Journal of Number Theory 165 (2016) 58–66

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Quartic polynomials and the Hasse norm theorem modulo squares

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ARTICLE INFO

Article history:
Received 4 November 2015
Received in revised form 22 January 2016
Accepted 23 January 2016
Available online 4 March 2016
Communicated by J.-L. Colliot-Thélène

MSC:
11E04
11E81

Keywords:
Brauer group
Cup-product
Field extension
The u-invariant
The Hasse norm theorem

ABSTRACT

Let F be a field, char $F \neq 2$, L/F a quartic field extension. Define by $G_{L/F}$ the group of elements $r \in F^*$ such that $D \cup (r) = 0$ for any regular field extension K/F and any $D \in {}_2\mathrm{Br}(KL/K)$. We show that $G_{L/F} = F^{*2}N_{L/F}L^*$. As a consequence we prove that the Hasse norm theorem modulo squares holds for L/F.

1. The annihilator group for a quartic field extension

In the present paper we investigate universal annihilators of the 2-torsion part of the relative Brauer group for quartic field extensions. More precisely, let F be a field, char $F \neq 2$, L/F a finite field extension. Recall that a field extension E/F is called regular

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http://dx.doi.org/10.1016/j.jnt.2016.01.022

if F is algebraically closed in E, and E/F is separable [L, Ch. 8, §4]. Define by $G_{L/F}$ the group of elements $r \in F^*$ such that $D \cup (r) = 0$ for any regular field extension K/F and any $D \in {}_2\mathrm{Br}(KL/K)$ (here $\cup$ is the cup-product $H^2(K, \mathbb{Z}/2\mathbb{Z}) \otimes H^1(K, \mathbb{Z}/2\mathbb{Z}) \to H^3(K, \mathbb{Z}/2\mathbb{Z})$).

In general the problem of computation of the group $G_{L/F}$ seems to be very hard, because usually it is difficult to describe the group ${}_2\mathrm{Br}(KL/K)$, or, at least, some nontrivial elements in this group. However, if $r = N_{L/F}(s)$ for $s \in L^*$, and $D \in {}_2\mathrm{Br}(KL/K)$, then by the projection formula

$$D \cup (r) = D \cup (N_{KL/K}(s)) = N_{KL/K}(D_{KL} \cup (s)) = 0.$$

Therefore, $F^{*2} N_{L/F} L^* \subset G_{L/F}$. This observation and the examples below make reasonable the following

Conjecture. *For any finite field extension L/F one has $G_{L/F} = F^{*2} N_{L/F} L^*$.*

We do not know any counterexample to this equality. On the other hand, there are certain cases, which confirm the conjecture. We consider them one by one.

1) Let L/F be an extension of odd degree n. Then, since ${}_2\mathrm{Br}(KL/K) = 0$ and $F^{*n} \subset N_{L/F} L^*$, we get $G_{L/F} = F^{*2} N_{L/F} L^* = F^*$.
2) Let $L = F(\sqrt{a})$ be a quadratic extension, x an indeterminate. Assume that $r \in F^*$ is such an element that $(a, x) \cup (r) = 0$. Then $r \in N_{L/F} L^*$, hence $G_{L/F} = N_{L/F} L^* = F^{*2} N_{L/F} L^*$.
3) Let $L = F(\sqrt{a}, \sqrt{b})/F$ be a biquadratic extension. Similarly to case 2)

$$G_{F(\sqrt{a},\sqrt{b})/F} \subset N_{F(\sqrt{a})/F} F(\sqrt{a})^* \cap N_{F(\sqrt{b})/F} F(\sqrt{b})^*.$$

On the other hand, it is well known that

$$N_{F(\sqrt{a})/F} F(\sqrt{a})^* \cap N_{F(\sqrt{b})/F} F(\sqrt{b})^* = F^{*2} N_{F(\sqrt{a},\sqrt{b})/F} F(\sqrt{a}, \sqrt{b})^*,$$

which implies that as in the previous cases $G_{L/F} = F^{*2} N_{L/F} L^*$.

