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Explicit examples for the Breuil–Mézard conjecture



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ABSTRACT

In this paper, we compute some universal deformation rings for certain rank two Galois representations. We then study the relations between different deformation rings. These relations give explicit examples for the Breuil–Mézard conjecture.

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1. Introduction

Fix a prime number $p \geq 3$, a finite extension \mathbb{F}/\mathbb{F}_p . Let $\bar{r} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\mathbb{F})$ be a rank two continuous representation. In this paper, for certain non-split \bar{r} , we compute a Fontaine–Laffaille deformation ring of \bar{r} and a potentially Barsotti–Tate deformation ring of \bar{r} . We then construct a canonical isomorphism between these two deformation rings, which provides an explicit example for the geometric Breuil–Mézard conjecture.

1.1. The geometric Breuil–Mézard conjecture

Let L be a finite totally ramified extension of $W(\mathbb{F})[1/p]$ with ring of integers \mathcal{O} and uniformiser π . We assume that L is sufficiently large, in particular, we assume that

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$\#\mathbb{F} > 5$, so that $\mathrm{PGL}_2(\mathbb{F})$ is a simple group. (Note that this condition is not essential for the computation in this paper, but it is needed in [7] to prove Theorem 1.3 and in Section 4.1 for the global situation.) Let $\tau : I_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(L)$ be an inertial type, i.e., a representation with open kernel which extends to $W_{\mathbb{Q}_p}$. Let ϵ and ω be the p -adic cyclotomic character and the mod p cyclotomic character respectively. Fix integers a, b with $b \geq 0$ and a character $\psi : G_{\mathbb{Q}_p} \rightarrow \mathcal{O}^\times$ such that $\overline{\psi}\epsilon = \det \bar{r}$. We let $R^{\square, \psi}(a, b, \tau, \bar{r})$ and $R_{cr}^{\square, \psi}(a, b, \tau, \bar{r})$ be the framed deformation \mathcal{O} -algebras which are universal for framed deformations of \bar{r} with determinant $\psi\epsilon$, and are potentially semistable (respectively potentially crystalline) with Hodge–Tate weights $(a, a + b + 1)$ and inertia type τ . Let $\sigma(\tau)$ and $\sigma^{cr}(\tau)$ denote the finite-dimensional irreducible L -representation of $\mathrm{GL}_2(\mathbb{Z}_p)$ corresponding to τ via Henniart’s inertial local Langlands correspondence. We set $\sigma(a, b, \tau) = \sigma(\tau) \otimes_L \det^a \mathrm{Sym}^b L^2$ and $\sigma^{cr}(a, b, \tau) = \sigma^{cr}(\tau) \otimes_E \det^a \mathrm{Sym}^b L^2$. We let $L_{a, b, \tau}$ (respectively $L_{a, b, \tau}^{cr}$) be a $\mathrm{GL}_2(\mathbb{Z}_p)$ -stable \mathcal{O} -lattice in $\sigma(a, b, \tau)$ (respectively $\sigma^{cr}(a, b, \tau)$). Write $\sigma_{m, n}$ for the representation $\det^m \otimes \mathrm{Sym}^n \mathbb{F}^2$ of $\mathrm{GL}_2(\mathbb{F}_p)$, $0 \leq m \leq p - 2$, $0 \leq n \leq p - 1$. Then we may write

$$(L_{a, b, \tau} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \bigoplus_{m, n} \sigma_{m, n}^{a_{m, n}},$$

and

$$(L_{a, b, \tau}^{cr} \otimes_{\mathcal{O}} \mathbb{F})^{ss} \xrightarrow{\sim} \bigoplus_{m, n} \sigma_{m, n}^{a_{m, n}^{cr}},$$

for some integers $a_{m, n}$ and $a_{m, n}^{cr}$. The geometric Breuil–Mézard conjecture is the following.

Conjecture 1.1. *There are cycles $\mathcal{C}_{m, n}(\bar{r})$ depending only on m, n , and \bar{r} such that for any a, b, τ ,*

$$Z(R^{\square, \psi}(a, b, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n} \mathcal{C}_{m, n}(\bar{r})$$

and

$$Z(R_{cr}^{\square, \psi}(a, b, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n}^{cr} \mathcal{C}_{m, n}(\bar{r}).$$

Remark 1.2. See [7, Section 1.1] for the definition of cycles. As remarked in [7, Remark 3.1.5], or by [7, Lemma 4.3.1], the truth of the conjecture is independent of the choice of ψ . We may assume that ψ is crystalline. If Conjecture 1.1 is true for all a, b, τ , then $\mathcal{C}_{m, n}(\bar{r}) = 0$ unless $\det \bar{r}|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$. Furthermore, if $\det \bar{r}|_{I_{\mathbb{Q}_p}} = \omega^{2m+n+1}$, we must have

$$\mathcal{C}_{m, n}(\bar{r}) = Z(R_{cr}^{\square, \psi}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi),$$

where \tilde{m} is chosen so that $\psi|_{I_{\mathbb{Q}_p}} = \epsilon^{2\tilde{m}+n}$ and $\tilde{m} \equiv m \pmod{p-1}$.

Using the numerical Breuil–Mézard conjecture, which is proved in many cases by Kisin in [9], Emerton and Gee proved the following result in [7].

Theorem 1.3 (Emerton, Gee). *If $\bar{r} \approx \begin{pmatrix} \omega\chi & * \\ 0 & \chi \end{pmatrix}$ for any χ , then Conjecture 1.1 holds for \bar{r} .*

1.2. A special case of the geometric Breuil–Mézard conjecture

Let P be the property of being crystalline, or being semistable, or having Hodge–Tate weights W . We say that a representation $r : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}(V)$ in characteristic p has property P if there is a representation $\tilde{r} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}(\tilde{V})$ in characteristic 0 with property P , such that r is the reduction of \tilde{r} .

