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More infinite families of congruences modulo 5 for broken 2-diamond partitions

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Abstract. The notion of broken k-diamond partitions was introduced by Andrews and Paule. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Recently, Chan, and Paule and Radu proved some congruences modulo 5 for $\Delta_2(n)$. In this paper, we prove several new infinite families of congruences modulo 5 for $\Delta_2(n)$ by using an identity due to Newman. Our results generalize the congruences proved by Paule and Radu.

Keywords: broken k-diamond partition, congruence, theta function.

AMS Subject Classification: 11P83, 05A17

1 Introduction

The aim of this paper is to prove several new infinite families of congruences modulo 5 for broken 2-diamond partitions, which generalize some congruence results due to Paule and Radu [12].

Let us begin with some notation and terminology on q-series and partitions. We use the standard notation

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$$

and often write

$$(a_1,a_2,\ldots,a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty.$$

Recall that the Ramanujan theta function $f(a,b)$ is defined by

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \tag{1.1}$$

where $|ab|<1$. The Jacobi triple product identity can be restated as

$$f(a,b)=(-a,-b,ab;ab)_\infty. \tag{1.2}$$

One special case of (1.1) is defined by

$$\psi(q)=f(q,q^3)=\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}. \tag{1.3}$$

By (1.1), (1.2) and (1.3),

$$\psi(q) = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}. \tag{1.4}$$

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [2] to the construction of a new class of directed graphs called broken k-diamond partitions. Let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n for a fixed positive integer k. Andrews and Paule [2] established the following generating function of $\Delta_k(n)$:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{4k+2};q^{4k+2})_\infty}. \tag{1.5}$$

Employing generating function manipulations, Andrews and Paule [2] proved that for all integers $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod{3}.$$

Since then, a number of congruences satisfied by $\Delta_k(n)$ for small values of k have been proved. See, for example, Chan [4], Chen, Fan and Yu [5], Hirschhorn and Sellers [7], Lin [8], Lin and Wang [9], Paule and Radu [12], Radu and Sellers [13, 14, 15], Xia [17, 18] and Yao [20].

In 2008, Chan [4] proved that for $n,\ k \geq 0$,

$$\Delta_2(10n+2) \equiv \Delta_2(10n+6) \equiv 0 \ (\mathrm{mod}\ 2),$$

and

$$\Delta_2\left(5^{k+1}(5n+\omega) + \frac{3 \times 5^{k+1}+1}{4}\right) \equiv 0 \ (\mathrm{mod}\ 5), \tag{1.6}$$

where $\omega \in \{2,\ 4\}$. Congruences $\Delta_2(10n+2) \equiv 0$ (mod 2) and $\Delta_2(25n+14) \equiv 0$ (mod 5) were conjectured by Andrews and Paule[2] and proved by Chan [4]. In 2010, Paule and Radu [12] proved several infinite families of congruences modulo 5 for $\Delta_2(n)$. They proved that for $k \geq 0$,

$$\Delta_2\left(\frac{15 \times 29^k + 1}{4}\right) \equiv 1 + k \ (\mathrm{mod}\ 5) \tag{1.7}$$

and if $p \nmid (4n+3)$, then

$$\Delta_2\left(5pn + \frac{15p+1}{4}\right) \equiv 0 \ (\mathrm{mod}\ 5), \tag{1.8}$$

where p is a prime with $p \equiv 13,\ 17$ (mod 20).

In this paper, we prove many new infinite families of congruences modulo 5 for $\Delta_2(n)$ by using an identity of Newman [11] and several theta function identities. Our results generalize congruences proved by Paule and Radu [12].

The following theorem states three non-standard infinite families of congruences modulo 5 for $\Delta_2(n)$.

