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Journal of Number Theory 100 (2003) 326–331

JOURNAL OF Number Theory

http://www.elsevier.com/locate/jnt

On the parity of exponents in the standard factorization of $n!$ ☆

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Received 4 April 2002; revised 31 October 2002
Communicated by D. Goss

Abstract

Let $p_1, p_2, \ldots$ be the sequence of all primes in ascending order. The following result is proved: for any given positive integer k and any given $\varepsilon_i \in \{0, 1\}$ $(i = 1, 2, \ldots, k)$, there exist infinitely many positive integers n with

$$e_1(n!) \equiv \varepsilon_1 (\bmod 2), e_2(n!) \equiv \varepsilon_2 (\bmod 2), \ldots, e_k(n!) \equiv \varepsilon_k (\bmod 2),$$

where $e_i(n!)$ denotes the exponent of the prime p_i in the standard factorization of positive integer $n!$. In 1997 Berend proved a conjecture of Erdős and Graham, that is, the conclusion with all $\varepsilon_i = 0$.

MSC: 11N25; 11B50

Keywords: Erdős problems; Parity; Prime factorization; Factorials

1. Introduction

Let $p_1, p_2, \ldots$ be the sequence of all primes in ascending order. For a positive integer n, let $e_i(n)$ be the nonnegative integer with $p_i^{e_i(n)} \mid n$ and $p_i^{e_i(n)+1} \nmid n$. In 1997, Berend [1] proved a conjecture of Erdős and Graham (cf. [3, p. 77]) by showing that

☆ Supported by the National Natural Science Foundation of China, Grant No. 10171046 and the "333 Project" Foundation of Jiangsu Province of China. The result has been announced at the International Congress of Mathematicians, Beijing, 20–28 August 2002.

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0022-314X/03/$ - see front matter
doi:10.1016/S0022-314X(03)00013-1

for every positive integer k there exist infinitely many positive integers n with

$$e_1(n!) \equiv 0 (\mathrm{mod}\, 2), e_2(n!) \equiv 0 (\mathrm{mod}\, 2), \ldots, e_k(n!) \equiv 0 (\mathrm{mod}\, 2).$$

It is clear that $n = 1$ is a solution. The initial value $n = 1$ is very useful in Berend's proof. For any other pattern, we do not know if an initial value exists. An interesting generalization is (see [2]).

Problem. Given a positive integer k and $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$, does there exist some $n > 1$ with

$$e_i(n!) \equiv \varepsilon_i (\mathrm{mod}\, 2), \quad i = 1, 2, \ldots, k?$$

For $2 \leqslant k \leqslant 5$ Chen and Zhu has verified that every pattern of length k appears, and believes that for each positive integer k, every pattern of length k appears (see [2, p. 2]). Chen and Zhu [2] showed that if there is an initial value n, then the initial value is bounded by an explicit bound depending on k and there are infinitely many such n with the difference of adjacent n less than an explicit bound depending on k. Recently, Sander [4] posed the following conjecture:

Conjecture. *Let $q_1, q_2, \ldots, q_k$ be distinct primes, and let $\varepsilon_i \in \{0, 1\}$ $(i = 1, 2, \ldots, k)$. Then there are infinitely many positive integers n such that*

$$e'_{q_i}(n!) \equiv \varepsilon_i (\mathrm{mod}\, 2),$$

where $e'_{q_i}(n!)$ is the exponent of q_i in the standard factorization of $n!$.

The conjecture is equivalent to a similar conjecture with the assumption that $q_1, q_2, \ldots, q_k$ are the first k primes if we do not fix k. Sander [4] proved the conjecture for $k = 2$. In the present paper, we improve the method in Sander [4] and show that for any pattern there exists an initial value n. This implies that the answer to the above problem is affirmative and the above conjecture is true for all k.