Let us consider the following special case. Assume that τ is a type such that in the right hand side of the equality

$$Z(R^{\square, \psi}(0, 0, \tau, \bar{r})/\pi) = \sum_{m, n} a_{m, n} \mathcal{C}_{m, n}(\bar{r})$$

only one term with $a_{m, n} = 1$ is nonzero. Then we should have an equality like

$$Z(R^{\square, \psi}(0, 0, \tau, \bar{r})/\pi) = Z(R_{cr}^{\square, \psi}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi). \quad (1.1)$$

This means that, in characteristic p , a deformation of \bar{r} with determinant $\overline{\epsilon\psi}$ is potentially Barsotti–Tate of type τ if and only if this deformation of \bar{r} is crystalline of Hodge–Tate weights $(\tilde{m}, \tilde{m} + n + 1)$. This kind of statement is certainly not true in characteristic 0, since the Hodge–Tate weights do not change by restricting to a subgroup of finite index.

Theorem 1.3 is proved by an abstract argument. Although there is a close relation between weights and types, it is not obvious that equality (1.1) holds. In this paper, for a certain type of representations \bar{r} , we construct an explicit isomorphism

$$R_{cr}^{\square}(\tilde{m}, n, \mathbf{1}, \bar{r})/\pi \rightarrow R^{\square}(0, 0, \tau, \bar{r})/\pi,$$

which gives us equality (1.1). In the following, we write $R(a, b, \tau, \bar{r})$ or $R_{cr}(a, b, \tau, \bar{r})$ for the universal deformation rings without frame.

We recall the definition of the category $\mathrm{MF}_{\mathcal{O}}^{p-2}$ of Fontaine–Laffaille modules. (See [6, Section 2.4.1] for more details.) A Fontaine–Laffaille module over \mathcal{O} is a finite free \mathcal{O} -module M together with a decreasing filtration $\mathrm{Fil}^i M$ by \mathcal{O} -submodules and Frobenius-linear, \mathcal{O} -linear maps $\phi_i : \mathrm{Fil}^i M \rightarrow M$ for all $0 \leq i \leq p-2$ such that

- (1) $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^{p-1} M = 0$.
- (2) For all $0 \leq i \leq p-3$, we have $\phi_i|_{\mathrm{Fil}^{i+1} M} = p\phi_{i+1}$.
- (3) $\sum_{i=0}^{p-2} \phi_i(\mathrm{Fil}^i M) = M$.

There is an exact functor T_{cris} from the category $\mathrm{MF}_{\mathcal{O}}^{p-2}$ to the category of $G_{\mathbb{Q}_p}$ -representations on free \mathcal{O} -modules defined by

$$T_{cris}(M) = \mathrm{Hom}_{\mathrm{Fil}, \mathrm{Frob}, \mathcal{O}}(M, A_{cris})^*.$$

This gives us an equivalence of categories between $\mathrm{MF}_{\mathcal{O}}^{p-2}$ and the category of $G_{\mathbb{Q}_p}$ -stable \mathcal{O} -lattices in crystalline L -representations of $G_{\mathbb{Q}_p}$ with all Hodge–Tate weights in $[0, p-2]$. Let $\mathrm{MF}_{\mathbb{F}}^{p-2}$ be the category of Fontaine–Laffaille modules over \mathbb{F} . Note that in $\mathrm{MF}_{\mathbb{F}}^{p-2}$, we have $\phi_i|_{\mathrm{Fil}^{i+1} M} = p\phi_{i+1} = 0$ for all $0 \leq i \leq p-3$.

Fix an integer $k \in [1, p-2]$. Let M be the object in $\mathrm{MF}_{\mathbb{F}}^{p-2}$ given by the following form:

$$\begin{aligned} M &= \mathbb{F}\langle E, F \rangle, \\ \mathrm{Fil}^i M &= \begin{cases} M & \text{if } i \leq 0, \\ \mathbb{F}\langle F \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\ \phi_0(E) &= \alpha E, \quad \phi_k(F) = \gamma E + \beta F. \end{aligned}$$

In the above formulae, if we replace \mathbb{F} by \mathcal{O} , α, β, γ by their Teichmüller lifts in \mathcal{O} , we obtain an object $M' \in \mathrm{MF}_{\mathcal{O}}^{p-2}$ such that $M' \otimes_{\mathcal{O}} \mathbb{F} = M$. Define

$$\bar{r} = T_{cris}(M) = \mathrm{Hom}_{\mathrm{Fil}, \mathrm{Frob}, \mathbb{F}}(M, A_{cris})^*,$$

which is a rank two \mathbb{F} -representation of $G_{\mathbb{Q}_p}$. We assume further that $\alpha\beta\gamma \neq 0$, in which case, \bar{r} is a nontrivial extension of $\mathrm{unr}(\beta) \cdot \omega^k$ by $\mathrm{unr}(\alpha)$, where $\mathrm{unr}(x) : G_{\mathbb{Q}_p} \rightarrow \mathbb{F}^\times$ is the unramified character which sends the geometric Frobenius to x . (See [3, Example 5.1.(1)].) Let $\tilde{\omega}$ be the Teichmüller lift of ω . The main result of this paper is the following one.

Theorem 1.4. *Let $j = p - k$. There is a natural isomorphism*

$$R_{cr}(0, k-1, \mathbf{1}, \bar{r})/\pi \rightarrow R(0, 0, \mathbf{1} \oplus \tilde{\omega}^j, \bar{r})/\pi,$$

which induces an isomorphism between the corresponding Galois representations. In other words, in characteristic p , a deformation of \bar{r} is crystalline with Hodge–Tate weights $(0, k)$ if and only if it is potentially Barsotti–Tate with type $\mathbf{1} \oplus \tilde{\omega}^j$.

Remark 1.5. As mentioned before, this result is a special case of the Breuil–Mézard conjecture. The existence of the isomorphism is a consequence of the results (for example [7, Theorem 5.5.4]) of Emerton and Gee. Nevertheless, the proof here is new and more explicit.

