



ELSEVIER

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt



On Drinfeld modular forms of higher rank II

Ernst-Ulrich Gekeler

ARTICLE INFO

Article history:

Received 11 December 2017

Accepted 10 November 2018

Available online xxxx

Communicated by F. Pellarin

To the memory of David Goss

Keywords:

Drinfeld modular forms

Bruhat–Tits building

Eisenstein series

ABSTRACT

We show that the absolute value $|f|$ of an invertible holomorphic function f on the Drinfeld symmetric space Ω^r ($r \geq 2$) is constant on fibers of the building map to the Bruhat–Tits building \mathcal{BT} . Its logarithm $\log |f|$ is an affine map on the realization of \mathcal{BT} . These results are used to study the vanishing loci of modular forms (coefficient forms, Eisenstein series, para-Eisenstein series) and to determine their images in \mathcal{BT} .

© 2018 Elsevier Inc. All rights reserved.

0. Introduction

The present paper continues the work reported in [11]. There we started with the investigation of growth/decay properties of (Drinfeld) modular forms for the group $\Gamma = \mathrm{GL}(r, \mathbb{F}_q[T])$, where \mathbb{F}_q is the field with q elements and $r \geq 3$. (The case $r = 2$ is well understood since the 1990s, see [6], [7], [9]). Several results not essentially needed for the reasoning of [11] were stated there without proof, e.g.: If f is an invertible holomorphic function on the Drinfeld symmetric space Ω^r on which Γ acts, then

- $\log_q |f(\omega)|$ is constant on fibers of the building map

$$\lambda: \Omega^r \longrightarrow \mathcal{BT}$$

to the Bruhat–Tits building \mathcal{BT} ;

E-mail address: gekeler@math.uni-sb.de.

<https://doi.org/10.1016/j.jnt.2018.11.011>

0022-314X/© 2018 Elsevier Inc. All rights reserved.

- regarded as a function on the set $\mathcal{BT}(\mathbb{Q})$ of points of \mathcal{BT} with rational barycentric coordinates, this function is affine (that is, interpolates linearly in simplices).

As these fundamental facts turn out to be crucial for subsequent work, we give here complete proofs: see Theorems 2.4 and 2.6. We use these and some results about functional determinants to study the vanishing loci $V(f)$ of several classes of modular forms f for Γ . We are able to determine the image $\lambda(V(f))$ under λ for

- f one of the coefficient forms g_1, \dots, g_{r-1} (which together with the discriminant $\Delta = g_r$ generate the algebra of modular forms of type 0: Theorem 4.2;
- f an Eisenstein series E_k : Theorem 4.5;
- f a para-Eisenstein series α_k : Theorem 4.8.

We finally show that (after some normalization) the k -th para-Eisenstein series α_k is a locally uniform limit of coefficient forms ${}_a\ell_k$, when the degree of $a \in \mathbb{F}_q[T]$ tends to infinity: Theorem 4.8.

Here we take the opportunity to point to recent articles of Basson [2] and Basson–Breuer [3] about higher rank Drinfeld modular forms, whose results are complementary to the present work.

The plan of the paper is as follows.

In Section 1, apart from recalling basic facts and definitions, we use successive minimum bases of lattices in C_∞ (the characteristic- p analogue of the complex numbers) to describe the fundamental domain \mathbf{F} for Γ and to define the spectrum $\text{spec}(\Lambda)$, a fundamental invariant of the \mathbb{F}_q -lattice Λ .

In Section 2 we give a geometric description of the fiber $\lambda^{-1}(\mathbf{x})$ of $\mathbf{x} \in \mathcal{BT}(\mathbb{Q})$. It turns out to be an affinoid with the absolute value property AVP: If f is an invertible function on $\lambda^{-1}(\mathbf{x})$, then the absolute value $|f(\omega)|$ is constant: Theorem 2.4. The fact that $\log_q |f|$ is an affine function on $\mathcal{BT}(\mathbb{Q})$ (Theorem 2.6) is obtained by investigating the restriction of f to $\lambda^{-1}(\mathbf{s})$, where \mathbf{s} is a line segment inside a simplex σ of \mathcal{BT} . In this situation we may reduce the assertion to a known fact of one-dimensional geometry due to Motzkin [14], see also [4] I.8.5.

In Section 3 we study functional determinants

$$\det_{1 \leq i, j \leq r'} \left(\frac{\partial f_i}{\partial \omega_j} \right), \quad 1 \leq r' \leq r-1, \quad \omega_1, \dots, \omega_{r-1} \text{ the coordinates on } \Omega^r,$$

where the f_i are either

- the para-Eisenstein series $\alpha_i = \alpha_i(\Lambda)$, or
- the Eisenstein series $E_{q^{i-1}}(\Lambda)$, or
- the coefficient forms ${}_a\ell_i$ for some fixed $a \in \mathbb{F}_q[T]$ (in particular, ${}_T\ell_i = g_i$, the basic coefficient forms which describe the generic Drinfeld module ϕ^ω of rank r).

From considering the T -torsion $T\phi^\omega \cong \mathbb{F}_q^r$ of ϕ^ω , we reduce the calculation of the functional determinant to evaluating certain Moore determinants. This notably shows the non-vanishing (Proposition 3.15).

In the fourth section we apply the preceding to locate the vanishing sets $V(f)$ inside \mathbf{F} and their images $\lambda(V(f))$.

For each function f of type g_i , E_k , α_i , there is a natural and explicitly computable range $R(f)$ such that $\lambda(V(f))$ is contained in $R(f)$. It is fairly easy to verify that in fact

- $\lambda(V(g_i)) = R(g_i) := \mathcal{W}_{r-i}$ (Theorem 4.2), where \mathcal{W}_{r-i} is the $(r-i)$ -th wall of $\mathcal{W} = \lambda(\mathbf{F})$

and

- $\lambda(V(E_k)) = R(E_k) := \mathcal{W}_{r-1}$ (Theorem 4.5; E_k is the Eisenstein series of weight k with $0 < k \equiv 0 \pmod{q-1}$),

but difficult to show that

- $\lambda(V(\alpha_i)) = R(\alpha_i) := \mathcal{W}(i)$ (Theorem 4.8).

Here $\mathcal{W}(i) = \lambda(\mathcal{F}(i))$ is defined through spectral properties of the corresponding lattices.

These theorems also include smoothness and intersection properties of the $V(f)$. Together with Theorems 2.4 and 2.6 and Remarks 2.9 and 4.15, they allow a precise description of $|f(\omega)|$ or $\|f\|_{\mathbf{x}}$ for any of these modular forms f , $\omega \in \mathbf{F}$ and $\mathbf{x} \in \mathcal{W}(\mathbb{Q})$, where $\|f\|_{\mathbf{x}}$ is the spectral norm of f on $\lambda^{-1}(\mathbf{x})$.

Notation. We use essentially the same notation as in [11], that is:

$\mathbb{F} = \mathbb{F}_q$ = finite field with q elements, of characteristic p

$\overline{\mathbb{F}}$ = algebraic closure of \mathbb{F} , $\mathbb{F}^{(n)} = \{x \in \overline{\mathbb{F}} \mid x^{q^n} = x\}$

$A = \mathbb{F}[T]$ the polynomial ring over \mathbb{F} , $K = \text{Quot}(A) = \mathbb{F}(T)$

$K_\infty = \mathbb{F}((T^{-1}))$ the completion of K w.r.t. the absolute value $|\cdot|$ at infinity, normalized by $|T| = q$

C_∞ = completed algebraic closure of K_∞ , $O_\infty \subset K_\infty$ and $O_{C_\infty} \subset C_\infty$ the rings of integers

$\log = -v_\infty : C_\infty^* \rightarrow \mathbb{Q}$ the map $z \mapsto \log_q |z|$

$\Omega^r = \{\omega = (\omega_1 : \cdots : \omega_r) \in \mathbb{P}^{r-1}(C_\infty) \mid \text{the } \omega_i \text{ or } K_\infty\text{-l.i.}\}$, where $r \geq 2$ and l.i. is short for linearly independent

$\Omega^r(R) = \{\omega \in \Omega^r \mid \text{the } \omega_i \text{ lie in the subring } R \text{ of } C_\infty\}$

$\Gamma = \text{GL}(r, A)$ the modular group with center $Z \cong \mathbb{F}^*$

$\Gamma(T) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{T}\}$ the T -th congruence subgroup

τ is a non-commutative variable subject to $\tau c = c^q \tau$ for $c \in C_\infty$. We identify the \mathbb{F} -algebra $C_\infty\{\tau\}$ of “polynomials” in τ with the \mathbb{F} -algebra $\text{End}_{\mathbb{F}}(\mathbb{G}_a/C_\infty) =$

$\{\sum_{\text{finite}} a_i X^{q^i} \mid a_i \in C_\infty\}$ of q -additive polynomials via $\tau^i \leftrightarrow X^{q^i}$; ditto for “power series”
 $C_\infty\{\{\tau\}\} \stackrel{!}{=} \{\sum_{i \geq 0} a_i X^{q^i}\}.$

Given an \mathbb{F} -lattice Λ in C_∞ ,

$$\begin{aligned} e_\Lambda(X) &= X \prod'_{\lambda \in \Lambda} (1 - X/\lambda) = \sum_{0 \leq i \leq \dim_{\mathbb{F}} \Lambda} \alpha_i(\Lambda) X^{q^i} = \sum \alpha_i(\Lambda) \tau^i \\ \log_\Lambda(X) &= \sum \beta_i(\Lambda) \tau^i = \text{inverse of } e_\Lambda(X) \text{ in } C_\infty\{\{\tau\}\} \\ E_k(\Lambda) &= \sum'_{\lambda \in \Lambda} \lambda^{-k} \end{aligned}$$

are the exponential function, the logarithm function, the k -th Eisenstein series of Λ , respectively. Here, as usual, a primed product \prod' or sum \sum' means the product or sum over the non-vanishing elements of the index set.

If $\omega_1, \dots, \omega_r$ are K_∞ -l.i. with lattice $\Lambda = \Lambda_\omega = \sum A\omega = \sum A\omega_i$ then $e_\omega := e_\Lambda$, and $\phi^\omega := \phi^\Lambda$ denotes the attached Drinfeld module with operator polynomial

$$\phi_a^\omega(X) = \sum_{0 \leq k \leq r \cdot \deg a} {}_a\ell_k(\omega) X^{q^k} = \sum {}_a\ell_k(\omega) \tau^k$$

and a -torsion submodule ${}_a\phi^\omega$ ($a \in A$).

Reduced analytic subspaces \mathbf{X} of $\mathbb{P}^n(C_\infty)$ or $\mathbb{A}^n(C_\infty)$ are usually described through their sets $\mathbf{X}(C_\infty)$ of C_∞ -valued points. Similarly, we often don't distinguish in notation between a simplicial complex \mathcal{S} and its realization $\mathcal{S}(\mathbb{R})$.

Finally, the cardinality of the finite set X is denoted by $\#(X)$, the multiplicative group of the ring R by R^* .

1. Modular forms

(1.1) Recall that the Drinfeld symmetric space for $r \geq 2$ is

$$\begin{aligned} \Omega^r &:= \{\omega = (\omega_1 : \dots : \omega_r) \in \mathbb{P}^{r-1}(C_\infty) \mid \omega_1, \dots, \omega_r \text{ l.i. over } K_\infty\} \\ &\quad (\text{l.i.} = \text{linearly independent}) \\ &= \mathbb{P}^{r-1}(C_\infty) \setminus \bigcup_{H \text{ hyperplane defined over } K_\infty} H, \end{aligned}$$

which carries a natural structure as a rigid analytic space defined over K_∞ . The orbit space $\Gamma \setminus \Omega^r$ is (the set of C_∞ -points of) the moduli space for Drinfeld A -modules of rank r , as is explained below. Such a Drinfeld module ϕ is given through the operator polynomial

$$\begin{aligned} \phi_T(X) &= TX + g_1 X^q + \dots + g_{r-1} X^{q^{r-1}} + \Delta X^{q^r} \\ &= T\tau^0 + g_1 \tau + \dots + g_{r-1} \tau^{r-1} + \Delta \tau^r, \end{aligned} \tag{1.1.1}$$

where τ denotes the operator ($x \mapsto x^q$) in $\text{End}_{\mathbb{F}}(\mathbb{G}_a)$, with $g_i, \Delta \in C_{\infty}$ and $\Delta \neq 0$. We also put $g_0 := T$ and $g_r := \Delta$. Each such $\phi = \phi^{\omega}$ comes from a uniquely determined A -lattice $\Lambda = \Lambda_{\omega} = A\omega_1 + \cdots + A\omega_r$ in C_{∞} , where $\omega = (\omega_1, \dots, \omega_r) \in C_{\infty}^r$ determines a point $(\omega_1 : \dots : \omega_r) \in \Omega^r$. In this way, A -lattices of rank r in C_{∞} (resp. homothety classes of such lattices) correspond to rank- r Drinfeld modules (resp. isomorphism classes of such modules). We normalize projective coordinates on Ω^r such that $\omega_r = 1$. Then Γ acts on Ω^r through

$$\begin{aligned} \gamma\omega &= \omega', \quad \omega'_i = \text{aut}(\gamma, \omega)^{-1} \sum_{1 \leq j \leq r} \gamma_{i,j} \omega_j, \quad \text{where} \\ \text{aut}(\gamma, \omega) &= \gamma_{r,1} \omega_1 + \cdots + \gamma_{r,r} \omega_r \quad (\gamma = (\gamma_{i,j})), \end{aligned} \quad (1.1.2)$$

and the $g_i = g_i(\omega)$ become functions on Ω^r via

$$\phi_T^{\omega} = \sum_{0 \leq i \leq r} g_i(\omega) \tau^i. \quad (1.1.3)$$

In fact, g_i is a modular form of weight $q^i - 1$ and type 0 for the modular group Γ .