Theorem 1.1 *If p is a prime with $p \equiv 1,\ 9 \pmod{20}$, then for $k \geq 0$,*

$$\Delta_2\left(\frac{15p^k+1}{4}\right) \equiv 1+k \pmod{5}. \tag{1.9}$$

If p is a prime with $p \equiv 13,\ 17 \pmod{20}$, then for $k \geq 0$,

$$\Delta_2\left(\frac{15p^{2k}+1}{4}\right) \equiv 0 \pmod{5}, \tag{1.10}$$

and

$$\Delta_2\left(\frac{15p^{2k+1}+1}{4}\right) \equiv 1 \pmod{5}. \tag{1.11}$$

It should be noted that if we set $p = 29$ in (1.9), we obtain (1.7). Therefore, (1.9) is a generalization of (1.7).

In order to state the following theorem, we introduce the Legendre symbol. Let $p \geq 3$ be a prime. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0, & \text{if } p | a. \end{cases}$$

Theorem 1.2 *Let $\alpha,\ \beta,\ \nu$ be nonnegative integers and let $p_1,\ \ldots, p_\alpha,\ p_{\alpha+1}, q_1,\ \ldots, q_\beta, r_1,\ \ldots, r_\nu,\ r_{\nu+1}$ be primes with $p_i \equiv 13,\ 17 \pmod{20}$, $q_j \equiv 1,\ 9 \pmod{20}$ and $\left(\frac{-5}{r_s}\right) = -1$ for $1 \leq i \leq \alpha+1$, $1 \leq j \leq \beta$ and $1 \leq s \leq \nu+1$. Define*

$$A = 5^k p_1^2 p_2^2 \cdots p_\alpha^2 q_1^5 q_2^5 \cdots q_\beta^5 r_1^2 r_2^2 \cdots r_\nu^2. \tag{1.12}$$

If $p_{\alpha+1} \nmid (4n+3)$, then

$$\Delta_2\left(5Ap_{\alpha+1}n + \frac{15Ap_{\alpha+1}+1}{4}\right) \equiv 0 \pmod{5}. \tag{1.13}$$

If $r_{\nu+1} \nmid n$, then

$$\Delta_2\left(5Ar_{\nu+1}n + \frac{15Ar_{\nu+1}^2+1}{4}\right) \equiv 0 \pmod{5}. \tag{1.14}$$

Remark. If we set $\alpha = \beta = \nu = k = 0$ in (1.13), we obtain (1.8). From Theorem 1.2, we can obtain many new congruences modulo 5 for $\Delta_2(n)$. For example, setting $k = \beta = \nu = 0$, $\alpha = 1$ and $p_1 = p_2 = 13$, we deduce that if $13 \nmid (4n+3)$, then

$$\Delta_2(10985n + 8239) \equiv 0 \pmod{5}. \tag{1.15}$$

If we set $p = 13$ in (1.8), we find that if $13 \nmid (4n+3)$, then

$$\Delta_2(65n + 49) \equiv 0 \pmod{5}. \tag{1.16}$$

It should be noted that we can not replace n by $169n + 126$ in (1.16) to get (1.15) since $13 | (4(169n + 126) + 3)$. Therefore, Congruence (1.16) does not imply (1.15).

2 Proof of Theorem 1.1

In order to prove Theorem 1.1, we first prove two lemmas.

Lemma 2.1 *For $n \geq 0$,*

$$\Delta_2(5n+4) \equiv t_6(n) \pmod 5, \tag{2.1}$$

where $t_6(n)$ is the number of representations of n as a sum of six triangular numbers.

Proof. Setting $n = 2$ in (1.5), we get

$$\sum_{n=0}^{\infty} \Delta_2(n) q^n = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty}{(q;q)_\infty^3 (q^{10};q^{10})_\infty}. \tag{2.2}$$

By the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5. \tag{2.3}$$

Thanks to (2.2) and (2.3),

$$\sum_{n=0}^{\infty} \Delta_2(n) q^n \equiv \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^4} \pmod 5. \tag{2.4}$$

It follows from [3, Entry 10 (iv), p. 262] that

$$\frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^2} = \frac{(q^5;q^4)_\infty^4}{(q^{10};q^{10})_\infty^2} - 4q \frac{(q;q)_\infty (q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty (q^5;q^5)_\infty}. \tag{2.5}$$

