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Journal of Number Theory

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Higher order SPT functions for overpartitions, overpartitions with smallest part even, and partitions with smallest part even and without repeated odd parts



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ARTICLE INFO

Article history:

Received 23 March 2014

Received in revised form 24 June 2014

Accepted 7 October 2014

Available online 8 December 2014

Communicated by David Goss

Keywords:

Number theory

Partitions

Overpartitions

Rank moments

Crank moments

Andrews' spt-function

Smallest parts function

Higher order spt functions

ABSTRACT

We consider the symmetrized moments of three ranks and cranks, similar to the work of Garvan in [17] for the rank and crank of a partition. By using Bailey pairs and elementary rearrangements, we are able to find useful expressions for these moments. We then deduce inequalities between the corresponding ordinary moments. In particular we prove that the crank moment for overpartitions is always larger than the rank moment for overpartitions, $\overline{M}_{2k}(n) > \overline{N}_{2k}(n)$; with recent asymptotics this was known to hold for sufficiently large values of n for each fixed k . Lastly we provide higher order spt functions for overpartitions, overpartitions with smallest part even, and partitions with smallest part even and no repeated odds.

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<http://dx.doi.org/10.1016/j.jnt.2014.10.011>

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1. Introduction

Here we consider certain rank and crank moments for partition like functions and how they relate to smallest parts functions. In particular we prove inequalities between certain moments and define higher order smallest parts functions as the difference of symmetrized moments.

We begin by looking at the ordinary partition function. We recall a partition of n is a non-increasing sequence of positive integers that sum to n . We denote the number of partitions of n by $p(n)$. We see $p(4) = 5$ since the partitions of 4 are 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. In [3] Andrews defined $\text{spt}(n)$ to be the total number of occurrences of the smallest parts in the partitions of n . Thus from the partitions of 4, we see $\text{spt}(4) = 10$.

We recall the rank of a partition is the largest parts minus the number of parts. The crank of a partition is the largest part if there are no ones and otherwise is the number of parts larger than the number of ones minus the number of ones. The first point of interest of the rank and crank of a partition is that the rank gives a combinatorial explanation of the well known congruences $p(5n+4) \equiv 0 \pmod{5}$ and $p(7n+5) \equiv 0 \pmod{7}$ and the crank gives a combinatorial explanation of $p(5n+4) \equiv 0 \pmod{5}$, $p(7n+5) \equiv 0 \pmod{7}$, and $p(11n+6) \equiv 0 \pmod{11}$. Specifically, if we group the partitions of $5n+4$ according to their rank (or crank) modulo 5, we get 5 equally sized sets and so $p(5n+4) \equiv 0 \pmod{5}$. Similarly if we group the partitions of $7n+5$ according to their rank (or crank) modulo 7, we get 7 equally sized sets. If we group the partitions of $11n+6$ according to their rank modulo 11 we do not in general get 11 equally sized sets, however we do get 11 equally sized sets if we group by the crank modulo 11. As we will see shortly, the rank and crank have other uses as well. We let $N(m, n)$ denote the number of partitions of n with rank m and $M(m, n)$ denote the number of partitions of n with crank m . After suitably altering the interpretations for $n = 0$ and $n = 1$, one has that

$$C(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m, n) z^m q^n = \frac{(q; q)_{\infty}}{(zq, z^{-1}q; q)_{\infty}}.$$

For the rank we have

$$\begin{aligned} R(z, q) &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, z^{-1}q; q)_n} \\ &= \frac{1}{(q; q)_{\infty}} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2} (1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right]. \end{aligned}$$

Here and throughout the rest of this paper we are using the standard product notation,

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),$$

$$\begin{aligned}
 (a; q)_\infty &= \prod_{j=0}^{\infty} (1 - aq^j), \\
 (a_1, \dots, a_k; q)_n &= (a_1; q)_n \dots (a_k; q)_n, \\
 (a_1, \dots, a_k; q)_\infty &= (a_1; q)_\infty \dots (a_k; q)_\infty.
 \end{aligned}$$

We can now introduce the rank and crank moments

$$\begin{aligned}
 N_k(n) &= \sum_{m=-\infty}^{\infty} m^k N(m, n), \\
 M_k(n) &= \sum_{m=-\infty}^{\infty} m^k M(m, n).
 \end{aligned}$$

Both of these sums are actually finite since $N(m, n) = M(m, n) = 0$ for $|m| > n$. Also the odd moments are zero since $N(-m, n) = N(m, n)$ and $M(-m, n) = M(m, n)$. These moments were first considered by Atkin and Garvan in [7]. By Andrews [3] $\text{spt}(n) = np(n) - \frac{1}{2}N_2(n)$ and by Dyson [15] $np(n) = \frac{1}{2}M_2(n)$, thus

$$\text{spt}(n) = \frac{1}{2}M_2(n) - \frac{1}{2}N_2(n).$$

We then see a useful way to study smallest parts functions is to consider the related rank and crank moments. Rather than immediately working with these moments, it has proved fruitful to consider a symmetrized version (for examples of this see [2,11,14,17,23]). In [17] Garvan used symmetrized moments of the rank and crank functions given by

$$\begin{aligned}
 \eta_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n), \\
 \mu_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n),
 \end{aligned}$$

to define a higher order analog of the spt function given by

$$\text{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n).$$

One can use $N(-m, n) = N(m, n)$ and $M(-m, n) = M(m, n)$ with the proof of Theorem 1 of [2] to find that the odd symmetrized moments are also zero. In Theorem 4.3 of [17], Garvan found the following formulas relating the ordinary and symmetrized moments,

$$\begin{aligned}\eta_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) N(m, n), \\ \mu_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) M(m, n), \\ N_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \eta_{2j}(n), \\ M_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \mu_{2j}(n).\end{aligned}$$

Here

$$g_k(x) = \prod_{j=0}^{k-1} (x^2 - j^2)$$

and the sequence $S^*(n, k)$ is defined recursively by $S^*(n+1, k) = S^*(n, k-1) + k^2 S^*(n, k)$ and the boundary conditions $S^*(1, 1) = 1$, $S^*(n, k) = 0$ for $k \leq 0$ or $k > n$.

In particular we have $\mu_2(n) = \frac{1}{2} M_2(n)$ and $\eta_2(n) = \frac{1}{2} N_2(n)$ so that $\text{spt}_1(n) = \text{spt}(n)$. The functions $\text{spt}_k(n)$ are interesting on their own, as they have a clever combinatorial interpretation and satisfy various congruences. Also in developing the functions $\text{spt}_k(n)$ we get information on the rank and crank moments.

In [13] Bringmann, Mahlburg, and Rhoades derived asymptotics for M_{2k} , N_{2k} , and $M_{2k} - N_{2k}$. In particular this showed that for each $k \geq 1$, for sufficiently large n one has $M_{2k}(n) - N_{2k}(n) > 0$. In considering the symmetrized moments that lead to $\text{spt}_k(n)$, Garvan in [17] proved that indeed $M_{2k}(n) - N_{2k}(n) > 0$ for all $k \geq 1$ and all $n \geq 1$.

In [12] Bringmann, Mahlburg, and Rhoades derived asymptotics for M_k^+ , N_k^+ , and $M_k^+ - N_k^+$, where

$$\begin{aligned}N_k^+(n) &= \sum_{m=1}^{\infty} m^k N(m, n), \\ M_k^+(n) &= \sum_{m=1}^{\infty} m^k M(m, n).\end{aligned}$$

We note $N_{2k}(n) = 2N_{2k}^+(n)$ and $M_{2k}(n) = 2M_{2k}^+(n)$. In [4] Andrews, Chan, and Kim deduced $M_k^+(n) > N_k^+(n)$ for all $k \geq 1$ and all $n \geq 1$.

