Journal of Number Theory 152 (2015) 156–181

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Determination of elliptic curves by their adjoint p-adic L-functions

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A R T I C L E I N F O

Article history:
Received 10 June 2014
Received in revised form 20 October 2014
Accepted 2 December 2014
Available online 4 February 2015
Communicated by Ph. Michel

Keywords:
L-function
p-adic L-function
Elliptic curve

A B S T R A C T

Fix p an odd prime. Let E be an elliptic curve over $\mathbb{Q}$ with semistable reduction at p. We show that the adjoint p-adic L-function of E evaluated at infinitely many integers prime to p completely determines up to a quadratic twist the isogeny class of E. To do this, we prove a result on the determination of isobaric representations of $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ by certain L-values of p-power twists.

1. Introduction

In this paper we will prove the following result concerning the p-adic L-function of the symmetric square of an elliptic curve over $\mathbb{Q}$, denoted $L_p(Sym^2 E, s)$ for $s \in \mathbb{Z}_p$. More specifically, Theorem 1 gives a generalization of the result obtained in [14] concerning p-adic L-functions of elliptic curves over $\mathbb{Q}$:

Theorem 1. *Let p be an odd prime and E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p. Suppose*

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http://dx.doi.org/10.1016/j.jnt.2014.12.012
0022-314X/

$$L_p(Sym^2\, E, n) = C L_p(Sym^2\, E', n) \tag{1.1}$$

for all integers n prime to p in an infinite set Y and some constant $C \in \overline{\mathbb{Q}}$. Then E' is isogenous to a quadratic twist E_D of E. If E and E' have square free conductors, then E and E' are isogenous over $\mathbb{Q}$.

Suppose E has good reduction at p. We follow the definition in [5] of the p-adic L-function for the symmetric square of an elliptic curve E over $\mathbb{Q}$ which is defined as the Mazur–Mellin transform of a p-adic measure $\mu_p := \mu_p(E)$. If $\chi : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ is a non-trivial wild p-adic character of conductor p^{m_χ}, which can be identified with a primitive Dirichlet character, then

$$L_p(Sym^2\, E, \chi) = \int\limits_{\mathbb{Z}_p^\times} \chi \mathrm{d}\mu_p = C_E \cdot \alpha_p^{-2m_\chi} \tau(\overline{\chi})^2 p^{m_\chi} L(Sym^2\, E, \chi, 2) \tag{1.2}$$

where C_E is a constant that depends on E, $\tau(\chi)$ is the Gauss sum of χ and α_p is a root of the polynomial $X^2 - a_p X + p$, with $a_p = p + 1 - \#E(\mathbb{F}_p)$. It is proved in [5] that if E has good ordinary reduction at p then $\mu_p(E)$ is a bounded measure on $\mathbb{Z}_p^\times$, while if E has good supersingular reduction at p then $\mu_p(E)$ is h-admissible with $h = 2$ (cf. [23]).

Similarly, if E has bad multiplicative reduction at p, then for a non-trivial even Dirichlet character as above we have

$$L_p(Sym^2\, E, \chi) = \int\limits_{\mathbb{Z}_p^\times} \chi \mathrm{d}\mu_p = C'_E \tau(\overline{\chi})^2 p^{m_\chi} L(Sym^2\, E, \chi, 2), \tag{1.3}$$

with $\mu_p(E)$ a bounded measure on $\mathbb{Z}_p^\times$.

Set

$$L_p(Sym^2\, E, \chi, s) := L_p(Sym^2\, E, \chi \cdot \langle x \rangle^s)$$

where $\langle \cdot \rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$, with $\langle x \rangle = \frac{x}{\omega(x)}$ and $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the Teichmüller character.

Using the theory of h-admissible measures developed in [23], by Lemma 4 in Section 6 identity (1.1) implies that

$$L_p(Sym^2\, E, \chi, s) = C L_p(Sym^2\, E', \chi, s)$$

holds for all $s \in \mathbb{Z}_p$ and χ a wild p-adic character.

Let f, f' be the newforms of weight 2 associated to E and E', and π, π' the unitary cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ generated by f and f' respectively. Then

$$L(Sym^2\, E, s) = L(Sym^2\, \pi, s - 1) \tag{1.4}$$

where $Sym^2\,\pi$ is the automorphic representation of $\mathrm{GL}(3,\mathbb{A}_\mathbb{Q})$ associated to π by Gelbart and Jacquet in [6]. $Sym^2\,\pi$ is cuspidal only if E is non-CM, otherwise it is an isobaric sum of unitary cuspidal automorphic representations. Note that the critical strip for $L(Sym^2\,E,s)$ is $1 < \mathrm{Re}(s) < 2$, with the center at $s = 3/2$. This corresponds to the critical strip $0 < \mathrm{Re}(s) < 1$ of $L(Sym^2\,\pi, s)$, with the center at $s = 1/2$.

Theorem 1 is then a consequence of the following result on the determination of isobaric automorphic representations of GL(3) over $\mathbb{Q}$:

Theorem 2. *Suppose π and π' are two isobaric sums of unitary cuspidal automorphic representations of $GL(3,\mathbb{A}_\mathbb{Q})$ with the same central character ω. Let $X^w_{(p)}$ be the set of p-power order characters of conductor p^a for some a. Suppose $L(\pi\otimes\chi,s)$ and $L(\pi'\otimes\chi,s)$ are entire for all $\chi\in X^w_{(p)}$, and that there exist constants $B,C\in\mathbb{C}$ such that*

$$L(\pi\otimes\chi,\beta) = B^aCL(\pi'\otimes\chi,\beta) \tag{1.5}$$

for some $1\geq\beta>\frac{2}{3}$ and for all $\chi\in X^w_{(p),a}$ primitive p-power order characters of conductor p^a for all but a finite number of a. Then $\pi\cong\pi'$. Moreover, if π and π' are isobaric sums of tempered unitary cuspidal automorphic representations then the same result holds if (1.5) is satisfied for some $1\geq\beta>\frac{1}{2}$ (if the generalized Ramanujan conjecture is true this condition is automatically satisfied).

Note that in [17] a result was proved concerning the determination of GL(3) forms by twists of characters of almost prime modulus of the central L-values. In our case, we twist over a more sparse set of characters.

Using Theorem 4.1.2 in [18], the following is a consequence of Theorem 2:

Theorem 3. *Suppose π and π' are two unitary cuspidal automorphic representations of $GL(2,\mathbb{A}_\mathbb{Q})$ with the same central character ω. Suppose there exist constants $B,C\in\mathbb{C}$ such that*

$$L(Ad(\pi)\otimes\chi,\beta) = B^aCL(Ad(\pi')\otimes\chi,\beta) \tag{1.6}$$

for some $1\geq\beta>\frac{2}{3}$ and for all $\chi\in X^w_{(p),a}$ primitive p-power order characters of conductor p^a for all but a finite number of a. Then there exists a quadratic character ν such that $\pi\cong\pi'\otimes\nu$. If π and π' are tempered then the same result holds if (1.6) is true for some $1\geq\beta>\frac{1}{2}$.

To prove Theorem 2, we will show the following more general result on isobaric sums of unitary cuspidal automorphic representations of $\mathrm{GL}(n,\mathbb{A}_\mathbb{Q})$ for $n\geq 3$:

Theorem 4. *Let π be an isobaric sum of unitary cuspidal automorphic representations of $GL(n,\mathbb{A}_\mathbb{Q})$ with $n\geq 3$ and s, r be integers relatively prime to p. If $L(\pi\otimes\chi,s)$ and*

$L(\pi' \otimes \chi, s)$ are entire for all χ p-power order characters of conductor p^a for some a, then

$$\lim_{a\to\infty} p^{-a} \sum\nolimits^{*}_{\chi \ mod \ p^a} \overline{\chi}(s)\chi(r)L(\pi\otimes\chi,\beta) = \frac{1}{p}\left(1-\frac{1}{p}\right)\frac{a_\pi(s/r)}{(s/r)^\beta} \tag{1.7}$$

where $\sum^*$ denotes the sum over primitive p-power order characters of conductor p^a and $1 \geq \beta > \frac{n-1}{n+1}$ if π is an isobaric sum of tempered unitary cuspidal automorphic representations and $1 \geq \beta > \frac{n-1}{n}$ in general. Here the elements $a_\pi(s/r)$ represent the coefficients of the Dirichlet series that defines $L(\pi, s)$ in the right half-plane $Re(s) > 1$, with $a_\pi(1) = 1$ and $a_\pi(s/r) := 0$ if $r \nmid s$.

