, then $\tilde{L}$ is contained in $L_1^* \oplus \cdots \oplus L_m^*$. The quotient $\tilde{L}/(L_1 \oplus \cdots \oplus L_m)$ is a $\mathbb{Z}$-submodule (or equivalently, a subgroup) of $L_1^*/L_1 \oplus \cdots \oplus L_m^*/L_m$. This submodule is called the glue code for $\tilde{L}$ (with respect to $L_1 \oplus \cdots \oplus L_m$).

4. CLASSIFICATION OF GAUSSIAN LATTICES

In this section, we will classify the even unimodular Gaussian lattices of rank 12. At the end of this section, we also show that there are three even unimodular Gaussian lattices of rank 8 up to equivalence. Recall that an even unimodular lattice of rank 24 is called a Niemeier lattice. By Theorem 2.5, it suffices to classify conjugacy classes of elements τ with $\tau^2 = -1$ in $\mathrm{Aut}(N)$ for each Niemeier lattice N. Every Niemeier lattice is uniquely characterized by its root system, which is one of the following:

$$\emptyset, A_1^{24}, A_2^{12}, A_3^8, A_4^6, D_4^6, A_5^4D_4, A_6^4, A_7^2D_5^2, A_8^3, D_6^4, A_9^2D_6, E_6^4,$$
$$A_{11}D_7E_6, A_{12}^2, D_8^3, A_{15}D_9, D_{10}E_7^2, A_{17}E_7, D_{12}^2, A_{24}, E_8^3, D_{16}E_8, D_{24}.$$

We denote by $N(R)$ the Niemeier lattice with root system R. The lattice $N(\emptyset)$ is isomorphic to the Leech lattice and $\mathrm{Aut}(N(\emptyset))$ is the Conway group $\cdot O$. By Conway *et al.* [2], we have the following:

PROPOSITION 4.1. *There exists just one conjugacy class of elements τ with $\tau^2 = -1$ in $\cdot O$ which corresponds to the class of $2B$-elements in the simple group Co_1.*

For the remainder of this section, we assume $R \neq \emptyset$, and $N = N(R)$ will denote the Niemeier lattice with root system R. Then R is written as a union of irreducible root systems A_n, D_n, E_6, E_7, E_8, which are called the components of R. Let R' be the union of the components of type A_1, D_{2k}, E_7, E_8, and $R'' = R \backslash R'$.

By Chapters 4 and 16 in [3], $\mathrm{Aut}(N)$ contains normal subgroups $G_{01} > G_0$, where G_0 is the subgroup generated by the reflections with respect to the elements of R and G_{01} is the component-wise stabilizer of R. The group G_0 is the direct product of the Weyl groups for the components of R. Set $G_1 = G_{01}/G_0$ and $G_2 = \mathrm{Aut}(N)/G_{01}$. The group G_2 is a permutation group on the

Table I

Table 16.1 in [3] and our Results

Root system R	$\|G_\infty\|$	$\|G_1\|$	$\|G_2\|$	Case 1	Case 2
D_{24}	2	1	1	1	
$D_{16}E_8$	2	1	1	1	
E_8^3	1	1	6	1	1
A_{24}	5	2	1		
D_{12}^2	4	1	2	1	1
$A_{17}E_7$	6	2	1		
$D_{10}E_7^2$	4	1	2		1
$A_{15}D_9$	8	2	1		
D_8^3	8	1	6	1	1
A_{12}^2	13	2	2		1×2
$A_{11}D_7E_6$	12	2	1		
E_6^4	9	2	24		1
$A_9^2D_6$	20	2	2		1×2
D_6^4	16	1	24		2
A_8^3	27	2	6		
$A_7^2D_5^2$	32	2	4		1
A_6^4	49	2	12		1
$A_5^4D_4$	72	2	24		1
D_4^6	64	3	720	1	2
A_4^6	125	2	120		1×2
A_3^8	256	2	1344		1
A_2^{12}	729	2	$\|M_{12}\|$		1
A_1^{24}	4096	1	$\|M_{24}\|$		1
$\emptyset$	—	1	1	1	
Total				28	

set of the components. The orders of G_1, G_2 are listed in Table 16.1 of [3]. We will list them in Table I together with our results.