The content of this paper is as follows. In Section 2, we construct a Breuil module with descent data \mathcal{M} such that the generic fiber of \mathcal{M} is isomorphic to \bar{r} . By the relation between Breuil modules and finite flat group schemes, this proves Theorem 1.4 for the trivial deformation of \bar{r} . In Section 3, we compute the deformations of M and \mathcal{M} , which give us $R_{cr}(0, k-1, \mathbf{1}, \bar{r})/\pi$ and $R(0, 0, \mathbf{1} \oplus \tilde{\omega}^j, \bar{r})/\pi$. With the explicit description of the rings, we then construct the isomorphism. In Section 4, we construct a commutative diagram for a global Galois representation by applying Theorem 1.4 and explain how we may prove similar results in more general settings.

2. M and \mathcal{M} have isomorphic generic fibers

2.1. Breuil modules with descent data

Let us first recall the definition of Breuil modules with descent data. (See also for example [2] and [8].) Let $K = \mathbb{Q}_p((-p)^{\frac{1}{p-1}})$. Then K is a totally ramified extension of \mathbb{Q}_p with degree $d = p-1$. Let $G = \text{Gal}(K/\mathbb{Q}_p)$ and $\mathcal{S} = \mathbb{F}[u]/u^{dp}$. We may consider ω as a function on G . For any $g \in G$, we write $[g] : \mathcal{S} \rightarrow \mathcal{S}$ to be the \mathbb{F} -linear endomorphism of \mathcal{S} as an \mathbb{F} -algebra such that $[g](u) = \omega(g)u$. Let $\phi : \mathcal{S} \rightarrow \mathcal{S}$ be the \mathbb{F} -linear endomorphism of \mathcal{S} such that $\phi(u) = u^p$.

Definition 2.1. Let $\kappa \in [2, p-1]$ be an integer. The category $\text{BrMod}_{dd, K/\mathbb{Q}_p}^{\kappa-1}$ consists of quintuples $(\mathcal{M}, \text{Fil}^{\kappa-1} \mathcal{M}, \phi_{\kappa-1}, \{[g]\}, N)$ where:

- (1) \mathcal{M} is a finitely generated free \mathcal{S} -module.
- (2) $\text{Fil}^{\kappa-1} \mathcal{M}$ is an \mathcal{S} -submodule of \mathcal{M} containing $u^{d(\kappa-1)} \mathcal{M}$.
- (3) $\phi_{\kappa-1} : \text{Fil}^{\kappa-1} \mathcal{M} \rightarrow \mathcal{M}$ is an \mathbb{F} -linear and ϕ -semi-linear map with image generating \mathcal{M} as an \mathcal{S} -module.
- (4) $N : \mathcal{M} \rightarrow u\mathcal{M}$ is an \mathbb{F} -linear map such that

$$\begin{aligned} N(ux) &= uN(x) - ux \quad \forall x \in \mathcal{M}, \\ u^d N(\text{Fil}^{\kappa-1} \mathcal{M}) &\subset \text{Fil}^{\kappa-1} \mathcal{M}, \\ \phi_{\kappa-1}(u^d N(x)) &= N(\phi_{\kappa-1}(x)) \quad \forall x \in \text{Fil}^{\kappa-1} \mathcal{M}. \end{aligned}$$

- (5) $[g] : \mathcal{M} \rightarrow \mathcal{M}$ are additive bijections for each $g \in G$, preserving $\text{Fil}^{\kappa-1} \mathcal{M}$, commuting with the $\phi_{\kappa-1}$ -, N -, and \mathbb{F} -actions, and satisfying $[g_1] \circ [g_2] = [g_1 g_2]$ for all $g_1, g_2 \in G$, and $[1]$ is the identity map. Furthermore, if $a \in \mathbb{F}$, $m \in \mathcal{M}$, then

$$[g](au^i m) = a\omega(g)^i u^i [g](m).$$

Remark 2.2. (1) If $\kappa = 2$, the category $\text{BrMod}_{dd, K/\mathbb{Q}_p}^1$ is equivalent to the category of finite flat group schemes over \mathcal{O}_K together with an \mathbb{F} -action and descent data on the generic fiber from K to \mathbb{Q}_p . In this case it follows from other axioms that there is always a unique N which satisfies the required properties. See for example [2, Proposition 5.1.3].

(2) If $\kappa \leq \kappa'$, then there is a fully faithful functor $J : \text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa-1} \rightarrow \text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa'-1}$ which identifies $\text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa-1}$ as a full subcategory of $\text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa'-1}$. More precisely, if $\mathcal{M} = (\mathcal{M}, \text{Fil}^{\kappa-1} \mathcal{M}, \phi_{\kappa-1}, \{[g]\}, N)$ is an object in $\text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa-1}$, then

$$J(\mathcal{M}) = (J(\mathcal{M}), \text{Fil}^{\kappa'-1} J(\mathcal{M}), \phi_{\kappa'-1}, \{[g]\}, N)$$

where $J(\mathcal{M}) = \mathcal{M}$, $\text{Fil}^{\kappa'-1} J(\mathcal{M}) = u^{d(\kappa'-\kappa)} \text{Fil}^{\kappa-1} \mathcal{M}$, $\phi_{\kappa'-1}(u^{d(\kappa'-\kappa)} x) = \phi_{\kappa-1}(x)$, and $N, [g]$ remain the same.

(3) Let $\text{Rep}_{\mathbb{F}}(G_{\mathbb{Q}_p})$ be the category of representations of $G_{\mathbb{Q}_p}$ over \mathbb{F} -vector spaces. There is a covariant functor

$$T_{st} : \text{BrMod}_{dd,K/\mathbb{Q}_p}^{\kappa-1} \rightarrow \text{Rep}_{\mathbb{F}}(G_{\mathbb{Q}_p}).$$

We call the Galois representation $T_{st}(\mathcal{M})$ the *generic fiber* of \mathcal{M} .

Let \mathcal{M} be the object in $\text{BrMod}_{dd,K/\mathbb{Q}_p}^1$ given by the following data:

$$\begin{aligned} \mathcal{M} &= \mathcal{S}\langle e, f \rangle, \\ \text{Fil}^1 \mathcal{M} &= \langle u^d e, f + te \rangle, \\ \phi_1(u^d e) &= \alpha e, \quad \phi_1(f + te) = \beta f, \\ [g](e) &= e, \quad [g](f) = \omega(g)^j f, \\ N(e) &= 0, \quad N(f) = ne. \end{aligned}$$

Here $j = p - k$, $t = -\frac{\gamma}{\alpha} u^j$, $n = \frac{\gamma}{\beta} j u^{pj}$, α, β, γ, k are the numbers we used to define the Fontaine–Laffaille module M . It is easy to check that \mathcal{M} is a well defined Breuil module.

