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Journal of Number Theory 107 (2004) 287–297

JOURNAL OF
**Number
Theory**

<http://www.elsevier.com/locate/jnt>

On a theorem of Levinson

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Received 17 September 2003

Communicated by B. Conrey

Abstract

Levinson investigated the number of real zeros of the real or imaginary part of

$$\pi^{-\frac{\sigma}{2}-\frac{it}{2}}\Gamma\left(\frac{\sigma}{2}+\frac{it}{2}\right)\zeta(\sigma+it),$$

where $\sigma > 0$ and $\zeta(s)$ is the Riemann zeta function. By the functional equation,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

we may assume $\sigma > \frac{1}{2}$. In this paper, we consider

$$\pi^{-\frac{s+\lambda}{2}}\Gamma\left(\frac{s+\lambda}{2}\right)\zeta(s+\lambda) \pm \pi^{-\frac{s-\lambda}{2}}\Gamma\left(\frac{s-\lambda}{2}\right)\zeta(s-\lambda)$$

for any complex number s and any $\lambda > 0$, as general forms of the real or imaginary part of the above function, and then we further study the zeros of the functions.

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MSC: Primary 11M06

Keywords: Lindelöf hypothesis; Riemann hypothesis; Riemann zeta function

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1. Introduction

For the Riemann zeta function $\zeta(s)$, we let

$$F(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Denote by $\eta_\lambda(T)$ the number of zeros of $\operatorname{Re} F(\frac{1}{2} + \lambda + it)$ or the number of zeros of $\operatorname{Im} F(\frac{1}{2} + \lambda + it)$ with $0 < t \leq T$. Levinson [L] proved the following.

Theorem A (Levinson). *For any $\lambda > 0$,*

$$\eta_\lambda(T) \geq \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T + \lambda + T^{1-\frac{1}{4}\lambda} \log T).$$

In Levinson's demonstration [L] which used the argument principle, the number of zeros of the Riemann zeta function in $\operatorname{Re}(s) > \delta$ and $0 < \operatorname{Im}(s) \leq T$ for $\delta > \frac{1}{2}$ and $T > 0$, played a key role. Let $N_\lambda(T)$ denote the number of zeros of $\zeta(s)$ in $\operatorname{Re}(s) > \frac{1}{2} + \lambda$ and $0 < \operatorname{Im}(s) \leq T$. He in fact showed

$$\begin{aligned} \eta_\lambda(T) &\geq \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) - 2N_\lambda(T) \quad \left(0 < \lambda < \frac{3}{2}\right), \\ \eta_\lambda(T) &\geq \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\lambda) \quad \left(\lambda \geq \frac{3}{2}\right). \end{aligned}$$

In his time, Levinson used $N_\lambda(T) = O(T^{1-\frac{1}{4}\lambda} \log T)$. This bound has been improved (see [T, p. 253]).

We let

$$\Xi(z) = \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{s(s-1)}{2} F(s),$$

where $s = \frac{1}{2} + iz$. We note that

$$\begin{aligned} \operatorname{Re} F\left(\frac{1}{2} + \lambda + it\right) &= -\operatorname{Re} \frac{2\Xi(t - i\lambda)}{(t - i\lambda)^2 + \frac{1}{4}} = -\frac{\Xi(t - i\lambda)}{(t - i\lambda)^2 + \frac{1}{4}} - \frac{\Xi(t + i\lambda)}{(t + i\lambda)^2 + \frac{1}{4}}, \\ \operatorname{Im} F\left(\frac{1}{2} + \lambda + it\right) &= -\operatorname{Im} \frac{2\Xi(t - i\lambda)}{(t - i\lambda)^2 + \frac{1}{4}} = i \frac{\Xi(t - i\lambda)}{(t - i\lambda)^2 + \frac{1}{4}} - i \frac{\Xi(t + i\lambda)}{(t + i\lambda)^2 + \frac{1}{4}}. \end{aligned}$$

We set

$$F_\lambda(z) = \frac{\Xi(z - i\lambda)}{(z - i\lambda)^2 + \frac{1}{4}} + \frac{\Xi(z + i\lambda)}{(z + i\lambda)^2 + \frac{1}{4}} \quad \text{or} \quad \frac{\Xi(z - i\lambda)}{(z - i\lambda)^2 + \frac{1}{4}} - \frac{\Xi(z + i\lambda)}{(z + i\lambda)^2 + \frac{1}{4}}.$$