By (2.3) and (2.5),

$$\begin{aligned}
\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty^4} &= \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} \frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^2} \\
&= \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2} \left(\frac{(q^5;q^5)_\infty^4}{(q^{10};q^{10})_\infty^2} - 4q \frac{(q;q)_\infty (q^{10};q^{10})_\infty^3}{(q^2;q^2)_\infty (q^5;q^5)_\infty} \right) \\
&= \frac{(q^5;q^5)_\infty^4}{(q;q)_\infty^2 (q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2} - 4q \frac{(q^{10};q^{10})_\infty^3}{(q;q)_\infty (q^2;q^2)_\infty^3 (q^5;q^5)_\infty} \\
&\equiv \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^3 (q^5;q^5)_\infty^3}{(q^{10};q^{10})_\infty^3} - 4q \frac{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2}{(q;q)_\infty (q^5;q^5)_\infty} \pmod 5. \qquad (2.6)
\end{aligned}$$

In view of (2.4) and (2.6),

$$\sum_{n=0}^{\infty} \Delta_2(n) q^n \equiv \frac{(q;q)_\infty^3 (q^2;q^2)_\infty^3 (q^5;q^5)_\infty^3}{(q^{10};q^{10})_\infty^3} - 4q \frac{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2}{(q;q)_\infty (q^5;q^5)_\infty} \pmod 5. \tag{2.7}$$

We have the well-known result of Jacobi [1, p.176] which states that

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{\frac{n(n+1)}{2}}. \tag{2.8}$$

By (2.8), it is trivial to check that

$$(q;q)_\infty^3 \equiv (q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty \pmod 5. \tag{2.9}$$

From [3, Corollary (ii), p. 49],

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = (-q^{10},-q^{15},q^{25};q^{25})_\infty + q(-q^5,-q^{20},q^{25};q^{25})_\infty + q^3\frac{(q^{50};q^{50})_\infty^2}{(q^{25};q^{25})_\infty}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.7), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta_2(n)q^n \equiv& \frac{(q^5;q^5)_\infty^3}{(q^{10};q^{10})_\infty^3}\big((q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty\big)\\ &\times\big((q^{20},q^{30},q^{50};q^{50})_\infty - 3q^2(q^{10},q^{40},q^{50};q^{50})_\infty\big)\\ &- 4q\frac{(q^{10};q^{10})_\infty^2}{(q^5;q^5)_\infty}\bigg((-q^{10},-q^{15},q^{25};q^{25})_\infty + q(-q^5,-q^{20},q^{25};q^{25})_\infty\\ &+ q^3\frac{(q^{50};q^{50})_\infty^2}{(q^{25};q^{25})_\infty}\bigg) \pmod 5. \end{aligned} \tag{2.11}$$

If we extract the terms of the form q^{5n+4} in both sides of (2.11), divide by q^4, replace q^5 by q and then apply (2.3), we obtain

$$\sum_{n=0}^{\infty}\Delta_2(5n+4)q^n \equiv \frac{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^2}{(q;q)_\infty(q^5;q^5)_\infty} \equiv \frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^6} \pmod 5. \tag{2.12}$$

By (1.3) and (1.4), the generating function of $t_6(n)$ is

$$\sum_{n=0}^{\infty}t_6(n)q^n = \psi^6(q) = \frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^6}. \tag{2.13}$$

Congruence (2.1) follows from (2.12) and (2.13). This completes the proof of this lemma.

Lemma 2.2 *For $n,\ k \geq 0$,*

$$t_6\left(p^k n + \frac{3(p^k-1)}{4}\right) = \frac{p^{2k}-1}{p^2-1}t_6\left(pn + \frac{3(p-1)}{4}\right) + \frac{p^2-p^{2k}}{p^2-1}t_6(n), \tag{2.14}$$

where p is a prime with $p \equiv 1 \pmod 4$.