We apply the idea of a higher order spt function to three other smallest parts functions. We use $\overline{\text{spt}}(n)$, the number of smallest parts in the overpartitions of n , $\overline{\text{spt}}_2(n)$ the number of smallest parts in the overpartitions of n with smallest part even, and $\text{M2spt}(n)$ the number of smallest parts in the partitions of n with smallest part even and without repeated odd parts.

For each function we consider a certain partition type function and introduce a rank and crank. The smallest parts function will agree with the difference of the second symmetrized moments of the crank and rank. We define a higher order smallest parts function by the difference of the symmetrized moments. The purpose of this paper is to find expressions for the rank and crank moments, define the higher order smallest parts functions, and deduce the higher order smallest parts functions are non-negative. From this non-negativity we prove inequalities between the ordinary crank and rank moments. In Section 2 we give the statements and proofs of our Theorems. We give combinatorial interpretations of the higher order smallest parts functions in Section 3. In Section 4 we prove two congruences for $\overline{\text{spt}}_2(n)$, a higher order analog of $\overline{\text{spt}}(n)$. All the machinery from [17] can be reused for these purposes. In the following subsections we discuss each of the three smallest parts functions.

1.1. The number of smallest parts in overpartitions

An overpartition of n is a partition of n in which the first occurrence of a part may be overlined. We denote the number of overpartitions of n by $\bar{p}(n)$. Thus while $p(4) = 5$ we have instead $\bar{p}(4) = 14$ since the overpartitions of 4 are $4, \bar{4}, 3 + 1, 3 + \bar{1}, \bar{3} + 1, \bar{3} + \bar{1}, 2 + 2, \bar{2} + 2, 2 + 1 + 1, 2 + \bar{1} + 1, \bar{2} + 1 + 1, \bar{2} + \bar{1} + 1, 1 + 1 + 1 + 1$, and $\bar{1} + 1 + 1 + 1$.

In [10] Bringmann, Lovejoy, and Osburn defined $\overline{\text{spt}}(n)$ as the number of smallest parts in the overpartitions of n . We use the convention of only including the overpartitions where the smallest part is not overlined. We see then $\overline{\text{spt}}(4) = 13$.

As in [10] and others, for an overpartition π of n we define the Dyson rank of π to be the largest part minus the number of parts of π . We let $\bar{N}(m, n)$ denote the number of overpartitions of n with Dyson rank equal to m . As in Proposition 1.1 and the proof of Proposition 3.2 of [20], the generating function for $\bar{N}(m, n)$ is given by

$$\bar{R}(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \bar{N}(m, n) z^m q^n = \sum_{n=0}^{\infty} \frac{(-1; q)_n q^{n(n+1)/2}}{(zq; q)_n (z^{-1}q; q)_n} \quad (1.1)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \right]. \quad (1.2)$$

The second equality is an application Watson's transformation. We recall Watson's transformation is

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(aq/bc, d, e; q)_n \left(\frac{aq}{de}\right)_n}{(q, aq/b, aq/c; q)_n} \\ &= \frac{(aq/d, aq/e; q)_{\infty}}{(aq, aq/de; q)_{\infty}} \sum_{n=0}^{\infty} \frac{(a, \sqrt{aq}, -\sqrt{aq}, b, c, d, e; q)_n (aq)^{2n} (-1)^n q^{n(n-1)/2}}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, aq/d, aq/e; q)_n (bcde)^n}. \end{aligned}$$

As in [10], for an overpartition π of n we define a residual crank of π by the crank of the subpartition of π consisting of the non-overlined parts of π . We let $\overline{M}(m, n)$ denote the number of overpartitions of n with this residual crank equal to m . The generating function for $\overline{M}(m, n)$ is then given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M}(m, n) z^m q^n = \frac{(-q; q)_{\infty} (q; q)_{\infty}}{(zq; q)_{\infty} (z^{-1}q; q)_{\infty}}. \quad (1.3)$$

Of course this interpretation is not quite correct, as $\frac{(q; q)_{\infty}}{(zq, z^{-1}q; q)_{\infty}}$ does not agree at q for the crank of the partition consisting of a single one. Thus the interpretation of this residual crank is not quite correct for overpartitions whose non-overlined parts consist of a single one. However, this is the generating function we must use.

We have the ordinary and symmetrized moments defined by

$$\begin{aligned} \overline{N}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{N}(m, n), \\ \overline{M}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{M}(m, n), \\ \overline{\eta}_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N}(m, n), \\ \overline{\mu}_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M}(m, n). \end{aligned}$$

Again these sums are actually finite sums and the odd moments are zero due to the symmetry $\overline{N}(-m, n) = \overline{N}(m, n)$ and $\overline{M}(-m, n) = \overline{M}(m, n)$. We find the proof of Theorem 4.3 of [17] works to give that

$$\begin{aligned} \overline{\eta}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{N}(m, n), \\ \overline{\mu}_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{M}(m, n), \\ \overline{N}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\eta}_{2j}(n), \\ \overline{M}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\mu}_{2j}(n). \end{aligned}$$

Although it is not immediately apparent, similar to $\text{spt}(n)$ we do have $\overline{\text{spt}}(n) = \overline{\mu}_2(n) - \overline{\eta}_2(n)$. We then define the higher order spt function $\overline{\text{spt}}_k(n) = \overline{\mu}_{2k}(n) - \overline{\eta}_{2k}(n)$.

That $\overline{\text{spt}}(n)$ is indeed the difference of the symmetrized moments follows by the combinatorial interpretation of the higher order $\overline{\text{spt}}_k(n)$ in Section 3.

In Corollary 2.8 we find $\overline{\text{spt}}_k(n)$ has the generating function

$$\begin{aligned} \sum_{n=1}^{\infty} \overline{\text{spt}}_k(n) q^n &= \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \bar{\eta}_{2k}(n)) q^n \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(-q^{n_1+1}; q)_{\infty}}{(q^{n_1+1}; q)_{\infty}}. \end{aligned}$$

In Corollary 2.10 we use this to prove the inequality $\overline{M}_{2k}(n) > \overline{N}_{2k}(n)$ for all $k \geq 1$ and all $n \geq 1$. Previously this inequality was known to hold for each fixed k for sufficiently large n , due to the work of Zapata Rolon [26] in determining the asymptotics for \overline{M}_k^+ , \overline{N}_k^+ , and $\overline{M}_k^+ - \overline{N}_k^+$. Here \overline{M}_k^+ and \overline{N}_k^+ are defined in the same fashion as M_k^+ and N_k^+ . In [5] Andrews, Chan, Kim, and Osburn established $\overline{N}_1^+(n) > \overline{N}_1(n)$. However, it is still only conjectured that $\overline{N}_k^+(n) > \overline{N}_k(n)$ for all $k \geq 1$ and $n \geq 1$.

Unlike in [18], here $k = 1, 2$ as a subscript in $\overline{\text{spt}}(n)$ does not specify the smallest part being odd or even.

1.2. The number of smallest parts in overpartitions with smallest part even

Next we restrict to overpartitions where the smallest part is even. We denote the number of overpartitions of n with smallest part even by $\overline{p2}(n)$. Thus $\overline{p2}(n) = 4$ since such overpartitions of 4 are $4, \bar{4}, 2+2$, and $\bar{2}+2$. In [10] Bringmann, Lovejoy, and Osburn defined the associated smallest parts function $\overline{\text{spt}2}(n)$. As with $\overline{\text{spt}}(n)$, we only include the overpartitions where the smallest part is not overlined. Thus $\overline{\text{spt}2}(4) = 3$.