Then we have

$$F_\lambda(t) = -\operatorname{Re} F\left(\frac{1}{2} + \lambda + it\right) \quad \text{or} \quad -i \operatorname{Im} F\left(\frac{1}{2} + \lambda + it\right).$$

In this article, we further study the zeros of $F_\lambda(z)$; we supply a relationship between the zeros of $F_\lambda(z)$ and the zeros of $\zeta(s)$ ($s = \frac{1}{2} + iz$) in $\operatorname{Re}(s) > \frac{1}{2} + \lambda$. Namely we provide information on zeros of $F_\lambda(z)$ subject to the zeros of the Riemann zeta function in $\operatorname{Re}(s) > \frac{1}{2} + \lambda$.

In order to state our theorem we introduce some notation. Let $\lambda > 0$. In $\operatorname{Re}(s) > \frac{1}{2} + \lambda$ and $\operatorname{Im}(s) > 0$, denote by s_1, s_2, \dots the zeros of $\zeta(s)$ with $\operatorname{Im}(s_1) \leq \operatorname{Im}(s_2), \dots$, in case they exist. For each $\kappa > 0$ and $n = 1, 2, \dots$, we define $\mathcal{R}_\lambda(\kappa, n)$ as follows:

$$\begin{aligned} \mathcal{R}_\lambda(\kappa, n) = \{z: |\operatorname{Re}(z)| \geq |\operatorname{Im}(s_n)| \text{ and } |\operatorname{Re}(z) - \operatorname{Im}(s_k^*)| > \kappa, \\ s_k^* = s_k \text{ or } \bar{s}_k \text{ for } k = 1, 2, \dots\}. \end{aligned}$$

We see that

$$\mathcal{R}_\lambda(\kappa, n+1) \subseteq \mathcal{R}_\lambda(\kappa, n)$$

for $n = 1, 2, 3, \dots$. We set

$$\mathcal{R}_\lambda(\kappa) = \mathcal{R}_\lambda(\kappa, 1).$$

We note that for $T > 0$

$$N_\lambda(T) = \#\{n: \operatorname{Im}(s_n) \leq T\}.$$

Thus it is easy to see that

$$\limsup_{n \rightarrow \infty} \operatorname{Im}(s_{n+1} - s_n) = \infty$$

for we know that

$$N_\lambda(T) = O(T^{1-\frac{1}{4\lambda}} \log T).$$

In this paper, we shall show the following theorem which complements Levinson's theorem.

Theorem B. *Let $\lambda > 0$. There exists a constant $\kappa > 0$ such that all zeros of $F_\lambda(z)$ which are in $\mathcal{R}_\lambda(\kappa)$ are real. In particular if $\lambda \geq \frac{1}{2}$, then all zeros of $F_\lambda(z)$ are real.*

Because of the nature of the argument principle, Levinson [L] could check only the sign change of $\operatorname{Re} F(\frac{1}{2} + \lambda + it)$ (or $\operatorname{Im} F(\frac{1}{2} + \lambda + it)$) in a given interval. Thus we do not know the behavior of zeros of $F_\lambda(z)$, in terms of the location of zeros of $\zeta(s)$ in

$\operatorname{Re}(s) > \frac{1}{2} + \lambda$. Namely, how do the zeros of $\zeta(s)$ in $\operatorname{Re}(s) > \frac{1}{2} + \lambda$ force zeros of $F_\lambda(z)$ to be nonreal zeros? What are the intervals where $F_\lambda(z)$ has real zeros only? Theorem B provides information on zeros of $F_\lambda(z)$ according to the location of zeros of $\zeta(s)$ in $\operatorname{Re}(s) > \frac{1}{2} + \lambda$.

With the standard method [T, p. 212] one can show that for any $\lambda > 0$ the number of zeros of $F_\lambda(z)$ within $0 < \operatorname{Re}(z) \leq T$ is

$$\frac{T}{2\pi} \log \frac{\sqrt{4(\lambda+1)^2 + T^2}}{2\pi} - \frac{T}{2\pi} + \left(\frac{\lambda}{\pi} + \frac{1}{2\pi} \right) \tan^{-1} \frac{T}{2\lambda+2} + O\left(\frac{\log T}{\log(2\lambda+2)} \right).$$

For this, see Proposition 2.2. Thus it is easy to see that for any $0 < \lambda < \frac{1}{2}$ the number of zeros of $F_\lambda(z)$ within $T \leq \operatorname{Re}(z) \leq T + \kappa$ is $O(\log T)$. By aid of this fact, Theorem B immediately implies the following.

