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Diophantine approximation over Piatetski-Shapiro primes

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ABSTRACT

Let $c > 1$ and $0 < \gamma < 1$ be real. In this paper, we study the solubility of the Diophantine inequality

$$\left|p_1^c + p_2^c + \cdots + p_s^c - N\right| < \varepsilon$$

in Piatetski-Shapiro primes $p_1, p_2, \ldots, p_s$ of the form $p_j = [m^{1/\gamma}]$ for some $m \in \mathbb{N}$, and improve the previous results of Kumchev and Petrov [20].

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1. Introduction

Let $0 < \gamma < 1$ be a fixed real number. In [24] Piatetski-Shapiro considered the question whether the sequence

$$\mathcal{N}_\gamma = \{n \in \mathbb{N} : n = [m^{1/\gamma}] \text{ for some } m \in \mathbb{N}\}$$

contains infinitely many primes. For the number $\pi_\gamma(N)$ of primes $p \leq N$ that belong to $\mathcal{N}_\gamma$, he proved that when $\gamma > 11/12$, one has the asymptotic formula

$$\pi_\gamma(N) = \frac{N^\gamma}{\log N}\big(1 + O(\log N)^{-1}\big). \tag{1.1}$$

Such prime numbers $p \in \mathcal{N}_\gamma$ have become known as *Piatetski-Shapiro primes* (of index γ), and the work on it has attracted a lot of attention [1,10,12–15,17,22,25] culminating in the result of Rivat and Wu [26] who proved that $\pi_\gamma(N) \to \infty$ for $\gamma > 205/243$.

Another problem proposed by Piatetski-Shapiro [23] around the time he proved (1.1) investigates the solubility of the Diophantine inequality

$$\big|p_1^c + p_2^c + \cdots + p_s^c - N\big| < \varepsilon \tag{1.2}$$

in primes $p_1, p_2, \ldots, p_s$, where the exponent $c > 1$ is not an integer, $\varepsilon > 0$ is a fixed small number, and N is a large real number. Denote by $H(c)$ the least integer s such that (1.2) has solutions for sufficiently large N. It was proved in [23] that

$$H(c) \leq c(4\log c + O(\log\log c))$$

for large c, and $H(c) \leq 5$ for $1 < c < 3/2$. About forty years later Tolev [27] obtained $H(c) \leq 3$ for $1 < c < 15/14$. Afterwards it has motivated a series of improvements on sums of both three and five powers of primes. In particular, one can chart the developments in [3–7,16,19] and [2,8,21,28–30] for the two problems, respectively. For instance, the best result to date for three powers of primes was built in a recent work of Cai [6] with a statement that $H(c) \leq 3$ for $1 < c < 43/36$.

Note that the sequence of Piatetski-Shapiro primes of index γ is a "thin" set of primes, and gets thinner as γ decreases. As researchers in additive number theory often ask whether different additive questions can be resolved in prime numbers from thin sets, Piatetski-Shapiro primes have become very favorite to accomplish this goal. Very recently, Kumchev and Petrov [20] proposed to explore the solubility of the Diophantine inequality (1.2) in Piatetski-Shapiro primes. More precisely, they established in the ternary case the following variant of Tolev's result in [27], i.e. for sufficiently large N, the inequality

$$\big|p_1^c + p_2^c + p_3^c - N\big| < (\log N)^{-1} \tag{1.3}$$

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has solutions in prime numbers $p_1, p_2, p_3 \in \mathcal{N}_\gamma$, with $\gamma < 1 < c$ and $15(c-1) + 28(1-\gamma) < 1$. In this paper, we are able to extend the range of γ and c to $8(c-1)+21(1-\gamma) < 1$.

Theorem 1. *Let $\gamma < 1 < c$ and $8(c-1) + 21(1-\gamma) < 1$. Then for sufficiently large N, the inequality (1.3) has solutions in prime numbers $p_1, p_2, p_3 \in \mathcal{N}_\gamma$.*

Actually the proof of Theorem 1 can be easily adapted to establish the following companion results on the binary and quaternary inequalities.

Theorem 2. *Let $(c,\gamma) \in \big\{(c,\gamma) : \gamma < 1 < c,\ 21(c-1)+33(1-\gamma) < 4,\ 5(c-1)+15(1-\gamma) < 1,\ 9(c-1) + 35(1-\gamma) < 2\big\}$. Then for sufficiently large N, the inequality*

$$\big|p_1^c + p_2^c + p_3^c + p_4^c - N\big| < (\log N)^{-1}$$

has solutions in prime numbers $p_1, \dots, p_4 \in \mathcal{N}_\gamma$.

Theorem 3. *Let $(c,\gamma) \in \big\{(c,\gamma) : \gamma < 1 < c,\ 21(c-1)+33(1-\gamma) < 4,\ 5(c-1)+15(1-\gamma) < 1,\ 9(c-1) + 35(1-\gamma) < 2\big\}$. For a large Z, let $\mathcal{E}(Z)$ denote the set of $N \in (Z/2, Z]$ for which the inequality*

$$\big|p_1^c + p_2^c - N\big| < (\log N)^{-1}$$

has no solutions in prime numbers $p_1, p_2 \in \mathcal{N}_\gamma$. Then the Lebesgue measure of the exceptional set $\mathcal{E}(Z)$ is $O(Z\exp(-(\log Z)^{1/4}))$.