Proof. Newman [11] proved that for $n \geq 0$,

$$t_6\left(pn + \frac{3(p-1)}{4}\right) = t_6\left(\frac{3(p-1)}{4}\right) t_6(n) - p^2 t_6\left(\frac{n}{p} - \frac{3(p-1)}{4p}\right), \tag{2.15}$$

where p is a prime with $p \equiv 1 \pmod 4$. The following formula for $t_6(n)$ was proved by Liu [10] by using an identity of Ramanujan:

$$t_6(n) = \frac{1}{8}\left(\sum_{d|(4n+3),\ d\equiv 3 \pmod 4,\ d>0} d^2 - \sum_{d|(4n+3),\ d\equiv 1 \pmod 4,\ d>0} d^2\right). \tag{2.16}$$

By (2.16),

$$t_6\left(\frac{3(p-1)}{4}\right) = (1+p^2). \tag{2.17}$$

Replacing n by $pn + \frac{3(p-1)}{4}$ in (2.15) and utilizing (2.17), we obtain

$$t_6\left(p^2 n + \frac{3(p^2-1)}{4}\right) = (1+p^2) t_6\left(pn + \frac{3(p-1)}{4}\right) - p^2 t_6(n). \tag{2.18}$$

Now, we are ready to prove Lemma 2.2 by induction.

It is easy to check that (2.14) is true when $k=0$ and $k=1$. Assume that (2.14) holds when $k=m$ and $k=m+1$ ($m \geq 0$), that is,

$$t_6\left(p^m n + \frac{3(p^m-1)}{4}\right) = \frac{p^{2m}-1}{p^2-1} t_6\left(pn + \frac{3(p-1)}{4}\right) + \frac{p^2-p^{2m}}{p^2-1} t_6(n) \tag{2.19}$$

and

$$t_6\left(p^{m+1} n + \frac{3(p^{m+1}-1)}{4}\right) = \frac{p^{2m+2}-1}{p^2-1} t_6\left(pn + \frac{3(p-1)}{4}\right) + \frac{p^2-p^{2m+2}}{p^2-1} t_6(n) \tag{2.20}$$

Replacing n by $p^m n + \frac{3(p^m-1)}{4}$ in (2.18) and employing (2.19) and (2.20), we have

$$\begin{aligned}
t_6\left(p^{m+2} n + \frac{3(p^{m+2}-1)}{4}\right) =& (1+p^2) t_6\left(p^{m+1} n + \frac{3(p^{m+1}-1)}{4}\right) - p^2 t_6\left(p^m n + \frac{3(p^m-1)}{4}\right) \\
=& (1+p^2)\left(\frac{p^{2m+2}-1}{p^2-1} t_6(pn + \frac{3(p-1)}{4}) + \frac{p^2-p^{2m+2}}{p^2-1} t_6(n)\right) \\
& - p^2\left(\frac{p^{2m}-1}{p^2-1} t_6(pn + \frac{3(p-1)}{4}) + \frac{p^2-p^{2m}}{p^2-1} t_6(n)\right) \\
=& \left(\frac{(1+p^2)(p^{2m+2}-1)}{p^2-1} - \frac{p^2(p^{2m}-1)}{p^2-1}\right) t_6\left(pn + \frac{3(p-1)}{4}\right) \\
& + \left(\frac{(1+p^2)(p^2-p^{2m+2})}{p^2-1} - \frac{p^2(p^2-p^{2m})}{p^2-1}\right) t_6(n)
\end{aligned}$$

$$=\frac{p^{2m+4}-1}{p^2-1}t_6\left(pn+\frac{3(p-1)}{4}\right)+\frac{p^2-p^{2m+4}}{p^2-1}t_6(n),$$

which implies that (2.14) is true when $k=m+2$ and this lemma is proved by induction.

Now, we turn to prove Theorem 1.1.