Corollary. *Let $0 < \lambda < \frac{1}{2}$. Then the number of real zeros of $F_\lambda(z)$ within $0 < \operatorname{Re}(z) \leq T$ is*

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(N_\lambda(T) \log T).$$

We remark that for $\lambda \geq \frac{1}{2}$, our result is stronger than Levinson's theorem, because $F_\lambda(z)$ has real zeros only and the explicit formula of the number of zeros of the function is given. Also the argument in the second part of Theorem B can be employed to prove that all zeros of $F_\lambda(z)$ are real for any $\lambda > 0$, provided that the Riemann hypothesis is valid.

Under the Lindelöf hypothesis, we could have a better result than Theorem B.

Theorem C. *Assume the Lindelöf hypothesis. For each $\kappa > 0$ there exists n such that all zeros of $F_\lambda(z)$ which are in $\mathcal{R}_\lambda(\kappa, n)$ are real.*

Our method in the paper was invoked by the author's paper [K].

In Section 2, we introduce some properties concerning the Riemann zeta function $\zeta(s)$ and $F_\lambda(z)$. In Section 3, we will prove Theorems B and C.

2. Basic facts

We need some standard facts on the Riemann zeta function $\zeta(s)$. Namely $\zeta(s)$ satisfies the following.

Proposition 2.1.

(1) $F(s) = F(1-s)$ and $\Xi(z)$ is an entire function of order 1.

(2) We have

$$\Xi(z) = \Xi(0) \prod_{\rho} \left(1 - \frac{z^2}{\rho^2}\right),$$

where ρ runs through all zeros of $\Xi(z)$ in $0 < \operatorname{Re}(\rho) < \frac{1}{2}$ and $\sum_{\rho} \frac{1}{|\rho|^2} < \infty$.

(3) There exists $H_0 > 0$ such that for $T > 0$,

$$\#\{z : \Xi(z) = 0, \operatorname{Im}(z) \geq 0 \text{ and } T \leq \operatorname{Re}(z) \leq T + 1\} \leq H_0 \log T.$$

(4) There exists $H_1 > 0$ such that for a sufficiently large T ,

$$H_1 \log T \leq \#\{z : \Xi(z) = 0, \operatorname{Im}(z) \geq 0, \text{ and } T \leq \operatorname{Re}(z) \leq T + 1\}.$$

(5) Let $\sigma \in \mathbb{R}$. Then there is $k \in \mathbb{R}$ such that $\zeta(s) = O(|s|^k)$ for $\operatorname{Re}(s) \geq \sigma$.

Proof. For (1), (3)–(5) see [T, Theorem 2.1, p. 214, Theorem 9.14, and p. 95]. For (2), see [T, pp. 29–30 and Chapter III]. \square

Proposition 2.2. Let $\lambda > 0$ and $T > 0$. Then the number of zeros of $F_{\lambda}(z)$ within $0 < \operatorname{Re}(z) \leq T$ is

$$\frac{T}{2\pi} \log \frac{\sqrt{4(\lambda+1)^2 + T^2}}{2\pi} - \frac{T}{2\pi} + \left(\frac{\lambda}{\pi} + \frac{1}{2\pi}\right) \tan^{-1} \frac{T}{2\lambda+2} + O\left(\frac{\log T}{\log(2\lambda+2)}\right).$$

Proof. We recall that

$$\begin{aligned} -F_{\lambda}(z) &= F(s+\lambda) + F(s-\lambda) \\ &= \pi^{-\frac{s+\lambda}{2}} \Gamma\left(\frac{s+\lambda}{2}\right) \zeta(s+\lambda) + \pi^{-\frac{s-\lambda}{2}} \Gamma\left(\frac{s-\lambda}{2}\right) \zeta(s-\lambda), \end{aligned}$$

where $s = \frac{1}{2} + iz$. By Proposition 2.1(1) we obtain

$$F(s+\lambda) + F(s-\lambda) = F(1-s+\lambda) + F(1-s-\lambda). \quad (2.1)$$

We set

$$H(s) = (s+\lambda)(s+\lambda-1)(s-\lambda)(s-\lambda-1)(F(s+\lambda) + F(s-\lambda)).$$