We use the M_2 -rank of an overpartition π . This rank is given by

$$M_2\text{-rank} = \left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi) + \#(\pi_o) - \chi(\pi), \quad (1.4)$$

where $l(\pi)$ is the largest part of π , $\#(\pi)$ is the number of parts of π , $\#(\pi_o)$ is the number of odd non-overlined parts of π , and $\chi(\pi) = 1$ if the largest part of π is odd and non-overlined and $\chi(\pi) = 0$ otherwise. The M_2 -rank for overpartitions was introduced by Lovejoy in [21]. We let $\overline{N2}(m, n)$ denote the number of overpartitions of n with M_2 -rank m . Lovejoy found the generating function for $\overline{N2}$ is given by

$$\overline{R2}(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{N2}(m, n) z^m q^n = \sum_{n=0}^{\infty} q^n \frac{(-1; q)_{2n}}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \quad (1.5)$$

$$= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+2n}}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right]. \quad (1.6)$$

We also use the second residual crank from [10]. For an overpartition π of n we take the crank of the partition $\frac{\pi_e}{2}$ obtained by taking the subpartition π_e , of the even non-overlined parts of π , and halving each part of π_e . We let $\overline{M2}(m, n)$ denote the number of overpartitions π of n and such that the partition $\frac{\pi_e}{2}$ has crank m . Then the generating function for $\overline{M2}$ is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \overline{M2}(m, n) z^m q^n = \frac{(-q; q)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty}}. \quad (1.7)$$

Again this interpretation fails for overpartitions whose only even non-overlined parts are a single two.

We have the ordinary and symmetrized moments defined by

$$\begin{aligned} \overline{N2}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{N2}(m, n), \\ \overline{M2}_k(n) &= \sum_{m \in \mathbb{Z}} m^k \overline{M2}(m, n), \\ \overline{\eta}2_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{N2}(m, n), \\ \overline{\mu}2_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} \overline{M2}(m, n). \end{aligned}$$

Again these sums are finite sums, the odd moments are zero, and

$$\begin{aligned} \overline{\eta}2_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{N2}(m, n), \\ \overline{\mu}2_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) \overline{M2}(m, n), \\ \overline{N2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\eta}2_{2j}(n), \\ \overline{M2}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \overline{\mu}2_{2j}(n). \end{aligned}$$

As with $\overline{\text{spt}}(n)$, we define the higher order smallest parts function $\overline{\text{spt}2}_k(n) = \overline{\mu}2_{2k}(n) - \overline{\eta}2_{2k}(n)$. Based on the combinatorial interpretations of Section 3, we have $\overline{\text{spt}2}(n) = \overline{\mu}2_{2k}(n) - \overline{\eta}2_{2k}(n)$. In Corollary 2.9 we find a generating function for $\overline{\text{spt}2}_k(n)$ to be given by

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\text{spt}}_{2k}(n) q^n &= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2(1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&\quad \times \frac{(-q^{2n_1+1}; q)_{\infty}}{(q^{2n_1+1}; q)_{\infty}}.
\end{aligned}$$

From this generating function, in [Corollary 2.10](#), we deduce that $\overline{M}_{2k}(n) > \overline{N}_{2k}(n)$ for $n = 2$ and $n \geq 4$. In [\[24\]](#), Mao derived asymptotics for \overline{N}_{2k} (as well as \overline{N}_{2k}), however we do not yet have asymptotics for $\overline{M}_{2k}(n)$ nor $\overline{M}_{2k}(n) - \overline{N}_{2k}(n)$.

It is important to note that in [\[19\]](#) Larsen, Rust, and Swisher proved a stronger result than $\overline{M}_{2k}(n) > \overline{N}_{2k}(n)$. In particular they proved that

$$\overline{M}_{2k}^+(n) > \overline{N}_{2k}^+(n),$$

where $\overline{N}_{2k}^+(n)$ and $\overline{M}_{2k}^+(n)$ are defined similarly to $N_k^+(n)$ and $M_k^+(n)$. The methods used to handle when the series are only over $m \geq 1$ are quite different than the methods used here. Also they extend a result of Mao [\[23\]](#) that $\overline{N}_{2k} > \overline{N}_{2k}$ to $\overline{N}_k^+ > \overline{N}_{2k}^+$.

1.3. The number of smallest parts in partitions with smallest part even and without repeated odd parts

Lastly we consider partitions with smallest part even and without repeated odd parts. We let $p_2(n)$ denote the number of such partitions of n . We see $p_2(4) = 2$ from the partitions 4 and $2 + 2$. In [\[1\]](#) Ahlgren, Bringmann, and Lovejoy defined $M_2\text{spt}(n)$ to be the number of smallest parts in the partitions of n without repeated odd parts and with smallest part even. We see $M_2\text{spt}(4) = 3$.

We recall the M_2 -rank of a partition π without repeated odd parts is given by

$$M_2\text{-rank} = \left\lceil \frac{l(\pi)}{2} \right\rceil - \#(\pi), \quad (1.8)$$

where $l(\pi)$ is the largest part of π and $\#(\pi)$ is the number of parts of π . The M_2 -rank was introduced by Berkovich and Garvan in [\[8\]](#). We let $N_2(m, n)$ denote the number of partitions of n with distinct odd parts and M_2 -rank m . By Lovejoy and Osburn [\[22\]](#) the generating function for $N_2(m, n)$, which we further rearrange as in [\[18\]](#) (using Watson's transformation), is given by

$$R_2(z, q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m, n) z^m q^n = \sum_{n=0}^{\infty} q^{n^2} \frac{(-q; q^2)_n}{(zq^2; q^2)_n (z^{-1}q^2; q^2)_n} \quad (1.9)$$

$$= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(2n+1)}(1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right]. \quad (1.10)$$

We use another residual crank that was defined in [18]. For a partition π of n with distinct odd parts we take the crank of the partition $\frac{\pi_e}{2}$ obtained by taking the subpartition π_e , of the even parts of π , and halving each part of π_e . We let $M2(m, n)$ denote the number of partitions π of n with distinct odd parts and such that the partition $\frac{\pi_e}{2}$ has crank m . Then the generating function for $M2(m, n)$ is given by

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M2(m, n) z^m q^n = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(zq^2; q^2)_{\infty} (z^{-1}q^2; q^2)_{\infty}}. \quad (1.11)$$

Of course this interpretation is not quite correct, here it fails for partitions with distinct odd parts whose only even parts are a single two.

We have the ordinary and symmetrized moments defined by

$$\begin{aligned} N2_k(n) &= \sum_{m \in \mathbb{Z}} m^k N2(m, n), \\ M2_k(n) &= \sum_{m \in \mathbb{Z}} m^k M2(m, n), \\ \eta2_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N2(m, n), \\ \mu2_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M2(m, n). \end{aligned}$$

Again these sums are finite sums, the odd moments are zero, and

$$\begin{aligned} \eta2_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) N2(m, n), \\ \mu2_{2k}(n) &= \frac{1}{(2k)!} \sum_{m \in \mathbb{Z}} g_k(m) M2(m, n), \\ N2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \eta2_{2j}(n), \\ M2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) \mu2_{2j}(n). \end{aligned}$$

We define the higher order smallest parts function $M2spt_k(n) = \mu2_{2k}(n) - \eta2_{2k}(n)$, which in Section 3 we find does agree with $M2spt(n)$ so that $M2spt_1(n) = M2spt(n)$. In Corollary 2.7 we find a generating function for $M2spt_k(n)$ is

$$\sum_{n=1}^{\infty} M2spt_k(n) q^n = \sum_{n=1}^{\infty} (\mu2_{2k}(n) - \eta2_{2k}(n)) q^n$$

$$\begin{aligned}
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2(1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&\quad \times \frac{(-q^{2n_1+1}; q^2)_\infty}{(q^{2n_1+2}; q^2)_\infty}.
\end{aligned}$$

Again we use the generating function in [Corollary 2.10](#) to deduce the inequality between ordinary moments, $M_{2k}(n) > N_{2k}(n)$ for $n = 2$ and $n \geq 4$. No one has yet given asymptotics for these rank and crank moments. Also no one has yet investigated the corresponding M_k^+ and N_k^+ . Numerical evidence suggests $M_k^+ > N_k^+$ for all $k \geq 1$ and $n \geq 4$.