(1.2) A modular form of weight $k \in \mathbb{N}_0$ and type $m \in \mathbb{Z}/(q-1)$ for Γ is a function $f : \Omega^r \rightarrow C_{\infty}$ that

- (i) is holomorphic;
- (ii) satisfies

$$f(\gamma\omega) = \frac{\text{aut}(\gamma, \omega)^k}{(\det \gamma)^m} f(\omega) \quad \text{for } \gamma \in \Gamma$$

and

- (iii) satisfies a certain boundary condition (see (1.7), (1.8)).

(Apart from the considerations of (1.7) and (1.8), modular forms of non-trivial types will play no role in this article.)

(1.3) Let \overline{M}^r be the weighted projective space $\text{Proj } C_{\infty}[X_1, \dots, X_r]$, where the weight of X_i is defined as $wt(X_i) := w_i := q^i - 1$. Then the moduli scheme for rank- r Drinfeld modules is the open subscheme M^r of \overline{M}^r defined by the non-vanishing of X_r :

$$M^r := (\text{Proj } C_{\infty}[X_1, \dots, X_r])_{X_r \neq 0},$$

the map

$$\begin{aligned} \Omega^r &\longrightarrow M^r(C_{\infty}) \\ \omega &\longmapsto (g_1(\omega) : \cdots : g_r(\omega)) \end{aligned}$$

is analytic and Γ -invariant, and defines an isomorphism of analytic spaces

$$\Gamma \backslash \Omega^r \xrightarrow{\cong} M^r(C_{\infty}).$$

(1.4) For each \mathbb{F} -lattice (= discrete \mathbb{F} -subspace) Λ in C_∞ , we write its exponential function e_Λ and its inverse \log_Λ in the non-commutative ring $C_\infty\{\{\tau\}\}$ as

$$\begin{aligned} e_\Lambda &= \sum_{i \geq 0} \alpha_i \tau^i = \sum \alpha_i X^{q^i} = X \prod_{\lambda \in \Lambda} ' (1 - X/\lambda) \\ \log_\Lambda &= \sum_{i \geq 0} \beta_i \tau^i. \end{aligned}$$

Then

$$\alpha_0 = \beta_0 = 1, \quad \sum_{i+j=k} \alpha_i \beta_j^{q^i} = \sum_{i+j=k} \alpha_i^{q^j} \beta_j = 0 \text{ for } k > 0, \quad (1.4.1)$$

and $-\beta_j$ agrees with the Eisenstein series E_{q^j-1} , where

$$E_k = \sum_{\lambda \in \Lambda} ' \lambda^{-k} \quad (k > 0) \text{ and } E_0 = -1. \quad (1.4.2)$$

The quantities α_i, β_j, E_k depend on Λ and are written as $\alpha_i(\Lambda), \dots$ (or $\alpha_i(\boldsymbol{\omega}), \dots$ if Λ happens to be an A -lattice with A -basis $\{\omega_1, \dots, \omega_r\}$). Given such an A -lattice Λ and $a \in A$, we write the a -th operator polynomial of the associated Drinfeld module $\phi = \phi^\omega$ as

$$\phi_a^\omega = \sum_{0 \leq i \leq r \cdot \deg a} a \ell_i \tau^i, \quad (1.4.3)$$

with $a \ell_i = a \ell_i(\Lambda) = a \ell_i(\boldsymbol{\omega})$, $\boldsymbol{\omega} = (\omega_1, \dots, \omega_r)$. In particular, ${}_T \ell_i = g_i$. All the functions α_k, β_k, E_k are modular forms of type 0 for Γ , with weight $q^k - 1$ for α_k, β_k and k for E_k .

(1.5) A *successive minimum basis* (SMB) of the A -lattice Λ of rank r is an ordered (we (ab)use the curly brackets notation of non-ordered sets) A -basis $\{\omega_1, \dots, \omega_r\}$ which satisfies for $1 \leq i \leq r$:

$|\omega_i|$ is minimal among $\{|\lambda| \mid \lambda \in \Lambda \setminus A\text{-span of } \{\omega_1, \dots, \omega_{i-1}\}\}$. Such an SMB exists for each A -lattice Λ , and it has the properties ([10], Proposition 3.1):

- (i) for $a_1, \dots, a_r \in K_\infty$, $|\sum_{1 \leq i \leq r} a_i \omega_i| = \max_i |a_i \omega_i|$;
- (ii) the series $|\omega_1|, \dots, |\omega_r|$ is an invariant of Λ and doesn't depend on the choice of the SMB $\{\omega_1, \dots, \omega_r\}$.

(1.6) Let \mathbf{F} be the set

$$\mathbf{F} = \{\boldsymbol{\omega} \in \Omega^r \mid \{\omega_r, \dots, \omega_1\} \text{ is an SMB of } \Lambda_{\boldsymbol{\omega}} = \sum_{1 \leq i \leq r} A \omega_i\}$$

(note the reverse order!). It is an admissible open subspace of the analytic space Ω^r , and each $\boldsymbol{\omega} \in \Omega^r$ is Γ -equivalent with at least one and at most finitely many $\boldsymbol{\omega}' \in \mathbf{F}$. We call

\mathbf{F} the fundamental domain for Γ on Ω^r . As modular forms f are uniquely determined on \mathbf{F} , we will focus our study to the restriction of f to \mathbf{F} .

Remark. Of course, the defining condition for $\omega \in \mathbf{F}$ doesn't depend on the choice of projective coordinates for ω . This notion of fundamental domain is weaker than the requirements on classical fundamental domains, as the almost uniqueness of representatives $\omega' \in \mathbf{F}$ cannot be achieved. Let e.g. $\gamma \in \mathrm{GL}(r, \mathbf{F}) \hookrightarrow \Gamma$ be an upper triangular matrix. Then with ω also $\gamma\omega$ belongs to \mathbf{F} .

(1.7) We may now specify the missing boundary condition (iii) in (1.2). Let $f : \Omega^r \rightarrow C_\infty$ be a function that satisfies conditions (i) and (ii), with the type $m = 0$ to fix ideas. We denote by f^* the unique extension of weight k ($f^*(c\omega) = c^{-k}f^*(\omega)$ for $c \in C_\infty^*$) of f to

$$\Omega^{r,*} := \{\omega = (\omega_1, \dots, \omega_r) \in C_\infty^r \mid \text{the } \omega_i \text{ are } K_\infty\text{-l.i.}\}.$$

Then f^* is Γ -invariant, and there exists a holomorphic function $F : \{(g_1, \dots, g_r) \in C_\infty^r \mid g_r \neq 0\} \rightarrow C_\infty$ such that

$$f^*(\omega) = F(g_1(\omega), \dots, g_r(\omega)). \quad (1.7.1)$$

The boundary condition (iii) now requires that F admit a holomorphic extension to $C_\infty^r \setminus \{0\}$. Suppose this holds. Then by GAGA (or the non-archimedean Chow lemma, or by direct proof), F is a polynomial in the g_i , necessarily isobaric of weight k (where $wt(g_i) = w_i = q^i - 1$). Let such a polynomial F be given. Then the function

$$\begin{aligned} f : \Omega^r &\longrightarrow C_\infty \\ \omega &\longmapsto F(g_1(\omega), \dots, g_r(\omega)) \end{aligned}$$

is bounded on \mathbf{F} , as the $g_i(\omega)$ are ([11], Corollary 4.16).

Next, let f with (i), (ii) be given and suppose it is bounded on \mathbf{F} . Then $F : C_\infty^{r-1} \times C_\infty^* \rightarrow C_\infty$ defined by (1.7.1) extends to $C_\infty^r \setminus \{0\}$, due to the non-archimedean analogue of the Riemann removable singularities theorem [1].

Now we allow the type m to be non-trivial, $0 \leq m < q - 1$. Let h be the function defined in [11], Theorem 3.8 that satisfies

$$(-1)^{r-1} h^{q-1} = T^{-1} g_r = T^{-1} \Delta.$$

It is a modular form of weight $w'_r := (q^r - 1)/(q - 1)$ and type 1. Some f subject to (1.2)(i) and (ii) with (k, m) arbitrary satisfies

$$f^*(\omega) = F(g_1(\omega), \dots, g_{r-1}(\omega), h(\omega))$$

with some holomorphic F , where $f^*(\gamma\omega) = (\det \gamma)^m f^*(\omega)$ for $\gamma \in \Gamma$. If F extends to $C_\infty^r \setminus \{0\}$, then F is an isobaric polynomial of weight k , where $wt(h) = w'_r$. The rest of the argument is as in the case where $m = 0$. Therefore we have shown:

1.8 Proposition. *Let $f : \Omega^r \rightarrow C_\infty$ be a function subject to conditions (i) and (ii) of (1.2). The following are equivalent:*

- (a) f is regular along the divisor $X_r = 0$ of \overline{M}^r (that is, the associated F extends);
- (b) f is a polynomial F in the forms g_1, \dots, g_{r-1}, h ;
- (c) f is bounded on the fundamental domain \mathbf{F} . \square

Remarks. (i) The polynomial F in (b) is necessarily isobaric of weight k (weights w_i for the g_i , weight w'_r for h), and of type m , that is $F(g_1, \dots, g_{r-1}, h) = h^m F'(g_1, \dots, g_r)$, with some isobaric F' of weight $k - m \cdot w'_r$.

(ii) While (a) and (b) are specific to the case considered (where the acting group is the full modular group $\Gamma = \mathrm{GL}(r, A)$ and the moduli scheme is easy to describe), condition (c) naturally generalizes. If Γ' is some congruence subgroup of Γ , modular forms f for Γ' may be defined by the conditions (i), (ii Γ'), (iii Γ'), where

- (ii Γ') the modular equation (ii) holds for $\gamma \in \Gamma'$;
- (iii Γ') $f_{[\gamma]_k}$ is bounded on \mathbf{F} for all γ in a system of representatives of Γ/Γ' ; here $f_{[\gamma]_k}$ denotes the usual right action of $\gamma \in \Gamma$ on f .

This definition has the advantage that it doesn't require a precise description of the corresponding moduli scheme.

(1.9) Next, we consider arbitrary \mathbb{F} -lattices Λ in C_∞ , that is, discrete (finite- or infinite-dimensional) \mathbb{F} -subspaces of C_∞ . A *successive minimum basis* of X over \mathbb{F} (or \mathbb{F} -SMB for short) is an ordered \mathbb{F} -basis $\{\lambda_1, \lambda_2, \dots\}$ with the property analogous with (1.5): For each $i \in \mathbb{N}$ less than or equal to $\dim_{\mathbb{F}}(\Lambda)$,

$$|\lambda_i| \text{ is minimal among } \{|\lambda| \mid \lambda \in \Lambda \setminus \mathbb{F} - \text{span of } \{\lambda_1, \dots, \lambda_{i-1}\}\}.$$

As is easily seen, each Λ possesses an \mathbb{F} -SMB $\{\lambda_1, \lambda_2, \dots\}$, and

- (i) $|\sum a_i \lambda_i| = \max_i \{|\lambda_i| \mid a_i \neq 0\}$ for $a_1, a_2, \dots \in \mathbb{F}$, almost all vanishing;
- (ii) the series $|\lambda_1|, |\lambda_2|, \dots$ is an invariant of Λ and independent of the choice of the \mathbb{F} -SMB $\{\lambda_1, \lambda_2, \dots\}$

We call that series the *spectrum* $\mathrm{spec}(\Lambda)$ of Λ . Further, Λ is *separable* if $\mathrm{spec}(\Lambda)$ is multiplicity free, i.e., $|\lambda_1| < |\lambda_2| < \dots$, and *inseparable* otherwise, k -inseparable if $|\lambda_k| = |\lambda_{k+1}|$. We put on record the observation:

(1.10) Knowing $\text{spec}(\Lambda)$ is the same as knowing the Newton polygon $NP(e_\Lambda)$ of e_Λ (as defined in [15] II Sect. 6), as by (i) both are equivalent to knowing the numbers of elements of Λ of given sizes. In particular, Λ is separable if and only if the segments of $NP(e_\Lambda)$ have lengths $(q-1)$, $(q-1)q$, $(q-1)q^2$, ..., as in this case there are precisely $(q-1)q^{i-1}$ elements $\lambda \in \Lambda$ with $|\lambda| = |\lambda_i|$.