Then by (2.1) we get

$$H(s) = H(1-s).$$

Since $\Xi(z)$ is entire, so is $H(s)$. We adopt the same method as in [T, p. 212]. Namely we apply the method to the function $H(s)$. We choose the polygonal path $[\lambda + 2 - iT, \lambda + 2 + iT, -\lambda - 1 + iT, -\lambda - 1 - iT]$. As in [T, p. 212], we have

$$\pi \cdot \text{the number of zeros of } F_\lambda(z) \text{ within } 0 < \operatorname{Re}(z) \leq T = \Delta \arg H(s),$$

where Δ denotes the variation from $2 + \lambda$ to $\frac{1}{2} + iT$ via $2 + \lambda + iT$. For the proof of Proposition 2.2 it suffices to calculate

$$\Delta \arg \pi^{-(s+2)/2}, \quad \Delta \arg \Gamma\left(\frac{s+\lambda}{2}\right) \quad \text{and} \quad \Delta \arg \left(\zeta(s+\lambda) + \pi^\lambda \frac{\Gamma\left(\frac{s-\lambda}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}\right)} \zeta(s-\lambda) \right).$$

Clearly we have

$$\Delta \arg \pi^{-(s+2)/2} = -\frac{T}{2} \log \pi. \quad (2.2)$$

We recall the Stirling formula:

$$\log \Gamma(s) = \left(s - \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi - \int_0^\infty \frac{P(t)}{z+t} dt,$$

where $P(t) = t - [t] - \frac{1}{2}$, $[t]$ denoting the largest integer $\leq t$. See [S, p. 405]. Using the Stirling formula we get

$$\Delta \arg \Gamma\left(\frac{s+\lambda}{2}\right) = \frac{T}{2} \log \frac{\sqrt{4(\lambda+1)^2 + T^2}}{2} - \frac{T}{2} + \left(\lambda + \frac{1}{2}\right) \tan^{-1} \frac{T}{2\lambda+2} + O(1). \quad (2.3)$$

We put

$$h(s) = \zeta(s+\lambda) + \pi^\lambda \frac{\Gamma\left(\frac{s-\lambda}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}\right)} \zeta(s-\lambda).$$

One can see that using Proposition 2.1(5) and the Stirling formula there exist $\sigma_0 > 1$ and $\alpha \in \mathbb{R}$ (independent on λ) such that

$$h(s) = O(|s|^\alpha) \quad \text{and} \quad \operatorname{Re}(h(s)) > \frac{1}{2} \quad (2.4)$$

for $\operatorname{Re}(s) > \lambda + \sigma_0$. Under condition (2.4) we use the method as in [T, p. 213] and then we get

$$\Delta \arg h(s) = O\left(\frac{\log T}{\log(2\lambda + 2)}\right). \quad (2.5)$$

From (2.2), (2.3), and (2.5) Proposition 2.2 follows. \square

3. Proof of Theorems B and C

We fix $\lambda > 0$. We may deal with the function

$$F_\lambda(z) = \frac{\Xi(z - i\lambda)}{(z - i\lambda)^2 + \frac{1}{4}} + \frac{\Xi(z + i\lambda)}{(z + i\lambda)^2 + \frac{1}{4}},$$

because the proof of Theorem B for this function works for the other too.

3.1. The region of real zeros of $F_\lambda(z)$

In this section, we will prove the first part of the Theorems B and C. We adopt the idea from the author's paper [K] for the proof of this part.

We recall Proposition 2.1(2). We calculate

$$\left| \frac{\Xi(\bar{z} - i\lambda)(z - i\lambda)^2 + \frac{1}{4}}{\Xi(z - i\lambda)(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right| = \left| \frac{(z - i\lambda)^2 + \frac{1}{4}}{(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right| \prod_\rho \left| \frac{\rho^2 - (\bar{z} - i\lambda)^2}{\rho^2 - (z - i\lambda)^2} \right| \quad (3.1)$$

for $\operatorname{Im}(z) < 0$. We observe that for $-\lambda \leq \operatorname{Im}(\rho) < 0$,

$$\left| \frac{\rho^2 - (\bar{z} - i\lambda)^2}{\rho^2 - (z - i\lambda)^2} \right| \leq 1.$$

Thus we get

$$\left| \frac{\Xi(\bar{z} - i\lambda)}{\Xi(z - i\lambda)} \right| \leq \prod_{\operatorname{Im}(\rho) < -\lambda \text{ or } \operatorname{Im}(\rho) \geq 0} \left| \frac{\rho^2 - (\bar{z} - i\lambda)^2}{\rho^2 - (z - i\lambda)^2} \right|. \quad (3.2)$$