Remark. The case of a multiquadratic field extension $L = F(\sqrt{a_1}, \dots, \sqrt{a_n})$ $(n \geq 3)$ is much subtler. Obviously, $F^{*2} N_{L/F} L^* \subset G_{L/F} \subset \bigcap_{1 \leq i \leq n} N_{F(\sqrt{a_i})/F} F(\sqrt{a_i})^*$. However, generally $F^{*2} N_{L/F} L^* \neq \bigcap_{1 \leq i \leq n} N_{F(\sqrt{a_i})/F} F(\sqrt{a_i})^*$, and it is unclear how to prove (disprove) the conjecture even in the case of a triquadratic extension.

4) Let L/F be a cyclic Galois field extension (a generalization of example 2)). Then $G_{L/F} = F^{*2} N_{L/F} L^*$ as well.

Indeed, in view of example 1) we may assume that $[L : F] = 2n$. Let $\chi : Gal(L/F) \to \mathbb{Q}/\mathbb{Z}$ be a character of order $2n$, z an indeterminate. Let further $L_0 = F(\theta)$ be an intermediate field $F \subset L_0 \subset L$ such that $[L : L_0] = 2$. Consider the element $A = \chi \cup N_{L_0(z)/F(z)}(z - \theta) \in \mathrm{Br}(L(z)/F(z))$. Clearly, $2A = 0$.

Let $r \in G_{L/F}$, so $\chi \cup N_{L_0(z)/F(z)}(z - \theta) \cup (r) = 0$. Taking the residue at $z - \theta$, we get that $\chi \cup (r)_{L_0} = 0$. Since $\chi^2 : Gal(L_0/F) \to \mathbb{Q}/\mathbb{Z}$ is a character of order n, and the extension L_0/F is cyclic, we get that $\chi \cup (r) = \chi^2 \cup (s)$ for some $s \in F^*$ [GS, Cor. 4.7.4]. Therefore, $\chi \cup (rs^{-2}) = 0$, which implies that $rs^{-2} \in N_{L/F}L^*$ [GS, Cor. 4.7.4]. Thus, $r \in F^{*2}N_{L/F}L^*$.

In this section we prove the conjecture for all field extensions of degree 4. As an application we prove in section 2 the Hasse norm theorem modulo squares for these extensions.

A few words about our notation. All the fields below are assumed to be of characteristic distinct from 2. A quadratic form over a field will be called merely form. The diagonal form with coefficients $a_1, \dots, a_n$ is denoted by $\langle a_1, \dots, a_n \rangle$. Frequently, slightly abusing notation, we will identify the form and the corresponding element in the Witt ring $W(F)$. For instance, the equality $\varphi = 0$ means that the form φ is hyperbolic. The symbol (a, b) stands for the class in the Brauer group $\mathrm{Br}(k)$ of the quaternion algebra with generators i, j, and the relations $i^2 = a$, $j^2 = b$, $ij = -ji$. The group ${}_2\mathrm{Br}(k)$ is the 2-torsion part of $\mathrm{Br}(k)$, and $\mathrm{Br}(l/k)$ is the kernel of the restriction map $\mathrm{Br}(k) \to \mathrm{Br}(l)$.

For a field E and $n \geq 0$ put $H^n(E) = H^n(E, \mathbb{Z}/2\mathbb{Z})$. In particular, we have $H^1(E) = E^*/E^{*2}$, and $H^2(E) = {}_2\mathrm{Br}(E)$. For any $a \in E^*$ denote by (a) the corresponding element in $H^1(E)$. For any $a_1, \dots, a_n \in E^*$ put $(a_1, \dots, a_n) = (a_1) \cup \dots \cup (a_n) \in H^n(E)$.

One of the main tools used below is the exact cohomology group sequence for the rational function field [GS, 6.9.3]

$$0 \to H^n(F) \xrightarrow{\mathrm{res}} H^n(F(x)) \xrightarrow{\amalg \partial_p} \coprod_{p \in \mathbb{A}^1_F} H^{n-1}(F_p) \to 0.$$

We consider here a point $p \in \mathbb{A}^1_F$ as a monic irreducible polynomial over F, $F_p = F[x]/p$ is the corresponding residue field, and $\partial_p : H^n(F(x)) \to H^{n-1}(F_p)$ is the residue homomorphism. Recall that if $f_i \in F[x]$, then

$$\partial_p(f_1, \dots, f_n) = \begin{cases} 0 & \text{if all } f_i \text{ are not divisible by } p \\ (\overline{f_1}, \dots, \overline{f_{n-1}}) & \text{if } v_p(f_n) = 1 \text{ and } v_p(f_i) = 0 \text{ for } 1 \leq i \leq n-1 \end{cases}$$

Define also $\partial_\infty(f_1, \dots, f_n) = \partial_u(f_1(u^{-1}), \dots, f_n(u^{-1}))$, where $u = x^{-1}$.