LEMMA 4.2. *If $R \neq D_4^6$, then G_{01} is generated by G_0 and -1. Moreover if $|G_1| = 2$, then G_{01} is the direct product of G_0 and $\langle -1 \rangle$.*

Proof. Suppose $R \neq D_4^6$. If $R = R'$, then the Weyl group G_0 contains -1 and furthermore $|G_1| = 1$ by Table 1. On the other hand, if $R \neq R'$, then G_0 does not contain -1 and $|G_1| = 2$ by Table I. Hence -1 is contained in $G_{01} \setminus G_0$ and the statement holds. ∎

Remark 4.3. For each component R_0 of R', G_0 contains $-1_{\langle R_0 \rangle} \oplus 1_{\langle R \setminus R_0 \rangle}$. Moreover G_{01} contains $1_{\langle R' \rangle} \oplus -1_{\langle R'' \rangle}$. Lemma 4.2 implies that these elements generate the center of G_{01}.

We will classify the automorphisms $\tau \in \mathrm{Aut}(N)$ satisfying $\tau^2 = -1$.

Case 1. $\tau \in G_{01}$: First suppose $\tau \in G_{01}$. Then Lemma 4.2 forces $|G_1| \neq 2$. By Table 1, we see that $|G_1|$ is odd, and hence $\tau \in G_0$. By Lemma 3.1 the following proposition is obtained.

PROPOSITION 4.4 *If R is one of*

$$D_4^6, D_8^3, D_{12}^2, E_8^3, D_{16}E_8, D_{24},$$

then G_{01} contains just one conjugacy class of τ with $\tau^2 = -1$. If R is not any of them, then G_{01} does not contain such automorphisms.

Case 2. $\tau \notin G_{01}$: Next suppose $\tau \notin G_{01}$. We may assume

$$R \neq A_{11}D_7E_6, A_{15}D_9, A_{17}E_7, A_{24}, D_{16}E_8, D_{24},$$

because $G_2 = 1$ in these cases. Moreover, we may assume

$$R \neq A_8^3$$

by the following lemma, which is an easy consequence of Lemma 3.1.

LEMMA 4.5. *Let R_0 be a component fixed by τ. Then $R_0 = E_8$ or D_{2k} for some integer $k > 1$. Moreover, the action of τ on R_0 is uniquely determined up to the conjugation of an element of $W(R_0)$. In particular, τ is conjugate to $-1_{R_0}\tau$ in* $\mathrm{Aut}(N)$.

LEMMA 4.6. *Suppose that $\tau_1, \tau_2 \in \mathrm{Aut}(N)\backslash G_{01}$ satisfy $\tau_i^2 = -1$ and $\tau_1 G_{01} = \tau_2 G_{01}$. If $|G_1| = 1$ or 3, then τ_2 is conjugate to τ_1. If $|G_1| = 2$, then τ_2 is conjugate to τ_1 or $-\tau_1$.*

Proof. Let τ_1, τ_2 be as in the statement. Then $\tau_2\tau_1^{-1} = -\tau_2\tau_1 \in G_{01}$. By Lemma 4.5, we may assume $\tau_1|_{R_0} = \tau_2|_{R_0}$ for any fixed component R_0. Let R_1, R_2 be any components such that $\tau_1(R_1) = R_2$. Let Φ_1 be a fundamental root system of R_1, and set $\Phi_2 = \tau_1(\Phi_1)$. By the regularity of the action of the Weyl group on its fundamental systems, there exists a unique element $u \in W(R_1)$ ($\subset \mathrm{Aut}(N)$) satisfying $-\tau_2\tau_1(\Phi_1) = u(\Phi_1)$. Since $u|_{R_2} = 1_{R_2}$, we have $u^{-1}\tau_2 u\tau_1^{-1}(\Phi_1) = \Phi_1$ and

$$\begin{aligned} u^{-1}\tau_2 u\tau_1^{-1}(\Phi_2) &= u^{-1}\tau_2 u(\Phi_1) \\ &= u^{-1}\tau_2(\tau_2\tau_1^{-1}(\Phi_1)) \\ &= u^{-1}\tau_1(\Phi_1) \\ &= \Phi_2. \end{aligned}$$

Hence, we may assume $\tau_2\tau_1^{-1}$ preserves a fundamental system of each component of R.

(i) If $|G_1| = 1$ then $G_{01} = G_0$ and we have $\tau_1 = \tau_2$ by the regularity of the action of the Weyl group on its fundamental systems.