We recall s_n 's. For each $n = 1, 2, 3, \dots$, we let $s_n = \frac{1}{2} + i(x_n - y_n i)$, where x_n, y_n 's are real. Then $y_n > \lambda$. Then we may write

$$\prod_{\operatorname{Im}(\rho) < -\lambda \text{ or } \operatorname{Im}(\rho) \geq 0} \frac{\rho^2 - (\bar{z} - i\lambda)^2}{\rho^2 - (z - i\lambda)^2} = \prod_{n=1}^{\infty} \frac{(x_n - iy_n)^2 - (\bar{z} - i\lambda)^2}{(x_n - iy_n)^2 - (z - i\lambda)^2} \prod_{n=1}^{\infty} \frac{(a_n + ib_n)^2 - (\bar{z} - i\lambda)^2}{(a_n + ib_n)^2 - (z - i\lambda)^2}, \quad (3.3)$$

where $a_n > 0$ and $b_n \geq 0$ for each n . By Proposition 2.1(3) and (4), we have

$$\#\{n : T \leq x_n \leq T + 1\} \leq H_0 \log T \quad (\text{for } T > 0) \quad (3.4)$$

and

$$\#\{n : T \leq a_n \leq T + 1\} \geq H_1 \log T \quad (T \rightarrow \infty). \quad (3.5)$$

Let $\kappa > 0$ and n be a positive integer. Let $z = x - iy \in \mathcal{R}_\lambda(\kappa, n)$ with $x > 1$ and $y > 0$. By (2.4), $F_\lambda(z)$ has no zeros in $|\operatorname{Im}(z)| > 1 + \lambda + \sigma_0$. So, we may assume $y \leq 1 + \lambda + \sigma_0$ for investigating the zero free region of $F_\lambda(z)$. Set $\sigma_1 = 2 + 2\lambda + \sigma_0$. By (3.5) we see that

$$\begin{aligned} \log \prod_{n=1}^{\infty} \left| \frac{(x_n - iy_n)^2 - (\bar{z} - i\lambda)^2}{(x_n - iy_n)^2 - (z - i\lambda)^2} \right|^2 &\leq \log \prod_{n=1}^{\infty} \left(1 + \frac{4y(y_n - \lambda)}{(x_n - x)^2 + (y_n - y - \lambda)^2} \right) \\ &\leq 4y \sum_{n=1}^{\infty} \frac{y_n - \lambda}{(x_n - x)^2 + (y_n - y - \lambda)^2} \\ &\leq 2y \sum_{x_n < x - \kappa \text{ or } x_n > x + \kappa} \frac{1}{(x_n - x)^2} \\ &\leq 4y \left(H_0 \sum_{n=0}^{\infty} \frac{\log(\kappa + n + x)}{(\kappa + n)^2} + O(1) \right) \\ &= 4y H_0 \left(\frac{\log(\kappa + x)}{\kappa^2} + \sum_{n=1}^{\infty} \frac{\log x}{(\kappa + n)^2} + O(1) \right) \end{aligned} \quad (3.6)$$

for a sufficiently large x . By (3.4) we also see that

$$\begin{aligned} \log \prod_{n=1}^{\infty} \left| \frac{(a_n + ib_n)^2 - (\bar{z} - i\lambda)^2}{(a_n + ib_n)^2 - (z - i\lambda)^2} \right|^2 &\leq \log \prod_{n=1}^{\infty} \left(1 + \frac{4y(b_n - \lambda)}{(a_n + x)^2 + (b_n - y - \lambda)^2} \right) \\ &\quad + \log \prod_{n=1}^{\infty} \left(1 - \frac{4y(b_n + \lambda)}{(a_n - x)^2 + (b_n + y + \lambda)^2} \right) \\ &\leq 4y \sum_{n=1}^{\infty} \frac{1}{(a_n + x)^2} - \frac{\lambda}{(a_n - x)^2 + \sigma_1^2} \\ &\leq 4y \left(o(1) - \sum_{x \leq a_n \leq x+1} \frac{\lambda}{(a_n - x)^2 + \sigma_1^2} \right) \\ &\leq 4y \left(o(1) - \frac{\lambda H_1 \log x}{1 + \sigma_1^2} \right), \end{aligned} \quad (3.7)$$

where $o(1)$ goes to 0 as $x \rightarrow \infty$. It is easy to see that

$$\log \left| \frac{(z - i\lambda)^2 + \frac{1}{4}}{(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right|^2 \leq \frac{4y(2\lambda + 1)}{x^2}. \quad (3.8)$$