First we consider the case of tower of two quadratic extensions, which is crucial in the proof of the general case.

Proposition 1.1. *Let F be a field, $u, v, w \in F$, x an indeterminate, $L = F(\sqrt{v + 2w\sqrt{u}})$ a tower of two quadratic extensions. Then*

1) *If $a \in F^*$ and $(ux^2 + vx + w^2, -x, a) = 0$, then $a \in F^{*2}N_{L/F}F^*$.*
2) *$G_{L/F} = F^{*2}N_{L/F}F^*$.*

Proof. Note first that 1) implies 2), since by [LLT, Th. 3.9] $(ux^2+vx+w^2,-x) \in {}_2\mathrm{Br}(L/F)$. To verify 1) consider any $a \in F^*$ such that $(ux^2+vx+w^2,-x,a)=0$. Then

$$(a,u) = \partial_\infty(a,-x,ux^2+vx+w^2) = 0.$$

Therefore, we may suppose that $a = N_{F(\sqrt{u})/F}(\alpha+\sqrt{u}) = \alpha^2-u$ for some $\alpha \in F$. Obviously, to show that $a \in F^{*2}N_{L/F}L^*$ it suffices to find an element $r \in F^*$ such that $(\alpha+\sqrt{u})r \in N_{L/F(\sqrt{u})}L^*$. Indeed, if $(\alpha+\sqrt{u})r = N_{L/F(\sqrt{u})}(z)$, where $z \in L^*$, then

$$ar^2 = N_{F(\sqrt{u})/F}((\alpha+\sqrt{u})r) = N_{L/F}(z) \in N_{L/F}L^*,$$

and we are done.

Let us consider the norm form

$$(x_1+y_1\sqrt{u})^2 - (v+2w\sqrt{u})(x_2+y_2\sqrt{u})^2 = f_1 + f_2\sqrt{u},$$

where

$$f_1(x_1,y_1,x_2,y_2) = x_1^2 + uy_1^2 - vx_2^2 - uvy_2^2 - 4uwx_2y_2,$$
$$f_2(x_1,y_1,x_2,y_2) = 2x_1y_1 - 2vx_2y_2 - 2w(x_2^2+uy_2^2).$$

Notice that, since $v+2w\sqrt{u} \notin F(\sqrt{u})^{*2}$, the forms f_1 and f_2 have no common zero. It suffices to show that the form $f_1-\alpha f_2$ is isotropic. Indeed, if this is the case, then there exists a vector $(x_1,y_1,x_2,y_2) \neq 0$ such that $f_1(x_1,y_1,x_2,y_2) = \alpha r$ and $f_2(x_1,y_1,x_2,y_2) = r$ for some $r \in F^*$, hence $(\alpha+\sqrt{u})r = N_{L/F(\sqrt{u})}(z)$, where $z = x_1+y_1\sqrt{u}+(x_2+y_2\sqrt{u})\sqrt{v+2w\sqrt{u}} \in L^*$.

The matrix of the form $f_1-\alpha f_2$ is $\begin{pmatrix} 1 & -\alpha & 0 & 0 \\ -\alpha & u & 0 & 0 \\ 0 & 0 & -v+2w\alpha & -2uw+v\alpha \\ 0 & 0 & -2uw+v\alpha & -uv+2uw\alpha \end{pmatrix}$. Hence

$$f_1 \perp -\alpha f_2 \simeq \langle 1, -(\alpha^2-u)\rangle \perp (-v+2w\alpha)\langle 1, -(\alpha^2-u)(v^2-4uw^2)\rangle.$$