Theorem 1. *For any given positive integer k and any $\varepsilon_i \in \{0, 1\}$ $(i = 1, 2, \ldots, k)$, there exist infinitely many positive integers n with*

$$e_1(n!) \equiv \varepsilon_1 (\mathrm{mod}\, 2), e_2(n!) \equiv \varepsilon_2 (\mathrm{mod}\, 2), \ldots, e_k(n!) \equiv \varepsilon_k (\mathrm{mod}\, 2).$$

2. Proof

Lemma 1 (Sander [4]). *Let n be a positive integer with p_i-adic expansion $n = n_s p_i^s + \cdots + n_1 p_i + n_0$, $0 \leqslant n_j < p_i$ $(j = 0, 1, \ldots, s)$. Then*

$$e_i(n!) \equiv \begin{cases} \sum_{j \geqslant 1} n_j \pmod 2 & \text{for } i = 1, \\ \sum_{2 \nmid j} n_j \pmod 2 & \text{for } i > 1. \end{cases}$$

Lemma 2. *If there exist* k *integers* $n_1, n_2, \dots, n_k$ *and* 2^k *integers* $m_1, m_2, \dots, m_{2^k}$ *with* $n_i + m_j > 0$ $(i = 1, 2, \dots, k;\ j = 1, 2, \dots, 2^k)$ *and*

$$(e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \dots, e_k((n_k + m_j)!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2, *then there exist infinitely many positive integers* n *for each of which all* $n + m_j > 0$ *and*

$$(e_1((n + m_j)!), e_2((n + m_j)!), \dots, e_k((n + m_j)!)), \quad j = 1, 2, \dots, 2^k$$

represent all parities modulo 2.

Proof. Assume that $2^{2t} > \max_{i,j} \{n_i + m_j\}$. By the Chinese Remainder Theorem, there exist infinitely many positive integers $n > \max_i n_i$ such that

$$n \equiv n_i (\operatorname{mod} p_i^{2t+1}), \quad i = 1, 2, \dots, k.$$

Noting that

$$n + m_j = \frac{n - n_i}{p_i^{2t}} p_i^{2t} + n_i + m_j, \quad 0 < n_i + m_j < p_i^{2t},$$

by Lemma 1, we have

$$e_i((n + m_j)!) \equiv e_i\left(\left(\frac{n - n_i}{p_i^{2t}}\right)!\right) + e((n_i + m_j)!) \pmod 2.$$

Thus, for $j = 1, 2, \dots, 2^k$, we have

$$\begin{aligned} &(e_1((n + m_j)!), e_2((n + m_j)!), \dots, e_k((n + m_j)!)) \\ &\quad \equiv \left(e_1\left(\left(\frac{n - n_1}{p_1^{2t}}\right)!\right), e_2\left(\left(\frac{n - n_2}{p_2^{2t}}\right)!\right), \dots, e_k\left(\left(\frac{n - n_k}{p_k^{2t}}\right)!\right)\right) \\ &\qquad + (e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \dots, e_k((n_k + m_j)!)) \pmod 2. \end{aligned}$$

Lemma 2 follows from the above congruent equality by the observation that for any given $(a_1, a_2, \dots, a_k) \in \mathbf{Z}^k$,

$$(a_1, a_2, \dots, a_k) + (x_1, x_2, \dots, x_k)$$

goes through all 2^k parities modulo 2 as $(x_1, x_2, \dots, x_k)$ goes through all 2^k parities modulo 2. This completes the proof of Lemma 2. $\square$

Theorem 2. *For any given $k \geqslant 1$, there exist 2^k positive integers $l_1, l_2, \ldots, l_{2^k}$ such that*

$$(e_1(l_j!), e_2(l_j!), \ldots, e_k(l_j!)), \quad j = 1, 2, \ldots, 2^k$$

represent all parities modulo 2.

Proof. We use induction on k. For $k = 1$, Theorem 2 is trivial by taking $l_1 = 1$ and $l_2 = 2$. Suppose that Theorem 2 is true for k. That is, there exist 2^k positive integers $l_1, l_2, \ldots, l_{2^k}$ such that

$$(e_1(l_j!), e_2(l_j!), \ldots, e_k(l_j!)), \quad j = 1, 2, \ldots, 2^k$$

represent all parities modulo 2. Let t be an even integer with

$$2^t > \max_j \ l_j,$$

and let

$$n_1 = 2^{t+1} + 2^t,$$

$$n_i = p_i^t \quad (2 \leqslant i \leqslant k), \quad n_{k+1} = p_{k+1}^{t+1},$$

$$m_j = \begin{cases} -1 - l_j & \text{for } 1 \leqslant j \leqslant 2^k, \\ l_{j-2^k} & \text{for } 2^k + 1 \leqslant j \leqslant 2^{k+1}. \end{cases}$$