Remark. We note here that the area of pair (c,γ) in Theorems 2 and 3, i.e. the set $\big\{(c,\gamma) : \gamma < 1 < c,\ 21(c-1) + 33(1-\gamma) < 4,\ 5(c-1) + 15(1-\gamma) < 1,\ 9(c-1) + 35(1-\gamma) < 2\big\}$, can be expressed explicitly as

$$(c,\gamma) \in \begin{cases} \big\{(c,\gamma) : 4.5(c-1) + 17.5(1-\gamma) < 1\big\}, & \text{if } 0 < c-1 \le 1/8; \\ \big\{(c,\gamma) : 5(c-1) + 15(1-\gamma) < 1\big\}, & \text{if } 1/8 < c-1 \le 9/50; \\ \big\{(c,\gamma) : 5.25(c-1) + 8.25(1-\gamma) < 1\big\}, & \text{if } 9/50 < c-1 < 4/21. \end{cases} \tag{1.4}$$

The range (1.4) for γ and c satisfied in Theorems 2 and 3 can be compared with the previous result in [20], where the same conclusions were established but for $\gamma < 1 < c$ and

$$8(c-1) + 21(1-\gamma) < 1.$$

The theorems are proved by the circle method, and our improvement benefits from efforts in handling the minor arcs. For the number s of prime variables in the inequality (1.2) under consideration, traditional treatment as in [20] reveals that one can take out $s-2$ exponential sum(s) (over prime numbers) from the integral on minor arcs to borrow its upper bound estimate, and then inject the mean value estimate of the remaining integral of Vinogradov type shown in (2.9). By a different approach, we first fix one exponential sum and exchange the order of the summation and the integration. Then an appeal to Cauchy's inequality brings out a double integral with the integrands of exponential sums $S(\theta; X)$ and $T(\theta; X)$. Note that the exponential sum $T(\theta; X)$ takes values over integers rather than prime numbers. Such an argument hence allows a more flexible choice and relaxes the restriction over the exponential sums. As a result, we can get a better estimate and enlarge the area of pair (c, γ). See §3.2 below and the argument between (3.7) and (3.12) for details.

To conclude this section, we would like to mention that Kumchev and Petrov also established in [20] a general result for $c > 5$, $c \notin \mathbb{N}$ and $1 - (8c^2 + 12c + 12)^{-1} < \gamma < 1$, which states that when $s \geq 4c \log c + 4c/3 + 10$ the inequality (1.2) with $\varepsilon = (\log N)^{-1}$ has solutions in Piatetski-Shapiro primes.

As usual, the letter p, with or without subscripts, is always reserved for prime numbers. The letter δ denote a fixed positive number which can be chosen arbitrarily small; its value need not be the same in all occurrences. Sometimes we use $x \sim X$ as an abbreviation for $x \in (X/2, X]$.

We write $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ and $\Psi_\gamma(n) = \psi(-(n+1)^\gamma) - \psi(-n^\gamma)$, with $\psi(x) = x - [x] - 1/2$. We also write $\mathcal{N}_\gamma(X) = \mathcal{N}_\gamma \cap (X/2, X]$ and define several generating functions:

$$S(\theta; X) = \sum_{p \in \mathcal{N}_\gamma(X)} (\log p) e(\theta p^c), \qquad T(\theta; X) = \sum_{n \in \mathcal{N}_\gamma(X)} e(\theta n^c),$$

$$S_0(\theta; X) = \sum_{p \sim X} \gamma p^{\gamma-1} (\log p) e(\theta p^c), \qquad T_0(\theta; X) = \sum_{n \sim X} \gamma n^{\gamma-1} e(\theta n^c),$$

$$S_1(\theta; X) = \sum_{p \sim X} \Psi_\gamma(p) (\log p) e(\theta p^c), \qquad T_1(\theta; X) = \sum_{n \sim X} \Psi_\gamma(n) e(\theta n^c),$$

$$V(\theta; X) = \gamma \int_{X/2}^{X} u^{\gamma-1} e(\theta u^c) \mathrm{d}u.$$

2. Auxiliary lemmas

Lemma 2.1 *([20, Lemma 1]). Let (a_n) be a sequence of complex numbers with $|a_n| \leq A$. Then*

$$\sum_{n \in \mathcal{N}_\gamma(X)} a_n = \gamma \sum_{n \sim X} a_n n^{\gamma-1} + \sum_{n \sim X} a_n \Psi_\gamma(n) + O(AX^{\gamma-1}).$$