Setting $n=0$ in (2.14) and employing (2.17), we find that for $k\geq 0$,

$$t_6\left(\frac{3(p^k-1)}{4}\right)=\frac{p^{2k}-1}{p^2-1}(1+p^2)+\frac{p^2-p^{2k}}{p^2-1}=\frac{p^{2k+2}-1}{p^2-1}. \tag{2.21}$$

If p is a prime with $p\equiv 1,\ 9 \pmod{20}$, then for $k\geq 0$,

$$\frac{p^{2k+2}-1}{p^2-1}\equiv 1+k \pmod{5} \tag{2.22}$$

If p is a prime with $p\equiv 13,\ 17 \pmod{20}$, then for $k\geq 0$,

$$\frac{p^{4k+4}-1}{p^2-1}\equiv 0 \pmod{5} \tag{2.23}$$

and

$$\frac{p^{4k+2}-1}{p^2-1}\equiv 1 \pmod{5}. \tag{2.24}$$

In view of (2.21), (2.22), (2.23) and (2.24), we deduce that if p is a prime with $p\equiv 1,\ 9 \pmod{20}$, then for $k\geq 0$,

$$t_6\left(\frac{3(p^k-1)}{4}\right)\equiv 1+k \pmod{5}, \tag{2.25}$$

and if p is a prime with $p\equiv 13,\ 17 \pmod{20}$, then for $k\geq 0$,

$$t_6\left(\frac{3(p^{2k+1}-1)}{4}\right)\equiv 0 \pmod{5} \tag{2.26}$$

and

$$t_6\left(\frac{3(p^{2k}-1)}{4}\right)\equiv 1 \pmod{5}. \tag{2.27}$$

Replacing n by $\frac{3(p^k-1)}{4}$ in (2.1) and utilizing (2.25), we get (1.9). Similarly, Congruences (1.10) and (1.11) follow from (2.1), (2.26) and (2.27). The proof is complete.

3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we first prove some lemmas.

Lemma 3.1 *If p is a prime with $p \equiv 13,\ 17 \pmod{20}$ and $p \nmid (4n+3)$, then*

$$t_6\left(pn + \frac{3(p-1)}{4}\right) \equiv 0 \pmod{5}. \tag{3.1}$$

If p is a prime with $p \equiv 13,\ 17 \pmod{20}$, then

$$t_6\left(p^2 n + \frac{3(p^2-1)}{4}\right) \equiv t_6(n) \pmod{5}. \tag{3.2}$$

If p is a prime with $p \equiv 1,\ 9 \pmod{20}$, then

$$t_6\left(p^5 n + \frac{3(p^5-1)}{4}\right) \equiv t_6(n) \pmod{5}. \tag{3.3}$$

Proof. If p is a prime with $p \equiv 13,\ 17 \pmod{20}$, then $1 + p^2 \equiv 0 \pmod{5}$. By (2.17), we can rewrite (2.15) as

$$t_6\left(pn + \frac{3(p-1)}{4}\right) \equiv t_6\left(\frac{n}{p} - \frac{3(p-1)}{4p}\right) \pmod{5}. \tag{3.4}$$

Therefore, if $p \nmid (4n+3)$, then $\frac{n}{p} - \frac{3(p-1)}{4p}$ is not an integer and

$$t_6\left(\frac{n}{p} - \frac{3(p-1)}{4p}\right) = 0. \tag{3.5}$$

Congruence (3.1) follows from (3.4) and (3.5).

Replacing n by $pn + \frac{3(p-1)}{4}$ in (3.4), we obtain (3.2).

If p is a prime with $p \equiv 1,\ 9 \pmod{20}$, then

$$\frac{p^{10}-1}{p^2-1} \equiv 0 \pmod{5} \tag{3.6}$$

and

$$\frac{p^2-p^{10}}{p^2-1} \equiv 1 \pmod{5}. \tag{3.7}$$

Setting $k = 5$ in (2.14) and using (3.6) and (3.7), we get (3.3). This completes the proof of this lemma.