1.11 Proposition. *Let Λ be an \mathbb{F} -lattice with $\alpha_k(\Lambda) = 0$ for some $k < \dim_{\mathbb{F}}(\Lambda)$. Then Λ is k -inseparable. Conversely, if Λ is k -inseparable, there exists an isospectral \mathbb{F} -lattice Λ' (i.e., $\text{spec}(\Lambda) = \text{spec}(\Lambda')$) such that $\alpha_k(\Lambda') = 0$.*

Proof. If $\alpha_k(\Lambda) = 0$ then the abscissa q^k cannot be a break point of $NP(e_\Lambda)$. Therefore, $\#\{\lambda \in \Lambda \mid |\lambda| = |\lambda_k|\} = \#\{\lambda \in \Lambda \mid |\lambda| = |\lambda_{k+1}|\}$, that is, $|\lambda_k| = |\lambda_{k+1}|$. Let now Λ be k -inseparable, $e_\Lambda = \sum_{i \geq 0} \alpha_i \tau^i$. Its Newton polygon doesn't change if we replace α_k with $\alpha'_k = 0$. The lattice $\Lambda' = \ker(e_{\Lambda'})$ with $e_{\Lambda'} = \sum_{\substack{i \geq 0 \\ i \neq k}} \alpha_i \tau^i$ is as wanted. \square

Remark. The same argument shows: If Λ is k -inseparable for all $k \in S$, where S is a possibly infinite subset of \mathbb{N} , there exists an isospectral lattice Λ' with $\alpha_k(\Lambda') = 0$ for all $k \in S$.

(1.12) Suppose we are given an A -lattice Λ with SMB $\{\omega_r, \omega_{r-1}, \dots, \omega_1\}$. Then $\{T^j \omega_i \mid 1 \leq i \leq r, j \in \mathbb{N}_0\}$ is an \mathbb{F} -basis of Λ , from which we may construct an \mathbb{F} -SMB by conveniently ordering the indices (j, i) :

$$(j, i) \prec (j', i') \text{ if } |T^j \omega_i| < |T^{j'} \omega_{i'}| \text{ or } (|T^j \omega_i| = |T^{j'} \omega_{i'}| \text{ and } i > i').$$

Hence the \mathbb{F} -SMB starts

$$\lambda_1 = \omega_r, \lambda_2 = T\omega_r, \dots, \lambda_j = T^{j-1}\omega_r, \lambda_{j+1} = \omega_{r-1},$$

where j is maximal such that $|T^{j-1}\omega_r| \leq |\omega_{r-1}|$.

Recall that $\log z = \log_q |z|$ for $z \in C_\infty^*$, so $\log T = 1$. In particular, Λ is separable if and only if the $\log \omega_i \in \mathbb{Q}$ are all incongruent modulo \mathbb{Z} and Λ is 1-inseparable if and only if $|\omega_{r-1}| = |\omega_r|$. For $r \geq 3$, Λ is 2-inseparable if

$$\begin{array}{ll} \text{either} & |\omega_{r-2}| = |\omega_{r-1}| < q|\omega_r| \quad (\lambda_1 = \omega_r, \lambda_2 = \omega_{r-1}, \lambda_3 = \omega_{r-2}) \\ \text{or} & |\omega_{r-1}| = q|\omega_r| \quad (\lambda_1 = \omega_r, \lambda_2 = T\omega_r, \lambda_3 = \omega_{r-1}). \end{array}$$

2. A closer look to the building map

(2.1) Let $\lambda : \Omega^r \rightarrow \mathcal{BT}(\mathbb{Q})$ be the building map as described in [11] (2.3) onto the points with rational barycentric coordinates of the Bruhat–Tits building \mathcal{BT} of $\text{PGL}(r, K_\infty)$. The apartment \mathcal{A} of \mathcal{BT} is the full subcomplex defined by the standard torus T of diagonal matrices of $\text{GL}(r, K_\infty)$, with set of vertices

$$\mathcal{A}(\mathbb{Z}) = \{[L_{\mathbf{k}}] \mid \mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r\},$$

where $[L_{\mathbf{k}}]$ is the homothety class of the O_∞ -lattice $L_{\mathbf{k}} = (T^{k_1}O_\infty, \dots, T^{k_r}O_\infty)$ in K_∞^r . We have $[L_{\mathbf{k}}] = [L_{\mathbf{k}'}] \Leftrightarrow \mathbf{k}' - \mathbf{k} = (k, k, \dots, k)$ for some $k \in \mathbb{Z}$. The realization $\mathcal{A}(\mathbb{R})$ (for which we henceforth briefly write \mathcal{A}) is an Euclidean affine space with translation group

$$(T(K_\infty)/K_\infty^*T(O_\infty)) \otimes \mathbb{R} \xrightarrow{\cong} \mathbb{R}^r/\mathbb{R}(1, 1, \dots, 1) \xrightarrow{\cong} \{\mathbf{x} \in \mathbb{R}^r \mid x_r = 0\} \quad (2.1.1)$$

and with the natural choice of origin $\mathbf{o} = [L_{\mathbf{o}}]$. That is, we use that isomorphism as a description of $\mathcal{A} = \mathcal{A}(\mathbb{R})$. The choice of the Borel subgroup of upper triangular matrices in $\mathrm{GL}(r, K_\infty)$ determines the Weyl chamber

$$\mathcal{W} = \{\mathbf{x} \in \mathcal{A} \mid x_i \geq x_{i+1} \text{ for } 1 \leq i < r\} \quad (2.1.2)$$

with walls

$$\mathcal{W}_i = \{\mathbf{x} \in \mathcal{W} \mid x_i = x_{i+1}\} \quad (1 \leq i < r).$$

Then \mathcal{W} is a fundamental domain (in the classical sense) for Γ on \mathcal{BT} , that is, each $\mathbf{x} \in \mathcal{BT}(\mathbb{R})$ is Γ -equivalent with a unique $\mathbf{x} \in \mathcal{W}$. The relationship between the fundamental domains \mathbf{F} in Ω^r and \mathcal{W} in \mathcal{BT} is

$$\lambda(\mathbf{F}) = \mathcal{W}(\mathbb{Q}), \quad \lambda^{-1}(\mathcal{W}) = \mathcal{F}. \quad (2.1.3)$$

We define

$$\mathbf{F}_i := \lambda^{-1}(\mathcal{W}_i) = \{\boldsymbol{\omega} \in \mathbf{F} \mid |\omega_i| = |\omega_{i+1}|\}$$

and for $\mathbf{x} \in \mathcal{W}(\mathbb{Q})$

$$\mathbf{F}_{\mathbf{x}} := \lambda^{-1}(\mathbf{x}) = \{\boldsymbol{\omega} \in \mathbf{F} \mid \log \omega_i = x_i, 1 \leq i \leq r\}.$$

These are admissible open subspaces of \mathbf{F} , and $\mathbf{F}_{\mathbf{x}}$ even affinoid (see (2.4)).

2.2 Definition. Let $\mathbf{X} = \mathrm{Sp}(B)$ be an open affinoid subspace of some affine or projective space over C_∞ . The spectral norm of $f \in B$ is $\|f\|_{\mathbf{X}} := \sup\{|f(x)| \mid x \in \mathbf{X}\}$. The space \mathbf{X} satisfies the absolute value property AVP if and only if each unit $f \in B^*$ has constant absolute value on \mathbf{X} .

2.3 Examples.

(0) It is well-known that the r -dimensional unit ball

$$\{(\omega_1, \dots, \omega_r) \in C_\infty^r \mid |\omega_i| \leq 1 \text{ for all } i\}$$

satisfies AVP.

- (i) Let $C_a := \{\omega \in C_\infty \mid |\omega| = q^a\}$ be the circumference with radius q^a , $a \in \mathbb{Q}$. Then C_a satisfies AVP ([12] p. 93).
- (ii) For $r \geq 2$ we put

$$\mathbf{X}^r := \{\omega = (\omega_1 : \dots : \omega_r) \in \mathbb{P}^{r-1}(O_{C_\infty}) \mid |\ell_H(\omega_1, \dots, \omega_r)| < 1\},$$

where H runs through the finite set of hyperplanes of $\mathbb{P}^{r-1}(\mathbb{F})$ and $\ell_H : \mathbb{F}^r \rightarrow \mathbb{F}$ is a linear form with kernel H , uniquely extended to an O_{C_∞} -linear form $O_{C_\infty}^r \rightarrow O_{C_\infty}$. Then $\mathbf{X}^r = \lambda^{-1}(\mathbf{0})$, where $\mathbf{0} = (0, \dots, 0)$ is the origin of $\mathcal{W}(\mathbb{Q}) \subset \mathcal{A}(\mathbb{Q})$, and \mathbf{X}^r satisfies AVP ([11] (2.5), (2.7)).

- (iii) If $\mathbf{X}_1, \dots, \mathbf{X}_s$ satisfy AVP, so does $\mathbf{X} := \mathbf{X}_1 \times \dots \times \mathbf{X}_s$. This is obvious, since a unit of \mathbf{X} gives rise to units on the \mathbf{X}_i by fixing the j -coordinates, $j \neq i$.
- (iv) AVP is far from being satisfied for general affinoids. Let for example \mathbf{X} be the annulus

$$\mathbf{A}_{a,b} = \{\omega \in C_\infty \mid a \leq \log \omega \leq b\}$$

with rational numbers $a < b$. The coordinate function “ ω ” is a unit with non-constant absolute value.

2.4 Theorem. *For each $\mathbf{x} \in \mathcal{BT}(\mathbb{Q})$, the inverse image $\lambda^{-1}(\mathbf{x})$ is an open affinoid subspace of Ω^r that satisfies AVP.*

Proof. (i) We may assume $\mathbf{x} \in \mathcal{W}(\mathbb{Q})$ and thus $\lambda^{-1}(\mathbf{x}) = \mathbf{F}_{\mathbf{x}} = \{\omega \in \mathbf{F} \mid \log \omega_i = x_i, \text{ all } i\}$ with $\mathbf{x} = (x_1, x_2, \dots, x_r)$, $x_1 \geq \dots \geq x_r = 0$. Write $\{1, 2, \dots, r\}$ as the disjoint union of classes S , where i, j are in the same class if and only if $x_i \equiv x_j \pmod{\mathbb{Z}}$. For each class S let

$$\mathbf{C}_{\mathbf{x},S} := \{\omega_S = (\dots, \omega_i, \dots)_{i \in S} \mid \omega_i \in C_\infty, \log \omega_i = x_i, \omega_i = 1 \text{ if } i = r\}.$$

Then

$$\mathbf{F}_{\mathbf{x}} = \{\omega \in \prod_S \mathbf{C}_{\mathbf{x},S} \mid \omega \in \mathbf{F}\},$$

where the condition “ $\omega \in \mathbf{F}$ ” is equivalent with

- (a) $\omega \in \Omega^r$ (i.e., $\omega_1, \dots, \omega_r$ are K_∞ -linearly independent) and
- (b) $\{\omega_r = 1, \omega_{r-1}, \dots, \omega_1\}$ forms an SMB of $\Lambda_\omega = \sum_{1 \leq i \leq r} A\omega_i$.

- (ii) For each class S , consider the conditions on ω :

- (a_S) the ω_i ($i \in S$) are K_∞ -l.i. and
- (b_S) the ω_i ($i \in S$) in reverse order form an SMB of $\Lambda_{\omega,S} := \sum_{i \in S} A\omega_i$.

It follows from (1.5)(i) and the construction of $\mathbf{C}_{\mathbf{x},S}$ that (a) holds for ω whenever (a_S) holds for all S .

Suppose that (a_S) and (b_S) hold for all S . By the above, $\omega_1, \dots, \omega_r$ are K_∞ -l.i. and $|\omega_r| \leq |\omega_{r-1}| \leq \dots |\omega_1|$. We claim that, moreover, they form an SMB of Λ_ω .

For, suppose that there exists some $i < r$ and $\omega \in \Lambda_\omega \setminus \sum_{i < j \leq r} A\omega_j$ with $|\omega_i| > |\omega|$.

Write

$$\omega = \sum_{1 \leq j \leq r} a_j \omega_j = \sum_S \omega_S \text{ with } \omega_S = \sum_{j \in S} a_j \omega_j, a_j \in A.$$

Then $|\omega_i| > |\omega| = \max_S |\omega_S|$.

If $i \in S$, then $\omega_S \neq 0$ contradicts (b_S) . If $i \notin S$ then $\omega_S \in \sum_{\substack{j \in S \\ |\omega_j| < |\omega_i|}} A\omega_j \subset \sum_{\substack{j \in S \\ j > i}} A\omega_j$.