In particular, Lemma 2.1 yields

$$S(\theta; X) = S_0(\theta; X) + S_1(\theta; X) + O(1), \tag{2.1}$$

$$T(\theta; X) = T_0(\theta; X) + T_1(\theta; X) + O(1). \tag{2.2}$$

Lemma 2.2 *([20, Lemma 2]). Let (a_n) be a sequence of complex numbers with $|a_n| \le A$. When $0 < \sigma < (2\gamma - 1)/3$ and $X^{1-\gamma+\sigma} \le H \le X^{\gamma-2\sigma}$, one has*

$$\sum_{n\sim X} a_n \Psi_\gamma(n) \ll \sup_{\substack{Y\sim X \\ u\in\{0,1\}}} \sum_{1\le|h|\le H} \min\left(X^{\gamma-1}, |h|^{-1}\right)\left|\sum_{Y<n\le X} a_n e(h(n+u)^\gamma)\right| + AX^{\gamma-\sigma}.$$

Lemma 2.3. *Let γ, c, h, θ be real with $\gamma, c > 0$. For $u \in \{0,1\}$ and $Y \sim X$, one has*

$$\sum_{Y<n\le X} e(h(n+u)^\gamma + \theta n^c) \ll F^{1/2} + F^{1/6}X^{1/2} + F^{-1/3}X,$$

with $F \asymp |h|X^\gamma + |\theta|X^c$.

Proof. One can quote Lemma 3 with $r = 2$ and Lemma 4 with $r = 3$ in [20], or Lemma 12 in [20] with $M = 1$, to establish the desired bound. □

Lemma 2.4. *Let $\gamma < 1 < c$ and $7(c-1) + 10(1-\gamma) < 1$, and $|\theta| \le X^\delta$ for a sufficiently small $\delta > 0$. Then one has*

$$T_1(\theta; X) \ll X^{3\gamma-2c-\delta}.$$

Proof. We prove this lemma by considering two cases $X^{\gamma-c-\delta} \le |\theta| \le X^\delta$ and $|\theta| < X^{\gamma-c-\delta}$.

First we handle the case $X^{\gamma-c-\delta} \le |\theta| \le X^\delta$. Using Lemma 2.2, one has

$$T_1(\theta; X) \ll \sup_{\substack{Y\sim X \\ u\in\{0,1\}}} \sum_{1\le|h|\le H} \min\left(X^{\gamma-1}, |h|^{-1}\right)\left|\sum_{Y<n\le X} e(h(n+u)^\gamma + \theta n^c)\right| + X^{\gamma-\sigma}.$$

Then by Lemma 2.3, the first term on the right hand side is

$$\begin{aligned} &\ll \sum_{1\le|h|\le H} \min\left(X^{\gamma-1}, |h|^{-1}\right)F^{1/2} + \sum_{1\le|h|\le H} \min\left(X^{\gamma-1}, |h|^{-1}\right)F^{1/6}X^{1/2} \\ &\quad + \sum_{1\le|h|\le H} \min\left(X^{\gamma-1}, |h|^{-1}\right)F^{-1/3}X \\ &=: \Sigma_1 + \Sigma_2 + \Sigma_3, \end{aligned}$$

where $F \asymp |h|X^\gamma + |\theta|X^c$. By noting $X^{\gamma-c-\delta} \le |\theta| \le X^\delta$, one has

$$\begin{aligned} &\Sigma_1 \ll X^\delta\left((HX^\gamma)^{1/2} + X^{c/2}\right), \\ &\Sigma_2 \ll X^{1/2+\delta}\left((HX^\gamma)^{1/6} + X^{c/6}\right), \\ &\Sigma_3 \ll X^{1-\gamma/3}\Bigg(\sum_{|h|\le X^{1-\gamma}} X^{\gamma-1}|h|^{-1/3} + \sum_{|h|>X^{1-\gamma}} |h|^{-4/3}\Bigg) + X^{1-\gamma/3+\delta} \end{aligned}$$

$$\ll X^{1-\gamma/3+\delta}.$$

Taking $H = X^{1-\gamma+\sigma}$ with $\sigma = 2(c-\gamma)+\delta$, and putting Σ_1, Σ_2 and Σ_3 together, we have

$$\begin{aligned} T_1(\theta; X) &\ll X^{\delta}\big(X^{(1+\sigma)/2} + X^{c/2} + X^{(4+\sigma)/6} + X^{(3+c)/6} + X^{1-\gamma/3}\big) + X^{\gamma-\sigma} \\ &\ll X^{3\gamma-2c-\delta}, \end{aligned} \tag{2.3}$$

provided that $7(c-1)+10(1-\gamma) < 1$.