This result generalizes Proposition 2.2 in [14]. Theorem 4, together with the Generalized Strong Multiplicity One Theorem (see Section 2) can be used to prove Theorem 2.

Even though the identity in Theorem 4 holds for isobaric sums of unitary cuspidal automorphic representation of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$, we cannot generalize the result of Theorem 2 to $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$, because knowing the coefficients $a_\pi(n)$ will no longer be enough to determine the two representations if $n > 3$. Moreover, if π is an arbitrary cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$ with F a number field, it is known that $Sym^m\,\pi$ is automorphic only when $m \leq 4$ (see [10,11]). If $F = \mathbb{Q}$ and π is holomorphic of weight 2, these results have been extended to small $m \geq 5$ by recent work of Clozel and Thorne, and of Dieulefait. The author is not aware of any constructions of p-adic L-functions for $Sym^m\,E$ with $m \geq 3$. However, in [1] a p-adic L-function was constructed for certain automorphic representations π of $\mathrm{GL}(2n, \mathbb{A}_\mathbb{Q})$ under certain conditions, such as the non-vanishing of the twisted complex L-function $L(\pi \otimes \chi, 1/2)$ by some Hecke character χ trivial at infinity.

As a consequence of Theorem 4, the following non-vanishing result holds:

Corollary 1. *Let π be an isobaric sum of unitary cuspidal automorphic representations of $GL(n, \mathbb{A}_\mathbb{Q})$ with $n \geq 3$. There are infinitely many primitive p-power order characters χ of conductor p^a for some a, such that if $L(\pi\otimes\chi, s)$ is entire for all such characters then $L(\pi \otimes \chi, \beta) \neq 0$ for all $\beta \notin \left[\frac{2}{n+1}, 1 - \frac{2}{n+1}\right]$ if π is an isobaric sum of tempered unitary cuspidal automorphic representations and for $\beta \notin \left[\frac{1}{n}, 1-\frac{1}{n}\right]$ in general.*

A similar nonvanishing result involving p-power twists of cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ was proved in [24] for $\beta \notin \left[\frac{2}{n+1}, 1 - \frac{2}{2n+1}\right]$. In [2] a nonvanishing result for β in the same intervals as in Corollary 1 was proved for all twists of L-functions of $\mathrm{GL}(n)$, instead of just for p-power twists. In [13], the result in [2] was further improved to the interval $\beta \notin \left[\frac{2}{n}, 1-\frac{2}{n}\right]$. Note that the set of primitive characters of p-power order and conductor p^a for some a is more sparse than the set of characters considered in [2] and [13].

We should also note that for $n = 2$ Rohrlich [19] proves that if f is a newform of weight 2, then for all but finitely many twists by Dirichlet characters the L-function is nonvanishing at $s = 1$.

We now present an outline of the rest of the paper. In Section 2 we give an overview of the basic properties of the standard L-function associated to an isobaric representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$. In Section 3 we prove a simple lemma involving Gauss sums. In Section 4 we present a proof of Theorem 4, while in Section 5 we provide proofs of Theorem 2 and Theorem 3. In Section 6 we recall the basic properties of the symmetric square p-adic L-function of an elliptic curve and prove a lemma that will be crucial for the proof of Theorem 1. In Section 7 we provide a proof for Theorem 1.

2. Preliminaries

2.1. The standard L-function of GL(n)

Let π be an irreducible automorphic representation of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ and $L(\pi, s)$ its associated standard L-function. Write $\pi = \otimes_v' \pi_v$ as a restricted direct product with π_v admissible irreducible representations of the local groups $\mathrm{GL}(n, \mathbb{Q}_v)$. The Euler product

$$L(\pi, s) = \prod_v L(\pi_v, s) \tag{2.1}$$

converges for $\mathrm{Re}(s)$ large. There exist conjugacy classes of matrices $A_v(\pi) \in \mathrm{GL}(n, \mathbb{C})$ such that the local L-functions at finite places v with π_v unramified are

$$L(\pi_v, s) = \det(1 - A_v(\pi) v^{-s})^{-1} \tag{2.2}$$

We can take $A_v(\pi) = [\alpha_{1,v}(\pi), \cdots, \alpha_{n,v}(\pi)]$ to be diagonal representatives of the conjugacy classes.

For S a set of places of $\mathbb{Q}$ we can define

$$L^S(\pi, s) = \prod_{v \notin S} L_v(\pi, s) \tag{2.3}$$

called the incomplete L-function associated to set S.

Let $\boxplus$ be the isobaric sum introduced in [9]. We can define an irreducible automorphic representation, called an isobaric representation, $\pi_1 \boxplus \cdots \boxplus \pi_m$ of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, $n = \sum_{i=1}^m n_i$, for m cuspidal automorphic representations $\pi_i \in \mathrm{GL}(n_i, \mathbb{A}_{\mathbb{Q}})$. Such a representation satisfies

$$L^S(\boxplus_{j=1}^m \pi_j, s) = \prod_{j=1}^m L^S(\pi_j, s)$$

with S a finite set of places.

We say that an isobaric representation is tempered if each π_i in the isobaric sum $\pi = \pi_1 \boxplus \cdots \boxplus \pi_m$ is a tempered cuspidal automorphic representation, or more specifically if each local factor $\pi_{i,v}$ is tempered.

Since we will want bound (2.6) on the coefficients of the Dirichlet series (2.5) to hold, we will consider a subset of the set of isobaric representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$, more specifically those given by an isobaric sum of unitary cuspidal automorphic representations. We denote this subset by $\mathcal{A}_u(n)$. We will also consider the case when the unitary cuspidal automorphic representations in the isobaric sum are tempered, which is expected to always hold given the generalized Ramanujan conjecture.

The following generalization of the Strong Multiplicity One Theorem for isobaric representations is due to Jacquet and Shalika [9]:

Theorem *(Generalized Strong Multiplicity One). Consider two isobaric representations π_1 and π_2 of $GL(n, \mathbb{A}_{\mathbb{Q}})$ and S a finite set of places of $\mathbb{Q}$ that contains ∞, such that π_1 and π_2 are unramified outside set S. Then $\pi_{1,v} \cong \pi_{2,v}$ for all $v \notin S$ implies $\pi_1 \cong \pi_2$.*

Let $n \geq 3$ and let $\pi \in \mathcal{A}_u(n)$ be an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_{\mathbb{Q}})$ with (unitary) central character ω_π and contragredient representation $\tilde{\pi}$. We have

$$L(\pi_\infty, s) = \prod_{j=1}^{n} \pi^{-\frac{s-\mu_j}{2}} \Gamma\left(\frac{s-\mu_j}{2}\right), \quad L(\tilde{\pi}_\infty, s) = \prod_{j=1}^{n} \pi^{-\frac{s-\overline{\mu_j}}{2}} \Gamma\left(\frac{s-\overline{\mu_j}}{2}\right) \tag{2.4}$$

for some $\mu_j \in \mathbb{C}$, with π in this context denoting the transcendental number.

The L-function is defined for $\mathrm{Re}(s) > 1$ by the absolutely convergent Dirichlet series

$$L(\pi, s) = \sum_{m=1}^{\infty} \frac{a_\pi(m)}{m^s} \tag{2.5}$$

with $a_\pi(1) = 1$. This extends to a meromorphic function on $\mathbb{C}$ with a finite number of poles.

It is known that the coefficients $a_\pi(m)$ of the Dirichlet series satisfy

$$\sum_{m \leq M} |a_\pi(m)|^2 \ll_\epsilon M^{1+\epsilon} \tag{2.6}$$

for $M \geq 1$ (cf. Theorem 4 in [16], see also [8,9,20,21]). For this property to hold, it is necessary that π be an isobaric sum of unitary cuspidal automorphic representations, rather than any unitary isobaric representation.

If π is in fact an isobaric sum of tempered cuspidal automorphic representations, then we have that the coefficients $a_\pi(m)$ satisfy

$$|a_\pi(m)| \ll_\epsilon m^\epsilon.$$

The completed L-function $\Lambda(\pi, s) = L(\pi_\infty, s)L(\pi, s)$ obeys the functional equation

$$\Lambda(\pi, s) = \epsilon(\pi, s)\Lambda(\tilde{\pi}, 1 - s) \tag{2.7}$$

where the ϵ-factor is given by

$$\epsilon(\pi, s) = f_\pi^{1/2-s} W(\pi) \tag{2.8}$$

and f_π and $W(\pi)$ are the conductor and the root number of π.