(ii) If $|G_1| = 2$, then there is another possibility $\tau_2\tau_1^{-1} \in G_{01}\backslash G_0$. By Lemma 4.2, there exists some $w \in G_0$ satisfying $\tau_2\tau_1^{-1} = -w$. Here w is uniquely determined as the element satisfying $w(\Phi) = -\Phi$ for the fundamental system Φ of each component. Let $w_i = w|_{R_i}$ $(i = 1, 2)$ and regard w_i as elements of G_0. Moreover set $\tau_2' = w_2^{-1}\tau_2 w_2$. Then we have

$$\begin{aligned}
\tau_2'\tau_1^{-1}|_{R_1} &= w_2^{-1}\tau_2 w_2(-\tau_1)|_{R_1}\\
&= w_2^{-1}\tau_2 w_2(w^{-1}\tau_2)|_{R_1}\\
&= w_2^{-1}\tau_2 w_2(w_2^{-1}\tau_2)|_{R_1}\\
&= -w_2^{-1}|_{R_1}\\
&= -1_{R_1}.
\end{aligned}$$

This means $\tau_2'|_{R_1} = -\tau_1|_{R_1}$ and $\tau_2'|_{R_2} = -\tau_1|_{R_2}$. Hence, we may assume $\tau_2' = -\tau_1$ on any component which is not fixed by τ_1. By Lemma 4.5, it is easily proved that τ_2' is conjugate to $-\tau_1$ as required.

(iii) If $|G_1| = 3$, then the other possibility is $\tau_2 = \gamma\tau_1$, where γ is an element of order 3 that gives a generator of G_1. By $\tau_2^2 = -1$, we have $(\gamma\tau_1)^2 = -1$ and thus we have

$$\gamma^{-1}\tau_1\gamma = \gamma(\gamma\tau_1)^2\tau_1^{-1} = -\gamma\tau_1^{-1} = \gamma\tau_1 = \tau_2.$$

This means that τ_2 is conjugate to τ_1. ∎

PROPOSITION 4.7. (1) *If R is one of*

$$A_1^{24}, A_2^{12}, A_3^8, A_5^4 D_4, A_6^4, A_7^2 D_5^2, E_6^4, D_8^3, D_{10}E_7^2, D_{12}^2, E_8^3,$$

then $\mathrm{Aut}(N)\backslash G_{01}$ *contains just one conjugacy class of* τ with $\tau^2 = -1$.

(2) *If $R = A_4^6, A_9^2 D_6$, or A_{12}^2, then* $\mathrm{Aut}(N)\backslash G_{01}$ *contains just two conjugacy classes of τ with $\tau^2 = -1$, and moreover τ is not conjugate to $-\tau$.*

(3) *If $R = D_6^4$ or D_4^6, then* $\mathrm{Aut}(N)\backslash G_{01}$ *contains just two conjugacy classes of τ with $\tau^2 = -1$, and moreover τ is conjugate to $-\tau$.*

Proof. First of all we will give the outline of our proof.

(i) We will determine the action of τ on the components (i.e., the image of τ in G_2), in terms of a conjugacy class of involutions in G_2.

(ii) Let t be such an involution in G_2. Then we will define τ as a transformation which preserves R and whose image in G_2 is equal to t.

(iii) We will prove $\tau \in \mathrm{Aut}(N)$ by showing that τ preserves the glue code $N/L(R)$, whose order is listed as $|G_\infty|$ in Table 1, and whose generators are described in [7]. Then, by Lemma 4.6, the uniqueness of τ up to conjugacy will follow if $|G_1| = 1$ or 3.

(iv) If $|G_1| = 2$, then we will check whether τ is conjugate to $-\tau$ or not. Then the proof will be completed.

In the following proof, we regard $L(A_k)$ (resp. $L(E_6), L(E_7)$) as a lattice embedded in $\mathbb{R}^k$ (resp. $\mathbb{R}^6, \mathbb{R}^7$). We denote by $s(k)$ the transformation represented by

$$\begin{pmatrix} O_k & -I_k \\ I_k & O_k \end{pmatrix}$$

which exchanges two (isomorphic) components of rank k, and denote by $t(k)$ the transformation represented by the same matrix which preserves a component of rank $2k$. Moreover, in the proof of (3), we use the transformation $h(k)$ represented by

$$\begin{pmatrix} O_k & -H_k \\ H_k & O_k \end{pmatrix}, \quad \text{where} \quad H_k = \begin{pmatrix} -1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 0 & 1 \end{pmatrix},$$

which exchanges two components isomorphic to D_k $(k = 4, 6)$. Here notice that H_k is a representation of the graph automorphism of D_k.