In the case $|\theta| < X^{\gamma-c-\delta}$, we actually have $F \asymp |h|X^{\gamma}$. Then following a similar argument to the first case, we can also get $T_1(\theta; X) \ll X^{3\gamma-2c-\delta}$, which completes the proof of Lemma 2.4. □

Lemma 2.5. *Suppose the pair (c, γ) satisfies $\gamma < 1 < c$ and runs over $\{(c, \gamma) : 13(c-1) + 12(1-\gamma) < 2,\ 6(c-1) + 7(1-\gamma) < 1\}$, and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$ for a sufficiently small $\delta > 0$. Then one has*

$$T_0(\theta; X) \ll X^{3\gamma-2c-\delta}.$$

Proof. By Lemma 2.3 with $h = 0$, we have

$$\sum_{n\sim X} e(\theta n^c) \ll F^{1/2} + F^{1/6}X^{1/2} + F^{-1/3}X,$$

where $F \asymp |\theta|X^c$. Noting that $c < 7/6 < 3/2$ and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$, we obtain

$$\sum_{n\sim X} e(\theta n^c) \ll X^{c/6+1/2+\delta/6} + X^{1-\gamma/3+\delta/3}. \tag{2.4}$$

Then by (2.4) and partial summation, we have

$$\begin{aligned} T_0(\theta; X) &\ll X^{\gamma+c/6-1/2+\delta/6} + X^{2\gamma/3+\delta/3} \\ &\ll X^{3\gamma-2c-\delta}, \end{aligned} \tag{2.5}$$

provided that $(c, \gamma) \in \{(c, \gamma) : 13(c-1) + 12(1-\gamma) < 2, 6(c-1) + 7(1-\gamma) < 1\}$. Then the lemma follows. □

Lemma 2.6. *Suppose the pair (c, γ) satisfies $\gamma < 1 < c$ and runs over $\{(c, \gamma) : 21(c-1) + 33(1-\gamma) < 4,\ 9(c-1) + 17(1-\gamma) < 2\}$, and $|\theta| \le X^{\delta}$ for a sufficiently small $\delta > 0$. Then one has*

$$T_1(\theta; X) \ll X^{(5\gamma-3c)/2-\delta}.$$

Proof. The proof is essentially same as that of Lemma 2.4, but taking $\sigma = 3(c-\gamma)/2+\delta$ in (2.3), then the desired bound follows under the hypothesis $(c,\gamma) \in \{(c,\gamma) : 21(c-1)+33(1-\gamma) < 4, 9(c-1)+17(1-\gamma) < 2\}$. □

Lemma 2.7. *Suppose the pair (c,γ) satisfies $\gamma < 1 < c$ and runs over $\{(c,\gamma) : 10(c-1)+9(1-\gamma) < 2,\ 9(c-1)+11(1-\gamma) < 2\}$, and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$ for a sufficiently small $\delta > 0$. Then one has*

$$T_0(\theta; X) \ll X^{(5\gamma-3c)/2-\delta}.$$

Proof. By (2.5), we get $T_0(\theta; X) \ll X^{(5\gamma-3c)/2-\delta}$ provided that $(c,\gamma) \in \{(c,\gamma) : 10(c-1)+9(1-\gamma) < 2, 9(c-1)+11(1-\gamma) < 2\}$. □

Lemma 2.8. *Let $|\theta| > 0$, and $M < N \le 2M$. Then for any exponent pair (a,b) with $0 \le a \le 1/2 \le b \le 1$, one has*

$$\sum_{M<n\le N} e(\theta n^c) \ll (|\theta| M^c)^a M^{b-a} + M^{1-c}|\theta|^{-1}.$$

Proof. By a splitting argument, one can essentially discuss three cases according as $|\theta|M^{c-1} < 1/2$, $1/2 \le |\theta|M^{c-1} \le 1$ or $|\theta|M^{c-1} > 1$, and quote the treatment of [9, Theorems 2.1 & 2.2] and of [11, §2.3], respectively, to achieve the desired estimate. One can also refer to [6, Lemma 2.2] for a detailed explanation. □

Next we quote the following estimates on $S_0(\theta; X)$ and $S_1(\theta; X)$ from [20, Lemmas 11 & 14].

Lemma 2.9. *Let $6\rho < \gamma < 1 < c < 3/2 - 6\rho$ and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$ for a sufficiently small $\delta > 0$ and a fixed $\rho \in (0, 1/12)$. Then one has*

$$S_0(\theta; X) \ll X^{\gamma-\rho+\delta}.$$

Lemma 2.10. *Let $1-\rho < \gamma < 1 < c$ and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$ with a sufficiently small $\delta > 0$ and $\rho > 0$ that satisfies*

$$c + 14\rho < 2, \quad 2\gamma + 14\rho < 3, \quad 2c + 12\rho < 3. \tag{2.6}$$

Then one has

$$S_1(\theta; X) \ll X^{1-\rho+\delta}.$$

Lemma 2.11. *Suppose the pair (c,γ) satisfies $\gamma < 1 < c$ and runs over $\{(c,\gamma) : 9(c-1)+35(1-\gamma) < 2,\ 5(c-1)+15(1-\gamma) < 1\}$, and $X^{\gamma-c-\delta} \le |\theta| \le X^{\delta}$ for a sufficiently small $\delta > 0$. Then one has*

$$S(\theta; X) \ll X^{(5\gamma-c)/4-\delta}. \tag{2.7}$$

Proof. By (2.1), it suffices to show that

$$S_j(\theta;X) \ll X^{(5\gamma-c)/4-\delta} \quad (j=0,1).$$

Lemma 2.9 with $\rho=(c-\gamma)/4+2\delta$ yields the bound on $S_0(\theta;X)$, provided that

$$5(c-1)+3(1-\gamma)<1.$$

On the other hand, we appeal to Lemma 2.10 with $\rho=(4+c-5\gamma)/4+2\delta$ to estimate $S_1(\theta;X)$, hence the conditions (2.6) turn into

$$9(c-1)+35(1-\gamma)<2, \quad 7(c-1)+31(1-\gamma)<2, \quad 5(c-1)+15(1-\gamma)<1.$$