The assertion that the form $f_1-\alpha f_2$ is isotropic is equivalent to the equality $(\alpha^2-u, v-2w\alpha)_{F(\sqrt{v^2-4uw^2})} = 0$. We have

$$(a, \frac{v-\sqrt{v^2-4uw^2}}{2u}) = \partial_{ux^2+vx+w}(a,-x,ux^2+vx+w) = 0.$$

(If the polynomial ux^2+vx+w^2 is reducible, i.e. the extension L/F is biquadratic, then by ∂_{ux^2+vx+w} we mean the residue map at any linear factor of ux^2+vx+w.) Since $(a,u)=0$, we get $(a, 2(v-\sqrt{v^2-4uw^2}))=0$. On the other hand, it can be easily checked that since $a=\alpha^2-u$, the following equality holds:

$$(a, 2(v - \sqrt{v^2 - 4uw^2})) = (a, v - 2w\alpha)_{F(\sqrt{v^2-4uw^2})} = (\alpha^2 - u, v - 2w\alpha)_{F(\sqrt{v^2-4uw^2})}.$$

This finishes the proof of the proposition. □

Proposition 1.1 can be generalized to arbitrary field extensions of degree 4.

Corollary 1.2. *Let F be a field, L/F a field extension of degree 4. Then $G_{L/F} = {F^*}^2 N_{L/F}L^*$.*

Proof. Since $\operatorname{char} F \neq 2$, the extension L/F is separable, hence we may assume that $L = F[x]/p(x)$, where the polynomial $p(x) = x^4 + ax^2 + bx + c$ is irreducible. By Proposition 1.1 we may assume that the extension L/F has no intermediate subextension, hence the resolvent cubic of L/F is irreducible. By [S, Cor. 4] we have $(-x, x(x-a)^2 - 4cx + b^2) \in {}_2\mathrm{Br}(L(x)/F(x))$. Let $e \in G_{L/F}$. In particular, $(e, -x, x(x-a)^2 - 4cx + b^2) = 0$. Hence, by specialization we get that

$$(e, -\alpha, \alpha(\alpha - a)^2 - 4c\alpha + b^2) = 0$$

for any field extension E/F and $\alpha \in E$. Put $K = F[x]/x(x-a)^2 - 4cx + b^2$. It is easy to check that the polynomial $x(x-a)^2 - 4cx + b^2$ is the resolvent cubic of the extension L/F. Therefore, the extension KL/K is a tower of two quadratic extensions, say $KL = K(\sqrt{v + 2w\sqrt{u}})$, where $u, v, w \in K$ (here KL is usual compositum of two finite field extensions of F). We have

$$(-x, ux^2 + vx + w^2) \in {}_2\mathrm{Br}(KL(x)/K(x)),$$

where x is an indeterminate. Note that $(-x, ux^2 + vx + w^2)_{K(x)(\sqrt{d})} \neq 0$ for any $d \in K^*$, because otherwise $-dx$ would be a square in the residue field K_{ux^2+vx+w}, which is not the case.

Therefore, by [S, Cor. 4] there is $\alpha \in K(x)$ such that

$$(-x, ux^2 + vx + w^2) = (-\alpha, \alpha(\alpha - a)^2 - 4c\alpha + b^2).$$

Hence

$$(e, -x, ux^2 + vx + w^2) = (e, -\alpha, \alpha(\alpha - a)^2 - 4c\alpha + b^2) = 0.$$

Proposition 1.1 shows that $e \in {K^*}^2 N_{KL/K}(KL)^*$. Hence

$$e^3 = N_{K/F}(e) \in N_{K/F}({K^*}^2 N_{KL/K}(KL)^*) \subset {F^*}^2 N_{KL/F}(KL)^* \subset {F^*}^2 N_{L/F}L^*.$$

Thus, $e = e^{-2}e^3 \in {F^*}^2 N_{L/F}L^*$, which completes the proof. □

Corollary 1.3. *Let F be a field, $L = F[x]/p(x)$, where $p(x) = x^4 + ax^2 + bx + c$, $a, b, c \in F$. Suppose that the F-algebra L is separable. Let $f(x) = x(x-a)^2 - 4cx + b^2$, and $e \in F^*$. Then the following conditions are equivalent:*