(1) If R is one of

$$A_1^{24}, A_2^{12}, A_3^8, A_6^4, A_7^2D_5^2, E_6^4, D_{12}^2,$$

then τ acts fixed-point-freely on the components of R, and if R is one of

$$A_5^4D_4, D_8^3, D_{10}E_7^2, E_8^3,$$

then τ fixes only one component (D_4, D_8, D_{10}, E_8, respectively). By the structure of G_2, the action on the components (the image in G_2) is uniquely determined. In fact, if $R \neq A_1^{24}, A_2^{12}, A_3^8$, the proof is easy because G_2 is a small group. If $R = A_1^{24}$ (resp. A_2^{12}, A_3^8), the group G_2 is M_{24} (resp. $M_{12}, 2^3 : PSL_2(7)$) and its structure is well known, and there is a unique conjugacy class of fixed-point-free involutions.

Now we will prove the assertion case by case.

$R = A_1^{24}$: The glue code is the binary Golay code of length 24. The group $G_2 \cong M_{24}$ contains a unique class of t as above. We may assume $t = (1\ 2)(3\ 4)\cdots(21\ 22)(23\ 24)$. Since $W(A_1)$ contains -1, $\mathrm{Aut}(N)$ contains

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus 12},$$

which is a desired element.

$R = A_2^{12}$: The glue code is the ternary Golay code whose automorphism group is the non-split extention $2 \cdot M_{12}$. This group can be regarded as a subgroup complementary to G_0 in $\mathrm{Aut}(N)$. By the character table in [2], the group $2 \cdot M_{12}$ contains a unique class of the desired element τ. In particular, τ is conjugate to $-\tau$.

$R = A_3^8$: The glue code is a type II $\mathbb{Z}_4$-code of length 8 whose image modulo 2 is the extended Hamming code. Such a $\mathbb{Z}_4$-code is uniquely determined up to isomorphism (and is called the octacode). Its generator matrix is

$$\begin{pmatrix} {}^t x_1 \\ {}^t x_2 \\ {}^t x_3 \\ {}^t x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 3 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 3 & 3 & 1 & 2 \end{pmatrix}.$$

Let $\tau = s(3) \oplus s(3) \oplus s(3) \oplus s(3)$. Then the action of τ on the glue code is represented by the 8×8 matrix

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus 4}.$$

Since $\tau(x_1) = x_2$ and $\tau(x_3) = x_4$, the transformation τ preserves the glue code and thus $\tau \in \mathrm{Aut}(N)$. Moreover, let

$$\sigma = \begin{pmatrix} O_3 & O_3 & I_3 & O_3 \\ O_3 & O_3 & O_3 & -I_3 \\ -I_3 & O_3 & O_3 & O_3 \\ O_3 & I_3 & O_3 & O_3 \end{pmatrix}^{\oplus 2}.$$

Then $\sigma \in \mathrm{Aut}(N)$, since $\sigma(x_1) = -x_3$, $\sigma(x_2) = x_4$. Moreover, it is easily verified that $\tau^\sigma = -\tau$ as required.

$R = A_5^4 D_4$: The glue code is the subgroup generated by $x_1 ={}^t(0,1,2,-1,\omega)$, $x_2 ={}^t(1,1,1,3,0)$, $x_3 ={}^t(3,3,0,0,1)$ of $\mathbb{Z}_6^4 \times \mathbb{F}_4$. Set $\tau = s(5) \oplus s(5) \oplus t(2)$. Since $t(2)$ is an element of the Weyl group, $t(2)$ acts trivially on $\mathbb{F}_4 \cong D_4^*/D_4$. Hence, we have $\tau(x_1) = x_1 + 5x_2$, $\tau(x_2) = 2x_1 + 5x_2$, $\tau(x_3) = x_3$, as desired. Moreover, let

$$\sigma = \begin{pmatrix} O_5 & O_5 & I_5 & O_5 \\ O_5 & O_5 & O_5 & -I_5 \\ -I_5 & O_5 & O_5 & O_5 \\ O_5 & I_5 & O_5 & O_5 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Then $\sigma(x_1) = 5x_1 + 2x_2$, $\sigma(x_2) = 2x_1 + x_2$, $\sigma(x_3) = 3x_2 + x_3$ and further $\tau^\sigma = -\tau$ as required.