It then follows that (2.7) holds under the hypothesis $(c,\gamma)\in\{(c,\gamma): 9(c-1)+35(1-\gamma)<2, 5(c-1)+15(1-\gamma)<1\}$. □

Lemma 2.12. *Let $\gamma<1<c$ and $8(c-1)+21(1-\gamma)<1$, and $X^{\gamma-c-\delta}\le|\theta|\le X^{\delta}$ for a sufficiently small $\delta>0$. Then one has*

$$S(\theta;X)\ll X^{(3\gamma-c)/2-\delta}. \tag{2.8}$$

Proof. Actually (2.8) has been established in [20, Corollary 16]. Alternatively, we can repeat the proof of Lemma 2.11, but alter the choices of ρ: namely, we appeal to Lemma 2.9 with $\rho=(c-\gamma)/2+2\delta$ and to Lemma 2.10 with $\rho=(2+c-3\gamma)/2+2\delta$, to achieve the bound of $S(\theta;X)$ under the hypothesis $8(c-1)+21(1-\gamma)<1$. □

Lemma 2.13. *Let $L=\log X$, and I is an interval in $\mathbb{R}$. Then one has*

$$\begin{aligned}\int_I |V(\theta;X)|^2\mathrm{d}\theta &\ll X^{2\gamma-c}L,\\ \int_I |S(\theta;X)|^2\mathrm{d}\theta &\ll |I|X^{\gamma}L^2+X^{2\gamma-c}L^3.\end{aligned} \tag{2.9}$$

Proof. The two inequalities are (35) and (32) in [20]. □

3. Proof of the theorems

Let $X=(N/2)^{1/c}$ and $\tau=(\log N)^{-1}$. Since $\log X\asymp\log N$, we use L to denote both $\log N$ and $\log X$. We consider

$$|p_1^c+\cdots+p_s^c-N|<\tau,$$

with $p_j\in\mathcal{N}_\gamma(X),\ j=1,\ldots,s$, and $s=3$ or 4. Let us fix a kernel $K\in\mathbf{C}^\infty(\mathbb{R})$ such that

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$$\widehat{K}(t) \geq 0, \qquad \frac{1}{4}\mathbf{1}_I(4x) \leq K(x) \leq \mathbf{1}_I(x),$$

where $\mathbf{1}_I$ is the indicator function of the interval $I = [-1, 1]$. We can ensure these conditions by choosing K to be a convolution of the form $K = \widetilde{K} * \widetilde{K}$, where $\widetilde{K} \in \mathbf{C}^\infty(\mathbb{R})$ is even and satisfies $\mathbf{1}_I(4x) \leq \widetilde{K}(x) \leq \mathbf{1}_I(2x)$. We consider the quantity

$$R(N) = \sum_{p_1,\dots,p_s \in \mathcal{N}_\gamma(X)} \left\{ \prod_{j=1}^{s} (\log p_j) \right\} K_\tau(p_1^c + \cdots + p_s^c - N),$$

where $K_\tau = K(x/\tau)$. By Fourier inversion,

$$R(N) = \int\limits_{\mathbb{R}} S(\theta; X)^s e(-N\theta) \mathrm{d}_\tau \theta,$$

where $\mathrm{d}_\tau\theta = \widehat{K_\tau}(\theta)\mathrm{d}\theta$. Note that because of the compact support of the kernel K, we have $\widehat{K_\tau}(\theta) \ll \tau \widehat{K}(\tau\theta) \ll \tau$. We will analyze the last integral to show that

$$R(N) \gg \tau X^{s\gamma - c}. \tag{3.1}$$

Let $\delta > 0$ be sufficiently small, and set

$$\mathfrak{M} = (-X^{\gamma-c-\delta}, X^{\gamma-c-\delta}), \qquad \mathfrak{m} = \{\theta : X^{\gamma-c-\delta} \leq |\theta| \leq X^\delta\}.$$

Then $R(N)$ can be expressed as

$$R(N) = \left\{ \int\limits_{\mathfrak{M}} + \int\limits_{\mathfrak{m}} + \int\limits_{|\theta| \geq X^\delta} \right\} S(\theta; X)^s e(-N\theta) \mathrm{d}_\tau\theta.$$

In general, the contribution of large θ is the easy part. Actually we can obtain

$$\int\limits_{|\theta| \geq X^\delta} |S(\theta; X)|^s \mathrm{d}_\tau\theta \ll \tau X^{s\gamma - c - 1}, \tag{3.2}$$

by the same argument leading to [20, (39)]. Therefore, we shall concentrate on dealing with the contribution from the major arc $\mathfrak{M}$ and minor arcs $\mathfrak{m}$.