Let χ denote an even primitive Dirichlet character that is unramified at ∞ and with odd conductor q coprime to f_π. The twisted L-function obeys the functional equation (see for example [8])

$$\Lambda(\pi \otimes \chi, s) = \epsilon(\pi \otimes \chi, s)\Lambda(\tilde{\pi} \otimes \overline{\chi}, 1 - s) \tag{2.9}$$

where $\Lambda(\pi \otimes \chi, s) = L(\pi_\infty, s)L(\pi \otimes \chi, s)$. The ϵ-factor is given by

$$\epsilon(\pi \otimes \chi, s) = \epsilon(\pi, s)\omega_\pi(q)\chi(f_\pi)q^{-ns}\tau(\chi)^n \tag{2.10}$$

with $\tau(\chi)$ the Gauss sum of the character χ (cf. Proposition 4.1 in [2]).

Since $L(\pi \otimes \chi, s)$ does not vanish in the half-plane $\mathrm{Re}(s) > 1$, it is enough to consider $1/2 \leq \mathrm{Re}(s) \leq 1$. Twisting π by a unitary character $|\cdot|^{it}$ if needed, we can take $s \in \mathbb{R}$. Hence, from now on,

$$\frac{1}{2} \leq s \leq 1. \tag{2.11}$$

2.2. Approximate functional equation

We present a construction introduced in [13,14]. For a smooth function g with compact support on $(0, \infty)$, normalized such that $\int_0^\infty g(u)\frac{du}{u} = 1$, we can introduce an entire function k given by

$$k(s) = \int_0^\infty g(u)u^{s-1}du$$

such that $k(0) = 1$ by normalization and k decreases rapidly in vertical strips. We then define two functions for $y > 0$,

$$F_1(y) = \frac{1}{2\pi i}\int_{(2)} k(s)y^{-s}\frac{ds}{s}, \tag{2.12}$$

$$F_2(y) = \frac{1}{2\pi i}\int_{(2)} k(-s)G(-s + \beta)y^{-s}\frac{ds}{s}, \tag{2.13}$$

with $G(s) = \frac{L(\tilde{\pi}_\infty, 1-s)}{L(\pi_\infty, s)}$ and the integrals above over $\mathrm{Re}(s) = 2$. The functions $F_1(y)$ and $F_2(y)$ obey the following relations (see [13]):

1. $F_{1,2}(y) \ll C_m y^{-m}$ for all $m \geq 1$, as $y \to \infty$.
2. $F_1(y) = 1 + O(y^m)$ for all $m \geq 1$ for y small enough.
3. $F_2(y) \ll_\epsilon 1 + y^{1-\eta-\mathrm{Re}(\beta)-\epsilon}$ for any $\epsilon > 0$, where $\eta = \max_{1\leq j\leq n} \mathrm{Re}(\mu_j)$ and μ_j as in (2.4). If π is tempered then $\eta = 0$ and in general the following inequality holds (see [15]):

$$0 \leq \eta \leq \frac{1}{2} - \frac{1}{n^2+1}. \tag{2.14}$$

The following approximate functional equation was first used in [14] for cuspidal automorphic representations of $\mathrm{GL}(n)$ over $\mathbb{Q}$. It also holds for $\pi \in \mathcal{A}_u(n)$ such that $L(\pi \otimes \chi, s)$ is entire (see for example [7]). A similar approximate functional equation was proved in [3] for $L(\pi, \beta)$ at the center $\beta = \frac{1}{2}$, for slightly different rapidly decreasing functions.

Proposition. *If $\pi \in \mathcal{A}_u(n)$ and χ is a primitive Dirichlet character of conductor q such that $L(\pi \otimes \chi, s)$ is entire, then for any $\frac{1}{2} \leq \beta \leq 1$*

$$L(\pi \otimes \chi, \beta) = \sum_{m=1}^{\infty} \frac{a_\pi(m)\chi(m)}{m^\beta} F_1\left(\frac{my}{f_\pi q^n}\right) + \omega_\pi(q)\epsilon(0,\pi)\tau(\chi)^n (f_\pi q^n)^{-\beta} \sum_{m=1}^{\infty} \frac{a_{\tilde{\pi}}(m)\overline{\chi}(m f'_\pi)}{m^{1-\beta}} F_2\left(\frac{m}{y}\right),$$

where f'_π is the multiplicative inverse of f_π modulo q.

2.3. Dihedral representations

We now review some results on dihedral representations. Let π be a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ with conductor f_π. We have the symmetric square L-function $L(\pi, s, Sym^2)$ given by an Euler product with local factors

$$L_v(\pi, s, Sym^2) = (1 - \alpha_v^2 v^{-s})^{-1}(1 - \alpha_v\beta_v v^{-s})^{-1}(1 - \beta_v^2 v^{-s})^{-1}$$

for primes v with $v \nmid f_\pi$ and $A_v(\pi) = \{\alpha_v, \beta_v\}$ the diagonal representatives of the conjugacy classes attached to π_v.

By [6], there exists an isobaric automorphic representation $Sym^2(\pi)$ of $\mathrm{GL}(3, \mathbb{A}_\mathbb{Q})$ whose standard L-function agrees with $L(\pi, s, Sym^2)$ at least at primes v with $v \nmid f_\pi$. We have that $Sym^2(\pi)$ is cuspidal if and only if π is dihedral. A dihedral representation is a representation induced by an idele class character η of a quadratic extension K of $\mathbb{Q}$. If $\pi = I_K^{\mathbb{Q}}(\eta)$ is a dihedral representation then

$$L(I_K^{\mathbb{Q}}(\eta), s) = L(\eta, s).$$

Let π be a (unitary) cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Suppose π is dihedral, of the form $I_K^{\mathbb{Q}}(\eta)$ for a (unitary) character η of C_K. We can express $Sym^2\,\pi$ as follows (see also [12]). Let τ be the non-trivial automorphism of the degree 2 extension $K/\mathbb{Q}$. Note that

$$\eta\eta^\tau = \eta_0 \circ N_{K/\mathbb{Q}}, \tag{2.15}$$

where η_0 is the restriction of η to $C_{\mathbb{Q}}$. We have

$$I_K^{\mathbb{Q}}(\eta\eta^\tau) \cong \eta_0 \boxplus \eta_0\delta \tag{2.16}$$

where δ is the quadratic character of $\mathbb{Q}$ associated to $K/\mathbb{Q}$.

If λ, μ are characters of C_K, then by applying Mackey:

$$I_K^{\mathbb{Q}}(\lambda) \boxtimes I_K^{\mathbb{Q}}(\mu) \cong I_K^{\mathbb{Q}}(\lambda\mu) \boxplus I_K^{\mathbb{Q}}(\lambda\mu^\tau). \tag{2.17}$$

Taking $\lambda = \mu = \eta$ in (2.17) and using (2.15) and (2.16),

$$\pi \boxtimes \pi \cong I_K^{\mathbb{Q}}(\eta^2) \boxplus \eta_0 \boxplus \eta_0\delta.$$

Since $\pi \boxtimes \pi = Sym^2(\pi) \boxplus \omega$ with $\omega = \eta_0\delta$,

$$Sym^2(\pi) \cong I_K^{\mathbb{Q}}(\eta^2) \boxplus \eta_0. \tag{2.18}$$

3. A simple lemma involving Gauss sums

For an odd prime p, define the sets (following the notations in [14])

$$X_{(p)} = \{\chi \text{ a Dirichlet character of conductor } p^a \text{ for some } a\},$$
$$X_{(p)}^w = \{\chi \in X_{(p)} | \chi \text{ has } p\text{-power order}\}.$$

The characters of $X_{(p)}^w$ are called wild at p.

If $\chi \in X_{(p)}$, then $\chi : (\mathbb{Z}/p^a\mathbb{Z})^\times \to \mathbb{C}^\times$ for some a. Note that $(\mathbb{Z}/p^a\mathbb{Z})^\times \cong \mathbb{Z}/p^{a-1}\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}$. A character in $X_{(p)}$ is an element in $X_{(p)}^w$ if and only if it is trivial on the elements of exponent $p-1$.

We denote the integers mod p^a of exponent $p-1$ by S_a and the sum over all primitive wild characters of conductor p^a by $\sum^*_{\chi \bmod p^a}$.

Consider the set

$$G(p^a) := \ker((\mathbb{Z}/p^a\mathbb{Z})^\times \to (\mathbb{Z}/p)^\times) \cong \mathbb{Z}/p^{a-1}\mathbb{Z}. \tag{3.1}$$

Using the orthogonality of characters we get that summing over the primitive wild characters of conductor p^a gives (see [14])

$$\sum\nolimits^{*}_{\chi \bmod p^a} \chi = |G(p^a)|\delta_{S_a} - |G(p^{a-1})|\delta_{S_{a-1}}, \tag{3.2}$$

with $|G(p^a)| = p^{a-1}$ from (3.1) and δ_{S_a} the characteristic function of S_a.