$R = A_6^4$: The glue code is the $\mathbb{Z}_7$-code generated by $x_1 ={}^t(1,2,3,0)$, $x_2 ={}^t(0,3,-2,1)$. Set $\tau = s(6) \oplus (-s(6))$. Then we have $\tau(x_1) = 5x_1 + 4x_2$, $\tau(x_2) = 4x_1 + 2x_2$, as desired. Moreover, let

$$\sigma = \begin{pmatrix} O_6 & O_6 & -I_6 & O_6 \\ O_6 & O_6 & O_6 & -I_6 \\ I_6 & O_6 & O_6 & O_6 \\ O_6 & I_6 & O_6 & O_6 \end{pmatrix}.$$

Then $\sigma(x_1) = 4x_1 + 2x_2$, $\sigma(x_2) = 2x_1 + 3x_2$ and further $\tau^\sigma = -\tau$ as required.

$R = E_6^4$: The glue code is the $\mathbb{Z}_3$-code generated by $x_1 ={}^t(1,1,1,0)$, $x_2 ={}^t(0,-1,1,1)$. Set $\tau = s(6) \oplus s(6)$. Then we have $\tau(x_1) = 2x_1 + x_2, \tau(x_2) = x_1 + x_2$, as desired. Moreover, let

$$\sigma = \begin{pmatrix} O_6 & O_6 & I_6 & O_6 \\ O_6 & O_6 & O_6 & -I_6 \\ -I_6 & O_6 & O_6 & O_6 \\ O_6 & I_6 & O_6 & O_6 \end{pmatrix}.$$

Then $\sigma(x_1) = x_1 + x_2$, $\sigma(x_2) = x_1 + 2x_2$ and further $\tau^\sigma = -\tau$ as required.

$R = A_7^2 D_5^2$: The glue code is the subgroup generated by $x_1 ={}^t(3,1,1,0), x_2 ={}^t(2,0,-1,1)$ of $\mathbb{Z}_8^2 \times \mathbb{Z}_4^2$. Set $\tau = s(7) \oplus (-s(5))$. Then we have $\tau(x_1) =$

$3x_1 + 3x_2, \tau(x_2) = 2x_1 + 5x_2$, as desired. Moreover, let

$$\sigma = \begin{pmatrix} O_7 & I_7 \\ I_7 & O_7 \end{pmatrix} \oplus \begin{pmatrix} -I_5 & O_5 \\ O_5 & I_5 \end{pmatrix}.$$

Then $\sigma(x_1) = 3x_1, \sigma(x_2) = 2x_1 + x_2$ and further $\tau^\sigma = -\tau$ as required.

$R = D_8^3$: The glue code is the subgroup generated by $x_1 = {}^t(\omega, 1, 1)$, $x_2 = {}^t(1, \omega, 1)$, $x_3 = {}^t(1, 1, \omega)$ of $\mathbb{F}_4^3$. Notice that it is not an $\mathbb{F}_4$-submodule of $\mathbb{F}_4^3$. Set $\tau = t(4) \oplus s(8)$. Since $t(4)$ acts trivially on $\mathbb{F}_4 \cong D_8^*/D_8$, we have $\tau(x_1) = x_1, \tau(x_2) = x_3$, as desired.

$R = D_{10}E_7^2$: The glue code is the subgroup generated by $x_1 = {}^t(\omega, 1, 0), x_2 = {}^t(\omega^2, 0, 1)$ of $\mathbb{F}_4 \times \mathbb{Z}_2^2$. Set $\tau = t(5) \oplus s(7)$. Since $t(5)$ exchanges ω and ω^2, we have $\tau(x_1) = x_2, \tau(x_2) = x_1$, as desired.

$R = D_{12}^2$: The glue code is the subgroup generated by ${}^t(1, \omega), {}^t(\omega, 1)$ of $\mathbb{F}_4^2$. Clearly, $\tau = s(12)$ is a desired element.