2. Theorems and proofs

For $C2(z, q)$, $\overline{C}(z, q)$, and $\overline{C2}(z, q)$ we use that

$$\frac{(q; q)_\infty}{(zq, z^{-1}q; q)_\infty} = \frac{1}{(q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2} (1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right),$$

this is [\[16, Eq. \(7.15\)\]](#). Thus

$$\begin{aligned}
C2(z, q) &= \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(zq^2; q^2)_\infty (z^{-1}q^2; q^2)_\infty} \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right] \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2} \left(\frac{1-z}{1-zq^{2n}} + \frac{1-z^{-1}}{1-z^{-1}q^{2n}} \right) \right] \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1-z)}{1-zq^{2n}}.
\end{aligned}$$

And similarly we have

$$\begin{aligned}
\overline{C}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1-z)}{1-zq^n}, \\
\overline{C2}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2} (1-z)}{1-zq^{2n}}.
\end{aligned}$$

We find similar expressions for the ranks.

$$R2(z, q) = \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(2n+1)/2} (1+q^{2n})}{(1-zq^{2n})(1-z^{-1}q^{2n})} \right]$$

$$\begin{aligned}
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \left[1 + \sum_{n=1}^{\infty} (-1)^n q^{n(2n+1)} \left(\frac{1-z}{1-zq^{2n}} + \frac{1-z^{-1}}{1-z^{-1}q^{2n}} \right) \right] \\
&= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(2n+1)} (1-z)}{1-zq^{2n}}.
\end{aligned}$$

Next we have,

$$\begin{aligned}
\bar{R}(z, q) &= \frac{(-q; q)_\infty}{(q; q)_\infty} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n^2+n}}{(1-zq^n)(1-z^{-1}q^n)} \right] \\
&= \frac{(-q; q)_\infty}{(q; q)_\infty} \left[1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2+n}}{(1+q^n)} \left(\frac{1-z}{1-zq^n} + \frac{1-z^{-1}}{1-z^{-1}q^n} \right) \right] \\
&= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+n} (1-z)}{(1+q^n)(1-zq^n)}.
\end{aligned}$$

Similarly we have

$$\bar{R}2(z, q) = 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n} (1-z)}{(1+q^{2n})(1-zq^{2n})}.$$

Using that

$$\left(\frac{\partial}{\partial z} \right)^j \frac{1-z}{1-zq^n} = \frac{-j!(1-q^n)q^{n(j-1)}}{(1-zq^n)^{j+1}},$$

we find that the partial derivatives of the crank and rank generating functions are as follows,

$$C2^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j C2(z, q) = \frac{-j!(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)+2jn} (1-q^{2n})}{(1-zq^{2n})^{j+1}},$$

$$R2^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j R2(z, q) = \frac{-j!(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(2n-1)+2jn} (1-q^{2n})}{(1-zq^{2n})^{j+1}},$$

$$\bar{C}^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j \bar{C}(z, q) = \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)/2+jn} (1-q^n)}{(1-zq^n)^{j+1}},$$

$$\bar{R}^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j \bar{R}(z, q) = 2 \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n^2+jn} (1-q^n)}{(1+q^n)(1-zq^n)^{j+1}},$$

$$\overline{C2}^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j \overline{C2}(z, q) = \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n(n-1)+2jn}(1-q^{2n})}{(1-zq^{2n})^{j+1}},$$

$$\overline{R2}^{(j)}(z, q) = \left(\frac{\partial}{\partial z} \right)^j \overline{R2}(z, q) = 2 \frac{-j!(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^n q^{n^2+2jn}(1-q^{2n})}{(1+q^{2n})(1-zq^{2n})^{j+1}}.$$

We collect all expressions for the symmetrized moments in one theorem. Some of these have been used and proved before in the various papers about these moments.

Theorem 2.1. *For all $k \geq 1$*

$$\begin{aligned} \sum_{n=1}^{\infty} \mu_{2k}(n) q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2kn}}{(1-q^{2n})^{2k}} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \eta_{2k}(n) q^n &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(2n+1)+2kn}}{(1-q^{2n})^{2k}} \\ &= \frac{(-q; q^2)_\infty}{(q^2; q^2)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(2n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \bar{\mu}_{2k}(n) q^n &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)/2+kn}}{(1-q^n)^{2k}} \\ &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)/2+kn}(1+q^n)}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \bar{\eta}_{2k}(n) q^n &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n^2+n+kn}}{(1+q^n)(1-q^n)^{2k}} \\ &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n^2+kn}}{(1-q^n)^{2k}}, \\ \sum_{n=1}^{\infty} \bar{\mu}'_{2k}(n) q^n &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2kn}}{(1-q^{2n})^{2k}} \\ &= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn}(1+q^{2n})}{(1-q^{2n})^{2k}}, \\ \sum_{n=1}^{\infty} \bar{\eta}'_{2k}(n) q^n &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n^2+2n+2kn}}{(1+q^{2n})(1-q^{2n})^{2k}} \\ &= 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n^2+2kn}}{(1-q^{2n})^{2k}}. \end{aligned}$$

Proof. We follow the proof for a similar expression in Theorem 2 of [2].

$$\begin{aligned}
 \sum_{n=1}^{\infty} \mu_{2k}(n) q^n &= \frac{1}{(2k)!} \left(\left(\frac{\partial}{\partial z} \right)^{2k} z^{k-1} C2(z, q) \right) \Big|_{z=1} \\
 &= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (k-1) \dots (k-j) C2^{2k-j}(1, q) \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+2n(2k-j)} (1 - q^{2n})}{(1 - q^{2n})^{2k-j+1}} \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+4nk}}{(1 - q^{2n})^{2k}} \sum_{j=0}^{k-1} \binom{k-1}{j} (q^{-2n} (1 - q^{2n}))^j \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n-1)+4nk}}{(1 - q^{2n})^{2k}} (1 + q^{-2n} (1 - q^{2n}))^{k-1} \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \neq 0} \frac{(-1)^{n+1} q^{n(n+1)+2nk}}{(1 - q^{2n})^{2k}} \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1 + q^{2n})}{(1 - q^{2n})^{2k}}.
 \end{aligned}$$

We omit the proofs of the other identities, as they are near identical to the above, but with $C2(z, q)$ replaced with $R2(z, q)$, $\bar{C}(z, q)$, $\bar{R}(z, q)$, $\bar{C}2(z, q)$, and $\bar{R}2(z, q)$ respectively. \square

We recall two sequences of functions α_n and β_n are a Bailey pair relative to (a, q) if

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}.$$

The following is Theorem 3.3 of [17],

Theorem 2.2. Suppose α_n and β_n are a Bailey pair relative to $(1, q)$ and $\alpha_0 = \beta_0 = 1$, then

$$\begin{aligned}
 &\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1}^2 q^{n_1+n_2+\dots+n_k} \beta_{n_1}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} \\
 &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr} \alpha_r}{(1 - q^r)^{2k}}.
 \end{aligned}$$

The following is Corollary 3.4 of Theorem 3.3 from [17],

Corollary 2.3.