The following result for hyper-Kloosterman sums was proved in [25]:

Lemma 1. *Let p be a prime number, $1 < n < p$ and $q = p^a$ with $a > 1$. Let x' denote the inverse of x mod q and let $e(x) := e^{2\pi i x}$. Then for any integer z coprime to p the hyper-Kloosterman sum*

$$\Big| \sum_{\substack{x_1,\cdots,x_n (mod\ q)\\ (x_i,p)=1}} e\left(\frac{x_1 + \cdots + x_n + z x_1' \cdots x_n'}{q}\right)\Big|$$

is bounded by

$$\begin{cases} \le (n+1)q^{n/2} & \text{if } 1 < n < p-1, a > 1 \\ \le p^{1/2}q^{n/2} & \text{if } n = p-1, a \ge 5 \\ \le pq^{n/2} & \text{if } n = p-1, a = 4 \\ \le p^{1/2}q^{n/2} & \text{if } n = p-1, a = 3 \\ \le q^{n/2} & \text{if } n = p-1, a = 2. \end{cases} \tag{3.3}$$

As a consequence of Lemma 1 we prove the following result:

Lemma 2. *Let $\tau(\chi)$ denote the Gauss sum of the character χ. If $(r,p) = 1$, then the following bound holds:*

$$\Big|\sum\nolimits^{*}_{\chi\ mod\ p^a} \overline{\chi}(r)\tau^n(\chi)\Big| \ll p^{1/2+a(n+1)/2} \tag{3.4}$$

for $2 < n \le p$.

Proof. If χ is a primitive character of conductor p^a, then

$$\tau(\chi) = \sum_{m=0}^{p^a-1} \chi(m) e^{2\pi i m/p^a}.$$

Let

$$A := \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(r)\tau^n(\chi),$$

then

$$A = \sum\nolimits^*_{\chi \bmod p^a} \left[\overline{\chi}(r)\left(\sum_{m=0}^{p^a-1}\chi(m)e^{2\pi i m/p^a}\right)^n\right].$$

We rewrite the above sum as

$$A = \sum\nolimits^*_{\chi \bmod p^a} \left[\overline{\chi}(r)\left(\sum_{x_1=0}^{p^a-1}\chi(x_1)e^{2\pi i x_1/p^a}\right)\cdots\left(\sum_{x_n=0}^{p^a-1}\chi(x_n)e^{2\pi i x_n/p^a}\right)\right].$$

This in turn gives

$$A = \sum_{x_1=0}^{p^a-1}\cdots\sum_{x_n=0}^{p^a-1}\sum\nolimits^*_{\chi \bmod p^a}\chi(r')\chi(x_1)\cdots\chi(x_n)e\left(\frac{x_1+\cdots+x_n}{p^a}\right).$$

Hence,

$$A = \sum_{x_1=0}^{p^a-1}\cdots\sum_{x_n=0}^{p^a-1}\left[\sum\nolimits^*_{\chi \bmod p^a}\chi(r'x_1\cdots x_n)\right]e\left(\frac{x_1+\cdots+x_n}{p^a}\right)$$

which by Eq. (3.2) gives

$$A = \sum_{x_1=0}^{p^a-1}\cdots\sum_{x_n=0}^{p^a-1}e\left(\frac{x_1+\cdots+x_n}{p^a}\right)\left(p^{a-1}\delta_{S_a}(r'x_1\cdots x_n) - p^{a-2}\delta_{S_{a-1}}(r'x_1\cdots x_n)\right).$$

Thus,

$$A = p^{a-1}\sum_{b\in S_a}T(br,p^a) - p^{a-2}\sum_{c\in S_{a-1}}\sum_{i=0}^{p-1}T(cr+ip^{a-1},p^a) \tag{3.5}$$

where

$$T(u,p^a) = \sum_{\substack{x_1,\cdots,x_{n-1} (\bmod p^a)\\ (x_i,p)=1}} e\left(\frac{x_1+\cdots+x_{n-1}+ux_1'\cdots x_{n-1}'}{p^a}\right).$$

From Lemma 1, for $(u,p)=1$ and a sufficiently large

$$|T(u,p^a)| \ll p^{1/2+a(n-1)/2}. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$|A| \ll p^{a-1}(p-1)p^{1/2+a(n-1)/2} + p^{a-2}(p-1)^2 p^{1/2+a(n-1)/2}.$$

Thus $|A| \ll p^a p^{1/2+a(n-1)/2}$. □

4. Non-vanishing of p-power twists on $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$

Let s, r be integers relatively prime to p. For π an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_\mathbb{Q})$ define

$$S_{s/r}(p^a, \pi, \beta) = p^{-a} \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(s)\chi(r)L(\pi \otimes \chi, \beta) \tag{4.1}$$

where $\sum^*$ denotes the sum over primitive wild characters of conductor p^a.

In this section we prove Theorem 4, which states that:

$$\lim_{a\to\infty} S_{s/r}(p^a, \pi, \beta) = \frac{1}{p}\left(1 - \frac{1}{p}\right)\frac{a_\pi(s/r)}{(s/r)^\beta} \tag{4.2}$$

for $\beta > \frac{n-1}{n+1}$ if π is tempered, and for $\beta > \frac{n-1}{n}$ in general.

Note that in this section by π tempered we will mean an isobaric sum of tempered (unitary) cuspidal automorphic representations. If $r \nmid s$ above, then we define $a_\pi(s/r)$ to be zero.

Proof of Theorem 4. We generalize the proof of Proposition 2.2 in [14] and use methods also developed in [13,24]. The following approximate functional equation holds (see Section 2):

$$L(\pi \otimes \chi, \beta) = \sum_{m=1}^{\infty} \frac{a_\pi(m)\chi(m)}{m^\beta} F_1\left(\frac{my}{f_\pi p^{an}}\right) + \omega_\pi(p^a)\epsilon(0,\pi)\tau(\chi)^n(f_\pi p^{an})^{-\beta} \sum_{m=1}^{\infty} \frac{a_{\tilde{\pi}}(m)\overline{\chi}(mf'_\pi)}{m^{1-\beta}} F_2\left(\frac{m}{y}\right),$$

where χ is a character of conductor p^a and f'_π is the multiplicative inverse of f_π modulo p^a.

Define x such that $xy = p^{an}$. Write

$$S_{s/r}(p^a, \beta) = S_{1,s/r}(p^a, \beta) + S_{2,s/r}(p^a, \beta), \tag{4.3}$$

where

$$S_{1,s/r}(p^a, \beta) = p^{-a} \sum\nolimits^*_{\chi \bmod p^a} \sum_{m=1}^{\infty} \frac{a_\pi(m)\chi(ms'r)}{m^\beta} F_1\left(\frac{m}{f_\pi x}\right) \tag{4.4}$$

and

$$S_{2,s/r}(p^a,\beta) = p^{-a}\omega_\pi(p^a) \sum\nolimits^*_{\chi \bmod p^a} \epsilon(0,\pi)\tau(\chi)^n (f_\pi p^{an})^{-\beta} \times \sum_{m=1}^{\infty} \frac{a_{\tilde{\pi}}(m)\overline{\chi}(ms'rf'_\pi)}{m^{1-\beta}} F_2\left(\frac{m}{y}\right). \tag{4.5}$$

Let

$$Z_{s/r}(p^a,\beta) = \sum_{b\in S_a} \sum_{\substack{rm\equiv bs(p^a)\\ m\geq 1}} \frac{a_\pi(m)}{m^\beta} F_1\left(\frac{m}{f_\pi x}\right). \tag{4.6}$$

Then applying Eq. (3.2) gives

$$S_{1,s/r}(p^a) = p^{-a}\sum_{m=1}^{\infty} \frac{a_\pi(m)}{m^\beta} F_1\left(\frac{m}{f_\pi x}\right)\left[p^{a-1}\delta_{S_a}(ms'r) - p^{a-2}\delta_{S_{a-1}}(ms'r)\right],$$

hence

$$S_{1,s/r} = \frac{1}{p}\left[Z_{s/r}(p^a,\beta) - p^{-1}Z_{s/r}(p^{a-1},\beta)\right]. \tag{4.7}$$

If $r|s$, consider the term in (4.6) with $b = 1$ and $m = s/r$. This is a solution to the equation $rm \equiv bs(\bmod\ p^a)$ for all a. We will want to set the necessary condition for this to be the only dominant contribution. If $r \nmid s$ this term will not appear in the sum and the argument remains as below, requiring the condition that there is no dominant contribution and that the limit of $S_{s/r}(p^a,\pi,\beta)$ as $a \to \infty$ is zero.