1) $e \in F^{*2} N_{L/F} L^*$.
2) $(e, -x, f(x)) = 0$.
3) $(e, -\xi) = 0 \in {}_2\mathrm{Br}(F(\xi))$, *where ξ is an arbitrary root of f, in the case $b \neq 0$; $(e, -\xi) = 0 \in {}_2\mathrm{Br}(F(\xi))$, where ξ is an arbitrary nonzero root of f, and $(e, a^2 - 4c) = 0$ in the case $b = 0$.*

Proof. The equivalence of conditions 2) and 3) follows at once by computing the residues of the symbol $(e, -x, f(x))$. If p is irreducible, then $(-x, f(x)) \in {}_2\mathrm{Br}((L(x)/F(x))$. Hence the implication 1) $\Longrightarrow$ 2) follows from Corollary 1.2, while the implication 2) $\Longrightarrow$ 1) follows from the proof of Corollary 1.2.

If $p(x)$ is a product of polynomials of degree 1 and 3, then by [S, Cor. 4] $(-x, f(x)) = 0$, $N_{L/F} L^* = F^*$, so in this case the equivalence of 1) and 2) is obvious.

It remains to consider the case where p is a product of two irreducible quadratic polynomials. In this case $p(x) = (x^2 + qx + r_1)(x^2 - qx + r_2)$, and a straightforward computation shows that

$$f(x) = (x + q^2)(x^2 + (q^2 - 2r_1 - 2r_2)x + (r_2 - r_1)^2).$$

The discriminant of the factor $g(x) = x^2 + (q^2 - 2r_1 - 2r_2)x + (r_2 - r_1)^2$ equals $(q^2 - 4r_1)(q^2 - 4r_2)$. It is easy to check that the polynomial $f(x)$ is separable, and $\partial_g(-x, g(x)) = q^2 - 4r_1$. Hence condition 2) is equivalent to the equality $(e, q^2 - 4r_1)_{F(\sqrt{(q^2-4r_1)(q^2-4r_2)})} = 0$, which means that the form

$$\langle 1, -(q^2 - 4r_1), -e, (q^2 - 4r_2)e \rangle$$

is isotropic. On the other hand, clearly, the last condition is equivalent to

$$e \in N_{F(\sqrt{q^2-4r_1})/F} F(\sqrt{q^2 - 4r_1})^* N_{F(\sqrt{q^2-4r_2})/F} F(\sqrt{q^2 - 4r_2})^*,$$

which is just condition 1). □

Corollary 1.4. *Let K/F be a cubic field extension, $K = F(\theta)$, $\xi = -\theta^{-1} N_{K/F}\theta$. Then $N_{K/F}\xi = -b^2$, $b \in F^*$. Let $f(x) = x(x-a)^2 - 4cx + b^2$ be the characteristic polynomial of ξ. Denote by G the group consisting of all elements $v \in F^*$ such that $(v, \theta) \in \mathrm{res}_{K/F}\, {}_2\mathrm{Br}(F)$. Then $G = F^{*2} N_{L/F} L^*$, where $L = F[x]/p(x)$, $p(x) = x^4 + ax^2 + bx + c$.*

Proof. First consider the case where the extension K/F is separable. The assertion $N_{K/F}\xi = -b^2$, where $b = N_{K/F}\theta$, is obvious. Clearly, there exist unique $a, c \in F$ such that $f(x) = x(x-a)^2 - 4cx + b^2$ is the characteristic polynomial of ξ. Applying the norm, it is easy to see that the condition $(v,\theta) \in \mathrm{res}_{K/F}\, {}_2\mathrm{Br}(F)$ is equivalent to the condition $(v,\theta) = (v, N_{K/F}\theta)$, or, in other words, $(v,-\xi) = 0$. By Corollary 1.3 the last equality is equivalent to $v \in F^{*2} N_{L/F} L^*$, since one can check that $p(x)$ is separable.

If the extension K/F is not separable, then $a = c = 0$, $b \neq 0$, and, obviously, $F^{*2} N_{L/F} L^* = G = F^*$. □

Denote by $u(F)$ the u-invariant of the field F, i.e. the maximal number n such that there exists an n-dimensional anisotropic form over F.