Proposition 2.3. *The generic fiber of \mathcal{M} is isomorphic to \bar{r} , i.e., $T_{st}(\mathcal{M}) \cong \bar{r}$.*

2.2. Proof of Proposition 2.3

In general, it is not easy to describe explicitly the generic fibers attached to Fontaine–Laffaille modules or Breuil modules. Fortunately, in our case, we can compare the generic fibers without explicitly determining either. The idea is to embed the categories $\text{MF}_{\mathbb{F}}^{p-2}$ and $\text{BrMod}_{dd,K/\mathbb{Q}_p}^1$ into a larger category $\text{BrMod}_{dd,K/\mathbb{Q}_p}^{p-2}$, thus we obtain two objects \tilde{M} and $\tilde{\mathcal{M}}$. Then to show that M and \mathcal{M} have isomorphic generic fibers, it suffices to construct a certain morphism between \tilde{M} and $\tilde{\mathcal{M}}$. We carry out this construction in the following.

By the construction in [1, Section 2.4], we have a functor $\mathcal{F}^{p-2} : \text{MF}_{\mathbb{F}}^{p-2} \rightarrow \text{BrMod}^{p-2}$ given by $M \mapsto \mathbb{F}[u]/u^p \otimes_{\mathbb{F}} M$. Here BrMod^{p-2} is the category of Breuil modules with no descent data. More precisely, for our M , $\mathcal{F}^{p-2}(M)$ is given by the following data:

$$\begin{aligned}
\mathcal{F}^{p-2}(M) &= \mathbb{F}[u]/u^p \langle E, F \rangle, \\
\mathrm{Fil}^{p-2}(\mathcal{F}^{p-2}(M)) &= \mathbb{F}[u]/u^p \langle u^{p-2}E, u^{p-2-k}F \rangle, \\
\phi_{p-2}(u^{p-2}E) &= \alpha E, \phi_{p-2}(u^{p-2-k}F) = \gamma E + \beta F, \\
N &= 0.
\end{aligned}$$

By embedding BrMod^{p-2} into $\mathrm{BrMod}_{dd,K/\mathbb{Q}_p}^{p-2}$, we obtain \tilde{M} , which has the following form:

$$\begin{aligned}
\tilde{M} &= \mathcal{S} \langle E, F \rangle, \\
\mathrm{Fil}^{p-2} \tilde{M} &= \mathcal{S} \langle u^{d(p-2)}E, u^{d(p-2-k)}F \rangle, \\
\phi_{p-2}(u^{d(p-2)}E) &= \alpha E, \phi_{p-2}(u^{d(p-2-k)}F) = \gamma E + \beta F, \\
[g] &= 1 \text{ for all } g \in G, \\
N &= 0.
\end{aligned}$$

By Remark 2.2(2), it is easy to see that $\tilde{\mathcal{M}}$ has the following form:

$$\begin{aligned}
\tilde{\mathcal{M}} &= \mathcal{S} \langle e, f \rangle, \\
\mathrm{Fil}^{p-2} \tilde{\mathcal{M}} &= \mathcal{S} \langle u^{d(p-2)}e, u^{d(p-3)}(f + te) \rangle, \\
\phi_{p-2}(u^{d(p-2)}e) &= \alpha e, \phi_{p-2}(u^{d(p-3)}(f + te)) = \beta f, \\
[g](e) &= e, [g](f) = \omega(g)^j f, \\
N(e) &= 0, N(f) = ne,
\end{aligned}$$

with $j = p - k$, $t = -\frac{\gamma}{\alpha}u^j$, $n = \frac{\gamma}{\beta}ju^{pj}$ as before.

We define a morphism $T : \tilde{M} \rightarrow \tilde{\mathcal{M}}$ of \mathcal{S} -modules by letting

$$\begin{aligned}
E &\mapsto e, \\
F &\mapsto u^{pj'}f.
\end{aligned}$$

Here $j' = k - 1 = d - j$. We check that this is a morphism in $\mathrm{BrMod}_{dd,K/\mathbb{Q}_p}^{p-2}$.

(1) T sends $\mathrm{Fil}^{p-2} \tilde{M}$ to $\mathrm{Fil}^{p-2} \tilde{\mathcal{M}}$.

Indeed, we have $T(u^{d(p-2)}E) = u^{d(p-2)}e$, and

$$\begin{aligned}
T(u^{d(p-2-k)}F) &= u^{d(p-2-k)+pj'}f \\
&= u^{d(p-2)-j}(f + te) + \frac{\gamma}{\alpha}u^{d(p-2)}e.
\end{aligned}$$

The claim follows.

(2) T commutes with ϕ_{p-2} .

Indeed, $\phi_{p-2}(T(u^{d(p-2)}E)) = \phi_{p-2}(u^{d(p-2)}e) = \alpha e = T(\phi_{p-2}(u^{d(p-2)}E))$, and

$$\begin{aligned}\phi_{p-2}(T(u^{d(p-2-k)}F)) &= \phi_{p-2}(u^{d(p-2)-j}(f + te) + \frac{\gamma}{\alpha}u^{d(p-2)}e) \\ &= u^{pj'}\beta f + \gamma e = T(\phi_{p-2}(u^{d(p-2-k)}F)).\end{aligned}$$

The claim follows.

(3) T commutes with $[g]$ for $g \in G$.

Indeed, it is obvious that $T([g]E) = [g](T(E))$. Also, note that $\omega^p = \omega$, we have

$$[g](T(F)) = \omega(g)^{pj'+j}u^{pj'}f = u^{pj'}f = T([g]F).$$

The claim follows.

(4) T commutes with N .