$R = E_8^3$: The glue code is the trivial code, and $\tau = t(4) \oplus s(8)$ is a desired element.

(2) Let $R = A_4^6, A_9^2D_6$, or A_{12}^2. Then the action of τ on the components is uniquely determined up to conjugacy. We will show the existence of τ.

$R = A_4^6$: The glue code is the $\mathbb{Z}_5$-code generated by $x_1 = {}^t(0, 0, 1, 3, 2, 1)$, $x_2 = {}^t(2, 3, 0, 1, 1, 0)$, $x_3 = {}^t(1, 1, 2, 0, 0, 3)$ (these generators are obtained by exchanging the fourth and sixth entries from those in [7]). Let $\tau = s(4) \oplus s(4) \oplus s(4)$. Then we have $\tau(x_1) = 2x_1$, $\tau(x_2) = 2x_3$, $\tau(x_3) = 2x_2$, and thus $\tau \in \mathrm{Aut}(N)$.

We will prove that τ is not conjugate to $-\tau$. Suppose that there exists some $u \in \mathrm{Aut}(N)$ with $\tau^u = -\tau$. Without loss of generality, we may assume that the order of u is a power of 2. Let $\bar{\tau}, \bar{u}$ be the images in G_2 of τ, u, respectively, so that $\bar{\tau} = (1\ 2)(3\ 4)(5\ 6)$. Since $|G_2/G_2'| = 2$ ($G_2 \cong PGL_2(5) \cong S_5$) and $\bar{\tau} \notin G_2'$, one of $\bar{u}$, $\bar{u}\bar{\tau}$ belongs to G_2'. Hence we may assume $\bar{u} \in G_2'$. This implies that $\bar{u}$ is an even permutation, and hence $\bar{u}$ is the product of two transpositions. One can check easily that

$$\sigma = \begin{pmatrix} O_4 & O_4 & O_4 & O_4 & I_4 & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & I_4 \\ O_4 & -I_4 & O_4 & O_4 & O_4 & O_4 \\ I_4 & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & I_4 & O_4 & O_4 \\ O_4 & O_4 & -I_4 & O_4 & O_4 & O_4 \end{pmatrix}$$

is an element of $\mathrm{Aut}(N)$ centralizing τ. Replacing u by u^σ or u^{σ^2} if necessary, we may assume that $\bar{u}$ fixes 1 and 2. Then, as G_2 does not contain a transposition, we see that $\bar{u} \neq (3\ 4)(5\ 6)$. Since $\tau^u = -\tau$, u has the form

$$\begin{pmatrix} X & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & -X & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & O_4 & Y & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & -Y \\ O_4 & O_4 & Z & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & -Z & O_4 & O_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} X & O_4 & O_4 & O_4 & O_4 & O_4 \\ O_4 & -X & O_4 & O_4 & O_4 & O_4 \\ O_4 & O_4 & O_4 & O_4 & O_4 & Y \\ O_4 & O_4 & O_4 & O_4 & Y & O_4 \\ O_4 & O_4 & O_4 & Z & O_4 & O_4 \\ O_4 & O_4 & Z & O_4 & O_4 & O_4 \end{pmatrix},$$

so that the action of u on the glue code is represented by

$$\begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon' & 0 \\ 0 & 0 & 0 & 0 & 0 & -\varepsilon' \\ 0 & 0 & \varepsilon'' & 0 & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon'' & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon' \\ 0 & 0 & 0 & 0 & \varepsilon' & 0 \\ 0 & 0 & 0 & \varepsilon'' & 0 & 0 \\ 0 & 0 & \varepsilon'' & 0 & 0 & 0 \end{pmatrix},$$

respectively, where $\varepsilon, \varepsilon', \varepsilon'' = \pm 1$. One can check that such an element never preserves the glue code for any choice of $\varepsilon, \varepsilon', \varepsilon''$. This is a contradiction.