Now if $m \neq s/r$, then $m = bs/r + kp^a$. If $k = 0$ then $b \neq 1$ and since $b \in S_a$, it follows that $b \gg p^{a/(p-1)}$ which implies

$$m \gg p^{a/(p-1)}.$$

If $k \neq 0$, then $m \ll kp^a$.

Decompose

$$Z_{s/r}(p^a,\beta) = \Sigma_{1,a} + \Sigma_{2,a},$$

where

$$\Sigma_{1,a} = \frac{a_\pi(s/r)}{(s/r)^\beta} F_1\left(\frac{s}{rf_\pi x}\right) \tag{4.8}$$

and

$$\Sigma_{2,a} = \sum_{b \in S_a} \sum_{\substack{rm \equiv bs(p^a) \\ m \geq 1, m \neq s/r}} \frac{a_\pi(m)}{m^\beta} F_1\left(\frac{m}{f_\pi x}\right). \tag{4.9}$$

Since $F_1\left(\frac{m}{f_\pi x}\right) = 1 + O\left(\frac{m}{f_\pi x}\right)$,

$$\Sigma_{1,a} = \frac{a_\pi(s/r)}{(s/r)^\beta}\left(1 + O\left(\frac{1}{x}\right)\right). \tag{4.10}$$

Following [14], let

$$b_{m,a} := \begin{cases} 1 & \text{if } m = bs/r + kp^a \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Then

$$\Sigma_{2,a} \ll \Bigg|\sum_{\substack{1 \leq m \ll x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Bigg| + \Bigg|\sum_{\substack{m \gg x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Bigg|. \tag{4.12}$$

Define

$$P_{2,a} = \Bigg|\sum_{\substack{1 \leq m \ll x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Bigg| \quad \text{and} \quad Q_{2,a} = \Bigg|\sum_{\substack{m \gg x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a} F_1\left(\frac{m}{f_\pi x}\right)\Bigg|.$$

Since $F_1\left(\frac{m}{f_\pi x}\right) = 1 + O(x^\epsilon)$ for $m \ll x^{1+\epsilon}$ and $F_1\left(\frac{m}{f_\pi x}\right) \ll \frac{x^t}{m^t}$ for any integer t and $m \gg x^{1+\epsilon}$

$$P_{2,a} \ll x^\epsilon \Bigg|\sum_{\substack{1 \leq m \ll x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^\beta} b_{m,a}\Bigg| \quad \text{and} \quad Q_{2,a} \ll x^t \Bigg|\sum_{\substack{m \gg x^{1+\epsilon} \\ m \neq s/r}} \frac{a_\pi(m)}{m^{\beta+t}} b_{m,a}\Bigg|. \tag{4.13}$$

If π is tempered then by (4.13)

$$P_{2,a} \ll x^\epsilon \sum_{\substack{1 \leq m \ll x^{1+\epsilon} \\ m \neq s/r}} m^{\epsilon-\beta} b_{m,a} \ll p^{-a} x^{1-\beta+\epsilon} \quad \text{and} \quad Q_{2,a} \ll p^{-a} x^{1-\beta+\epsilon} \tag{4.14}$$

hence

$$\Sigma_{2,a} \ll p^{-a} x^{1-\beta+\epsilon}. \tag{4.15}$$

We want $\Sigma_{2,a} \to 0$ as $a \to \infty$. Substituting with $x = p^{an(1-\upsilon)}$ gives the condition

$$\upsilon > 1 - \frac{1}{n(1-\beta+\epsilon)}. \tag{4.16}$$

If π is not tempered, then applying Cauchy–Schwarz's inequality in (4.13) gives

$$P_{2,a} \ll \frac{x^{1/2+\epsilon}}{p^{a/2}} \left(\sum_{1 \leq m \ll x^{1+\epsilon}} \frac{|a_\pi(m)|^2}{m^{2\beta}} \right)^{1/2}.$$

By inequality (2.6) and summation by parts we get

$$P_{2,a} \ll p^{-a/2} x^{1-\beta+\epsilon}. \tag{4.17}$$

Write $t = t_1 + t_2$ in (4.13), with t_1, t_2 large integers, and apply Cauchy–Schwarz's inequality:

$$\begin{aligned} Q_{2,a} &\ll x^{t_1+t_2} \left(\sum_{m \gg x^{1+\epsilon}} \frac{|a_\pi(m)|^2}{m^{2\beta+2t_1}} \right)^{1/2} \left(\sum_{m \gg x^{1+\epsilon}} \frac{b_{m,a}^2}{m^{2t_2}} \right)^{1/2} \\ &\ll x^{t_1+t_2} \left(\sum_{i \gg (1+\epsilon)\log(x)} \sum_{2^{i-1} < m \leq 2^i} \frac{|a_\pi(m)|^2}{m^{2\beta+2t_1}} \right)^{1/2} \left(\sum_{k \gg \frac{x^{1+\epsilon}}{p^a}} \frac{1}{(kp^a)^{2t_2}} \right)^{1/2}. \end{aligned} \tag{4.18}$$

Using (2.6) gives

$$Q_{2,a} \ll p^{-at_2} x^{1-\beta+\epsilon} \tag{4.19}$$

hence

$$\Sigma_{2,a} \ll p^{-a/2} x^{1-\beta+\epsilon}. \tag{4.20}$$

Since we want $\Sigma_{2,a} \to 0$, we get the condition

$$\upsilon > 1 - \frac{1}{2n(1-\beta+\epsilon)}. \tag{4.21}$$

For υ as above,

$$\lim_{a\to\infty} S_{1,s/r}(p^a, \beta) = \frac{p-1}{p^2} \cdot \frac{a_\pi(s/r)}{(s/r)^\beta}. \tag{4.22}$$

In (4.5) write

$$|S_{2,s/r}| \ll A_{2,s/r} + B_{2,s/r}, \tag{4.23}$$

where

$$A_{2,s/r} = p^{-a} p^{-an\beta} \sum_{m \ll y^{1+\epsilon}} \left[\frac{|a_{\tilde\pi}(m)|}{m^{1-\beta}} F_2\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi (\mathrm{mod}\ p^a)} \overline{\chi}(ms' r f'_\pi) \tau^n(\chi) \right| \right] \tag{4.24}$$

and

$$B_{2,s/r} = p^{-a} p^{-an\beta} \sum_{m \gg y^{1+\epsilon}} \left[\frac{|a_{\tilde{\pi}}(m)|}{m^{1-\beta}} F_2\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi (\mathrm{mod}\ p^a)} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right| \right]. \quad (4.25)$$

If π is tempered then $|a_{\tilde{\pi}}(m)| \ll m^\epsilon$. Also, $F_2\left(\frac{m}{y}\right) \ll 1 + \left(\frac{m}{y}\right)^{1-\beta-\epsilon}$ for $m \ll y^{1+\epsilon}$, which gives $F_2\left(\frac{m}{y}\right) \ll y^{\epsilon(1-\beta)}$. Applying Lemma 2,

$$|A_{2,s/r}| \ll p^{-a} p^{-an\beta} p^{1/2+a(n+1)/2} y^{\epsilon(1-\beta)} \sum_{m=1}^{y^{1+\epsilon}} m^{\epsilon+\beta-1}$$

hence for any $\epsilon > 0$

$$|A_{2,s/r}| \ll p^{-an\beta+a(n-1)/2} y^{\epsilon+\beta}. \quad (4.26)$$

Assume now π is not tempered. By Cauchy–Schwarz's inequality we obtain

$$|A_{2,s/r}| \ll p^{-a} p^{-an\beta} y^\epsilon \left(\sum_{m \ll y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}(m)|^2}{m^{2-2\beta}} \right)^{1/2} \times \left(\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right|^2 \right)^{1/2},$$

where

$$H(u) := \frac{1}{\pi(1+u^2)}.$$

A simple computation shows that

$$\sum_{m \ll y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}(m)|^2}{m^{2-2\beta}} \ll y^{2\beta-1+\epsilon}. \quad (4.27)$$

Hence,

$$|A_{2,s/r}| \ll y^{\beta-1/2+\epsilon} p^{-a-an\beta} \times \left(\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left| \sum\nolimits^*_{\chi \bmod p^a} \overline{\chi}(ms'rf'_\pi)\tau^n(\chi) \right|^2 \right)^{1/2}. \quad (4.28)$$