From (3.1)–(3.3) and (3.6)–(3.8) we obtain

$$\begin{aligned} & \log \left| \frac{\Xi(\bar{z} - i\lambda)(z - i\lambda)^2 + \frac{1}{4}}{\Xi(z - i\lambda)(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right|^2 \\ & \leq 4y \left(H_0 \frac{\log(\kappa + x)}{\kappa^2} + H_0 \sum_{n=1}^{\infty} \frac{\log x}{(\kappa + n)^2} + O(1) - \frac{\lambda H_1 \log x}{1 + \sigma_1^2} \right). \end{aligned} \quad (3.9)$$

We choose $\kappa_0 > 0$ satisfying

$$H_0 \left(\frac{1}{\kappa_0} + \frac{1}{\kappa_0^2} \right) - \frac{\lambda H_1}{1 + \sigma_1} < 0.$$

For this κ_0 , we set

$$\delta = \frac{\lambda H_1}{1 + \sigma_1} - H_0 \left(\frac{1}{\kappa_0} + \frac{1}{\kappa_0^2} \right).$$

Then we can see from (3.9) that for the constant $\delta > 0$ we have

$$\left| \frac{\Xi(\bar{z} - i\lambda)(z - i\lambda)^2 + \frac{1}{4}}{\Xi(z - i\lambda)(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right| < e^{-4y\delta(1+o(1))\log x}$$

for $0 < y \leq 1 + \lambda + \sigma_0$ and $x - iy \in \mathcal{R}_\lambda(\kappa_0)$. Namely, for a sufficiently large n , we have

$$\begin{aligned} |F_\lambda(z)| &= \left| \frac{\Xi(z - i\lambda)}{(z - i\lambda)^2 + \frac{1}{4}} + \frac{\Xi(z + i\lambda)}{(z + i\lambda)^2 + \frac{1}{4}} \right| \\ &\geq \frac{|\Xi(z - i\lambda)|}{|(z - i\lambda)^2 + \frac{1}{4}|(z + i\lambda)^2 + \frac{1}{4}|} \left(1 - \left| \frac{\Xi(\bar{z} - i\lambda)(z - i\lambda)^2 + \frac{1}{4}}{\Xi(z - i\lambda)(\bar{z} - i\lambda)^2 + \frac{1}{4}} \right| \right) > 0 \end{aligned}$$

in $0 < y \leq 1 + \lambda + \sigma_0$ and $x - iy \in \mathcal{R}_\lambda(\kappa_0, n)$. We immediately conclude that $F_\lambda(z) \neq 0$ in the same region. Hence there exists $\kappa > 0$ ($\kappa \geq \kappa_0$) such that in $0 < y \leq 1 + \lambda + \sigma_0$ and $x - iy \in \mathcal{R}_\lambda(\kappa)$, $F_\lambda(z) \neq 0$. Thus this fact implies that $F_\lambda(z) \neq 0$ in $0 < y$ and $x - iy \in \mathcal{R}_\lambda(\kappa)$ for $F_\lambda(z)$ has no zeros in $|\operatorname{Im}(z)| > 1 + \lambda + \sigma_0$. Using $F_\lambda(z) = F_\lambda(-z)$ and $F_\lambda(z) = \overline{F_\lambda(\bar{z})}$, we show the first part of Theorem B.

We recall that a necessary and sufficient condition for the truth of the Lindelöf hypothesis is that for any $\lambda > 0$, the number of zeros of the zeta function $\zeta(s)$ within $T \leq \text{Im}(s) \leq T + 1$ and $\text{Re}(s) > \frac{1}{2} + \lambda$ is $o(\log T)$ as $T \rightarrow \infty$. For this see [T, Theorem 13.5]. Thus under the Lindelöf hypothesis, we can see that in (3.9) H_0 can be arbitrarily small as $x \rightarrow \infty$. Thus we can prove Theorem C as in the proof of the first part of Theorem B.