3.1. The estimation over $\mathfrak{M}$

Set

$$\mathfrak{J}(N) = \int\limits_{\mathbb{R}} V(\theta; X)^s e(-N\theta) \mathrm{d}_\tau\theta,$$

$$\mathfrak{J}_*(N) = \int_{\mathfrak{M}} V(\theta; X)^s e(-N\theta) \mathrm{d}_\tau \theta.$$

Since Lemma 3.1 in [9] gives

$$V(\theta; X) \ll X^{\gamma-c} |\theta|^{-1},$$

we deduce that

$$|\mathfrak{J}(N) - \mathfrak{J}_*(N)| \le \int_{X^{\gamma-c-\delta}}^{\infty} |V(\theta; X)|^s \mathrm{d}_\tau \theta \ll \tau X^{\gamma-c+\delta}. \tag{3.3}$$

On the other hand, (49) in [20] indicates that when $\theta \in \mathfrak{M}$, one has

$$S(\theta; X) = V(\theta; X) + O\big(X^{\gamma-2\eta(X)}\big),$$

with $\eta(X) = (\log X)^{-3/4}$. Applying this and Lemma 2.13, together with the trivial bound $S(\theta; X) \ll X^\gamma$, we obtain

$$\begin{aligned}
&\left| \int_{\mathfrak{M}} S(\theta; X)^s e(-N\theta) \mathrm{d}_\tau \theta - \mathfrak{J}_*(N) \right| \\
&\le \int_{\mathfrak{M}} \big| S(\theta; X)^s - V(\theta; X)^s \big| \mathrm{d}_\tau \theta \\
&\ll \tau \int_{\mathfrak{M}} \big| S(\theta; X) - V(\theta; X) \big| \big(|S(\theta; X)|^{s-1} + |V(\theta; X)|^{s-1} \big) \mathrm{d}\theta \\
&\ll \tau X^{\gamma-2\eta(X)} X^{(s-3)\gamma} \Bigg(\int_{\mathfrak{M}} |S(\theta; X)|^2 \mathrm{d}\theta + \int_{\mathfrak{M}} |V(\theta; X)|^2 \mathrm{d}\theta \Bigg) \\
&\ll \tau X^{s\gamma-c-\eta(X)}.
\end{aligned} \tag{3.4}$$

Moreover, a standard Fourier integral argument (similar to the proof of Lemma 6 in [27], for example) yields

$$\mathfrak{J}(N) \gg \tau X^{s\gamma-c}. \tag{3.5}$$

Now by (3.3), (3.4) and (3.5), we get

$$\int_{\mathfrak{M}} S(\theta; X)^s e(-N\theta) \mathrm{d}_\tau \theta \gg \tau X^{s\gamma-c}. \tag{3.6}$$

3.2. The estimation over $\mathfrak{m}$

Now we focus on the integral over $\mathfrak{m}$. We have

$$\begin{aligned}\Big|\int_{\mathfrak{m}} S(\theta;X)^s e(-N\theta)\mathrm{d}_\tau\theta\Big| &\ll \sum_{p\in\mathcal{N}_\gamma(X)}(\log p)\Big|\int_{\mathfrak{m}} S(\theta;X)^{s-1}e((p^c-N)\theta)\mathrm{d}_\tau\theta\Big|\\ &\ll L\sum_{n\in\mathcal{N}_\gamma(X)}\Big|\int_{\mathfrak{m}} S(\theta;X)^{s-1}e((n^c-N)\theta)\mathrm{d}_\tau\theta\Big|.\end{aligned}$$

It follows from Cauchy's inequality, (2.2) and (2.9) that

$$\begin{aligned}&\Big|\int_{\mathfrak{m}} S(\theta;X)^s e(-N\theta)\mathrm{d}_\tau\theta\Big|^2\\ &\ll X^\gamma L^2\sum_{n\in\mathcal{N}_\gamma(X)}\Big|\int_{\mathfrak{m}} S(\theta;X)^{s-1}e((n^c-N)\theta)\mathrm{d}_\tau\theta\Big|^2\\ &\ll X^\gamma L^2\int_{\mathfrak{m}}\int_{\mathfrak{m}}\Big|S(\theta_1;X)^{s-1}S(\theta_2;X)^{s-1}\sum_{n\in\mathcal{N}_\gamma(X)}e(n^c(\theta_1-\theta_2))\mathrm{d}_\tau\theta_1\mathrm{d}_\tau\theta_2\Big|\\ &\ll X^\gamma L^2\int_{\mathfrak{m}}\int_{\mathfrak{m}^*}\Big|S(\theta_2;X)^{s-1}S(\theta+\theta_2;X)^{s-1}\,T(\theta;X)\mathrm{d}_\tau\theta\mathrm{d}_\tau\theta_2\Big|\\ &\ll X^\gamma L^2\int_{\mathfrak{m}}\int_{\mathfrak{m}^*}\Big|S(\theta_2;X)^{s-1}S(\theta+\theta_2;X)^{s-1}\,T_0(\theta;X)\mathrm{d}_\tau\theta\mathrm{d}_\tau\theta_2\Big|\\ &\quad+X^\gamma L^2\int_{\mathfrak{m}}\int_{\mathfrak{m}^*}\Big|S(\theta_2;X)^{s-1}S(\theta+\theta_2;X)^{s-1}\,T_1(\theta;X)\mathrm{d}_\tau\theta\mathrm{d}_\tau\theta_2\Big|\\ &\quad+X^{3\gamma+\delta}\cdot\max_{\theta\in\mathfrak{m}}\big|S(\theta;X)\big|^{2(s-3)},\end{aligned}\tag{3.7}$$