Define

$$D := \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) \left|\sum\nolimits^{*}_{\chi \bmod p^a} \bar{\chi}(ms'rf'_\pi)\tau^n(\chi)\right|^2. \tag{4.29}$$

We have

$$D \ll \sum\nolimits^{*}_{\chi \bmod p^a} \sum\nolimits^{*}_{\psi \bmod p^a} \left|\tau^n(\chi)\tau^n(\bar{\psi}) \sum_{m=-\infty}^{\infty} \bar{\chi}\psi(ms'rf'_\pi) H\left(\frac{m}{y}\right)\right|.$$

Following the general approach of [13,24], we consider the diagonal and off-diagonal contributions separately. Let's first compute the terms corresponding to $\chi = \psi$:

$$\sum\nolimits^{*}_{\chi \bmod p^a} \left|\tau^n(\chi)\tau^n(\bar{\chi}) \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right)\right| \ll p^{a+na} \sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right)$$

since there are $\ll p^a$ primitive p-power characters and since $|\tau^n(\chi)| = p^{an/2}$ from the properties of the Gauss sum of a primitive character. Using the Fourier transform property $\mathcal{F}\{g(xA)\} = \frac{1}{A}\hat{g}\left(\frac{\nu}{A}\right)$ for $A > 0$ (see also [13,24]) we get that

$$\sum_{m=-\infty}^{\infty} H\left(\frac{m}{y}\right) = y \sum_{\nu=-\infty}^{\infty} T(y\nu).$$

Function $T(\nu)$ is the Fourier transform of $H(m)$ and is given by $T(\nu) = e^{-2\pi|\nu|}$, hence $\sum_{m\in\mathbb{Z}} H\left(\frac{m}{y}\right) \ll y$. Note we have used the Poisson summation formula. Thus the contribution to D is $\ll p^{a+na}y$.

For the terms in D that have $\chi \neq \psi$, even if χ and ψ are primitive the product $\bar{\chi}\psi$ can be non-primitive because the conductors are not relatively prime. We have that for $g : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$:

$$\sum_{m=-\infty}^{\infty} g(m) f\left(\frac{m}{q}\right) = \sum_{b \bmod q} g(b) F\left(\frac{b}{q}\right) = \sum_{\nu=-\infty}^{\infty} \hat{g}(-\nu)\hat{f}(\nu)$$

where $F(x) = \sum_{\nu=-\infty}^{\infty} \hat{f}(\nu)e^{-2\pi i\nu x}$. Applying this in our case,

$$\sum_{m=-\infty}^{\infty} \bar{\chi}\psi(m) H\left(\frac{m}{y}\right) = \frac{y}{p^a} \sum_{\nu=-\infty}^{\infty} \left(\sum_{b \bmod p^a} \bar{\chi}\psi(b) e^{-2\pi i\nu b/p^a}\right) T\left(\frac{y\nu}{p^a}\right).$$

The interior sum is $\ll p^a$ since the number of characters is $\ll p^a$, and for $\nu = 0$ it is zero since $\bar{\chi}\psi$ is non-trivial. Thus,

$$\left|\sum_{m=-\infty}^{\infty} \bar{\chi}\psi(m) H\left(\frac{m}{y}\right)\right| \ll y \sum_{\nu\in\mathbb{Z},\nu\neq 0} T\left(\frac{y\nu}{p^a}\right).$$

Assuming $\upsilon > \frac{1}{n}$ (which will be part of our constraint) gives that $y/p^a \to \infty$. We have

$$\sum_{\nu \in \mathbb{Z}, \nu \neq 0} T\left(\frac{y\nu}{p^a}\right) \ll \frac{2}{e^{2\pi y p^{-a}} - 1} \ll \frac{1}{y}.$$

Putting everything together, these terms of D contribute $\ll p^{2a+na}$. Thus, we conclude that the two contributions for $\chi = \psi$ and $\chi \neq \psi$ combined give

$$D \ll p^{a+na} y. \tag{4.30}$$

From (4.28) and (4.30), even if π is not tempered,

$$|A_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta + a(n-1)/2}. \tag{4.31}$$

For $m \gg y^{1+\epsilon}$, $F_2\left(\frac{m}{y}\right) \ll \frac{y^t}{m^t}$ for any integer $t \geq 1$, and applying Cauchy–Schwarz's inequality in (4.25) gives

$$|B_{2,s/r}| \ll p^{-a} p^{-an\beta} y^t \left(\sum_{m \gg y^{1+\epsilon}} \frac{|a_{\tilde{\pi}}|^2}{m^{2-2\beta+2t}}\right)^{1/2} D^{1/2}.$$

Using summation by parts and (2.6), as well as the bound in (4.30) gives

$$|B_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta + a(n-1)/2}. \tag{4.32}$$

From (4.23), (4.31) and (4.32) we conclude that

$$|S_{2,s/r}| \ll y^{\beta+\epsilon} p^{-an\beta + a(n-1)/2}. \tag{4.33}$$

We want $S_{2,s/r} \to 0$ as $a \to \infty$. Taking $y = p^{an\upsilon}$ in (4.33) gives the condition

$$\upsilon < \frac{1 - n + 2n\beta}{2n(\beta + \epsilon)}. \tag{4.34}$$

If π is tempered then we need to check that υ satisfies conditions (4.16) and (4.34). Thus, for a general n, the desired condition is

$$\beta > \frac{n-1}{n+1}. \tag{4.35}$$

If π is not tempered, then conditions (4.21) and (4.34) need to be satisfied. This gives the condition

$$\beta > \frac{n-1}{n}. \qquad \square$$

Proof of Corollary 1. Take $s = r = 1$ in Theorem 4 and use the functional equation. Note that if $\beta > 1$, $L(\pi \otimes \chi, \beta)$ has an Euler product expansion and hence is nonvanishing. $\square$

5. Determination of GL(3) cusp forms

Let $\pi \in \mathcal{A}_u(3)$ be an isobaric sum of unitary cuspidal automorphic representations of $\mathrm{GL}(3, \mathbb{A}_\mathbb{Q})$. The local components π_ℓ are determined by the set of nonzero complex numbers $\{\alpha_\ell, \beta_\ell, \gamma_\ell\}$, which we represent by the diagonal matrix $A_\ell(\pi)$.

The L-factor of π at a prime ℓ is given by

$$\begin{aligned} L(\pi_\ell, s) &= \det(I - A_l(\pi)\ell^{-s})^{-1} \\ &= \prod_{j=1}^{n} (1 - \alpha_\ell \ell^{-s})^{-1}(1 - \beta_\ell \ell^{-s})^{-1}(1 - \gamma_\ell \ell^{-s})^{-1}. \end{aligned} \tag{5.1}$$

Let $S_0 = \{\ell : \pi_\ell \text{ unramified and tempered}\}$, and let $S_1 = \{\ell : \pi_\ell \text{ is ramified}\}$. Note that S_1 is finite. Take the union

$$S = S_0 \cup S_1 \cup \{\infty\}.$$

Since π is unitary, π_ℓ is tempered iff $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$.

Lemma 3. *If $\ell \notin S$ then*

$$A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}, \tag{5.2}$$

with $|u| = |w| = 1$ and $t \neq 0$ a real number. If $\ell \in S_0$ then

$$A_\ell(\pi) = \{\alpha, \beta, \gamma\}$$

with $|\alpha| = |\beta| = |\gamma| = 1$.

Proof. Suppose first that $\ell \notin S$. We may assume that $|\alpha_\ell| \neq 1$. Then it can be written as $\alpha_\ell = u\ell^t$, for some $|u| = 1$ complex and $t \neq 0$ real. By unitarity,

$$\{\bar{\alpha}_\ell, \bar{\beta}_\ell, \bar{\gamma}_\ell\} = \{\alpha_\ell^{-1}, \beta_\ell^{-1}, \gamma_\ell^{-1}\}.$$

Clearly $\bar{\alpha}_\ell \neq \alpha_\ell^{-1}$. Without loss of generality, take $\beta_\ell^{-1} = \bar{\alpha}_\ell$. Hence, this gives $\beta_\ell = u \cdot \ell^{-t}$. So, we must have $\bar{\gamma}_\ell = \gamma_\ell^{-1}$, thus $\gamma_\ell = w$ with $|w| = 1$. Hence

$$A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}$$

with $|u| = |w| = 1$.

Now suppose that $\ell \in S_0$. Then $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$. $\square$

Proof of Theorem 2. Let $T = \{\ell | \pi_\ell \text{ or } \pi'_\ell \text{ is ramified}\}$. This is a finite set.