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} q^{n(n-1)/2+kn} (1+q^n)}{(1-q^n)^{2k}}. \end{aligned}$$

For η_2 we will use the following.

Corollary 2.4.

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1} q^{2n_1+2n_2+\dots+2n_k}}{(-q; q^2)_{n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(2n-1)+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}}. \end{aligned}$$

Proof. We have a Bailey pair for $(1, q^2)$ from [25, p. 468] given by

$$\begin{aligned} \alpha_n &= \begin{cases} 1 & n = 0 \\ (-1)^n q^{2n^2} (q^n + q^{-n}) & n \geq 1 \end{cases} \\ \beta_n &= \frac{1}{(-q, q^2; q^2)_n}. \end{aligned}$$

Applying Theorem 2.2 to this Bailey pair gives the identity. \square

For $\bar{\eta}$ we will use the following.

Corollary 2.5.

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1}^2 q^{n_1+n_2+\dots+n_k}}{(q^2; q^2)_{n_1} (1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2q^{n^2+kn}}{(1-q^n)^{2k}}. \end{aligned}$$

Proof. We have a Bailey pair for $(1, q)$ from [25, p. 469] given by

$$\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^n 2q^{n^2} & n \geq 1 \end{cases}$$

$$\beta_n = \frac{1}{(q^2; q^2)_n}.$$

Applying Theorem 2.2 to this Bailey pair gives the identity. \square

Lastly we will use the following corollary for $\overline{\eta^2}$.

Corollary 2.6.

$$\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1}^2 (q; q^2)_{n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1 - q^{2n_k})^2 (1 - q^{2n_{k-1}})^2 \dots (1 - q^{2n_1})^2}$$

$$= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1 - q^{2n_k})^2 (1 - q^{2n_{k-1}})^2 \dots (1 - q^{2n_1})^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2q^{n^2+2kn}}{(1 - q^{2n})^{2k}}.$$

Proof. As in the proof of Theorem 7 of [6], we have

$$\sum_{j=-L}^L \frac{z^j q^{j^2}}{(q^2; q^2)_{L-j} (q^2; q^2)_{L+j}} = \frac{(-zq, -q/z; q^2)_L}{(q^2; q^2)_{2L}}.$$

Setting $z = -1$ gives a Bailey pair relative to $(1, q^2)$ where α_n and β_n are

$$\alpha_n = \begin{cases} 1 & n = 0 \\ (-1)^n 2q^{n^2} & n \geq 1 \end{cases}$$

$$\beta_n = \frac{(q; q^2)_n^2}{(q^2; q^2)_{2n}}.$$

This Bailey pair is also given as Lemma 2.3 in [9]. Applying Theorem 2.2 to this Bailey pair gives the identity. \square

Next we find expressions for $M2\text{spt}_k(n)$, $\overline{\text{spt}}_k(n)$, and $\overline{\text{spt}^2}_k(n)$.

Corollary 2.7. For all $k \geq 1$,

$$\sum_{n=1}^{\infty} M2\text{spt}_k(n) q^n$$

$$= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n$$

$$= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1 - q^{2n_k})^2 (1 - q^{2n_{k-1}})^2 \dots (1 - q^{2n_1})^2} \frac{(-q^{2n_1+1}; q^2)_{\infty}}{(q^{2n_1+2}; q^2)_{\infty}}.$$

Proof. By Theorem 2.1, Corollary 2.3 with q replaced by q^2 , and Corollary 2.4, we have that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} \\
 & \quad + \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{2n^2-n+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
 & \quad + \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{2n^2-n+kn} (1+q^{2n})}{(1-q^{2n})^{2k}} \\
 &= \frac{(-q; q^2)_{\infty}}{(q^2; q^2)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1} q^{2n_1+2n_2+\dots+2n_k}}{(-q; q^2)_{n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
 &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q^2)_{\infty}}{(q^{2n_1+2}; q^2)_{\infty}}. \quad \square
 \end{aligned}$$

Corollary 2.8. For all $k \geq 1$,

$$\begin{aligned}
 \sum_{n=1}^{\infty} \text{spt}_k(n) q^n &= \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \bar{\eta}_{2k}(n)) q^n \\
 &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(-q^{n_1+1}; q)_{\infty}}{(q^{n_1+1}; q)_{\infty}}.
 \end{aligned}$$

Proof. By Theorem 2.1, Corollary 2.3, and Corollary 2.5, we have that

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (\bar{\mu}_{2k}(n) - \bar{\eta}_{2k}(n)) q^n \\
 &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)/2+kn} (1+q^n)}{(1-q^n)^{2k}} + 2 \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n q^{n^2+kn}}{(1-q^n)^{2k}} \\
 &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\
 & \quad + \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^n 2q^{n^2+kn}}{(1-q^n)^{2k}}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{n_1}^2 q^{n_1+n_2+\dots+n_k}}{(q^2; q^2)_{n_1} (1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(q^{2n_1+2}; q^2)_\infty}{(q^{n_1+1}; q)_\infty^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \frac{(-q^{n_1+1}; q)_\infty}{(q^{n_1+1}; q)_\infty}. \quad \square
\end{aligned}$$

Corollary 2.9. For all $k \geq 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \overline{\text{spt}} 2_k(n) q^n &= \sum_{n=1}^{\infty} (\mu \overline{2}_{2k}(n) - \eta \overline{2}_{2k}(n)) q^n \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&\quad \times \frac{(-q^{2n_1+1}; q)_\infty}{(q^{2n_1+1}; q)_\infty}.
\end{aligned}$$

Proof. By Theorem 2.1, Corollary 2.3 with q replaced by q^2 , and Corollary 2.6, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} (\mu \overline{2}_{2k}(n) - \eta \overline{2}_{2k}(n)) q^n \\
&= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^{n+1} q^{n(n-1)+2kn} (1+q^{2n})}{(1-q^{2n})^{2k}} + 2 \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^n q^{n^2+2kn}}{(1-q^{2n})^{2k}} \\
&= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&\quad + \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n \geq 1} \frac{(-1)^n 2q^{n^2+2kn}}{(1-q^{2n})^{2k}} \\
&= \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q^2; q^2)_{n_1}^2 (q; q^2)_{n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&= \frac{(q^2; q^2)_\infty}{(q; q)_\infty^2} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q; q)_{2n_1}^2 q^{2n_1+2n_2+\dots+2n_k}}{(q^2; q^2)_{2n_1} (1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(q^{4n_1+2}; q^2)_\infty}{(q^{2n_1+1}; q)_\infty^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{2n_1+2n_2+\dots+2n_k}}{(1-q^{2n_k})^2 (1-q^{2n_{k-1}})^2 \dots (1-q^{2n_1})^2} \frac{(-q^{2n_1+1}; q)_\infty}{(q^{2n_1+1}; q)_\infty}. \quad \square
\end{aligned}$$

It is now clear that $M2\text{spt}_k(n) \geq 0$, $\overline{\text{spt}}_k(n) \geq 0$, and $\overline{\text{spt}2}_k(n) \geq 0$. Next we consider inequalities between the ordinary moments.

Corollary 2.10. *Suppose $k \geq 1$. For $n = 2$ and $n \geq 4$ we have*

$$M2_{2k}(n) > N2_{2k}(n).$$

For $n \geq 1$ we have

$$\overline{M}_{2k}(n) > \overline{N}_{2k}(n).$$

For $n = 2$ and $n \geq 4$ we have

$$\overline{M}2_{2k}(n) > \overline{N}2_{2k}(n).$$

Proof. We know

$$\sum_{n \geq 1} (\mu_{2j}(n) - \eta_{2j}(n)) q^n = \frac{q^{2j}(-q^3; q^2)_\infty}{(1 - q^2)^{2j}(q^4; q^2)_\infty} + \dots$$

where the omitted terms also have non-negative coefficients. It is then apparent that

$$\mu_{2j}(n) > \eta_{2j}(n)$$

for $j \geq 1$ and $n \geq 2j + 2$. This inequality also holds when $n = 2j$, but we instead have equality at $2j + 1$.