where $\mathfrak{m}^*:=\mathfrak{m}-\theta_2=\{X^{\gamma-c-\delta}-\theta_2\le|\theta|\le X^\delta-\theta_2\}$. The argument in (3.7) is similar to Zhao's work. By using Lemmas 2.4 and 2.12 if $s=3$ or Lemmas 2.6 and 2.11 if $s=4$, together with (2.9) again, one can estimate the double integral associated with T_1 by

$$\begin{aligned}&\int_{\mathfrak{m}}\int_{\mathfrak{m}^*}\Big|S(\theta_2;X)^{s-1}S(\theta+\theta_2;X)^{s-1}\,T_1(\theta;X)\mathrm{d}_\tau\theta\mathrm{d}_\tau\theta_2\Big|\\ &\ll X^{2\gamma+\delta}\cdot\max_{\theta\in\mathfrak{m}}\big|S(\theta;X)\big|^{2(s-3)}\cdot\max_{|\theta|\le X^\delta}\big|T_1(\theta;X)\big|\\ &\ll \tau^2X^{(2s-1)\gamma-2c-\delta}.\end{aligned}\tag{3.8}$$

The last term in (3.7) can be bounded similarly by applying Lemma 2.12 or Lemma 2.11 to $S(\theta;X)$ according as $s=3$ or 4.

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In order to handle the integral associated with T_0, we write $\mathfrak{m}^* = \mathfrak{m}_1 \cup \mathfrak{m}_2$ with

$$\mathfrak{m}_1 = \mathfrak{m}^* \cap \mathfrak{m}, \qquad \mathfrak{m}_2 = \mathfrak{m}^* \cap \mathfrak{M}.$$

Thus the double integral turns into

$$\int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}^*} \Big| S(\theta_2; X)^2 S(\theta+\theta_2; X)^2\, T_0(\theta; X) \mathrm{d}_\tau \theta \mathrm{d}_\tau \theta_2 \Big| = \int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}_1} + \int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}_2}. \tag{3.9}$$

Similar to (3.8), but using Lemmas 2.5 and 2.7 instead of Lemmas 2.4 and 2.6 (respectively) to bound $T_0(\theta; X)$, we get

$$\int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}_1} \Big| S(\theta_2; X)^{s-1} S(\theta+\theta_2; X)^{s-1}\, T_0(\theta; X) \mathrm{d}_\tau \theta \mathrm{d}_\tau \theta_2 \Big| \ll \tau^2 X^{(2s-1)\gamma - 2c - \delta}. \tag{3.10}$$

On the other hand, we write the double integral with θ over $\mathfrak{m}_2$ as

$$\begin{aligned} &\int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}_2} \Big| S(\theta_2; X)^{s-1} S(\theta+\theta_2; X)^{s-1}\, T_0(\theta; X) \mathrm{d}_\tau \theta \mathrm{d}_\tau \theta_2 \Big| \\ &= \int\limits_{\mathfrak{m}} \int\limits_{|\theta| \le X^{-c}} + \int\limits_{\mathfrak{m}} \int\limits_{X^{-c} < |\theta| \le X^{\gamma - c - \delta}} \\ &=: \ J_1 + J_2. \end{aligned}$$

Note that in both of the inner integrals on the right hand side, the variable θ is restricted to run over the arcs $\mathfrak{m}_2$. By trivial estimate $T_0(\theta; X) \ll X^\gamma$ and (2.9), we obtain

$$\begin{aligned} J_1 &\ll X^\gamma \cdot \max_{\theta \in \mathfrak{m}} \big| S(\theta; X) \big|^{2s-4} \int\limits_{\mathfrak{m}} \big| S(\theta_2; X) \big|^2 \mathrm{d}_\tau \theta_2 \int\limits_{|\theta| \le X^{-c}} \mathrm{d}_\tau \theta \\ &\ll \tau^2 X^{2\gamma - c + \delta} \cdot \max_{\theta \in \mathfrak{m}} \big| S(\theta; X) \big|^{2s-4} \\ &\ll \tau^2 X^{(2s-1)\gamma - 2c - \delta}, \end{aligned} \tag{3.11}$$

where Lemmas 2.11 and 2.12 (depending on $s = 4$ or 3) are utilized in the last step. For $|\theta| > X^{-c}$, by Lemma 2.8 with the exponent pair $(1/2, 1/2)$, one has

$$T_0(\theta; X) \ll X^{\gamma - 1} \sum_{n \sim X} e(n^c \theta) \ll |\theta|^{1/2} X^{c/2 + \gamma - 1} + X^{\gamma - c} |\theta|^{-1}.$$