Consider $\ell \notin T$ an arbitrary finite place with $\ell \neq p$. Let $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell, \gamma_\ell\}$ and $A_\ell(\pi') = \{\alpha'_\ell, \beta'_\ell, \gamma'_\ell\}$. Applying Theorem 4, $a_\pi(n) = B^a C a_{\pi'}(n)$ for all $(n,p) = 1$ and all but finitely many a. Since $a_\pi(1) = a_{\pi'}(1)$, we conclude that $B = C = 1$. Thus, $a_\pi(\ell) = a_{\pi'}(\ell)$.

We want to show that $A_l(\pi) = A_l(\pi')$. Indeed,

$$\alpha_\ell + \beta_\ell + \gamma_\ell = \alpha'_\ell + \beta'_\ell + \gamma'_\ell \tag{5.3}$$

and since π and π' have the same central character

$$\alpha_\ell \beta_\ell \gamma_\ell = \alpha'_\ell \beta'_\ell \gamma'_\ell. \tag{5.4}$$

To show that $\{\alpha_\ell\, \beta_\ell, \gamma_\ell\} = \{\alpha'_\ell, \beta'_\ell, \gamma'_\ell\}$, by Vieta's formulas (cf. [22]) and the above two relations, it is enough to check that

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell + \alpha'_\ell\gamma'_\ell + \beta'_\ell\gamma'_\ell.$$

Suppose $A_\ell(\pi) = \{u\ell^t, u\ell^{-t}, w\}$ with $|u| = |w| = 1$. Then

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = u^2 + uw(\ell^t + \ell^{-t}) = \frac{1}{u^2} + \frac{1}{uw}(\ell^t + \ell^{-t}) = \frac{w + u(\ell^t + \ell^{-t})}{u^2 w},$$

hence $\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \frac{\alpha_\ell+\beta_\ell+\gamma_\ell}{\alpha_\ell\beta_\ell\gamma_\ell}$.

Now suppose that $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell, \gamma_\ell\}$ with $|\alpha_\ell| = |\beta_\ell| = |\gamma_\ell| = 1$. Then

$$\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \frac{1}{\alpha_\ell\beta_\ell} + \frac{1}{\alpha_\ell\gamma_\ell} + \frac{1}{\beta_\ell\gamma_\ell} = \frac{\alpha_\ell + \beta_\ell + \gamma_\ell}{\alpha_\ell\beta_\ell\gamma_\ell}.$$

Thus, whenever $\alpha_\ell + \beta_\ell + \gamma_\ell = \alpha'_\ell + \beta'_\ell + \gamma'_\ell$ and $\alpha_\ell\beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell\gamma'_\ell$, we obtain that $\alpha_\ell\beta_\ell + \alpha_\ell\gamma_\ell + \beta_\ell\gamma_\ell = \alpha'_\ell\beta'_\ell + \alpha'_\ell\gamma'_\ell + \beta'_\ell\gamma'_\ell$.

We have thus shown that for $\ell \notin T \cup \{p\} \cup \{\infty\}$, $A_l(\pi) = A_\ell(\pi')$, hence $\pi_\ell \cong \pi'_\ell$. Since $T \cup \{p\} \cup \{\infty\}$ is a finite set, this implies that $\pi \cong \pi'$ by the Generalized Strong Multiplicity One Theorem. $\square$

Let π be a unitary cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$ with $A_\ell(\pi) = \{\alpha_\ell, \beta_\ell\}$. At an unramified place ℓ, it has $a_\ell = \alpha_\ell + \beta_\ell$ and central character $\omega(\varpi_\ell) = \alpha_\ell\beta_\ell$, with ϖ_ℓ the uniformizer at ℓ. There exists an isobaric automorphic representation $Ad(\pi)$ of $\mathrm{GL}(3, \mathbb{A}_\mathbb{Q})$ (cf. [6]) such that at an unramified place ℓ,

$$a_\ell(Ad(\pi)) = \alpha_\ell/\beta_\ell + \beta_\ell/\alpha_\ell + 1.$$

Proof of Theorem 3. Theorem 2 implies that $Ad(\pi) \cong Ad(\pi')$. Then, by Theorem 4.1.2 in [18], we deduce that since π and π' have the same central character, there exists a quadratic character ν such that $\pi \cong \pi' \otimes \nu$. $\square$

6. Adjoint p-adic L-functions

Fix p an odd prime and let E be an elliptic curve over $\mathbb{Q}$ with semistable reduction at p. We now describe a p-adic analogue to $L(Sym^2 E, s)$ by the Mazur–Mellin transform of a p-adic measure μ_p on $\mathbb{Z}_p^\times$ as introduced in [5]. For a review of the complex L-function associated to the symmetric square of an elliptic curve see also [4].

Consider the real and imaginary periods of a Néron differential of a minimal Weierstrass equation for E over $\mathbb{Z}$ which we denote by $\Omega^\pm(E)$. Let

$$\Omega^+(Sym^2 E(1)) := (2\pi i)^{-1}\Omega^+(E)\Omega^-(E) \text{ and } \Omega^+(Sym^2 E(2)) := 2\pi i\Omega^+(E)\Omega^-(E)$$

be the periods for $Sym^2 E$ at the critical twists. In [5] two p-adic distributions $\mu_p(\Omega^+(Sym^2 E(1)))$ and $\mu_p(\Omega^+(Sym^2 E(2)))$ are defined. In the present paper we will use the latter distribution.

Let X_p be the set of continuous characters of $\mathbb{Z}_p^\times$ into $\mathbb{C}_p^\times$. For $\chi \in X_p$, let p^{m_χ} be the conductor of χ. Since $\mathbb{Z}_p^\times \cong (1+p\mathbb{Z}_p) \times (\mathbb{Z}/p)^\times$, we can write $X := X_p$ as the product of $X((\mathbb{Z}/p)^\times)$ with $X_0 = X(1+p\mathbb{Z}_p)$. The elements of X_0 are called wild p-adic characters. By Section 2.1 in [23] we can give X_0 a $\mathbb{C}_p$-structure through the isomorphism of X_0 to the disk

$$U := \{u \in \mathbb{C}_p^\times \,||u-1| < 1\} \tag{6.1}$$

constructed by mapping $\nu \in X_0$ to $\nu(1+p)$, with $1+p$ a topological generator of $1+p\mathbb{Z}_p$.

We follow the definition of the p-adic distribution $\mu_p(\Omega^+(Sym^2 E(2)))$ on $\mathbb{Z}_p^\times$ in [5]. Suppose E has good reduction at p. Let $\chi \in X_0$ be a non-trivial wild p-adic character, with conductor p^{m_χ} which can be identified with a primitive Dirichlet character. Then given $\alpha_p(E)$ the root of $X^2 - a_pX + p$ with a_p the trace of the Frobenius at p, we define

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_p(\Omega^+(Sym^2 E(2))) := \alpha_p(E)^{-2m_\chi} \cdot \tau(\overline{\chi})^2 p^{m_\chi} \cdot \frac{L(Sym^2 E, \chi, 2)}{\Omega^+(Sym^2 E(2))}. \tag{6.2}$$

If E has good ordinary reduction at p then the distributions $\mu_p(\Omega^+(Sym^2 E(2)))$ are bounded measures on $\mathbb{Z}_p^\times$. If E has supersingular reduction at p then the distributions $\mu_p(\Omega^+(Sym^2 E(2)))$ give h-admissible measures on $\mathbb{Z}_p^\times$, with $h = 2$. Note that the set of h-admissible measures with $h = 1$ is larger, but contains the bounded measures.

Now suppose that E has bad multiplicative reduction at p (either split or non-split). Let $\chi \in X_0$ denote a Dirichlet character of conductor p^{m_χ} as above. Then

$$\int_{\mathbb{Z}_p^\times} \chi d\mu_p(\Omega^+(Sym^2 E(2))) := \tau(\overline{\chi})^2 p^{m_\chi} \cdot \frac{L(Sym^2 E, \chi, 2)}{\Omega^+(Sym^2 E(2))} \tag{6.3}$$

and the distributions $\mu_p(\Omega^+(Sym^2 E(2)))$ are bounded measures on $\mathbb{Z}_p^\times$.

Consider μ an h-admissible measure as above. Then

$$\chi \to L_\mu(\chi) := \int\limits_{\mathbb{Z}_p^\times} \chi d\mu \tag{6.4}$$

is an analytic function of type $o(\log^h)$ (cf. [23]). Note that for an analytic function F to be of type $o(\log^h)$ it must satisfy

$$\sup_{|u-1|_p<r} \|F(u)\| = o\left(\sup_{|u-1|_p<r} |\log_p^h(u)| \right) \text{ for } r \to 1_-.$$

An h-admissible measure μ is determined by the values $L_\mu(\chi x_p^r)$, where χ is a wild p-adic character and x_p is the p-th cyclotomic character given by the action on the p-power roots of unity, with $r = 0, 1, \cdots, h-1$.