However, the $S^*(k, j)$ are integers and are positive for $1 \leq j \leq k$, thus

$$\begin{aligned} M2_{2k}(n) - N2_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\mu_{2j}(n) - \eta_{2j}(n)) \\ &\geq \mu_{22}(n) - \eta_{22}(n) \\ &> 0, \end{aligned}$$

for $n \geq 4$ and $n = 2$.

Next we have

$$\sum_{n \geq 1} (\bar{\mu}_{2j}(n) - \bar{\eta}_{2j}(n)) q^n = \frac{q^j(-q^2; q)_\infty}{(1 - q)^{2j}(q^2; q)_\infty} + \dots$$

where the omitted terms also have non-negative coefficients. It is then apparent that

$$\bar{\mu}_{2j}(n) > \bar{\eta}_{2j}(n)$$

for $j \geq 1$ and $n \geq j$. Similar to the previous case,

$$\begin{aligned}
\overline{M}_{2k}(n) - \overline{N}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\overline{\mu}_{2j}(n) - \overline{\eta}_{2j}(n)) \\
&\geq \overline{\mu}_2(n) - \overline{\eta}_2(n) \\
&> 0,
\end{aligned}$$

for $n \geq 1$.

Last we have

$$\sum_{n \geq 1} (\overline{\mu}_{2j}(n) - \overline{\eta}_{2j}(n)) q^n = \frac{q^{2j}(-q^3; q)_{\infty}}{(1 - q^2)^{2j}(q^3; q)_{\infty}} + \dots$$

where the omitted terms also have non-negative coefficients. Thus

$$\overline{\mu}_{2j}(n) > \overline{\eta}_{2j}(n)$$

for $j \geq 1$ and $n \geq 2j + 2$. This inequality also holds when $n = 2j$, but we instead have equality at $2j + 1$. As before

$$\begin{aligned}
\overline{M}_{2k}(n) - \overline{N}_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\overline{\mu}_{2j}(n) - \overline{\eta}_{2j}(n)) \\
&\geq \overline{\mu}_2(n) - \overline{\eta}_2(n) \\
&> 0,
\end{aligned}$$

for $n \geq 4$ and $n = 2$. \square

3. Combinatorial interpretations

As in [17], for a partition π where the different parts are

$$n_1 < n_2 < \dots < n_m,$$

we have $f_j = f_j(\pi)$ is the frequency of the part n_j .

Thinking of overpartition and partitions without repeated odd parts as pairs of partitions, we make the following definition. Suppose $\vec{\pi} = (\pi_1, \dots, \pi_r)$ is a vector partition of n , then $f_j^1 = f_j^1(\vec{\pi}) = f_j(\pi_1)$. We now view overpartitions as partition pairs where π_2 is a partition into distinct parts, and view partitions with distinct odd parts as partition pairs where π_1 has only even parts and π_2 has only distinct odd parts. For overpartitions with smallest part even, we use a slightly different idea. We view an overpartition with smallest part even as a vector partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ where π_1 are the non-overlined even parts, π_2 are the non-overlined odd parts, and π_3 are the overlined parts. Furthermore, in all three cases we require the smallest part to only occur in π_1 . We denote the set

of overpartitions with smallest part not overlined by \bar{S} , the set of partitions with smallest part even and non-repeated odds by $S2$, and the set of overpartitions with smallest part even and not overlined by $\bar{S}2$.

We note that

$$\begin{aligned} \text{M2spt}(n) &= \sum_{\vec{\pi} \in S2, |\vec{\pi}|=n} f_1^1(\vec{\pi}), \\ \overline{\text{spt}}(n) &= \sum_{\vec{\pi} \in \bar{S}, |\vec{\pi}|=n} f_1^1(\vec{\pi}), \\ \overline{\text{spt}2}(n) &= \sum_{\vec{\pi} \in \bar{S}2, |\vec{\pi}|=n} f_1^1(\vec{\pi}). \end{aligned}$$

For $k \geq 1$ we extend the weight ω_k of [17], for a partition pair $\vec{\pi} = (\pi_1, \pi_2)$ or vector partition $\vec{\pi} = (\pi_1, \pi_2, \pi_3)$ we let $\omega_k(\vec{\pi}) = \omega_k(\pi_1)$. That is,

$$\begin{aligned} \omega_k(\vec{\pi}) &= \sum_{\substack{m_1+m_2+\dots+m_r=k \\ 1 \leq r \leq k}} \binom{f_1^1 + m_1 - 1}{2m_1 - 1} \\ &\times \sum_{2 \leq j_2 < j_3 < \dots < j_r} \binom{f_{j_2}^1 + m_2}{2m_2} \binom{f_{j_3}^1 + m_3}{2m_3} \dots \binom{f_{j_r}^1 + m_r}{2m_r}. \end{aligned}$$

Theorem 3.1. *For all $k \geq 1$ and $n \geq 1$ we have*

$$\begin{aligned} \text{M2spt}_k(n) &= \sum_{\vec{\pi} \in S2} \omega_k(\vec{\pi}), \\ \overline{\text{spt}}_k(n) &= \sum_{\vec{\pi} \in \bar{S}} \omega_k(\vec{\pi}), \\ \overline{\text{spt}2}_k(n) &= \sum_{\vec{\pi} \in \bar{S}2} \omega_k(\vec{\pi}). \end{aligned}$$

Proof. The proof is near identical as that of Theorem 5.6 of [17], the only difficulty being how to write out the general case. We will fully write out the case when $k = 3$ for $\overline{\text{spt}}_k(n)$, go over the case of $k = 4$ for $\text{M2spt}_k(n)$, and explain the procedure for general k which will then be clear.

We use

$$\begin{aligned} \sum_{n=j}^{\infty} \binom{n+j-1}{2j-1} x^n &= \frac{x^j}{(1-x)^{2j}}, \\ \sum_{n=j}^{\infty} \binom{n+j}{2j} x^n &= \frac{x^j}{(1-x)^{2j+1}}. \end{aligned}$$

For the $k = 3$ case for $\overline{\text{spt}}_k(n)$, we have

$$\begin{aligned}
 & \sum_{n=1}^{\infty} (\bar{\mu}_6(n) - \bar{\eta}_6(n)) q^n \\
 &= \sum_{1 \leq m \leq k \leq n} \frac{q^{m+k+n} (-q^{m+1}; q)_{\infty}}{(1-q^m)^2 (1-q^k)^2 (1-q^n)^2 (q^{m+1}; q)_{\infty}} \\
 &= \sum_{1 \leq m=k=n} + \sum_{1 \leq m=k < n} + \sum_{1 \leq m < k=n} \\
 &\quad + \sum_{1 \leq m < k < n} \left(\frac{q^{m+k+n} (-q^{m+1}; q)_{\infty}}{(1-q^m)^2 (1-q^k)^2 (1-q^n)^2 (q^{m+1}; q)_{\infty}} \right) \\
 &= \sum_{1 \leq m} \frac{q^{3m}}{(1-q^m)^6} (-q^{m+1}; q)_{\infty} \prod_{i>m} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < n} \frac{q^{2m}}{(1-q^m)^4} \frac{q^n}{(1-q^n)^3} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < k} \frac{q^m}{(1-q^m)^2} \frac{q^{2k}}{(1-q^k)^5} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k}} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < k < n} \frac{q^m}{(1-q^m)^2} \frac{q^k}{(1-q^k)^3} \frac{q^n}{(1-q^n)^3} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{1-q^i} \\
 &= \sum_{1 \leq m} \sum_{f_1=3}^{\infty} \binom{f_1+3-1}{6-1} q^{mf_1} (-q^{m+1}; q)_{\infty} \prod_{i>m} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < n} \sum_{f_1=2}^{\infty} \binom{f_1+2-1}{4-1} q^{mf_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{nf_{j_2}} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < k} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{mf_1} \sum_{f_{j_2}=2}^{\infty} \binom{f_{j_2}+2}{4} q^{kf_{j_2}} (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k}} \frac{1}{1-q^i} \\
 &\quad + \sum_{1 \leq m < k < n} \sum_{f_1=1}^{\infty} \binom{f_1+1-1}{2-1} q^{mf_1} \sum_{f_{j_2}=1}^{\infty} \binom{f_{j_2}+1}{2} q^{kf_{j_2}} \sum_{f_{j_3}=1}^{\infty} \binom{f_{j_3}+1}{2} q^{nf_{j_3}} \\
 &\quad \times (-q^{m+1}; q)_{\infty} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{1-q^i}.
 \end{aligned}$$