Hence $\omega = \sum \omega_S \in \sum_{j > i} A\omega_j$, which conflicts with the assumption on ω .

(iii) By (i) and (ii) we have

$$\begin{aligned} \mathbf{F}_\mathbf{x} &= \{\omega = (\dots, \omega_S, \dots) \in \prod_S \mathbf{C}_{\mathbf{x},S} \mid \text{for each } S, \omega_S \text{ satisfies} \\ &\quad (a_S) \text{ and } (b_S)\} \\ &= \prod_S \mathbf{B}_{\mathbf{x},S}, \end{aligned}$$

where

$$\mathbf{B}_{\mathbf{x},S} = \{\omega_S = (\omega_i)_{i \in S} \in \mathbf{C}_{\mathbf{x},S} \mid (a_S) \text{ and } (b_S) \text{ hold}\}.$$

We will see that all the $\mathbf{B}_{\mathbf{x},S}$ are isomorphic with spaces treated in Example 2.3.

(iv) If $\boxed{S = \{r\}}$ then $\mathbf{B}_{\mathbf{x},S} = \{1\}$.

If $\boxed{r \in S \neq \{r\}}$ then $\mathbf{B}_{\mathbf{x},S} = \{\omega_S = (\omega_i)_{i \in S} \mid \log \omega_i = x_i, (a_S) \text{ and } (b_S) \text{ hold}\}$.

As $x_i = \log \omega_i \in \mathbb{N}_0$, we may scale $\omega'_i := T^{-x_i} \omega_i$ and find

$$\mathbf{B}_{\mathbf{x},S} \cong \{\omega_S = (\omega_i)_{i \in S} \mid |\omega_i| = 1, \omega_r = 1, (a_S) \text{ and } (b_S) \text{ hold}\},$$

which equals the space \mathbf{X}^s with $s := \#(S)$ of Example 2.3(ii), for which AVP is satisfied.

If $\boxed{S = \{i\}}$ with $i \neq r$ then (a_S) and (b_S) are trivially fulfilled, so $\mathbf{B}_{\mathbf{x},S} = \mathbf{C}_{\mathbf{x},S} = \{\omega \in C_\infty \mid \log \omega_i = x_i\} = \mathbf{C}_{x_i}$, a circumference, for which AVP holds by Example 2.3(i).

Let now S be such that $\boxed{r \notin S \text{ and } s := \#(S) > 1}$. As the $x_i = \log \omega_i$ for $i \in S$ are all congruent modulo \mathbb{Z} , we may scale the ω_i by integral powers of T such that they have the same absolute value. Again multiplying with a fractional power of T , we may achieve $|\omega_i| = 1$ for $i \in S$. Hence

$$\mathbf{B}_{\mathbf{x},S} \cong \{\omega_S = (\omega_i)_{i \in S} \mid |\omega_i| = 1, (a_S) \text{ and } (b_S) \text{ hold}\},$$

which, by projecting to the last coordinate, becomes isomorphic with $\mathbf{X}^s \times \mathbf{C}_0$ (see again Examples 2.3).

(v) Together, the analytic space $\mathbf{F}_{\mathbf{x}}$ is isomorphic with the product of affinoids each enjoying AVP, and thus enjoys the same properties. \square

Remark. Let $\mathbf{x} = (x_i)_{1 \leq i \leq r}$ be an element of $\mathcal{W}(\mathbb{Q})$. Then \mathbf{x} lies in the interior of a simplex σ of maximal dimension $r - 1$ if and only if all the x_i are incongruent modulo \mathbb{Z} , if and only if $\mathbf{F}_{\mathbf{x}}$ is a product of $(r - 1)$ circumferences. Congruences $(\bmod \mathbb{Z})$ of the x_i imply that \mathbf{x} belongs to a simplex of smaller dimension.

2.5 Corollary. *If $f \in \mathcal{O}(\Omega^r)^*$ is a global unit (i.e., an invertible holomorphic function on Ω^r) then $|f|$ factors through λ , and may thus be considered as a function on $\mathcal{BT}(\mathbb{Q})$. \square*

This holds e.g. for the discriminant function Δ and its root h . The values of $|\Delta|$ on $\mathcal{W}(\mathbb{Z})$ have been determined in [11]. Since \mathcal{W} is a fundamental domain for Γ and Δ is modular, this suffices to find $|\Delta(\omega)|$ for any ω , in view of the next result.

For a unit f we write $\log f(\mathbf{x})$ for the common value $\log_q |f(\omega)|$ of $\omega \in \mathbf{F}_{\mathbf{x}}$.

2.6 Theorem. *Let f be an invertible holomorphic function on Ω^r . Then $\mathbf{x} \mapsto \log f(\mathbf{x})$ is an affine function on $\mathcal{BT}(\mathbb{Q})$. That is, given a simplex $\sigma = \{\mathbf{k}_0, \dots, \mathbf{k}_n\}$ of \mathcal{BT} and barycentric coordinates $(t_0, \dots, t_n) \in \mathbb{Q}_{\geq 0}^{n+1}$ with $\sum t_i = 1$ for $\mathbf{x} = \sum_{0 \leq i \leq n} t_i \mathbf{k}_i$, then*

$$\log f(\mathbf{x}) = \sum_{0 \leq i \leq n} t_i \log f(\mathbf{k}_i). \quad (2.6.1)$$

Before proving the theorem, we need some auxiliary results.

2.7 Proposition. *Given rational numbers $a < b$, let $x_1, \dots, x_s \in C_a$ and $y_1, \dots, y_t \in C_b$ be finitely many points. For $1 \leq i \leq s$ (resp. $1 \leq j \leq t$) let*

$$\mathbf{B}_i := \{z \in C_{\infty} \mid |z - x_i| < q^{a_i}\}$$

be an open disc around x_i with $a_i \leq a$ (resp. \mathbf{B}'_j an open disc of radius q^{b_j} with $b_j \leq b$ around y_j). Suppose that all the $\mathbf{B}_i, \mathbf{B}'_j$ are disjoint. Let f be an invertible holomorphic function on

$$\mathbf{B} := A_{a,b} \setminus \left(\bigcup_i \mathbf{B}_i \cup \bigcup_j \mathbf{B}'_j \right) = \{\omega \in C_{\infty} \mid q^a \leq |\omega| \leq q^b, |\omega - x_i| \geq a_i, |\omega - y_j| \geq b_j\}.$$

Then there exists an invertible holomorphic function g on \mathbf{B} with constant absolute value and integers $m, m_1, \dots, m_s, n_1, \dots, n_t$ such that

$$f(\omega) = \omega^m \prod_i (\omega - x_i)^{m_i} \prod_j (\omega - y_j)^{n_j} g(\omega).$$

Proof. This is Theorem I.8.5 of [4] (which is due to Motzkin [14]), adapted to our framework. \square

2.8 Corollary. *In the above situation, the absolute value $|f(\omega)|$ depends only on $|\omega|$. Regarded as a function of $x := \log \omega$, $|\log f(\omega)|$ interpolates linearly between a and b .*

Proof. This results from the fact that ω^m , $(\omega - x_i)^{m_i}$, $(\omega - y_j)^{n_j}$ share these properties. \square

2.9 Remark. For further use, we note the following slight generalization. We suppress the condition of f to be invertible, but allow a finite number of zeroes in \mathbf{C}_a and \mathbf{C}_b . Replacing $|f(\omega)|$ with the spectral norm $\|f\|_x$ on \mathbf{C}_x , we still have

$$x \mapsto \log_q \|f\|_x \text{ interpolates linearly between } a \text{ and } b.$$

This results from applying 2.8 to $\mathbf{B}' = \mathbf{B}$ minus the union of small disjoint open discs around the zeroes of f . Note that $\log_q \|f\|_x = \log f(\omega)$ for $a < x = \log \omega < b$.

Proof of Theorem 2.6. (i) We assume without restriction that σ is the standard simplex of maximal dimension $r - 1$, with vertices $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{k}_i = (1, 1, \dots, 1, 0, \dots, 0)$ with i ones and $(r - i)$ zeroes ($1 \leq i < r$; we remind the reader that all points $\mathbf{x} = (x_1, \dots, x_{r-1}, 0)$ have a redundant 0 as last entry x_r). Then

$$\lambda^{-1}(\sigma) = \{\boldsymbol{\omega} \in \mathbf{F} \mid q \geq |\omega_1| \geq |\omega_2| \geq \dots \geq |\omega_{r-1}| \geq 1, \omega_r = 1\}.$$

Recall that $\boldsymbol{\omega} \in C_\infty^r$ gives rise to an element of \mathbf{F} if and only if (compare proof of Theorem 2.4):

- (a) the entries ω_i are K_∞ -l.i.;
- (b) $\{\omega_r, \omega_{r-1}, \dots, \omega_1\}$ forms an SMB of Λ_ω .

(ii) Instead of $\boldsymbol{\omega} = (\omega_1, \dots, \omega_{r-1})$ we use $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{r-1})$ as coordinates, where

$$\alpha_i = \omega_i / \omega_{i+1} \text{ and so } \omega_i = \prod_{i \leq j < r} \alpha_j. \quad (1)$$

Then $|\alpha_i| \geq 1$ and $|\prod_{1 \leq i < r} \alpha_i| \leq q$. Fix some $\boldsymbol{\omega}^{(o)} = (\omega_1^{(o)}, \dots, \omega_{r-1}^{(o)}) \in \lambda^{-1}(\sigma)$ and for some i , $1 \leq i < r$, its α -coordinates $(\alpha_1^{(o)}, \dots, \alpha_{i-1}^{(o)}, \alpha_{i+1}^{(o)}, \dots, \alpha_{r-1}^{(o)})$, where we suppose that

$$\tilde{q}_i := q / \left| \prod_{\substack{1 \leq j < r \\ j \neq i}} \alpha_j^{(o)} \right| > 1.$$

Consider the set of possible α_i :

$$\begin{aligned} \mathbf{A}'_i &:= \{\alpha \in \mathbb{C}_\infty \mid 1 \leq |\alpha| \leq \tilde{q}_i\} \text{ and} \\ \mathbf{A}_i &:= \{\boldsymbol{\omega} \in \mathbf{F} \mid \omega_j/\omega_{j+1} = \alpha_j^{(o)} \text{ for } j \neq i, \omega_i/\omega_{i+1} \in \mathbf{A}'_i\}, \end{aligned}$$

which, of course, depend on the choice of $\boldsymbol{\omega}^{(o)}$.

(iii) The map

$$\begin{aligned} u_i : \mathbf{A}_i &\longrightarrow \mathbf{A}'_i \\ \boldsymbol{\omega} &\longmapsto \alpha = \omega_i/\omega_{i+1} \end{aligned}$$

is well-defined and injective. *Claim:* u_i is an open embedding onto \mathbf{A}'_i minus a finite number of disjoint open discs of shape

$$\{\alpha \in \mathbb{C}_\infty \mid |\alpha - z_j| < r_j\} \text{ with } |z_j| = 1 = r_j \text{ or } |z_j| = \tilde{q}_i = r_j.$$

To prove the claim, we consider for each $\alpha \in \mathbf{A}'_i$ the point $\boldsymbol{\omega}$ corresponding via (1) to $(\alpha_1^{(o)}, \dots, \alpha_{i-1}^{(o)}, \alpha, \alpha_{i+1}^{(o)}, \dots, \alpha_{r-1}^{(o)})$. Then $\alpha \in \text{im}(u_i) \Leftrightarrow \boldsymbol{\omega} \in \mathbf{F} \Leftrightarrow$ conditions (a) and (b) hold for $\boldsymbol{\omega}$.

First, suppose that $\boxed{1 < |\alpha| < \tilde{q}_i}$. Then $|\omega_{i-1}^{(o)}| > |\omega_i| > |\omega_{i+1}^{(o)}|$, and it is easily seen that (a) and (b) turn over from $\boldsymbol{\omega}^{(o)}$ to $\boldsymbol{\omega}$, hence $\alpha \in \text{im}(u_i)$.

Next, assume $\boxed{|\alpha| = 1}$. Then $|\omega_{i-1}^{(o)}| > |\omega_i| = |\omega_{i+1}^{(o)}|$. As

$$\begin{aligned} \omega_j &= \omega_j^{(o)} \text{ for } j > i \\ \omega_j &= \omega_j^{(o)}(\omega_i/\omega_i^{(o)}) \text{ for } j < i, \end{aligned}$$

in particular, $|\omega_j| < q$ for all j , $\boldsymbol{\omega}$ fulfills (a) and (b) if and only if $\omega_i, \omega_{i+1}^{(o)}, \dots, \omega_k^{(o)}$ are K_∞ -linearly independent and form an SMB of their A -span, where k with $i \leq k \leq r$ is maximal with $|\omega_i| = |\omega_k^{(o)}|$. This is the case if and only if $|\omega_i - \omega_j^{(o)}| \geq |\omega_i|$ for $j = i+1, \dots, k$, if and only if $|\alpha - \omega_j^{(o)}/\omega_{i+1}^{(o)}| \geq 1$ for $j = i+1, \dots, k$.