Indeed,

$$N(u^{pj'}f) = u^{pj'}N(f) - pj'u^{pj'}f = u^{pj'}\frac{\gamma}{\beta}u^{pj}e = \frac{\gamma}{\beta}u^{pd}e = 0.$$

The claim follows easily.

We see that $T : \tilde{M} \rightarrow \tilde{\mathcal{M}}$ is a morphism in $\text{BrMod}_{dd,K/\mathbb{Q}_p}^{p-2}$. It induces a morphism on generic fibers

$$T_{st}(T) : \bar{r} = \bar{r}_{\tilde{M}} \rightarrow \bar{r}_{\tilde{\mathcal{M}}} = \bar{r}_{\mathcal{M}}.$$

It is easy to check that $T_{st}(T)$ is an isomorphism on the rank one sub representations and the rank one quotient representations, thus $T_{st}(T)$ is an isomorphism. The generic fibers of M and \mathcal{M} are isomorphic.

3. Deformations of M and \mathcal{M}

3.1. Deformations of M

To study the crystalline deformations of \bar{r} with Hodge–Tate weights $(0, k)$, it suffices to study the deformations of the Fontaine–Laffaille module M . By [6, Corollary 2.4.3], we know that $R_{cr}(0, k-1, \mathbf{1}, \bar{r}) = \mathcal{O}[[X, W]]$. Therefore, $R_{cr}(0, k-1, \mathbf{1}, \bar{r})/\pi = \mathbb{F}[[X, W]]$. We give an explicit description of a deformation of M over $\mathbb{F}[[X, W]]$ which induces the corresponding universal Galois representation.

Let $N \in \text{Ext}^1(M, M)$ be an extension of M by M in $\text{MF}_{\mathbb{F}}^{p-2}$, we may assume that N has the following form:

$$\begin{aligned}
N &= \mathbb{F}\langle E, F, E', F' \rangle, \\
\mathrm{Fil}^i N &= \begin{cases} N & \text{if } i \leq 0, \\ \mathbb{F}\langle F, F' \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\
\phi_0(E') &= \alpha E' + XE + YF, \quad \phi_k(F') = \gamma E' + \beta F' + ZE + WF.
\end{aligned}$$

Here, E, F are basis of M , E', F' are lifts of E, F in N , X, Y, Z, W are elements in \mathbb{F} .

Lemma 3.1. *With the above notation, we may assume that $Y = Z = 0$.*

Proof. Let $E'' = E' + \alpha^{-1}YF$. This does not change the form of Fil^i . Note that $\phi_0(E'') = \phi_0(E') = \alpha E'' + XE$, we may assume that $Y = 0$.

Let $\Delta = -\gamma^{-1}Z$ and $F'' = F' + \Delta F$. This does not change the form of Fil^i . Moreover, we have

$$\begin{aligned}
\phi_k(F'') &= \phi_k(F') + \phi_k(\Delta F) \\
&= \gamma E' + \beta F' + ZE + WF + \Delta(\gamma E + \beta F) \\
&= \gamma E' + \beta F'' + WF.
\end{aligned}$$

Thus we may assume that $Z = 0$. \square

Every infinitesimal deformation of M has the following form, with $X, W \in \mathbb{F}$.

$$\begin{aligned}
N &= \mathbb{F}\langle E, F, E', F' \rangle, \\
\mathrm{Fil}^i N &= \begin{cases} N & \text{if } i \leq 0, \\ \mathbb{F}\langle F, F' \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\
\phi_0(E') &= \alpha E' + XE, \quad \phi_k(F') = \gamma E' + \beta F' + WF.
\end{aligned}$$

3.2. Deformations of \mathcal{M}

Using the same idea as above, we compute the space $\mathrm{Ext}^1(\mathcal{M}, \mathcal{M})$. By [Remark 2.2\(1\)](#), we do not have to specify the monodromy N . See [\[4, Section 4\]](#) for the computation in a more general setting. Let \mathcal{N} be such an object with basis e, f, e', f' over \mathcal{S} .

Lemma 3.2. *We may assume that $\mathrm{Fil}^1 \mathcal{N} = \langle u^d e, f + te, u^d e', f' + te' \rangle$.*

Proof. In general, we can write $\mathrm{Fil}^1 \mathcal{N} = \langle u^d e, f + te, u^d e' + Ae + Bf, f' + te' + Ce + Df \rangle$ with $A, B, C, D \in \mathcal{S}$. Since $f + te \in \mathrm{Fil}^1 \mathcal{N}$, we may assume that $B = D = 0$. Since $u^d \mathcal{N} \subset \mathrm{Fil}^1 \mathcal{N}$, we have $u^d e' \in \mathrm{Fil}^1 \mathcal{N}$ and thus $Ae \in \mathrm{Fil}^1 \mathcal{N}$. Therefore $u^d \mid A$ and we

may assume that $A = 0$. Finally, replacing f' by $f' + Ce$, we may assume that $C = 0$. The lemma follows. \square

By [Lemma 3.2](#), we may assume that \mathcal{N} has the following form:

$$\begin{aligned}\mathcal{N} &= \mathcal{S}\langle e, f, e', f' \rangle, \\ \text{Fil}^1 \mathcal{N} &= \langle u^d e, f + te, u^d e', f' + te' \rangle, \\ \phi_1(u^d e') &= \alpha e' + xe + yf, \quad \phi_1(f' + te') = \beta f' + ze + wf, \\ [g]e' &= e' + Ae + Bf, \quad [g]f' = \omega(g)^j f' + Ce + Df.\end{aligned}$$

Here A, B, C, D, x, y, z, w are elements in \mathcal{S} .

Lemma 3.3. *We may assume that $A = B = C = D = 0$.*

Proof. This is a special case of [\[4, Lemma 4.4\]](#). \square

Lemma 3.4. *We may assume that x and w are elements in \mathbb{F} , $y \in \mathcal{S}$ is either 0 or a monomial of degree j' , $z \in \mathcal{S}$ is either 0 or a monomial of degree j .*

Proof. This is a special case of [\[4, Lemma 4.5\]](#). \square

Lemma 3.5. *We may assume that $y = z = 0$.*

Proof. Let $e'' = e' + \alpha^{-1}yf$. First, since $u^{j'}f = u^{j'}(f + te) - u^{j'}te \in \text{Fil}^1 \mathcal{M}$, this does not change the form of Fil^1 . Moreover,

$$\phi_1(u^d e'') = \phi_1(u^d e') + \phi_1(u^d(\alpha^{-1}yf)) = \phi_1(u^d e') = \alpha e'' + xe,$$

we may assume that $y = 0$.