Consider the p-adic distribution $\mu = \mu_p(\Omega^+(Sym^2 E(2)))$ as defined above. Denote by L_p the corresponding p-adic L-function. We have

$$L_p(Sym^2 E, \chi, s) := \int\limits_{\mathbb{Z}_p^\times} \chi(x)\langle x\rangle^s d\mu,$$

where $\langle\cdot\rangle : \mathbb{Z}_p^\times \to 1 + p\mathbb{Z}_p$, $\langle x\rangle = \frac{x}{\omega(x)}$, with $\omega : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ the Teichmüller character.

We prove the following lemma:

Lemma 4. *Let p be an odd prime. Let E, E' be elliptic curves over $\mathbb{Q}$ with semistable reduction at p such that $L_p(Sym^2 E, n) = CL_p(Sym^2 E', n)$, for an infinite number of integers n prime to p in some set Y, and some constant $C \in \overline{\mathbb{Q}}$. Then for every finite order wild p-adic character χ,*

$$L_p(Sym^2 E, \chi, s) = CL_p(Sym^2 E', \chi, s)$$

holds for all $s \in \mathbb{Z}_p$.

Proof. We follow the approach in [14]. Let

$$G(\nu) = L_p(Sym^2 E, \nu) - CL_p(Sym^2 E', \nu)$$

for every $\nu \in X_0$. G vanishes on $X_1 = \{\alpha_n = \langle x\rangle^n | n \in Y\}$ by hypothesis; we want to show that G vanishes on X_0. We use the fact that G is an analytic function on X_0 of type $o(\log^h)$ (as in (6.4)). G considered as an analytic function on U (see (6.1)) vanishes on the subset

$$U_1 = \{(1+p)^n | n \in Y\}.$$

There exists $r = 1/p$ such that the number of zeros z of G with $|z-1| = r$ is infinite. Indeed, for all $n \in Y$ elements in an infinite set with n relatively prime to p as above, $z_n := (1+p)^n \in U_1$ is a zero of G and

$$|z_n - 1| = |(1+p)^n - 1|_p = \left|\sum_{j=1}^{n} \binom{n}{j} p^j\right|_p = \frac{1}{p}.$$

By Section 2.5 in [23], G is identically zero on U. □

7. Proof of Theorem 1

7.1. Proof of Theorem 1 in the non-CM case

By Lemma 4, for every finite order wild p-power character χ, the identity

$$L_p(Sym^2\, E, \chi, s) = C L_p(Sym^2\, E', \chi, s) \tag{7.1}$$

holds for all $s \in \mathbb{Z}_p$. By Eq. (6.2), if E has good reduction at p then

$$\alpha_p(E)^{-2m_\chi} L(Sym^2\, E, \chi, 2) = C' \alpha_p(E')^{-2m_\chi} L(Sym^2\, E', \chi, 2) \tag{7.2}$$

for some $C' \in \overline{\mathbb{Q}}$. If E has bad multiplicative reduction at p, then by (6.3),

$$L(Sym^2\, E, \chi, 2) = C' L(Sym^2\, E', \chi, 2). \tag{7.3}$$

Let π and π' be the unitary cuspidal automorphic representations over $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ associated to $Sym^2\, E$ and $Sym^2\, E'$ respectively. Then the unitarized L-functions L_u corresponding to π and π' satisfy $L_u(\pi, s) = L(Sym^2\, E, s+1)$. Hence, if E has semistable reduction at p, from (7.2) and (7.3) there exist constants $C_1, C_2 \in \mathbb{C}$ such that

$$L(\pi \otimes \chi, 1) = C_1 C_2^{m_\chi} L(\pi' \otimes \chi, 1)$$

for all wild p-power characters χ of conductor p^{m_χ} with m_χ sufficiently large. Then by Theorem 2, we conclude that $\pi \cong \pi'$ and thus $\mathrm{Ad}(\eta) \cong \mathrm{Ad}(\eta')$ where η and η' are the unitary cuspidal automorphic representations of $\mathrm{GL}(2, \mathbb{Q})$ associated to E. By Theorem 4.1.2 in [18] we conclude that $\eta' = \eta \otimes \nu$ with ν a quadratic character since $\omega_\eta = \omega_{\eta'} = 1$. Write $\nu(\cdot) = \left(\frac{\cdot}{D}\right)$. It then follows by Faltings' isogeny theorem that E' is isogenous to E_D, where for the elliptic curve E given by the equation $y^2 = f(x)$ we have that E_D is given by the equation $Dy^2 = f(x)$. Clearly if the conductors of E and E' are square free, then E and E' are isogenous.

7.2. Proof of Theorem 1 in the CM case

An elliptic curve E over $\mathbb{Q}$ is of CM-type if $\mathrm{End}(E) \otimes \mathbb{Q} = K$, with $K = \mathbb{Q}(\sqrt{-D})$ an imaginary quadratic number field. We have that $L(E, s) = L(\eta, s - 1/2)$ for some unitary Hecke character η of the idele class group C_K. Let $\pi = I_K^{\mathbb{Q}}(\eta)$ be the associated dihedral representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. Denote by π' the cuspidal automorphic representation $I_K^{\mathbb{Q}}(\eta^2)$ of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$. By (2.18) we have

$$L(Sym^2\,\pi, s) = L(\pi', s)L(\eta_0, s),$$

where η_0 is the restriction of η to $C_{\mathbb{Q}}$. Twisting by some character χ gives

$$L(Sym^2\,\pi \otimes \chi, s) = L(\pi' \otimes \chi, s)L(\eta_0 \otimes \chi, s).$$

Note that $L(\pi' \otimes \chi, s)L(\eta_0 \otimes \chi, s)$ is entire unless $\eta_0 \otimes \chi$ is trivial, in which case

$$L(Sym^2\,\pi \otimes \eta_0^{-1}, s) = L(\pi' \otimes \eta_0^{-1}, s)\zeta(s)$$

has a pole at $s = 1$. Hence, we have that $L(Sym^2\,\pi \otimes \chi, s)$ is entire unless $\chi = \eta_0^{-1}$.

Proof of Theorem 1 in the CM case. Let π and π' be the isobaric sums of unitary cuspidal automorphic representations over $\mathrm{GL}(3, \mathbb{A}_{\mathbb{Q}})$ associated to $Sym^2\,E$ and $Sym^2\,E'$ respectively. Just as in the non-CM case, it follows that if E has semistable reduction at p we have that

$$L(\pi \otimes \chi, 1) = C_1 C_2^{m_\chi} L(\pi' \otimes \chi, 1)$$

for all wild p-power characters χ of conductor p^{m_χ} with m_χ sufficiently large and by the discussion above, the twisted L-functions are entire. Then by Theorem 2 we conclude that $\pi \cong \pi'$, and the proof proceeds as in the non-CM case. $\square$

Remark. Suppose E and E' are CM elliptic curves and let η and η' be their associated idele class characters over the imaginary quadratic number fields K and K' respectively. If we let π, π' be the representations induced by the characters η, η', then they are dihedral. Just as before,

$$L(Sym^2\,\pi, s) = L\left(I_K^{\mathbb{Q}}(\eta^2), s\right) L(\eta_0, s) \tag{7.4}$$

where η_0 denotes the restriction of η to $\mathbb{Q}$, and similarly for π'. If $K = K'$ then $\eta_0 = \eta_0'$. Hence, Theorem 1 for E, E' as above is a consequence of Lemma 4 and Theorem A in [14], since $I_K^{\mathbb{Q}}(\eta^2)$ is a cuspidal automorphic representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$.

It is unclear if for $K \neq K'$ Theorem 1 can be reduced to a consequence of a result on the determination of $\mathrm{GL}(2)$ cusp forms. The special values $L(\eta_0 \otimes \chi, 1)$ and $L(\eta_0 \otimes \chi, 2)$ can be expressed in terms of the generalized Bernoulli numbers $B_{1,\overline{\eta_0\chi}}$ and $B_{2,\overline{\eta_0\chi}}$ respectively, but there is no clear way to separate the contributions from η_0 and χ in $L(\eta_0 \otimes \chi, 1)$.

Acknowledgments

The author would like to thank her advisor Dinakar Ramakrishnan, Wenzhi Luo, Philippe Michel and Gergely Harcos for useful discussions. The author would also like to thank the referee for their comments which improved the presentation of this paper.

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