If $\boxed{|\alpha| = \tilde{q}_i}$, then $|\omega_{i-1}^{(o)}| = |\omega_i| > |\omega_{i+1}^{(o)}|$, and an analogous argument shows that $\alpha \in \text{im}(u_i)$ if and only if $|\alpha - \omega_j^{(o)}/\omega_{i+1}^{(o)}| \geq \tilde{q}_i$ for $j = k, \dots, i-1$, where k with $1 \leq k < i$ is minimal such that $|\omega_k^{(o)}| = |\omega_i|$.

Hence the claim is proved and \mathbf{A}_i is a one-dimensional analytic space isomorphic with some space that appears in Proposition 2.7.

(iv) Consider the image $\lambda(\mathbf{A}_i)$ in $\mathcal{BT}(\mathbb{Q})$. It is a maximal line segment in σ parallel with the vector $\mathbf{k}_i = (1, 1, \dots, 1, 0 \dots 0)$, and doesn't depend on $\boldsymbol{\omega}^{(o)}$ itself, but only on the absolute values $|\omega_i^{(o)}|$, or, what is the same, on $\lambda(\boldsymbol{\omega}^{(o)})$.

Given our data σ and f , an invertible function on $\lambda^{-1}(\sigma)$, a line segment \mathbf{s} in σ , (that is, in $\sigma(\mathbb{Q})$, which we always implicitly assume) is called *well-behaved* if $\mathbf{x} \mapsto \log f(\mathbf{x}) := \log_q \|f\|_{\mathbf{x}}$ is a linear function along \mathbf{s} . Applying Corollary 2.8 to \mathbf{A}_i , we find

that $\lambda(\mathbf{A}_i)$ is well-behaved. As the choices of $\omega^{(o)}$ and i were arbitrary (the excluded case $\tilde{q}_i = 1$ would lead to $\lambda(\mathbf{A}_i) = \lambda(\omega^{(o)})$), this shows that each line \mathbf{s} in the simplex σ which is parallel with one of the base vectors $\mathbf{k}_1, \dots, \mathbf{k}_{r-1}$ is well-behaved.

(v) With similar arguments one shows that each line segment \mathbf{s} in σ parallel with a 1-simplex facing σ and different from $(\mathbf{0}, \mathbf{k}_i)$ is well-behaved. The 1-simplices are of shape $\{\mathbf{k}_i, \mathbf{k}_j\}$ with $1 \leq i < j < r$, and one has to construct subspaces $\mathbf{A}_{i,j}$ similar to the \mathbf{A}_i of (ii) such that $\lambda(\mathbf{A}_{i,j})$ equals a maximal line segment in σ parallel with $\{\mathbf{k}_i, \mathbf{k}_j\}$. Alternatively, one could use the fact that $\mathrm{PGL}(r, K_\infty)$ acts transitively on the set of 1-simplices of \mathcal{BT} . We omit the details.

(vi) Now formula (2.6.1) holds along the 1-faces of σ , that is, along the boundary of each 2-face.

Let τ be an n -face of σ ($n \geq 2$), and suppose that

- (a) lines in $\tau(\mathbb{Q})$ parallel with one of its 1-faces are well-behaved;
- (b) formula (2.6.1) holds on the boundary of τ .

Then an easy calculation shows that (2.6.1) holds for all points $\mathbf{x} \in \tau(\mathbb{Q})$. Therefore, by induction on the dimension of faces of σ , we find that (2.6.1) holds for each $\mathbf{x} \in \sigma(\mathbb{Q})$. This finishes the proof. \square

3. Functional determinants

In this section, we deal with certain functional determinants related to the following families of modular forms:

- (3.1) (i) the coefficient forms $g_k(\omega)$ ($1 \leq k \leq r$), see (1.1);
- (ii) the Eisenstein series

$$E_k(\omega) = \sum'_{\lambda \in \Lambda_\omega} \lambda^{-k}, \text{ notably}$$

$$E_{q^k-1}(\omega) = -\beta_k(\omega), \text{ see (1.4);}$$

- (iii) the para-Eisenstein series $\alpha_k(\omega)$, see (1.4);

- (iv) the coefficient forms ${}_a\ell_k(\omega)$ (see (1.4); in particular ${}_T\ell_k = g_k$);

(v) the forms $\mu_i(\omega) := e_\omega(\omega_i/T)$ ($1 \leq i \leq r$, see [11] Sect. 3); they form an \mathbb{F} -basis of the space $V := {}_T\phi^\omega$ of T -division points of ϕ^ω ; their reciprocals are modular of weight 1 for the congruence subgroup $\Gamma(T) = \{\gamma \in \Gamma \mid \gamma \equiv 1 \pmod{T}\}$. In particular

$$e_V(X) = T^{-1}\phi_T^\omega(X) = X + T^{-1} \sum_{1 \leq j \leq r} g_j(\omega) X^{q^j}, \quad (3.2)$$

so the g_j are rational functions of the μ_i . In fact,

(3.3) $T^{-1}g_j$ is the $(q^j - 1)$ -th elementary symmetric function $s_{q^j-1}\{\mu^{-1}\}$ of the set $\{\mu^{-1} \mid 0 \neq \mu \in V\}$, $V = \text{set of } \mathbb{F}\text{-linear combinations of the } \mu_i \text{ } (1 \leq i \leq r)$.

These forms are subject to various relations; besides (3.3) we will use the following (see e.g. [5], Sect. 2; here and in the sequel, $\sum_{i+j=k} \dots$ is short for $\sum_{i,j \geq 0, i+j=k} \dots$):

(3.4) (i)

$$\sum_{i+j=k} \alpha_i^{q^j} E_{q^j-1} = 0 = \sum_{i+j=k} \alpha_i E_{q^j-1}^{q^i}, \quad k > 0, \text{ so}$$

$$\alpha_k = \sum_{0 \leq i < k} \alpha_i E_{q^{k-i}-1}^{q^i};$$

(ii)

$$\sum_{i+j=k} a \ell_i \alpha_j^{q^i} = a^{q^k} \alpha_k, \text{ that is}$$

$$\sum_{0 \leq j < k} a \ell_{k-j} \alpha_j^{q^{k-j}} = [a, k] \alpha_k, \text{ where } [a, k] := a^{q^k} - a.$$

(3.5) We let N^r be the (C_∞) -analytic space associated with the moduli scheme for rank- r Drinfeld modules ϕ with a structure of level T (i.e., the choice of an ordered \mathbb{F} -basis $\{\mu_1, \dots, \mu_r\}$ of the space of T -division points ${}_T\phi$). As we can choose the μ_i arbitrary subject only to the condition of \mathbb{F} -linear independence,

$$N^r = \mathbb{P}^{r-1}(C_\infty) \setminus \bigcup H,$$

where H runs through the finitely many hyperplanes defined over \mathbb{F} . Consider the commutative diagram

$$\begin{array}{ccc} \Omega^r & \longrightarrow & \Gamma(T) \setminus \Omega^r \xrightarrow{\cong} N^r = \{(\mu_1 : \dots : \mu_r) \in \mathbb{P}^{r-1}(C_\infty), \mu_1, \dots, \mu_r \text{ } \mathbb{F}\text{-l.i.}\} \\ \searrow & & \downarrow \\ \Gamma \setminus \Omega^r & \xrightarrow{\cong} & M^r = (\text{Proj}(C_\infty[g_1, \dots, g_r]))_{g_r \neq 0}, \end{array} \quad (3.6)$$

where the maps are the natural quotient maps, in coordinates

$$\omega \longmapsto (\mu_1(\omega) : \dots : \mu_r(\omega))$$

$$\searrow \quad \downarrow$$

$$(g_1(\omega) : \dots : g_r(\omega)).$$

The right hand map is given by (3.3), and the group of the Galois covering on the right hand side is

$$\Gamma/\Gamma(T) \cdot Z \xrightarrow{\cong} \text{GL}(r, \mathbb{F})/Z = \text{PGL}(r, \mathbb{F}),$$

which acts as matrix group on the column vector $(\mu_1, \dots, \mu_r)^T$.

3.7 Lemma. *The quotient map $\Omega^r \rightarrow N^r$ is étale.*

Proof. It suffices to show that $\Gamma(T)$ acts without fixed points on Ω^r . So let $\gamma \in \Gamma(T)$ be such that $\gamma\omega = \omega$ for some $\omega \in \Omega^r$. As γ fixes the image $\lambda(\omega) \in \mathcal{BT}(\mathbb{Q})$, γ has finite order. Since $\gamma \in \Gamma(T)$, its eigenvalues all equal 1, so γ is unipotent. But a unipotent $\gamma \neq 1$ cannot have fixed points on Ω^r (its eigenvectors on C_∞^r are K -rational), so $\gamma = 1$. \square

Note that any modular form f for Γ is a rational function of μ_1, \dots, μ_r , as follows from Proposition 1.8 and (3.3).

So $\frac{\partial f}{\partial \mu_j}$ is meaningful. In what follows, we regard “forms” as homogeneous functions on the canonical cones

$$\Omega^{r,*} := \{(\omega_1, \dots, \omega_r) \in C_\infty^r \mid (\omega_1 : \dots : \omega_r) \in \Omega^r\} \text{ over } \Omega^r,$$

$$N^{r,*} := \{(\mu_1, \dots, \mu_r) \in C_\infty^r \mid (\mu_1 : \dots : \mu_r) \in N^r\} \text{ over } N^r,$$

respectively. Then also $\frac{\partial f}{\partial \omega_r}$ is meaningful.

(3.8) Let now D be one of the differential operators $\frac{\partial}{\partial \omega_j}$ or $\frac{\partial}{\partial \mu_j}$ ($1 \leq j \leq r$). Then for $k > 0$

$$D(\alpha_k) = \sum_{0 \leq i < k} (E_{q^{k-i}-1})^{q^i} D(\alpha_i) + D(E_{q^k-1}),$$

as results from (3.4)(i). If $D(f)$ denotes either the vector $(\frac{\partial}{\partial \omega_1}(f), \dots, \frac{\partial}{\partial \omega_r}(f))$ or the vector $(\frac{\partial}{\partial \mu_1}(f), \dots, \frac{\partial}{\partial \mu_r}(f))$ then

$$(3.9) \quad D(E_{q^k-1}) = D(\alpha_k) + \text{a linear combination of the } D(\alpha_i) \text{ with } 1 \leq i < k.$$

In particular, we find

3.10 Proposition. *For each natural number $r' \leq r$ and each $a \in A$ of degree > 0 , the three functional determinants agree, where the indices i, j always range between 1 and r' :*

$$\begin{aligned} \det\left(\frac{\partial}{\partial \omega_j}(E_{q^i-1})\right) &= \det\left(\frac{\partial}{\partial \omega_j}(\alpha_i)\right) \\ &= ([a, r'] [a, r' - 1] \cdots [a, 1])^{-1} \det\left(\frac{\partial}{\partial \omega_j}(a\ell_i)\right). \end{aligned}$$

The statement remains true if all the $(\frac{\partial}{\partial \omega_j})$ are replaced with $(\frac{\partial}{\partial \mu_j})$.

Proof. The first equality comes from (3.9). The second equality follows similarly, starting with (3.4)(ii), which leads to

$$\begin{aligned} [a, k] D(\alpha_k) &= D(a\ell_k) + \text{a linear combination of the } D(a\ell_i) \\ &\text{with } 1 \leq i < k. \end{aligned}$$

The transfer - replacing the $\frac{\partial}{\partial \omega_j}$ with the $\frac{\partial}{\partial \mu_j}$ - is obvious. \square

Note that we can check the (non-) vanishing of these determinants with $a = T$, in which case ${}_a\ell_k = g_k$.

(3.11) Next, we let V be a finite \mathbb{F} -lattice in C_∞ with an ordered \mathbb{F} -basis $\{\lambda_1, \dots, \lambda_n\}$. All the quantities $\alpha_k = \alpha_k(V) = \alpha_k(\lambda_1, \dots, \lambda_n) = \alpha_k(\boldsymbol{\lambda}), \beta_k = \dots = \beta_k(\boldsymbol{\lambda}) = -E_{q^k-1}(\boldsymbol{\lambda}), D_j := \frac{\partial}{\partial \lambda_j}, \mathbf{D}(f) = (D_1(f), \dots, D_n(f))$ refer to V . From the identities in (1.4), we find for $n' \leq n$:

$$\det_{1 \leq i, j \leq n'} (D_j(\alpha_i)) = \det_{1 \leq i, j \leq n'} (D_j(E_{q^i-1})). \quad (3.12)$$

We now recycle and generalize the argument of [11] (6.3).