Lemma 3.2 *Let $p \geq 3$ be a prime with $\left(\frac{-5}{p}\right) = -1$. For $n \geq 0$,*

$$t_6\left(p^2 n + \frac{3(p^2-1)}{4}\right) \equiv t_6(n) \pmod{5}. \tag{3.8}$$

If $p \nmid n$,

$$t_6\left(pn + \frac{3(p^2-1)}{4}\right) \equiv 0 \pmod{5}. \tag{3.9}$$

Proof. Let $a(n)$ be defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2}{(q;q)_\infty (q^5;q^5)_\infty}. \tag{3.10}$$

By (2.3) and (2.13),

$$\sum_{n=0}^{\infty} t_6(n)q^n \equiv \frac{(q^2;q^2)_\infty^2 (q^{10};q^{10})_\infty^2}{(q;q)_\infty (q^5;q^5)_\infty} \pmod 5. \tag{3.11}$$

It follows from (3.10) and (3.11) that for $n \geq 0$,

$$t_6(n) \equiv a(n) \pmod 5. \tag{3.12}$$

Based on (1.3), (1.4) and (3.10),

$$\sum_{n=0}^{\infty} a(n)q^n = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} q^{\frac{k(k+1)}{2}+5\frac{m(m+1)}{2}},$$

which yields

$$a(n) = \sum_{\substack{0\leq k,\ m<+\infty,\\ \frac{k^2+k}{2}+\frac{5(m^2+m)}{2}=n}} 1 = \sum_{\substack{0\leq k,\ m<+\infty,\\ (2k+1)^2+5(2m+1)^2=8n+6}} 1. \tag{3.13}$$

Replacing n by $pn+\frac{3(p^2-1)}{4}$ in (3.13), we get

$$a\left(pn+\frac{3(p^2-1)}{4}\right) = \sum_{\substack{0\leq k,\ m<+\infty,\\ (2k+1)^2+5(2m+1)^2=8pn+6p^2}} 1. \tag{3.14}$$

Note that Identity $(2k+1)^2+5(2m+1)^2 \equiv 0 \pmod p$ implies $2k+1 \equiv 2m+1 \equiv 0 \pmod p$ since $p \geq 3$ is a prime and $\left(\frac{-5}{p}\right) = -1$. Setting $2k+1 = p(2k^{'}+1)$ and $2m+1 = p(2m^{'}+1)$ with $k^{'},\ m^{'} \geq 0$ in (3.14), we get

$$\begin{aligned} a\left(pn+\frac{3(p^2-1)}{4}\right) &= \sum_{\substack{0\leq\ k^{'},\ m^{'}<+\infty,\\ p^2(2k^{'}+1)^2+5p^2(2m^{'}+1)^2=8pn+6p^2}} 1 \\ &= \sum_{\substack{0\leq\ k^{'},\ m^{'}<+\infty,\\ (2k^{'}+1)^2+5(2m^{'}+1)^2=8n/p+6}} 1 \\ &= a(n/p). \end{aligned} \tag{3.15}$$

Replacing n by pn in (3.15), we have

$$a\left(p^2n+\frac{3(p^2-1)}{4}\right) = a(n). \tag{3.16}$$

Congruence (3.8) follows from (3.12) and (3.16). If $p \nmid n$, then $a(n/p) = 0$ and

$$a\left(pn + \frac{3(p^2-1)}{4}\right) = 0,$$

which yields (3.9) after using (3.12). The proof of this lemma is complete.

Now, we turn to prove Theorem 1.2.

Let α, β be nonnegative integers and let $p_1,\ p_2, \ldots, p_\alpha,\ p_{\alpha+1},\ q_1,\ q_2, \ldots, q_\beta$ be primes with $p_i \equiv 13,\ 17 \pmod{20}$ and $q_j \equiv 1,\ 9 \pmod{20}$ for $1 \le i \le \alpha+1$ and $1 \le j \le \beta$. Congruences (3.1), (3.2) and (3.3) imply that for $n \ge 0$,

$$t_6\left(p_1^2p_2^2\cdots p_\alpha^2 n + \frac{3(p_1^2p_2^2\cdots p_\alpha^2-1)}{4}\right) \equiv t_6(n) \pmod 5, \tag{3.17}$$

$$t_6\left(q_1^5q_2^5\cdots q_\beta^5 n + \frac{3(q_1^5q_2^5\cdots q_\beta^5-1)}{4}\right) \equiv t_6(n) \pmod 5, \tag{3.18}$$

and if $p_{\alpha+1} \nmid (4n+3)$, then

$$t_6\left(p_{\alpha+1} n + \frac{3(p_{\alpha+1}-1)}{4}\right) \equiv 0 \pmod 5. \tag{3.19}$$