By [Lemma 3.4](#), there exists $\Delta \in \mathbb{F}$ such that $z = \Delta\beta t$. Let $f'' = f' + \Delta(f + te)$. Certainly, this does not change the form of Fil^1 . Furthermore,

$$\phi_1(f'' + te') = \phi_1(f' + te') + \phi_1(\Delta(f + te)) = \beta f'' + wf.$$

Thus we may assume $z = 0$.

Finally, by [Lemma 3.4](#) again, the above changes of basis do not change the form of $[g]$. The lemma follows. \square

By [\[5, Equation \(6.10\)\]](#), we know that $\dim_{\mathbb{F}} \text{Ext}^1(\mathcal{M}, \mathcal{M}) = 2$. Every infinitesimal deformation of \mathcal{M} has the following form, with $x, w \in \mathbb{F}$.

$$\begin{aligned}
\mathcal{N} &= \mathcal{S}\langle e, f, e', f' \rangle, \\
\mathrm{Fil}^1 \mathcal{N} &= \langle u^d e, f + te, u^d e', f' + te' \rangle, \\
\phi_1(u^d e') &= \alpha e' + xe, \quad \phi_1(f' + te') = \beta f' + wf, \\
[g]e' &= e', \quad [g]f' = \omega(g)^j f', \\
N(e') &= 0, \quad N(f') = ne' + ce \text{ with } c = \left(\frac{\gamma}{\alpha}x - \frac{\gamma}{\beta}\right)j\beta^{-1}u^{pj}.
\end{aligned}$$

Here we give the formula of the monodromy operator for later computation. Note that this operator is uniquely determined by other data.

3.3. The isomorphism between the infinitesimal deformations

We have seen that the infinitesimal deformations of M and \mathcal{M} are parameterized by (X, W) and (x, w) respectively. In the following, we show that if we choose $X = x$ and $W = w$, then the corresponding N and \mathcal{N} have isomorphic generic fibers. Starting with the Fontaine–Laffaille module N , we obtain an object \tilde{N} in $\mathrm{BrMod}_{dd, K/\mathbb{Q}_p}^{p-2}$ given by the following data:

$$\begin{aligned}
\tilde{N} &= \langle E, F, E', F' \rangle, \\
\mathrm{Fil}^{p-2} \tilde{N} &= \langle u^{d(p-2)} E, u^{d(p-2-k)} F, u^{d(p-2)} E', u^{d(p-2-k)} F' \rangle, \\
\phi_{p-2}(u^{d(p-2)} E') &= \alpha E' + XE, \quad \phi_{p-2}(u^{d(p-2-k)} F') = \gamma E' + \beta F' + WF, \\
[g] &= 1 \text{ for all } g \in G, \\
N &= 0.
\end{aligned}$$

The Breuil module \mathcal{N} also gives us an object $\tilde{\mathcal{N}}$ in $\mathrm{BrMod}_{dd, K/\mathbb{Q}_p}^{p-2}$, which has the following form.

$$\begin{aligned}
\tilde{\mathcal{N}} &= \langle e, f, e', f' \rangle, \\
\mathrm{Fil}^{p-2} \tilde{\mathcal{N}} &= \langle u^{d(p-2)} e, u^{d(p-3)}(f + te), u^{d(p-2)} e', u^{d(p-3)}(f' + te') \rangle, \\
\phi_{p-2}(u^{d(p-2)} e') &= \alpha e' + xe, \quad \phi_{p-2}(u^{d(p-3)}(f' + te')) = \beta f' + wf, \\
[g]e' &= e', \quad [g]f' = \omega(g)^j f', \\
N(e') &= 0, \quad N(f') = ne' + ce \text{ with } c = \left(\frac{\gamma}{\alpha}x - \frac{\gamma}{\beta}\right)j\beta^{-1}u^{pj}.
\end{aligned}$$

Define a morphism of \mathcal{S} -modules $T : \tilde{N} \rightarrow \tilde{\mathcal{N}}$ by letting

$$\begin{aligned}
E &\mapsto e, \\
F &\mapsto u^{pj'} f, \\
E' &\mapsto e', \\
F' &\mapsto u^{pj'} f'.
\end{aligned}$$

We check that T is morphism of Breuil modules. Since the filtration and the descent data on \tilde{N} (resp. on $\tilde{\mathcal{N}}$) are direct sums of the filtration and the descent data on \tilde{M} (resp. on $\tilde{\mathcal{M}}$), T maps $\mathrm{Fil}^{p-2} \tilde{N}$ into $\mathrm{Fil}^{p-2} \tilde{\mathcal{N}}$ and T commutes with $[g]$.

(1) T commutes with ϕ_{p-2} .

Indeed, $\phi_{p-2}(T(u^{d(p-2)}E')) = \phi_{p-2}(u^{d(p-2)}e') = \alpha e' = T(\phi_{p-2}(u^{d(p-2)}E'))$, and

$$\begin{aligned} \phi_{p-2}(T(u^{d(p-2-k)}F')) &= \phi_{p-2}(u^{d(p-2)-j}(f' + te') + \frac{\gamma}{\alpha}u^{d(p-2)}e') \\ &= u^{pj'}\beta f' + \gamma e' + wu^{pj'}f = T(\phi_{p-2}(u^{d(p-2-k)}F)). \end{aligned}$$

The claim follows.

(2) T commutes with N .

Indeed, since $\deg_u n = \deg_u c = pj$, we have

$$N(u^{pj'}f') = u^{pj'}N(F') - pj'u^{pj'}F' = u^{pj'}(ne' + ce) = 0.$$

The claim follows easily.