Then l_j has the p_i-adic expansion

$$a_{jit} p_i^t + a_{ji(t-1)} p_i^{t-1} + \cdots + a_{ji1} p_i + a_{ji0}$$

with $a_{jit} = 0$ and $0 \leqslant a_{jiv} \leqslant p_i - 1$ for all j, i, v. For $1 \leqslant j \leqslant 2^k$, we have

$$n_1 + m_j = 2^{t+1} + (1 - a_{j1(t-1)}) 2^{t-1} + \cdots + (1 - a_{j11}) 2 + 1 - a_{j10},$$

$$n_1 + m_{2^k+j} = 2^{t+1} + 2^t + a_{j1(t-1)} 2^{t-1} + \cdots + a_{j11} 2 + a_{j10},$$

$$n_i + m_j = (p_i - 1 - a_{ji(t-1)}) p_i^{t-1} + \cdots + (p_i - 1 - a_{ji1}) p_i + p_i - 1 - a_{ji0},$$

$$n_i + m_{2^k+j} = p_i^t + a_{ji(t-1)} p_i^{t-1} + \cdots + a_{ji1} p_i + a_{ji0}, \quad 2 \leqslant i \leqslant k,$$

$$n_{k+1} + m_j = (p_{k+1} - 1 - a_{j(k+1)t})p_{k+1}^t + \cdots + (p_{k+1} - 1 - a_{j(k+1)1})p_{k+1} + p_{k+1} - 1 - a_{j(k+1)0},$$

$$n_{k+1} + m_{2^k+j} = p_{k+1}^{t+1} + a_{j(k+1)t}p_{k+1}^t + \cdots + a_{j(k+1)1}p_{k+1} + a_{j(k+1)0}.$$

By Lemma 1 and $2|t$, for $1 \leqslant i \leqslant k$, noting that $p - 1 - a \equiv a (\bmod 2)$ for prime $p > 2$, we have

$$e_i((n_i + m_j)!) \equiv e_i((l_j)!) \pmod 2,$$

$$e_i((n_i + m_{2^k+j})!) \equiv e_i((l_j)!) \pmod 2,$$

$$e_{k+1}((n_{k+1} + m_j)!) \equiv e_{k+1}((l_j)!) \pmod 2, \tag{1}$$

$$e_{k+1}((n_{k+1} + m_{2^k+j})!) \equiv 1 + e_{k+1}((l_j)!) \pmod 2. \tag{2}$$

Let $\varepsilon_i \in \{0, 1\} (i = 1, 2, \ldots, k+1)$. By the induction hypothesis we may take j, $1 \leqslant j \leqslant 2^k$, such that

$$e_i((l_j)!) \equiv \varepsilon_i \pmod 2, \quad i = 1, 2, \ldots, k.$$

For the j, by (1) and (2), we may choose $u = j$ or $2^k + j$ such that

$$e_{k+1}((n_{k+1} + m_u)!) \equiv \varepsilon_{k+1} \pmod 2.$$

Thus

$$e_i((n_i + m_u)!) \equiv \varepsilon_i \pmod 2, \quad i = 1, 2, \ldots, k+1.$$

This means that

$$(e_1((n_1 + m_j)!), e_2((n_2 + m_j)!), \ldots, e_k((n_k + m_j)!)), \quad j = 1, 2, \ldots, 2^{k+1}$$

represent all parities modulo 2. By Lemma 2 there exists a positive integer n with all $n + m_j > 0$ and

$$(e_1((n + m_j)!), e_2((n + m_j)!), \ldots, e_k((n + m_j)!)), \quad j = 1, 2, \ldots, 2^{k+1}$$

represent all parities modulo 2. This completes the proof of Theorem 2. $\square$

Theorem 1 follows from Theorem 2 and Lemma 2.

Acknowledgments

I am grateful to the referee for his/her kind comments.

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