$R = A_{12}^2$: The glue code is the $\mathbb{Z}_{13}$-code generated by $x = {}^t(1, 5)$. Let $\tau = s(12)$. Then $\tau(x) = 8x$ and thus $\tau \in \mathrm{Aut}(N)$. We will prove that τ is not conjugate to $-\tau$. Suppose that there exists some $u \in \mathrm{Aut}(N)$ with $\tau^u = -\tau$. Since $\mathrm{Aut}(N) = G_0\langle\tau\rangle$, we may assume $u \in G_0$. Hence u is represented by

$$\begin{pmatrix} X & O_{12} \\ O_{12} & Y \end{pmatrix}$$

for some 12×12 matrices X, Y. By $\tau^u = -\tau$, we have $Y = -X$. This means X and $-X$ are contained in the Weyl group $W(A_{12})$, and contradicts the fact $-1 \notin W(A_{12})$.

$R = A_9^2 D_6$: The glue code is the subgroup generated by $x_1 = {}^t(1, 2, \omega)$, $x_2 = {}^t(5, 5, 1)$ of $\mathbb{Z}_{10} \times \mathbb{F}_4$. Set $\tau = s(9) \oplus t(3)$. Since $t(3)$ is not contained in the Weyl group, $t(3)$ exchanges ω and ω^2. Hence we have $\tau(x_1) = 3x_1 + x_2, \tau(x_2) = x_2$, as desired. By the same argument as in the case $R = A_{12}^2$, we can prove that τ is not conjugate to $-\tau$.

(3) $R = D_6^4$: The group G_2 is isomorphic to S_4 and contains just two conjugacy classes of involutions. The glue code is the $\mathbb{F}_4$-code generated by $x_1 = {}^t(1, 1, 1, 1), x_2 = {}^t(0, 1, \omega^2, \omega)$.

First we define $\tau_1 = s(6) \oplus s(6)$, which acts linearly on the glue code. Then we have $\tau_1(x_1) = x_1, \tau_1(x_2) = x_1 + x_2$, as desired. Next we will define $\tau_2 = t(3) \oplus t(3) \oplus h(6)$, which acts semilinearly on the glue code. Since the matrix H_k gives a graph automorphism of D_6, the action of τ_2 on the glue code is the composition of the permutation (3 4) and the field automorphism on $\mathbb{F}_4$ (see Section 3). Hence we have $\tau_2(x_1) = x_1, \tau_2(x_2) = x_2$ as desired.

$R = D_4^6$: The group G_2 is isomorphic to S_6 and contains three conjugacy classes of involutions whose type of permutation are 2^3, $2^2 1^2$ and $2^1 1^4$.

The glue code is the hexacode over $\mathbb{F}_4$ with generator matrix

$$\begin{pmatrix} {}^t x_1 \\ {}^t x_2 \\ {}^t x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & \omega^2 & \omega \\ 0 & 1 & 0 & 1 & \omega & \omega^2 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We define τ_1 (resp. τ_2) corresponding to type 2^3 (resp. $2^2 1^2$) by $h(4) \oplus h(4) \oplus h(4)$ (resp. $s(4) \oplus s(4) \oplus t(2) \oplus t(2)$), whose action on the glue code is linear (resp. semilinear). Then we have

$$\tau_i(x_1) = x_2 + x_3, \quad \tau_i(x_2) = x_1 + x_3, \quad \tau_i(x_3) = x_3 \quad (i = 1, 2)$$

as desired.

Finally, we prove the non-existence of the automorphism τ_3 corresponding to the involution of type $2^1 1^4$. If it exists, τ_3 fixes four components and, by Lemma 3.1, its action on each component is contained in the Weyl group. Hence, $\bar{\tau_3}$ acts trivially on the corresponding four coordinates of the glue code. Since each codeword is uniquely determined by its three coordinates, the above property contradicts that $\bar{\tau_3}$ is an involution. ∎

By Propositions 4.1, 4.4 and 4.7 (see Table I as a summary), we have the following theorems.

THEOREM 4.8. *A Niemeier lattice is symplectic unless its root system is one of the following*:

$$A_8^3, A_{11}D_7E_6, A_{15}D_9, A_{17}E_7, A_{24}.$$

THEOREM 4.9. *Up to equivalence, there exist exactly* 28 *even unimodular Gaussian lattices of rank* 12.

Among them, just three lattices, which are obtained from E_8^3, $D_{16}E_8$, are decomposable. Such a decomposable lattice is obtained as a sum of two lattices of rank 4 and rank 8. Since the even unimodular Gaussian lattice of rank 4 is uniquely determined up to equivalence, we have proved the following:

COROLLARY 4.10. *Up to equivalence, there exist exactly three even unimodular Gaussian lattices of rank* 8.

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