Since T is a morphism of Breuil modules, we obtain a morphism $T_{st}(T)$ between the corresponding generic fibers. Note that $T_{st}(T)$ is an isomorphism on the rank two sub representations and on the rank two quotient representations, $T_{st}(T)$ is an isomorphism. This proves the claim.

3.4. The isomorphism between the universal deformations

By the discussion in Section 3.1, the universal deformation M^u of M over $\mathbb{F}[[X, W]]$ is given by the following data.

$$\begin{aligned} M^u &= \langle E, F \rangle, \\ \mathrm{Fil}^1 M^u &= \begin{cases} M^u & \text{if } i \leq 0, \\ \langle F \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\ \phi_0(E) &= (\alpha + X)E, \quad \phi_k(F) = \alpha^{-1}\gamma(\alpha + X)E + (\beta + W)F. \end{aligned}$$

By the discussion in Section 3.2 and the computation in [10, Prop. 6.9(2), Theorem 6.11(2)], the universal deformation \mathcal{M}^u of \mathcal{M} over $\mathbb{F}[[x, w]]$ is given by the following data.

$$\begin{aligned} \mathcal{M}^u &= \langle e, f \rangle, \\ \mathrm{Fil}^1 \mathcal{M}^u &= \langle u^d e, f + te \rangle, \\ \phi_1(u^d e) &= (\alpha + x)e, \quad \phi_1(f + te) = (\beta + w)f, \end{aligned}$$

$$[g]e = e, [g]f = \omega(g)^j f, \\ N(e) = 0, N(f) = \frac{j\gamma(\alpha + x)}{\alpha(\beta + w)} u^{pj} e.$$

By the same argument as in Section 2.2, if we identify $\mathbb{F}[[X, W]]$ and $\mathbb{F}[[x, w]]$ by sending X to x and W to w , then M^u and \mathcal{M}^u have isomorphic generic fibers. Theorem 1.4 follows.

4. Some remarks

4.1. A commutative diagram in the global case

Let $\bar{r} : G_{\mathbb{Q}_p} \rightarrow \mathrm{GL}_2(\mathbb{F})$ be a continuous representation. By [7, Proposition 3.2.1], there is a totally real field F and a continuous irreducible representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\mathbb{F})$ such that

- (1) p splits completely in F ;
- (2) $\bar{\rho}$ is totally odd;
- (3) $\bar{\rho}(G_F) = \mathrm{GL}_2(\mathbb{F})$;
- (4) if $v \nmid p$ is a place of F then $\bar{\rho}|_{G_{F_v}}$ is unramified;
- (5) if $v \mid p$ is a place of F then $\bar{\rho}|_{G_{F_v}} \cong \bar{r}$;
- (6) $[F : \mathbb{Q}]$ is even;
- (7) $\bar{\rho}$ is modular.

For the \bar{r} in Theorem 1.4, we fix a totally real field F and a Galois representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_2(\mathbb{F})$ with the properties as above. We choose a finite set S of finite places of F , containing places $v \mid p$ and at least one other place. Using [9, Lemma 2.2.1], we can and do choose S so that conditions (1)–(4) in [9, Section 2.2] hold.

Assume that $\bar{\rho}$ is absolutely irreducible. (In the general case, the following discussion still holds if we use framed deformations.) Let τ, j, k be as in Theorem 1.4. We consider two deformation problems for $\bar{\rho}$. First, we consider the deformations of $\bar{\rho}$, which have determinant $\psi\epsilon$, and are unramified outside S and potentially Barsotti–Tate of type τ at $v \mid p$. Let $R_{F,S}^\psi(0, 0, \tau)$ be the corresponding universal deformation ring. Secondly, we consider the deformations of $\bar{\rho}$, which have determinant $\psi\epsilon$, and are unramified outside S and crystalline with Hodge–Tate weights $(0, k)$ at $v \mid p$. Let $R_{F,S}^\psi(0, k - 1, 1)$ be the corresponding universal deformation ring.

Then we have the following commutative diagram.

$$\begin{array}{ccc} \otimes_{v|p} R(0, 0, \tau, \bar{\rho}|_{G_{F_v}})/\pi & \xrightarrow{\alpha} & R_{F,S}^\psi(0, 0, \tau)/\pi \\ \gamma \downarrow & & \downarrow \gamma' \\ \otimes_{v|p} R_{cr}(0, k - 1, \mathbf{1}, \bar{\rho}|_{G_{F_v}})/\pi & \xrightarrow{\beta} & R_{F,S}^\psi(0, k - 1, \mathbf{1})/\pi \end{array} \quad (4.1)$$

where γ is the isomorphism in [Theorem 1.4](#), γ' is the isomorphism induced from the universal properties of corresponding deformation rings as the two deformation conditions at $v \mid p$ are equivalent in characteristic p case, α and β are induced from the restriction of representations. The isomorphisms also induce isomorphisms between the corresponding Galois representations. One can also deduce this diagram from the results in [\[7\]](#).

The deformation rings $R_{F,S}^\psi(0, 0, \tau)$ and $R_{F,S}^\psi(0, k-1, \mathbf{1})$ are closely related to certain Hecke algebras ([\[9, Section 2\]](#) and [\[11, Section 2\]](#)). If we have $R = T$ results, i.e., $R_{F,S}^\psi(0, 0, \tau) \cong \mathbb{T}_1$ and $R_{F,S}^\psi(0, k-1, \mathbf{1}) \cong \mathbb{T}_2$ for Hecke algebras \mathbb{T}_1 and \mathbb{T}_2 , then γ' induces an isomorphism $\mathbb{T}_1 \otimes \mathbb{F} \cong \mathbb{T}_2 \otimes \mathbb{F}$.