For an \mathbb{F} -linear map $\varphi : V \rightarrow \mathbb{F}$ define

$$M(\varphi) := \sum_{\lambda \in V} {}'\varphi(\lambda)/\lambda.$$

Then

$$D_j(E_{q^i-1}) = \sum_{\mathbf{a} \in \mathbb{F}^n} {}'\frac{a_j}{(a_1\lambda_1 + \dots + a_n\lambda_n)} q^i = M(\varphi_j)^{q^i},$$

where

$$\varphi_j : (a_1\lambda_1 + \dots + a_n\lambda_n) \mapsto a_j.$$

Hence the determinant in (3.12) is the Moore determinant ([13] 1.9)

$$\det_{1 \leq i, j \leq n'} D_j(E_{q^i-1}(\boldsymbol{\lambda})) = \det_{1 \leq i, j \leq n'} (M(\varphi_j)^{q^i}),$$

which doesn't vanish if and only if the $M(\varphi_j)$ ($1 \leq j \leq n'$) are \mathbb{F} -linearly independent. The latter holds true for $1 \leq j \leq n$, as is shown by the argument ([11], Lemma 6.4). Therefore:

3.13 Proposition. *The determinants appearing in (3.12) never vanish. \square*

Remark. As the order of $\lambda_1, \dots, \lambda_n$ is arbitrary, the non-vanishing of the functional determinants holds for any given order.

Let now V be the r -dimensional \mathbb{F} -space ${}_T\phi^\omega$ with $\omega \in \Omega^{r,*}$. Then by (3.2), $\alpha_i(V) = T^{-1}g_i(\omega)$, and the $\mu_i(\omega)$ form an \mathbb{F} -basis of V . Combining (3.10) with (3.13) we find:

3.14 Proposition. *Neither of the following determinants vanishes:*

$$\det\left(\frac{\partial}{\partial \mu_j}(\alpha_i)\right), \det\left(\frac{\partial}{\partial \mu_j}(E_{q^i-1})\right), \det\left(\frac{\partial}{\partial \mu_j}({}_a\ell_i)\right).$$

Here “ \det ” = $\det_{1 \leq i, j \leq r'}$, and $1 \leq r' \leq r$. \square

Now we return to our usual convention: modular forms are functions on Ω^r , whose coordinates are normalized by $\omega_r = 1$. Then we are interested in functional determinants of rank $r' = r - 1$. Fixing $i_0 \in \{1, 2, \dots, r\}$, the μ_i ($1 \leq i \leq r$, $i \neq i_0$) are coordinates on N^r . From (3.7) we find that $\det(\frac{\partial \mu_i}{\partial \omega_j})$ never vanishes, where $1 \leq j \leq r - 1$ and $1 \leq i \leq r$, $i \neq i_0$. Together with (3.10) and (3.13) this yields:

3.15 Proposition. *Let $a \in A$ be non-constant. Then the three functional determinants (where $1 \leq i, j \leq r - 1$) never vanish:*

$$\det(\frac{\partial}{\partial \omega_j}(E_{q^{i-1}})), \det(\frac{\partial}{\partial \omega_j}(\alpha_i)), \det(\frac{\partial}{\partial \omega_j}(a\ell_i)). \quad \square$$

4. Vanishing loci of modular forms

Our aim is to show the smoothness and to determine the image under the building map λ of vanishing sets of modular forms. Obviously, it suffices to do so in the fundamental domain $\mathbf{F} \subset \Omega^r$ for Γ . Therefore we define for a modular form f the analytic space

$$V(f) := \{\omega \in \mathbf{F} \mid f(\omega) = 0\}. \quad (4.1)$$

We start with $f = g_i$, where $1 \leq i < r$ ($g_r = \Delta$ never vanishes on Ω^r). It has been shown in [10], Corollary 3.6 that $V(g_i)$ is contained in $\mathbf{F}_{r-i} = \{\omega \in \mathbf{F} \mid |\omega_{r-i}| = |\omega_{r-i+1}|\}$, that is, $\lambda(V(g_i)) \subset \mathcal{W}_{r-i}(\mathbb{Q})$. Besides that inclusion, the size $|g_i(\omega)|$ for $\omega \in \mathbf{F} \setminus \mathbf{F}_{r-i}$ as well as the spectral norm $\|g_i\|_{\mathbf{x}}$ for $\mathbf{x} \in \mathcal{W}_{r-1}(\mathbb{Q})$ has been determined in [11], Corollary 4.16. Here we show:

4.2 Theorem. (i) *For $1 \leq i < r$, $\lambda(V(g_i)) = \mathcal{W}_{r-i}(\mathbb{Q})$. More generally, let S be any non-empty subset of $\{1, 2, \dots, r - 1\}$. Then*

$$\lambda\left(\bigcap_{i \in S} V(g_i)\right) = \bigcap_{i \in S} \mathcal{W}_{r-i}(\mathbb{Q}).$$

(ii) *The $V(g_i)$ ($i \in S$) intersect transversally, and the analytic space $\bigcap_{i \in S} V(g_i)$ is smooth of dimension $r - 1 - \#(S)$.*

In what follows, we will at various places make use of the

4.3 Observation. Knowledge of the following data on $\omega = (\omega_1, \dots, \omega_{r-1}, 1) \in \mathbf{F}$ is equivalent:

- (a) $|\omega_1|, \dots, |\omega_{r-1}|$;
- (b) $|\mu_1|, \dots, |\mu_r|$, where $\mu_i = \mu_i(\omega)$;
- (c) $\mathbf{x} = \lambda(\omega)$;
- (d) the Newton polygon NP_{ω} of $T^{-1}\phi_T^{\omega}(X)$.

Proof. First note that $\{\mu_r, \dots, \mu_1\}$ is an \mathbb{F} -SMB of $V = {}_T\phi^\omega$ ([10], 3.4), so (b) determines the spectrum of V . Now (a) \Rightarrow (b) is [11], 4.2, (b) \Leftrightarrow (d) is (1.10), (a) \Leftrightarrow (c) is trivial, as $x_i = \log \omega_i$. A closer look to the equations [11], 4.2 shows that they may be solved for the $|\omega_i|$ if the $|\mu_i|$ are given, thus (b) \Rightarrow (a). \square

Proof of Theorem 4.2. (i) The inclusion of the left hand side into the right hand side is trivial. For the reverse inclusion, let $\mathbf{x} \in \bigcap_{i \in S} \mathcal{W}_{r-i}(\mathbb{Q})$ and $\boldsymbol{\omega} \in \mathbf{F}_{\mathbf{x}}$. Then $|\omega_{r-i}| = |\omega_{r-i+1}|$ for $i \in S$, thus also $|\mu_{r-i}| = |\mu_{r-i+1}|$ ([10], 3.4). The remark in (1.11) along with (3.2) shows that there is an $\boldsymbol{\omega}'$ with $NP_{\boldsymbol{\omega}} = NP_{\boldsymbol{\omega}'}$ such that $g_i(\boldsymbol{\omega}') = 0$ for all $i \in S$. (Note the reverse order in the \mathbb{F} -SMB $\{\mu_r, \dots, \mu_1\}$!) So $\mathbf{x} = \lambda(\boldsymbol{\omega}) = \lambda(\boldsymbol{\omega}')$ lies in the left hand side.

(ii) This follows from the non-vanishing of $\det_{1 \leq i, j \leq r-1} \left(\frac{\partial g_i}{\partial \omega_j} \right)$, cf. (3.15). \square

4.4 Examples. Let S be the full set $\{1, 2, \dots, r-1\}$. For $\boldsymbol{\omega} \in \mathbf{F}$, the following are equivalent:

- (a) $g_i(\boldsymbol{\omega}) = 0 \quad \forall i \in S$;
- (b) $\alpha_i(\boldsymbol{\omega}) = 0 \quad \forall i \in S$;
- (c) $E_{q^i-1}(\boldsymbol{\omega}) = 0 \quad \forall i \in S$;
- (d) $\boldsymbol{\omega} \in \Omega^r(\mathbb{F}^{(r)})$.

Here (a) \Leftrightarrow (b) \Leftrightarrow (c) follows from (3.4), and its equivalence with (d) is [10], Proposition 2.9. Correspondingly, $\lambda(\bigcap_{i \in S} V(g_i)) = \lambda(\Omega^r(\mathbb{F}^{(r)})) = \{\mathbf{0}\}$.

Next, we deal with the Eisenstein series $E_k(\boldsymbol{\omega})$, where we always suppose that $0 < k \equiv 0 \pmod{q-1}$, as E_k vanishes identically if $k \not\equiv 0 \pmod{q-1}$.

4.5 Theorem. (i) $\lambda(V(E_k)) = \mathcal{W}_{r-1}(\mathbb{Q})$;
 (ii) $V(E_{q^j-1})$ is smooth for $j > 0$;
 (iii) the same statement as (4.2)(ii), with g_i replaced by E_{q^i-1} .

Proof. (iii) follows from the non-vanishing of functional determinants as in (4.2).

(i) We have

$$E_k(\boldsymbol{\omega}) = \sum'_{\mathbf{a} \in A^r} (a_1 \omega_1 + \dots + a_r \omega_r)^{-k}. \quad (4.5.1)$$

Let i be minimal with $|\omega_i| = 1$. Then $i < r \Leftrightarrow \boldsymbol{\omega} \in \mathbf{F}_{r-1}$. The terms of largest absolute value in (4.5.1) are the $(a_i \omega_i + \dots + a_r \omega_r)^{-k}$ with $a_j \in \mathbb{F}$, all of absolute value 1. If $i = r$, they sum up to $\sum_{a \in \mathbb{F}^*} (a \omega_r)^{-k} = -1$, so $E_k(\boldsymbol{\omega}) \neq 0$ and $\lambda(V(E_k)) \subset \mathcal{W}_{r-1}$. Conversely, let $\mathbf{x} \in \mathcal{W}_{r-1}$, so $i < r$ for $\boldsymbol{\omega} \in \mathbf{F}_{\mathbf{x}}$. There do exist $\omega_i, \dots, \omega_r \in \overline{\mathbb{F}}$ linearly independent over \mathbb{F} such that $\sum'_{a_i, \dots, a_r \in \mathbb{F}} (a_i \omega_i + \dots + a_r \omega_r)^{-k} = 0$, since the vanishing set of this rational

function on $\overline{\mathbb{F}}^{r-i+1}$ cannot be contained in the union of the \mathbb{F} -rational hyperplanes. Therefore, the canonical reduction of $E_k(\omega)$ as a function on the affinoid \mathbf{F}_x has zeroes and $E_k(\omega)$ has non-constant absolute value on \mathbf{F}_x . By Theorem 2.4, E_k presents a zero on \mathbf{F}_x , which shows that $x \in \lambda(V(E_k))$.

(ii) For $j < r$, the smoothness of $V(E_{q^j-1})$ is covered by (iii). In the general case, let $\omega \in \mathbf{F}_{r-1}$ be a zero of E_{q^j-1} and $i < r$ as in (i), $D = \frac{\partial}{\partial \omega_i}$. Then

$$\begin{aligned} DE_{q^j-1}(\omega) &= \sum_{\mathbf{a} \in A^r} 'a_i(a_1\omega_1 + \dots + a_r\omega_r)^{-q^j} \\ &\equiv \sum_{a_i, \dots, a_r \in \mathbb{F}} 'a_i(a_i\omega_i, \dots, a_r\omega_r)^{-q^j} \end{aligned}$$

(where “ \equiv ” means up to terms of value strictly less than 1)

$$\equiv - \sum_{a_{i+1}, \dots, a_r \in \mathbb{F}} (\omega_i + a_{i+1}\omega_{i+1} + \dots + a_r\omega_r)^{-q^j}.$$

Let V be the \mathbb{F} -vector space spanned by $\omega_{i+1}, \dots, \omega_r$, with e -function e_V . Then

$$e_V(z)^{-1} = \sum_{v \in V} \frac{1}{z + v},$$

and the above is $-e_V(\omega_i)^{-q^j}$, of absolute value 1. Hence $DE_{q^j-1}(\omega) \neq 0$, which shows the smoothness in ω . \square

Remark. In the proof of (i) we have seen that $E_k(\omega) \equiv -1$ if $\omega \in \mathbf{F} \setminus \mathbf{F}_{r-1}$, i.e., if $\lambda(\omega) \notin \mathcal{W}_{r-1}(\mathbb{Q})$, while $\|E_k\|_x = 1$ for $x \in \mathcal{W}_{r-1}(\mathbb{Q})$. Concerning smoothness and intersection properties of their vanishing sets, in contrast with the “special” Eisenstein series of weight $q^j - 1$ for some j , the “non-special” ones behave rather erratic and unpredictably. This holds already in the case $r = 2$, see e.g. [8], Remark 6.7.

In order to treat the para-Eisenstein series α_k , we make the following definition.