The set of the 4 compositions of 3 is $A = \{(3), (2, 1), (1, 2), (1, 1, 1)\}$, thus we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\bar{\mu}_6(n) - \bar{\eta}_6(n)) q^n \\
&= \sum_{(m_1, \dots, m_r) = \vec{m} \in A} \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \dots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1 + m_1 - 1}{2m_1 - 1} \\
&\quad \times \binom{f_{j_2} + m_2}{2m_2} \dots \binom{f_{j_r} + m_r}{2m_r} q^{n_1 f_1 + n_{j_2} f_{j_2} + \dots + n_{j_r} f_{j_r}} (-q^{n_1+1}; q)_{\infty} \\
&\quad \times \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1 - q^i}.
\end{aligned}$$

This we recognize as the generating function for partition pairs $\vec{\pi} = (\pi_1, \pi_2) \in \bar{S}$ counted according to the weight ω_3 . This is the generating function obtained by summing according to the smallest part of π_1 being n_1 with frequency f_1 .

For the $k = 4$ case for $M2spt_k(n)$, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} (\mu_{28}(n) - \eta_{28}(n)) q^n \\
&= \sum_{1 \leq m \leq j \leq k \leq n} \frac{q^{2m+2j+2k+2n} (-q^{2m+1}; q^2)_{\infty}}{(1 - q^{2m})^2 (1 - q^{2j})^2 (1 - q^{2k})^2 (1 - q^{2n})^2 (q^{2m+2}; q^2)_{\infty}} \\
&= \sum_{1 \leq m=j=k=n} + \sum_{1 \leq m=j=k < n} + \sum_{1 \leq m=j < k=n} + \sum_{1 \leq m=j < k < n} + \sum_{1 \leq m < j=k=n} + \sum_{1 \leq m < j < k < n} \\
&\quad + \sum_{1 \leq m < j < k=n} + \sum_{1 \leq m < j < k < n} \frac{q^{2m+2j+2k+2n} (-q^{2m+1}; q^2)_{\infty}}{(1 - q^{2m})^2 (1 - q^{2j})^2 (1 - q^{2k})^2 (1 - q^{2n})^2 (q^{2m+2}; q^2)_{\infty}} \\
&= \sum_{1 \leq m} \frac{q^{8m}}{(1 - q^{2m})^8} (-q^{2m+1}; q^2)_{\infty} \prod_{i > m} \frac{1}{1 - q^{2i}} \\
&\quad + \sum_{1 \leq m < n} \frac{q^{6m}}{(1 - q^{2m})^6} \frac{q^{2n}}{(1 - q^{2n})^3} (-q^{2m+1}; q^2)_{\infty} \prod_{\substack{i > m \\ i \neq n}} \frac{1}{1 - q^{2i}} \\
&\quad + \sum_{1 \leq m < k} \frac{q^{4m}}{(1 - q^{2m})^4} \frac{q^{4k}}{(1 - q^{2k})^5} (-q^{2m+1}; q^2)_{\infty} \prod_{\substack{i > m \\ i \neq k}} \frac{1}{1 - q^{2i}} \\
&\quad + \sum_{1 \leq m < k < n} \frac{q^{4m}}{(1 - q^{2m})^4} \frac{q^{2k}}{(1 - q^{2k})^3} \frac{q^{2n}}{(1 - q^{2n})^3} (-q^{2m+1}; q^2)_{\infty} \prod_{\substack{i > m \\ i \neq k, n}} \frac{1}{1 - q^{2i}} \\
&\quad + \sum_{1 \leq m < j} \frac{q^{2m}}{(1 - q^{2m})^2} \frac{q^{6j}}{(1 - q^{2j})^7} (-q^{2m+1}; q^2)_{\infty} \prod_{\substack{i > m \\ i \neq j}} \frac{1}{1 - q^{2i}}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{1 \leq m < j < n} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{4j}}{(1-q^{2j})^5} \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i > m \\ i \neq j, n}} \frac{1}{1-q^{2i}} \\
& + \sum_{1 \leq m < j < k} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{2j}}{(1-q^{2j})^3} \frac{q^{4k}}{(1-q^{2k})^5} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i > m \\ i \neq j, k}} \frac{1}{1-q^{2i}} \\
& + \sum_{1 \leq m < j < k < n} \frac{q^{2m}}{(1-q^{2m})^2} \frac{q^{2j}}{(1-q^{2j})^3} \frac{q^{2k}}{(1-q^{2k})^3} \\
& \times \frac{q^{2n}}{(1-q^{2n})^3} (-q^{2m+1}; q^2)_\infty \prod_{\substack{i > m \\ i \neq j, k, n}} \frac{1}{1-q^{2i}}.
\end{aligned}$$

In order, the above eight terms correspond to the compositions of 4: (4), (3, 1), (2, 2), (2, 1, 1), (1, 3), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1).

Thus for each composition $m_1 + \dots + m_r = 4$ we have a sum of the form:

$$\sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \frac{q^{2n_1 m_1}}{(1-q^{2n_1})^{2m_1}} \frac{q^{2n_2 m_2}}{(1-q^{2n_2})^{2m_2+1}} \dots \frac{q^{2n_{j_r} m_r}}{(1-q^{2n_{j_r}})^{2m_r+1}} (-q^{2n_1+1}; q^2)_\infty \quad (3.1)$$

$$\begin{aligned}
& \times \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}} \\
& = \sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \sum_{f_1=m_1}^\infty \sum_{f_{j_2}=m_2}^\infty \dots \sum_{f_{j_r}=m_r}^\infty \binom{f_1+m_1-1}{2m_1-1} \\
& \times \binom{f_{j_2}+m_2}{2m_2} \dots \binom{f_{j_r}+m_r}{2m_r} q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \dots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q^2)_\infty \\
& \times \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}}. \quad (3.2)
\end{aligned}$$

Noting the f_{j_i} correspond to the frequencies of certain even parts, we see summing (3.1) over all compositions of 4 yields the generating function for partitions without repeated odd parts and smallest part even written as a partition pair $(\pi_1, \pi_2) \in S2$, counted according to the weight ω_4 . This is the generating function given by summing according to the smallest part being $2n_1$ with frequency f_1 .

For general k , we take the expression in Corollaries 2.7, 2.8, or 2.9 and break it into 2^{k-1} sums by turning the index bounds into $=$ or $<$. These correspond to the 2^{k-1} compositions of k . The sum with index bounds $n_1 \square_1 n_2 \square_2 \dots \square_{r-1} n_r$ where each \square_i is either “=” or “<” corresponds to the composition $(1\triangle_1 1\triangle_2 \dots \triangle_{r-1} 1)$ where \triangle_i is “+” if \square_i is “=” and \triangle_i is “,” if \square_i is “<”.