Corollary 1.5. *Let F be a field, $p(x) \in F[x]$ an irreducible quartic polynomial, $L = F[x]/p(x)$, $e \in F^*$, $e \notin F^{*2} N_{L/F} L^*$. Assume that the order of the Galois group of $p(x)$ is divisible by 3. Then there exists a regular field extension K/F such that $u(K) = 2$ and $e \notin K^{*2} N_{KL/K}(KL)^*$.*

Proof. By Corollary 1.3

$$(e,-\xi) \neq 0 \in {}_2\mathrm{Br}(F(\xi)),$$

where ξ is an arbitrary root of f. Since the order of the Galois group of $p(x)$ is divisible by 3, the extension $F(\xi)/F$ is of degree 3. Let $\varphi \simeq \langle 1, -v, -w \rangle$, where $v, w \in F^*$, be an anisotropic form. We claim that $(e,-\xi)_{F(\varphi)(\xi)} \neq 0$. Indeed, otherwise

$$(e,-\xi) = (v,w).$$

Then

$$(v,w) = N_{F(\xi)/F}(v,w) = N_{F(\xi)/F}(e,-\xi) = (e, b^2) = 0,$$

a contradiction, since the form $\varphi \simeq \langle 1, -v, -w \rangle$ is anisotropic.

Thus, we can subsequently split all 3-dimensional forms, coming to a field K such that $u(K) = 2$ and $(e,-\xi)_{K(\xi)} \neq 0$. Applying Corollary 1.3 again, we see that $e \notin K^{*2} N_{KL/K}(KL)^*$. □

In contrast to Corollary 1.5 we have the following

Proposition 1.6. *Let F be a field, $u(F) \leq 2$, $p(x) \in F[x]$ a separable monic quartic polynomial, $L = F[x]/p(x)$. Assume that $p(x)$ is either reducible, or the order of the Galois group of $p(x)$ is not divisible by 3. Then $F^* = N_{L/F} L^*$.*

Proof. If $p(x)$ has a linear factor, the claim is obvious. Now suppose that $p(x)$ is a product of two distinct irreducible quadratic polynomials, which determine quadratic extensions of F, say $F(\sqrt{d_1})/F$ and $F(\sqrt{d_2})/F$. Choose any $u \in F^*$. The form $\langle 1, -d_1, -u \rangle$ is isotropic, hence

$$u \in N_{F(\sqrt{d_1})/F} F(\sqrt{d_1})^* \subset N_{L/F} L^*.$$

Finally, in the case where $p(x)$ is irreducible, and the Galois group of $p(x)$ is not divisible by 3, there is a tower $F \subset E \subset L$, where E/F and L/E are quadratic field extensions. Since $u(F) \le 2$, we have $u(E) \le 2$ as well. Hence both norms $N_{L/E}$ and $N_{E/F}$ are surjective, which implies that $u \in N_{L/F}L^*$ for any $u \in F^*$. $\square$

2. The Hasse norm theorem modulo squares for quartic extensions

In this section we apply Proposition 1.1 and Corollary 1.3 for one arithmetic problem. Let F be a *global field*, and L/F a finite separable field extension. Let $\Omega(F)$ be the set of all valuations (archimedian and nonarchimedian) of F. For $v \in \Omega(F)$ denote by F_v the completion of F with respect to the valuation v. We say that the Hasse norm theorem modulo squares holds for the extension L/F if the following property holds:

Let $a \in F^*$ be such an element that $a \in F_v^{*2} N_{L\otimes_F F_v/F_v}(L \otimes_F F_v)^*$ for each $v \in \Omega(F)$. Then $a \in F^{*2} N_{L/F}L^*$.

This property was considered in [LW1] and [LW2] for certain Galois extensions. It was proved in [LW1] that the Hasse norm theorem modulo squares holds for any multi-quadratic extension. On the other hand, for any $k \ge 2$ examples of Galois extensions with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ such that this theorem does not hold were given in [LW2].