4.6 Definition. For $k \in \mathbb{N}$ let

$$\mathbf{F}(k) := \{\omega \in \mathbf{F} \mid \Lambda_\omega \text{ is } k\text{-inseparable}\}.$$

As this depends only on $\lambda(\omega)$, we also put

$$\mathcal{W}(k)(\mathbb{Q}) := \lambda(\mathbf{F}(k)) = \{x \in \mathcal{W}(\mathbb{Q}) \mid \Lambda_\omega \text{ is } k\text{-inseparable for one (thus all) } \omega \in \lambda^{-1}(x)\}.$$

It is the set of \mathbb{Q} -points of a full simplicial subcomplex $\mathcal{W}(k)$ of \mathcal{W} .

4.7 Examples. Let $\{\lambda_1, \lambda_2, \dots\}$ be the \mathbb{F} -SMB of Λ_ω constructed in (1.12). Hence $\omega \in \mathbf{F}(k)$ if and only if $|\lambda_k| = |\lambda_{k+1}|$.

(i) $\mathcal{W}(1) = \mathcal{W}_{r-1}$, for $|\lambda_1| = |\lambda_2| \Leftrightarrow |\omega_{r-1}| = |\omega_r| = 1$.

Assume now that $r = 3$. With some labor, we find the following descriptions of $\mathcal{W}(2)$, $\mathcal{W}(3)$, $\mathcal{W}(4)$ through their vertices. Note that the vertices of \mathcal{W} are the $[L_{\mathbf{k}}]$, where \mathbf{k} is an \mathbb{N}_0 -combination of the $\mathbf{k}_i = (1, 1, \dots, 1, 0, \dots, 0)$ (i ones, $1 \leq i < r$).

(ii) vertices of $\mathcal{W}(2)$: $\mathbf{k} = \mathbf{0}$ and $\mathbf{k} = \mathbf{k}_2 + n\mathbf{k}_1$ ($n \in \mathbb{N}_0$) (see (1.12)).

(iii) vertices of $\mathcal{W}(3)$: $\mathbf{k}_2, 2\mathbf{k}_2 + n\mathbf{k}_1$ ($n \in \mathbb{N}_0$), $n\mathbf{k}_1$ ($n \in \mathbb{N}$). This corresponds to the fact that $|\lambda_3| = |\lambda_4|$ if either of the conditions is satisfied:

- $|\omega_2| = |T^2\omega_3| = q^2$
- $q \leq |\omega_1| = |\omega_2| \leq q^2$
- $|\omega_1| = q$
- $|\omega_2| = |\omega_3| = 1$ and $|\omega_1| \geq q$.

(iv) vertices of $\mathcal{W}(4)$: $\mathbf{0}, \mathbf{k}_1, \mathbf{k}_1 + \mathbf{k}_2 + n\mathbf{k}_1$ ($n \in \mathbb{N}_0$), $2\mathbf{k}_2, 3\mathbf{k}_2 + n\mathbf{k}_1$ ($n \in \mathbb{N}_0$). This corresponds to: $|\lambda_4| = |\lambda_5|$ if and only if one of the following holds:

- $|\omega_2| = |\omega_3| = 1$ and $|\omega_1| \leq q$
- $|T\omega_2| = |\omega_1| \leq q^2$
- $|\omega_2| = |T\omega_3| = q$ and $|\omega_1| \geq q^2$
- $|\omega_1| = |T^2\omega_3| = q^2$ and $q \leq |\omega_2| \leq q^2$
- $q^2 \leq |\omega_1| = |\omega_2| \leq q^3$.

In the examples, $\mathcal{W}(k)$ is a full subcomplex of \mathcal{W} , which is everywhere of dimension $r - 2$ (that is, each simplex belongs to a simplex of maximal dimension $r - 2$), connected and contractible. These properties should hold in full generality. Certainly, the $\mathcal{W}(k)$ deserve more investigation!

4.8 Theorem. (i) $\lambda(V(\alpha_k)) = \mathcal{W}(k)(\mathbb{Q})$;

(ii) the same statement as (4.2)(ii) with g_i replaced by α_i .

Proof. Once again, (ii) follows from the non-vanishing of the functional determinant.

(i)(a) The fact that $V(\alpha_k)$ is contained in $\mathbf{F}(k)$ is a consequence of (1.11) and the definition of $\mathbf{F}(k)$. Suppose that $\mathbf{x} \in \mathcal{W}(k)(\mathbb{Q})$. We will show that $|\alpha_k|$ is not constant on $\mathbf{F}_{\mathbf{x}}$, which forces the existence of a zero of α_k in $\mathbf{F}_{\mathbf{x}}$ and gives the wanted equality.

(b) From $e_\omega(X) = \sum_{k \geq 0} \alpha_k(\omega) X^{q^k} = X \prod'_{\lambda \in \Lambda_\omega} (1 - X/\lambda)$ we see that

$$\alpha_k = \alpha_k(\omega) = s_{q^k-1}\{\lambda^{-1} \mid 0 \neq \lambda \in \Lambda_\omega\}$$

is the $(q^k - 1)$ -th elementary symmetric function in the λ^{-1} :

$$\alpha_k = \sum_S P(S),$$

where S runs through the family of $(q^k - 1)$ -subsets of $\Lambda_\omega \setminus \{0\}$ and

$$P(S) = \left(\prod_{\lambda \in S} \lambda \right)^{-1}.$$

(c) Let $m + 1$ (resp. n) be the least (resp. largest) subscript i such that $|\lambda_i| = |\lambda_k| = |\lambda_{k+1}|$, where $\{\lambda_1, \lambda_2, \dots\}$ is an \mathbb{F} -SMB of Λ_ω formed out of the $\{T^j \omega_i\}$ (see (1.12)). Then $m < k < n$, the ω_i appearing in $\lambda_{m+1}, \dots, \lambda_n$ are all different, and therefore $n - m \leq r$.

(d) Some $P(S)$ has largest absolute value if S contains all the $q^m - 1$ elements of $V' \setminus \{0\}$ and $q^k - q^m$ many elements of $V \setminus V'$, where $V = \sum_{1 \leq i \leq n} \mathbb{F} \lambda_i$, $V' = \sum_{1 \leq i \leq m} \mathbb{F} \lambda_i$.

The contribution of such S to α_k is

$$P := \left(\prod_{\lambda \in V'} \lambda^{-1} \right) \sum_{\substack{S' \subset V \setminus V' \\ \#(S') = q^k - q^m}} P(S'), \quad P(S') = \left(\prod_{\lambda \in S'} \lambda \right)^{-1}.$$

All the $P(S)$ of such S have the same absolute value $|P(S)| =: c$, which depends only on $|\lambda_1|, \dots, |\lambda_n|$ and therefore only on \mathbf{x} . We write $x \equiv y$ if $|x - y| < c$. Then

$$\alpha_k(\omega) \equiv P \equiv \alpha_k(V),$$

all of which are homogeneous functions of $\omega \in \mathbf{F}_x$ of weight $q^k - 1$. (That is, each of these functions f on \mathbf{F}_x may be regarded as a homogeneous function f^* on the cone $\mathbf{F}_x^* = \{(\omega_1, \dots, \omega_r) \in C_\infty^r \mid (\omega_1 : \dots : \omega_r) \in \mathbf{F}_x\}$, and $f^*(t\omega) = t^{1-q^k} f^*(\omega)$.)

(e) Therefore we must show that $|\alpha_k(V)|$ is not constant on \mathbf{F}_x , which, as $\alpha_n(V) = \left(\prod_{\lambda \in V} \lambda \right)^{-1}$ has constant absolute value, is equivalent with the fact that the k -th coefficient $\gamma_k(V)$ of

$$f_V(X) := \alpha_n(V)^{-1} e_V(X) = \prod_{\lambda \in V} (X - \lambda) = \sum_{0 \leq k \leq n} \gamma_k(V) X^{q^k}$$

has non-constant absolute value. This will be shown by varying the leading coefficients in $\overline{\mathbb{F}}$ of the $n - m$ different ω_i that appear in $\lambda_{m+1}, \dots, \lambda_n$ (see (c)) without changing $\mathbf{x} = \lambda(\omega)$. Dropping the requirement $\omega_r = 1$, we rescale the projective coordinates of ω such that $|\lambda_j| < 1$ for $1 \leq j \leq m$ and $|\lambda_j| = 1$ for $m + 1 \leq j \leq n$. Then the monic polynomial f_V gets coefficients in O_{C_∞} , and its reduction $\overline{f}_V \in \overline{\mathbb{F}}[X]$ satisfies

$$(\overline{f}_V) = (f_{\overline{V}})^{q^m},$$

where $\overline{V} \subset \overline{\mathbb{F}}$, spanned by $\overline{\lambda}_{m+1}, \dots, \overline{\lambda}_n$, is the reduction of V and $f_{\overline{V}}(X) = \prod_{\overline{\lambda} \in \overline{V}} (X - \overline{\lambda})$.

(f) Let $\ell := n - m = \dim \overline{V}$, which is larger or equal to 2. For an \mathbb{F} -subspace U of $\overline{\mathbb{F}}$ of dimension ℓ , write

$$f_U(X) = \prod_{u \in U} (X - u) = \sum_{0 \leq j \leq \ell} \gamma_j(U) X^{q^j}.$$

The $\gamma_j = \gamma_j(U)$ are homogeneous of weight $(q^j - q^\ell)$ (that is, $\gamma_j(tU) = t^{q^\ell - q^j} \gamma_j(U)$, $t \in \overline{\mathbb{F}}^*$) and may be considered as forms on $\Omega^\ell(\overline{\mathbb{F}}) = \{\boldsymbol{\nu} = (\nu_1 : \dots : \nu_\ell) \in \mathbb{P}^{\ell-1}(\overline{\mathbb{F}}) \mid \nu_1, \dots, \nu_\ell \text{ } \mathbb{F}\text{-l.i.}\}$. Further, for $0 < j < \ell$, $\gamma_j(\boldsymbol{\nu})$ vanishes somewhere on $\Omega^\ell(\overline{\mathbb{F}})$: take e.g. the ν_j as an \mathbb{F} -basis of $U := \mathbb{F}^{(\ell)}$; then $f_U(X) = X^{q^\ell} - X$.

(g) Putting $U := \overline{V}$, of dimension $\ell = m - n$, $j := k - m$, the fact that $\gamma_j(\overline{V})$ takes both zero and non-zero values on $\Omega^\ell(\overline{\mathbb{F}})$ implies that $|\alpha_k(V)|$ is non-constant on \mathbf{F}_x , provided we can find these zero/non-zero values inside the image of \mathbf{F}_x in $\Omega^\ell(\overline{\mathbb{F}})$.

(h) Let I be the set of those indices i with $1 \leq i \leq r$ and $\log \omega_i \in \mathbb{Z}$. Due to our normalization $|\lambda_{m+1}| = \dots = |\lambda_n| = 1$, $I' := \{m+1, \dots, k, k+1, \dots, n\} \subset I$. For $i \in I$ let $\theta_i \in \overline{\mathbb{F}}^*$ be the leading coefficient of ω_i , that is

$$\omega_i = \theta_i T^{\log \omega_i} (1 + n_i) \quad \text{with } |n_i| < 1.$$

Given $\boldsymbol{\omega} = (\omega_1 : \dots : \omega_r) \in \mathbf{F}_x$, with the scaling of projective coordinates as in (e), let $\boldsymbol{\omega}' = (\omega'_1 : \dots : \omega'_r)$ be defined by

$$\begin{aligned} \omega'_i &= \omega_i, & \text{if } i \notin I \\ &= \frac{\nu_i}{\theta_i} \omega_i, & \text{if } i \in I \end{aligned}$$

with some $\nu_i \in \overline{\mathbb{F}}^*$. Then $\boldsymbol{\omega}'$ belongs to \mathbf{F}_x at least if the ν_i ($i \in I$) are \mathbb{F} -linearly independent. Choosing first the ν_i ($i \in I'$) such that $(\nu_{m+1} : \dots : \nu_n) \in \Omega^\ell(\overline{\mathbb{F}})$ is a zero of γ_{k-m} and then the remaining ν_i ($i \in I \setminus I'$) such that $\{\nu_i \mid i \in I\}$ is \mathbb{F} -l.i., we find some $\boldsymbol{\omega}'$ where $|\alpha_k(V)|$ is strictly less than the spectral norm of $\boldsymbol{\omega} \mapsto \alpha_k(V)$ on \mathbf{F}_x .

This finishes the proof. \square

Let now $a \in A$ have positive degree, and consider the forms $a\ell_k$. Similar arguments as in the proof of Theorem 4.8, based on Proposition 1.11, allow to describe the set $\lambda(V(a\ell_k))$, that is, the analogue of 4.8(i). However, as both the statement and its proof are substantially more complex than its counterpart for α_k (see the relatively simple case of $r = 2$ treated in [9], Section 5), we restrict to stating the analogue of (4.2)(ii), (4.5)(iii) and (4.8)(ii), with identical proof.