For $\text{M2spt}_k(n)$ the sum corresponding to the fixed composition $m_1 + m_2 + \cdots + m_r = k$ is then rewritten as

$$\sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ \times q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \cdots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q^2)_{\infty} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^{2i}}. \quad (3.3)$$

Thus on the one hand summing (3.3) over all compositions of k gives $\sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n))q^n$, but also this is $\sum_{n=1}^{\infty} q^n \sum_{\vec{\pi} \in \mathcal{S}_2} \omega_k(\vec{\pi})$.

For $\overline{\text{spt}}_k(n)$, the general case follows the same idea, but differs in that the term for a fixed composition $m_1 + \cdots + m_r$ of k is

$$\sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ \times q^{n_1 f_1 + n_{j_2} f_{j_2} + \cdots + n_{j_r} f_{j_r}} (-q^{n_1+1}; q)_{\infty} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{1-q^i}.$$

Lastly, for $\overline{\text{spt}}_{2k}(n)$ the general term is

$$\sum_{1 \leq n_1 < n_{j_2} < \cdots < n_{j_r}} \sum_{f_1=m_1}^{\infty} \sum_{f_{j_2}=m_2}^{\infty} \cdots \sum_{f_{j_r}=m_r}^{\infty} \binom{f_1+m_1-1}{2m_1-1} \binom{f_{j_2}+m_2}{2m_2} \cdots \binom{f_{j_r}+m_r}{2m_r} \\ \times q^{2n_1 f_1 + 2n_{j_2} f_{j_2} + \cdots + 2n_{j_r} f_{j_r}} (-q^{2n_1+1}; q)_{\infty} \prod_{\substack{i > 2n_1 \\ i \notin \{2n_{j_2}, \dots, 2n_{j_r}\}}} \frac{1}{1-q^i}.$$

This finishes the proof. \square

To help illustrate what is being counted, we give the details to show $\overline{\text{spt}}_2(4) = 16$. Using the overpartitions of 4 listed in the introduction, we see the partition pairs of 4 from $\bar{\mathcal{S}}$ are $(4, \emptyset)$, $(3+1, \emptyset)$, $(1, 3)$, $(2+2, \emptyset)$, $(2+1+1, \emptyset)$, $(1+1, 2)$, and $(1+1+1+1, \emptyset)$. Since the compositions of 2 are just 2 and $1+1$ and for these partition pairs we have $f_j^1 = 0$ for $j > 2$, the weight ω_2 reduces to

$$\omega_2(\vec{\pi}) = \binom{f_1^1(\vec{\pi})+1}{3} + f_1^1(\vec{\pi}) \binom{f_2^1(\vec{\pi})+1}{2}.$$

For larger values of n , $\overline{\text{spt}}_2(n)$ and $\omega_2(\vec{\pi})$ would be slightly more complicated as there would be partition pairs with $f_j^1 \neq 0$ for j past 1 and 2. We collect the information for each partition pair in the following table.

\bar{S} -partition pair	f_1^1	f_1^2	ω_2
$(4, \emptyset)$	1	0	0
$(3 + 1, \emptyset)$	1	1	1
$(1, 3)$	1	0	0
$(2 + 2, \emptyset)$	2	0	1
$(2 + 1 + 1, \emptyset)$	2	1	3
$(1 + 1, 2)$	2	0	1
$(1 + 1 + 1 + 1, \emptyset)$	4	0	10

4. Congruences for $\overline{\text{spt}}_2(n)$

It appears that these higher order spt functions satisfy various congruences. We prove two of them.

Theorem 4.1. For $n \geq 0$,

$$\begin{aligned}\overline{\text{spt}}_2(5n + 1) &\equiv 0 \pmod{5}, \\ \overline{\text{spt}}_2(5n + 3) &\equiv 0 \pmod{5}.\end{aligned}$$

Proof. We have

$$\overline{\text{spt}}_2(n) = \frac{1}{24}(\overline{M}_4(n) - \overline{M}_2(n) - \overline{N}_4(n) + \overline{N}_2(n)). \quad (4.1)$$

Reducing equation (3.1) of [10] modulo 5 gives

$$\overline{N}_4(n) \equiv (2n + 4)\overline{N}_2(n) + (2n + 2)\overline{M}_2(n) + \overline{M}_4(n) + 2n\overline{M}_2(n) \pmod{5},$$

so (4.1) becomes

$$\overline{\text{spt}}_2(n) \equiv (3 + 2n)\overline{M}_2(n) + 2n\overline{M}_2(n) + (3 + 2n)\overline{N}_2(n) \pmod{5}. \quad (4.2)$$

The following are equation (4.4) and an equation out of the proof of Theorem 3.1 of [10]:

$$(2n^2 + n + 2)\overline{M}_2(n) + (n^2 + 4n + 2)\overline{M}_2(n) \equiv 0 \pmod{5}, \quad (4.3)$$

$$(4n^2 + n)\overline{M}_2(n) + (4n + 4)\overline{M}_2(n) + (3n^2 + 2)\overline{N}_2(n) \equiv 0 \pmod{5}. \quad (4.4)$$

In (4.3) we replace n by $5n + 1$ and in (4.4) we replace n by $5n + 3$ to get

$$\begin{aligned}\overline{M}_2(5n + 1) &\equiv 0 \pmod{5}, \\ 4\overline{M}_2(5n + 3) + \overline{M}_2(5n + 3) + 4\overline{N}_2(5n + 3) &\equiv 0 \pmod{5}.\end{aligned}$$

With (4.2) we then have

$$\overline{\text{spt}}_2(5n+1) \equiv 2\overline{M}_2(5n+1) \equiv 0 \pmod{5},$$

$$\overline{\text{spt}}_2(5n+3) \equiv 4\overline{M}_2(5n+3) + 4\overline{N}_2(5n+3) + \overline{M}_2(5n+3) \equiv 0 \pmod{5}. \quad \square$$

Congruences for $M_2\text{spt}_2(n)$ will be handled in a future paper.

5. Remarks

In [14] Dixit and Yee also generalized the spt function to Spt_j and generalized the higher order spt -function spt_k to ${}_j\text{spt}_k$. They used

$$\begin{aligned} \text{Spt}_j(n) &= \frac{1}{2}M_2(n) - \frac{1}{2}_{j+1}N_2(n), \\ {}_j\text{spt}_k(n) &= {}_j\mu_{2k}(n) - {}_{j+1}\mu_{2k}(n), \end{aligned}$$

where

$$\begin{aligned} {}_jN_k(n) &= \sum_{m \in \mathbb{Z}} m^k N_j(m, n), \\ {}_j\mu_k(n) &= \sum_{m \in \mathbb{Z}} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N_j(m, n), \end{aligned}$$

and $N_j(m, n)$ is the number of partitions of n with at least $j-1$ successive Durfee squares whose j -rank is m . It may be possible to work out generalizations of this form for the three spt functions we have investigated here.

It is worth mentioning that it is not \overline{R}_2 and \overline{C}_2 that were used in [18] to reprove certain congruences satisfied by $\overline{\text{spt}}_2(n)$. However, the methods in that paper can be used with \overline{R}_2 and \overline{C}_2 to prove the congruences $\overline{\text{spt}}_2(3n) \equiv \overline{\text{spt}}_2(3n+1) \equiv 0 \pmod{3}$. Yet those methods do no work to prove the congruence $\overline{\text{spt}}_2(5n+3) \equiv 0 \pmod{5}$ with \overline{R}_2 and \overline{C}_2 .

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