It then follows that

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$$\begin{aligned} J_2 &\ll X^{c/2+\gamma-1} \int\limits_{\mathfrak{m}} \int\limits_{X^{-c}<|\theta|\le X^{\gamma-c-\delta}} \left| S(\theta_2;X)^{s-1} S(\theta+\theta_2;X)^{s-1} |\theta|^{1/2} \mathrm{d}_\tau\theta \mathrm{d}_\tau\theta_2 \right| \\ &\quad + X^{\gamma-c} \int\limits_{\mathfrak{m}} \int\limits_{X^{-c}<|\theta|\le X^{\gamma-c-\delta}} \left| S(\theta_2;X)^{s-1} S(\theta+\theta_2;X)^{s-1} |\theta|^{-1} \mathrm{d}_\tau\theta \mathrm{d}_\tau\theta_2 \right| \\ &=: J_{21} + J_{22}, \end{aligned}$$

where $J_{21} \ll \tau^2 X^{7\gamma/2-1+\delta} \cdot \max_{\theta\in\mathfrak{m}} \left|S(\theta;X)\right|^{2s-6} \ll \tau^2 X^{(2s-1)\gamma-2c-\delta}$, and

$$\begin{aligned} J_{22} &\ll X^{\gamma-c} \cdot \max_{\theta\in\mathfrak{m}} \left|S(\theta;X)\right|^{2s-4} \int\limits_{\mathfrak{m}} \left|S(\theta_2;X)\right|^2 \mathrm{d}_\tau\theta_2 \int\limits_{X^{-c}<|\theta|\le X^{\gamma-c-\delta}} \frac{\mathrm{d}_\tau\theta}{|\theta|} \\ &\ll \tau^2 X^{2\gamma-c+\delta} \cdot \max_{\theta\in\mathfrak{m}} \left|S(\theta;X)\right|^{2s-4} \\ &\ll \tau^2 X^{(2s-1)\gamma-2c-\delta} \end{aligned}$$

with the same treatment as (3.11) when quoting Lemmas 2.11 and 2.12 to bound $S(\theta;X)$. Combining the estimates on J_1 and J_2, we deduce that

$$\int\limits_{\mathfrak{m}} \int\limits_{\mathfrak{m}_2} \left| S(\theta_2;X)^{s-1} S(\theta+\theta_2;X)^{s-1}\, T_0(\theta;X) \mathrm{d}_\tau\theta \mathrm{d}_\tau\theta_2 \right| \ll \tau^2 X^{(2s-1)\gamma-2c-\delta}. \tag{3.12}$$

Collecting (3.7)–(3.10) and (3.12) all together, we get

$$\int\limits_{\mathfrak{m}} S(\theta;X)^s e(-N\theta) \mathrm{d}_\tau\theta \ll \tau X^{s\gamma-c-\delta}. \tag{3.13}$$

The desired bound (3.1) now follows from (3.2), (3.6) and (3.13). This completes the proof of Theorems 1 and 2.

3.3. Comments on the proof of Theorem 3

We can prove Theorem 3 by following the similar argument in [20], but here we still present a brief explanation for completeness. When $s = 2$ and $N \sim Z$, we set $X = (2Z/3)^{1/c}$ and $\tau = (\log Z)^{-1}$, and then structure the proof similarly to the case $s = 4$, replacing the pointwise bounds (3.6) and (3.13) withe the mean-square inequalities

$$\int\limits_{Z/2}^{Z} \left| \int\limits_{\mathfrak{M}} \left(S(\theta;X)^2 - V(\theta;X)^2\right) e(-N\theta) \mathrm{d}_\tau\theta \right|^2 \mathrm{d}N \ll \tau^2 X^{4\gamma-c-\eta(X)} \tag{3.14}$$

and

$$\int_{Z/2}^{Z} \left| \int_{\mathfrak{m}} S(\theta; X)^2 e(-N\theta) \mathrm{d}_\tau \theta \right|^2 \mathrm{d}N \ll \tau^2 X^{4\gamma - c - \delta}. \tag{3.15}$$

From these inequalities, we see immediately that the bounds (3.6) and (3.13) with $s = 2$ fail for a set of Lebesgue measure $\ll Z^{1-\eta(X)}$. To complete the proof, we remark that an appeal to Plancherel's theorem (see [18, (4.6)]) deduces (3.14) and (3.15) from basic estimates for $S(\theta; X)$ (e.g., (2.7) for the case $s = 4$) that were used earlier to establish (3.6) and (3.13).

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