4.9 Theorem. *Let S be a non-empty subset of $\{1, 2, \dots, r-1\}$. The $V(a\ell_i)$ ($i \in S$) intersect transversally, and the analytic space $\bigcap_{i \in S} V(a\ell_i)$ is smooth of dimension $r-1-\#(S)$.* \square

We conclude with the relationship between α_k and the ${}_a\ell_k$.

(4.3) Let ρ be the Carlitz module, i.e., the rank-one Drinfeld module defined by $\rho_T(X) = TX + X^q$, and let ${}_ac_k$ be its coefficients:

$$\rho_a(X) = \sum_{0 \leq k \leq d} {}_ac_k X^{q^k} \quad (a \in A \text{ of degree } d, {}_ac_0 = a). \quad (4.10.1)$$

We have

$$\log {}_ac_k = (d - k)q^k, \quad 0 \leq k \leq d, \quad {}_ac_k = 0 \text{ for } k > d. \quad (4.10.2)$$

There exists $\bar{\pi} \in C_\infty$, well-defined up to a $(q - 1)$ -th root of unity, such that $\rho = \phi^{(L)}$ is the Drinfeld module corresponding to the rank-one lattice $L = \bar{\pi}A$, with

$$\log \bar{\pi} = q/(q - 1). \quad (4.10.3)$$

The exponential function of L is

$$\begin{aligned} e_L &= \sum_{k \geq 0} D_k^{-1} \tau^k, \quad D_k := [k][k - 1]^q \cdots [1]^{q^{k-1}}, \\ [k] &:= [T, k] = (T^{q^k} - T). \end{aligned} \quad (4.10.4)$$

Hence

$$\log \alpha_k(L) = -kq^k. \quad (4.10.5)$$

Replacing L with A yields the Drinfeld module $\phi^{(A)}$ with

$$\phi_T^{(A)} = TX + \bar{\pi}^{q-1} X^q$$

and

$$\begin{aligned} \alpha_k(A) &= \bar{\pi}^{(q^k-1)} \alpha_k(L), \\ \log \alpha_k(A) &= q(q^k - 1)/(q - 1) - kq^k. \end{aligned} \quad (4.10.6)$$

All of this is easily verified and may be found at different places, e.g. [5], [13], [16]. We also need the following lemma, whose proof is an exercise in manipulating the preceding formulas.

4.11 Lemma. *If $d = \deg a$ tends to infinity then $D_{ka}c_k/[a, k]$ tends to 1.*

Now we normalize the modular forms ${}_a\ell_k$ and α_k by dividing through the corresponding quantities of the Drinfeld module $\phi^{(A)}$.

$${}_a\tilde{\ell}_k := \bar{\pi}^{1-q^k} {}_ac_k^{-1} {}_a\ell_k; \quad \tilde{\alpha}_k := \bar{\pi}^{1-q^k} D_k \alpha_k. \quad (4.12)$$

This normalization is quite natural, see Remark 4.14.

4.13 Theorem. *As the degree d of a tends to infinity, ${}_a\tilde{\ell}_k$ tends to $\tilde{\alpha}_k$, locally uniformly on Ω^r .*

(Here locally uniform convergence means uniform convergence on the parts of an admissible covering of Ω^r .)

Proof. (i) Let V be the space of isobaric polynomials of weight $q^k - 1$ in the g_1, \dots, g_r (which by (1.8) is the space of modular forms of weight $q^k - 1$ and type 0). As $\dim_{C_\infty}(V) < \infty$, all norms on V agree. Since ${}_a\ell_k$, α_k and their normalizations belong to V , it suffices to show convergence with respect to one specific norm on V .

(ii) We let $\mathbf{F}^{(k)} := \{\omega \in \mathbf{F} \mid \log \omega_{r-1} \geq k\}$. It is an open admissible subspace, and we will use the norm $\|f\| := \sup_{\omega \in \mathbf{F}^{(k)}} f(\omega)$ (well-defined in view of (1.8)(c)). For

$\omega \in \mathbf{F}$ let $\{\lambda_1(\omega), \lambda_2(\omega), \dots\}$ be the \mathbb{F} -SMB of Λ_ω as in (1.12). Now if $\omega \in \mathbf{F}^{(k)}$ then for $1 \leq i \leq k$, $\lambda_i(\omega) = T^{i-1}\omega_r = T^{i-1}$. Hence the \mathbb{F} -span of these agrees with $A_{k-1} := \{a \in A \mid \deg a \leq k-1\}$. As in the proof of 4.8 we find for $1 \leq i \leq k$

$$\alpha_i(\omega) = \left(\prod'_{c \in A_{i-1}} c \right)^{-1} + \text{smaller terms},$$

that is

$$|\alpha_i(\omega)| = |\alpha_i(A)| = |\pi|^{q^i-1} |D_i|^{-1}.$$

(iii) Suppose that $\omega \in \mathbf{F}^{(k)}$ and $d = \deg a \geq k$. The elements

$$\mu_{i,j} := e_\omega(T^{j-1}\omega_i/a) \text{ with } 1 \leq i \leq r, 1 \leq j \leq d$$

form an \mathbb{F} -basis of ${}_a\phi^\omega$, the a -division points of ϕ^ω . We see from

$$|\mu_{i,j}(\omega)| = \left| \frac{T^{j-1}\omega_i}{a} \right| \prod'_{\substack{\lambda \in \Lambda_\omega \\ |a\lambda| < |T^{j-1}\omega_i|}} \left| 1 - \frac{T^{j-1}\omega_i}{a\lambda} \right|$$

that the $\lambda_j := \mu_{r,j}$ with $1 \leq j \leq k$ are the first k elements of an \mathbb{F} -SMB of ${}_a\phi^\omega$. Moreover, the next \mathbb{F} -SMB vector λ_{k+1} satisfies $|\lambda_{k+1}| > |\lambda_k|$, due to the assumption $\omega \in \mathbf{F}^{(k)}$. From

$$\phi_a^\omega(X) = \sum_{0 \leq i \leq rd} {}_a\ell_i(\omega) X^{q^i}$$

we find

$$\begin{aligned} a^{-1} {}_a\ell_i(\omega) &= \alpha_i({}_a\phi^\omega) = s_{q^i-1} \{ \mu^{-1} \mid 0 \neq \mu \in {}_a\phi^\omega \} \\ &= \left(\prod'_{c \in A_{i-1}} \left(\frac{c}{a} \right) \right) \cdot u \end{aligned}$$

with some $u \in C_\infty$ of absolute value 1. This implies

$$|{}_a\ell_i(\omega)| = |{}_a\ell_i(A)| = |\bar{\pi}|^{q^i-1} |{}_ac_i|,$$

valid for $i \leq k \leq d$ and $\omega \in \mathbf{F}^{(k)}$.

(iv) Next, consider the identity (3.4)(ii)

$$[a, k]\alpha_k(\omega) = \sum_{1 \leq i \leq k-1} {}_a\ell_i(\omega)\alpha_{k-i}^{q^i}(\omega) + {}_a\ell_k(\omega).$$

For $d = \deg a \geq k$, all the terms have constant absolute value on $\mathbf{F}^{(k)}$. Plugging in, we see that $\log([a, k]\alpha_k)$ and $\log({}_a\ell_k)$ grow of order $(d-k)q^k + q(q^k-1)/(q-1)$ with $d \rightarrow \infty$, while the log of the other terms grow of order less or equal to $(d-k+1)q^{k-1} + q(q^k-1)/(q-1)$. Upon normalization $f \rightsquigarrow \tilde{f}$, we find that ${}_a\tilde{\ell}_k$ tends to $[a, k]/({}_ac_k D_k)^{-1} \cdot \tilde{\alpha}_k$ uniformly on $\mathbf{F}^{(k)}$. The result now follows from Lemma 4.11. \square

4.14 Remarks. (i) In steps (ii) and (iii) of the preceding proof, in fact the stronger statements hold:

$$\lim \alpha_i(\omega) = \alpha_i(A), \lim {}_a\ell_i(\omega) = {}_a\ell_i(A),$$

where the limits are with respect to $|\omega_{r-1}| \rightarrow \infty$. This follows from a closer look to the arguments and estimates used there. Hence $\lim \tilde{\alpha}_i(\omega) = 1 = \lim {}_a\tilde{\ell}_i(\omega)$, which in the case $r = 2$ means $\tilde{\alpha}_i(\infty) = 1 = {}_a\tilde{\ell}_i(\infty)$.

(ii) Theorem 4.13 has been shown in the case $r = 2$ in [9], Theorem 6.16. The present proof isn't but a generalization of this special case.

4.15 Concluding remarks/questions. (i) Let f be one of the functions E_k or α_i on Ω^r . For $\mathbf{x} \in \lambda(V(f))$, f is given on $\mathbf{F}_{\mathbf{x}}$ as a convergent sum of terms

$$f(\omega) = \sum T_i + \sum U_j$$

with finitely many T_i all of constant absolute value $|T_i| = \|f\|_{\mathbf{x}}$ and terms U_j of strictly smaller value.

For $f = E_k$, the T_i are the $|a_i\omega_i + \cdots + a_r\omega_r|^{-k}$, $a_i, \dots, a_r \in \mathbb{F}$, see proof of (4.5); for $f = \alpha_i$, the T_i are certain $P(S)$, see proof of (4.8), part (d). A similar property may be shown for $f = {}_a\ell_k$. Hence $|f(\omega)| < \|f\|_{\mathbf{x}}$ arises from cancellations between the terms T_i . The proof of (2.6) combined with Remark 2.9 yields the following generalization of (2.6):

Given any of the modular forms f as above, the map

$$\mathbf{x} \mapsto \log_q \|f\|_{\mathbf{x}}$$

is an affine function on $\mathcal{BT}(\mathbb{Q})$ (which off $\lambda(V(f))$ agrees with $\log f(\omega)$, $\omega \in \mathbf{F}_{\mathbf{x}}$).

To which class of modular forms does this property generalize?

(ii) For all our distinguished modular forms f ($f = \alpha_{i,a}\ell_j, E_k$), $\lambda(V(f))$ is (the set of \mathbb{Q} -valued points of) a simplicial subcomplex of pure dimension $r - 2$ of \mathcal{W} . In the case $r = 2$ this means that the zeroes of f lie in the λ -preimage of vertices of the Bruhat–Tits tree \mathcal{BT} of $\mathrm{PGL}(2, K_\infty)$.

How can we characterize modular forms with this property?

References

- [1] Wolfgang Bartenwerfer, Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall, *J. Reine Angew. Math.* 286/287 (1976) 144–163.
- [2] Dirk Basson, A product formula for the higher rank Drinfeld discriminant function, *J. Number Theory* 178 (2017) 190–200.
- [3] Dirk Basson, Florian Breuer, On certain Drinfeld modular forms of higher rank, *J. Théor. Nombres Bordeaux* 29 (2017) 827–843.
- [4] Jean Fresnel, Marius van der Put, *Géométrie Analytique Rigide et Applications*, Progress in Mathematics, vol. 18, Birkhäuser Verlag, 1981.
- [5] Ernst-Ulrich Gekeler, On the coefficients of Drinfeld modular forms, *Invent. Math.* 93 (1988) 667–700.
- [6] Ernst-Ulrich Gekeler, On the Drinfeld discriminant function, *Compos. Math.* 106 (1997) 181–202.
- [7] Ernst-Ulrich Gekeler, A survey on Drinfeld modular forms, *Turkish J. Math.* 23 (1999) 485–518.
- [8] Ernst-Ulrich Gekeler, Finite modular forms, *Finite Fields Appl.* 7 (2001) 553–572.
- [9] Ernst-Ulrich Gekeler, Zero distribution and decay at infinity of Drinfeld modular coefficient forms, *Int. J. Number Theory* 7 (2011) 671–693.
- [10] Ernst-Ulrich Gekeler, Towers of $\mathrm{GL}(r)$ -type of modular curves, *J. Reine Angew. Math.* (2017), <https://doi.org/10.1515/Crelle-2017-0012>.
- [11] Ernst-Ulrich Gekeler, On Drinfeld modular forms of higher rank, *J. Théor. Nombres Bordeaux* 29 (2017) 875–902.
- [12] Lothar Gerritzen, Marius van der Put, *Schottky Groups and Mumford Curves*, Lect. Notes Math., vol. 817, Springer-Verlag, 1980.
- [13] David Goss, *Basic Structures of Function Field Arithmetic*, *Ergebnisse der Mathematik*, vol. 35, Springer-Verlag, 1996.
- [14] Elhanan Motzkin, La décomposition d'un élément analytique en facteurs singuliers, *Ann. Inst. Fourier* 27 (1977) 67–82.
- [15] Jürgen Neukirch, *Algebraic Number Theory*, *Grundlehren der math. Wissenschaften*, vol. 322, Springer-Verlag, 1999.
- [16] Dinesh Thakur, *Function Field Arithmetic*, World Scientific Publishing Co. Inc., 2004.