However, to our knowledge the case where the extension L/F is not Galois, was not investigated before. Below as a consequence of Proposition 1.1, Corollary 1.3, and the classical Hasse norm theorem for quadratic extensions we give two proofs of the following

Theorem 2.1. *The Hasse norm theorem modulo squares holds for any quartic field extension.*

First proof. Assume first that L/F is a tower of two quadratic extensions, say

$$L = F(\sqrt{v + 2w\sqrt{u}}) = F[x]/p(x),$$

where $p(x) = (x^2 - v)^2 - 4uw^2$. Let θ be any root of the polynomial $p(x)$. Put $g(x) = ux^2 + vx + w^2$. We have $\mathrm{res}_{F_v(x)(\theta)/F(x)(\theta)}(-x, g(x)) = 0$. Suppose that $e \in F^*$ is such that $e \in F_v^{*2} N_{L\otimes_F F_v/F_v}(L \otimes_F F_v)^*$ for each $v \in \Omega(F)$. Then by the projection formula $(e, -x, g(x))_{F_v(x)} = 0$ for each $v \in \Omega(F)$.

Consider the commutative diagram

$$\begin{array}{ccc} H^3(F(x), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\text{res}} & \bigoplus\limits_{v\in\Omega(F)} H^3(F_v(x), \mathbb{Z}/2\mathbb{Z}) \\ \partial\big\downarrow & & \partial\big\downarrow \\ \bigoplus\limits_{p} H^2(F_p, \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\text{res}} & \bigoplus\limits_{p}\bigoplus\limits_{w|v} H^2((F_p)_w, \mathbb{Z}/2\mathbb{Z}) \end{array}$$

where p runs over all monic irreducible separable polynomials over F. We have

$$\operatorname{res} \circ \partial(e, -x, g(x)) = \partial \circ \operatorname{res}(e, -x, g(x)) = \partial(0) = 0.$$

The lower horizontal map is injective [CF, Ch. 7], hence $\partial(e, -x, g(x)) = 0$, which implies that $(e, -x, g(x)) \in \operatorname{res}_{F(x)/F} H^3(F)$. Therefore, we get

$$(e, -x, g(x)) = \operatorname{res}_{F(x)/F} \circ s_x(e, -x, g(x)) = \operatorname{res}_{F(x)/F} \circ \partial_x(-x, e, -x, ux^2 + vx + w^2) = 0.$$

(Here $s_x : H^3(F(x)) \to H^3(F)$ is the specialization map associated with the zero point. We have $s_x(\alpha) = \partial_x((-x) \cup \alpha)$ for any $\alpha \in H^3(F(x))$ [GS, 6.8.6].) Now by Proposition 1.1 $e \in F^{*2} N_{L/F} L^*$.

In the general case let K/F be the resolvent cubic for L/F. Since the extension KL/K is a tower of two quadratic extensions, we get $e \in K^{*2} N_{KL/K}(KL)^*$. Hence

$$e^3 = N_{K/F}(e) \in F^{*2} N_{KL/F}(KL)^* \subset F^{*2} N_{L/F} L^*.$$

Therefore, $e = e^3 (e^{-1})^2 \in F^{*2} N_{L/F} L^*$, which proves the theorem. $\square$

Second proof. Let $L = F[x]/p(x)$, $p(x) = x^4 + ax^2 + bx + c$, $f(x) = x(x-a)^2 - 4cx + b^2$. Suppose that $e \in F^*$ is such that $e \in F_v^{*2} N_{L \otimes_F F_v / F_v}(L \otimes_F F_v)^*$ for each $v \in \Omega(F)$. Assume that $b \neq 0$. By Corollary 1.3 we get $(e, -\xi) = 0 \in {}_2\mathrm{Br}(F_v(\xi))$ for each root ξ of $f(x)$. In other words, $(e, -\xi) = 0 \in {}_2\mathrm{Br}(F(\xi)_w)$ for each $w \in \Omega(F(\xi))$. By the Hasse norm theorem for quadratic extensions $(e, -\xi) = 0 \in {}_2\mathrm{Br}(F(\xi))$ [CF, Ch. 7]. Now by Corollary 1.3 we conclude that $e \in F^{*2} N_{L/F} L^*$.

The case where $b = 0$ is treated similarly. $\square$

Acknowledgment

The author expresses his gratitude to the referee for his remarks, and, in particular, for the second proof of Theorem 2.1.

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