3.2. For $\lambda \geq \frac{1}{2}$, all zeros of $F_\lambda(z)$ are real

In this section, we shall prove the second part of Theorem B. Let $\lambda \geq \frac{1}{2}$. We recall

$$F_\lambda(z) = \frac{\Xi(z - i\lambda)}{(z - i\lambda)^2 + \frac{1}{4}} + \frac{\Xi(z + i\lambda)}{(z + i\lambda)^2 + \frac{1}{4}}.$$

By Proposition 2.1(2) and $\Xi(\bar{z}) = \overline{\Xi(z)}$ we may write

$$\Xi(z) = \Xi(0) \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{(\alpha_k + i\beta_k)^2} \right) \left(1 - \frac{z^2}{(\alpha_k - i\beta_k)^2} \right),$$

where $\alpha_k > 0$ and $\beta_k \geq 0$ for each $k = 1, 2, 3, \dots$. We set

$$W_\lambda(z) = \left((z + i\lambda)^2 + \frac{1}{4} \right) \Xi(z - i\lambda)$$

and for each $n = 1, 2, 3, \dots$

$$W_{\lambda,n}(z) = \Xi(0) \left((z + i\lambda)^2 + \frac{1}{4} \right) \prod_{k=1}^n \left(1 - \frac{(z - i\lambda)^2}{(\alpha_k + i\beta_k)^2} \right) \left(1 - \frac{(z - i\lambda)^2}{(\alpha_k - i\beta_k)^2} \right).$$

Then $W_{\lambda,n}(z)$ converges uniformly to $W_\lambda(z)$ on compact regions of the complex plane. We can write

$$W_{\lambda,n}(z) = u_n(z) + iv_n(z) \quad \text{and} \quad W_\lambda(z) = u(z) + iv(z),$$

where $u_n(z)$, $v_n(z)$, $u(z)$, and $v(z)$ are reals for any real z . Clearly we have

$$u_n(z) = \frac{W_{\lambda,n}(z) + \overline{W_{\lambda,n}(\bar{z})}}{2} \quad \text{and} \quad u(z) = \frac{W_\lambda(z) + \overline{W_\lambda(\bar{z})}}{2}.$$

Then it is easy to see that

$$u(z) = \left((z + i\lambda)^2 + \frac{1}{4} \right) \left((z - i\lambda)^2 + \frac{1}{4} \right) F_\lambda(z)$$

holds. Also $u_n(z)$ converges uniformly to $u(z)$ on compact regions of the complex plane, since $W_{\lambda,n}(z)$ converges uniformly to $W_\lambda(z)$ on compact regions of the

complex plane. Hence the real part of $W_{\lambda,n}(z)$ converges uniformly to

$$\left((z + i\lambda)^2 + \frac{1}{4}\right) \left((z - i\lambda)^2 + \frac{1}{4}\right) F_{\lambda}(z).$$

on compact regions of the complex plane

Lemma. *If $U(z)$ and $V(z)$ are real polynomials such that $W(z) = U(z) + iV(z)$ has n roots in the lower half-plane, then $U(z)$ has n pairs of conjugate complex roots at most.*

Proof. See [B, p. 215]. \square

We use the lemma to prove the second part of Theorem B. By Proposition 2.1(2) it is easy to see that $W_{\lambda,n}(z)$ has two zeros (one zero if $\lambda = \frac{1}{2}$) in the lower half-plane. Thus by the lemma the real part polynomial has 2 pairs (one pair if $\lambda = \frac{1}{2}$) of conjugate complex zeros at most. Namely, the real part polynomial has real zeros only except for four possible complex zeros (two possible complex zeros if $\lambda = \frac{1}{2}$). Thus,

$$\left((z + i\lambda)^2 + \frac{1}{4}\right) \left((z - i\lambda)^2 + \frac{1}{4}\right) F_{\lambda}(z)$$

has real zeros only except for four possible complex zeros (two possible complex zeros if $\lambda = \frac{1}{2}$) because the real part polynomial of $W_{\lambda,n}(z)$ converges uniformly to the function on compact regions of the complex plane. We note that

$$\left((z + i\lambda)^2 + \frac{1}{4}\right) \left((z - i\lambda)^2 + \frac{1}{4}\right)$$

has four complex zeros (two complex zeros if $\lambda = \frac{1}{2}$). Thus $F_{\lambda}(z)$ has real zeros only. Hence, we complete the proof of the second part of Theorem B.

Acknowledgments

I thank Professor Cem Yildirim for some valuable comments on the paper. Also I am indebted to the referee for his helpful suggestions.

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