4.2. Generalization to unramified extensions of \mathbb{Q}_p

It is possible to give such explicit examples for certain reducible rank two Galois representations of $G_{K'}$, where K' is an unramified extension of \mathbb{Q}_p . In this general case, we do not always have a nontrivial arrow $\tilde{M} \rightarrow \tilde{\mathcal{M}}$ in the category $\text{BrMod}_{dd, K/K'}^{p-2}$. To solve this problem, we need to construct a third object $\mathcal{P} \in \text{BrMod}_{dd, K/K'}^{p-2}$ and two arrows $\tilde{M} \rightarrow \mathcal{P}$ and $\tilde{\mathcal{M}} \rightarrow \mathcal{P}$ which both induce an isomorphism between generic fibers. See [\[8, Proposition 3.4.1\]](#) for a generalization of [Proposition 2.3](#). The space $\text{Ext}^1(M, M)$ is not difficult to compute. The space $\text{Ext}^1(\mathcal{M}, \mathcal{M})$ can be parameterized as in [\[4, Section 4\]](#). We do not give any details here because the computation is rather long and extra conditions are needed, yet the ideas are the same as in the \mathbb{Q}_p case.

4.3. An example for irreducible Galois representations

We can also give such explicit examples for certain irreducible rank two Galois representations. Fix an integer $k \in [1, p-2]$. Let $M \in \text{MF}_{\mathbb{F}}^{p-2}$ be the Fontaine–Laffaille module given by the following data:

$$\begin{aligned} M &= \mathbb{F}\langle E, F \rangle, \\ \text{Fil}^i M &= \begin{cases} M & \text{if } i \leq 0, \\ \mathbb{F}\langle F \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\ \phi_0(E) &= -F, \quad \phi_k(F) = E. \end{aligned}$$

Let $\bar{r} = T_{\text{cris}}(M)$. Then \bar{r} is a rank two irreducible Galois representation which is crystalline with Hodge–Tate weights $(0, k)$. It is easy to see that M is the reduction of a lattice in the (φ, N) -module D_μ given by

$$\begin{aligned} D_\mu &= L\langle E, F \rangle, \\ \varphi(E) &= -F + \mu E, \quad \varphi(F) = E, \\ \text{Fil}^k D_\mu &= \langle F \rangle, \quad N = 0, \quad \mu \in \mathfrak{m}_{\mathcal{O}_L}. \end{aligned}$$

See [3, Example 5.1(2)]. Let $\epsilon\psi$ be the determinant of the Galois representation attached to D_0 . In this case, using Breuil modules with coefficient $\mathbb{F} \otimes \mathbb{F}_{p^2}[u]/u^{p(p^2-1)}$ and descent data from $K = \mathbb{Q}_{p^2}((-p)^{\frac{1}{p^2-1}})$ to \mathbb{Q}_p (see [4, Section 1] for precise definition) we can prove the following result.

Theorem 4.1. *Let $j = p^2 - 1 - p(k - 1)$. Let $\omega_2 : \text{Gal}(K/\mathbb{Q}_p) \rightarrow \mathbb{F}_{p^2}^\times$ be the map defined by $\omega_2(g) = g((-p)^{\frac{1}{p^2-1}})/(-p)^{\frac{1}{p^2-1}}$. (We can also consider ω_2 as a level 2 fundamental character.) There is a natural isomorphism*

$$R_{cr}^\psi(0, k - 1, \mathbf{1}, \bar{r})/\pi \rightarrow R^\psi(0, 0, \tilde{\omega}_2^j \oplus \tilde{\omega}_2^{pj}, \bar{r})/\pi,$$

which induces an isomorphism between the corresponding Galois representations. In other words, in characteristic p , a deformation of \bar{r} with determinant $\overline{\epsilon\psi}$ is crystalline with Hodge–Tate weights $(0, k)$ if and only if it is potentially Barsotti–Tate with type $\tilde{\omega}_2^j \oplus \tilde{\omega}_2^{pj}$.

Proof. The proof is similar to the proof of Theorem 1.4. We sketch the main steps here. Let $d = p^2 - 1$.

Step 1: Let $\mathcal{M} \in \text{BrMod}_{dd, K/\mathbb{Q}_p}^1$ be the Breuil module given by the following data

$$\begin{aligned} \mathcal{M} &= \langle e, f \rangle, \\ \text{Fil}^1 \mathcal{M} &= \langle u^d e, f \rangle, \\ \phi_1(u^d e) &= -f, \quad \phi_1(f) = e, \\ [g]e &= (1 \otimes \omega_2(g)^j)e, \quad [g]f = (1 \otimes \omega_2(g)^{pj})f; \quad N = 0. \end{aligned}$$

Then M and \mathcal{M} have isomorphic generic fibers. The nontrivial morphism $\tilde{M} \rightarrow \tilde{\mathcal{M}}$ is given by

$$E \mapsto u^{p(k-1)}e, \quad F \mapsto u^{p^2(k-1)}f.$$

Step 2: We have $R_{cr}^\psi(0, k - 1, \mathbf{1}, \bar{r})/\pi \cong \mathbb{F}[[Z]]$ and the corresponding Fontaine–Laffaille module M^u over $\mathbb{F}[[Z]]$ is given by

$$\begin{aligned} M^u &= \langle E, F \rangle, \\ \text{Fil}^i M^u &= \begin{cases} M^u & \text{if } i \leq 0, \\ \langle F \rangle & \text{if } 1 \leq i \leq k, \\ 0 & \text{otherwise,} \end{cases} \\ \phi_0(E) &= -F, \quad \phi_k(F) = E + ZE. \end{aligned}$$

By [10, Prop. 6.10, Theorem 6.12(4)], we have $R^\psi(0, 0, \tilde{\omega}_2^j \oplus \tilde{\omega}_2^{pj}, \bar{r})/\pi = \mathbb{F}[[z]]$ and the corresponding Breuil module \mathcal{M}^u is given by

$$\begin{aligned}
\mathcal{M}^u &= \langle e, f \rangle, \\
\mathrm{Fil}^1 \mathcal{M}^u &= \langle u^d e, f \rangle, \\
\phi_1(u^d e) &= -f, \quad \phi_1(f) = e + ze, \\
[g]e &= (1 \otimes \omega_2(g)^j)e, \quad [g]f = (1 \otimes \omega_2(g)^{pj})f; \quad N = 0.
\end{aligned}$$

Step 3: The morphism $Z \mapsto z$ is the one that satisfies the properties in the theorem. \square

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