Replacing n by $q_1^5q_2^5\cdots q_\beta^5 n + \frac{3(q_1^5q_2^5\cdots q_\beta^5-1)}{4}$ in (3.17) and using (3.18), we have

$$t_6\left(p_1^2p_2^2\cdots p_\alpha^2 q_1^5q_2^5\cdots q_\beta^5 n + \frac{3(p_1^2p_2^2\cdots p_\alpha^2 q_1^5q_2^5\cdots q_\beta^5-1)}{4}\right) \equiv t_6(n) \pmod 5. \tag{3.20}$$

Let ν be a nonnegative integer and let $r_1, r_2, \ldots r_\nu, r_{\nu+1}$ be primes with $\left(\frac{-5}{r_s}\right) = -1$ for $1 \le s \le \nu+1$. Congruences (3.8) and (3.9) imply that for $n \ge 0$,

$$t_6(r_1^2r_2^2\cdots r_\nu^2 n + \frac{3(r_1^2r_2^2\cdots r_\nu^2-1)}{4}) \equiv t_6(n) \pmod 5 \tag{3.21}$$

and if $r_{\nu+1} \nmid n$, then

$$t_6\left(r_{\nu+1} n + \frac{3(r_{\nu+1}^2-1)}{4}\right) \equiv 0 \pmod 5. \tag{3.22}$$

Replacing n by $r_1^2r_2^2\cdots r_\nu^2 n + \frac{3(r_1^2r_2^2\cdots r_\nu^2-1)}{4}$ in (3.20) and utilizing (3.21), we get

$$t_6\left(p_1^2p_2^2\cdots p_\alpha^2 q_1^5q_2^5\cdots q_\beta^5 r_1^2r_2^2\cdots r_\nu^2 n + \frac{3(p_1^2p_2^2\cdots p_\alpha^2 q_1^5q_2^5\cdots q_\beta^5 r_1^2r_2^2\cdots r_\nu^2-1)}{4}\right)$$
$$\equiv t_6(n) \pmod 5. \tag{3.23}$$

Setting $p = 5$ in (2.15) and using (2.17), we get

$$t_6(5n+3) \equiv t_6(n) \pmod 5. \tag{3.24}$$

By (3.24) and mathematical induction, we see that for $n,\ k \geq 0$,

$$t_6\left(5^k n + \frac{3(5^k-1)}{4}\right) \equiv t_6(n) \pmod 5. \tag{3.25}$$

Replacing n by $5^k n + \frac{3(5^k-1)}{4}$ in (3.23) and employing (3.25), we deduce that for $n,\ k \geq 0$,

$$t_6\left(An + \frac{3(A-1)}{4}\right) \equiv t_6(n) \pmod 5, \tag{3.26}$$

where A is defined by (1.12).

Replacing n by $p_{\alpha+1}n + \frac{3(p_{\alpha+1}-1)}{4}$ $(p_{\alpha+1} \nmid (4n+3))$ in (3.26) and using (3.19), we have

$$t_6\left(Ap_{\alpha+1}n + \frac{3(Ap_{\alpha+1}-1)}{4}\right) \equiv 0 \pmod 5. \tag{3.27}$$

Congruence (1.13) follows from (2.1) and (3.27). Replacing n by $r_{\nu+1}n + \frac{3(r_{\nu+1}^2-1)}{4}$ $(r_{\nu+1} \nmid n)$ in (3.26) and utilizing (3.22), we find that

$$t_6\left(Ar_{\nu+1}n + \frac{3(Ar_{\nu+1}^2-1)}{4}\right) \equiv 0 \pmod 5, \tag{3.28}$$

which yields (1.14) after using (2.1). This completes the proof of Theorem 1.2.

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