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General Section

On the representations generated by Eisenstein series of weight $\frac{n+3}{2}$

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A R T I C L E I N F O

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A B S T R A C T

We consider the Eisenstein series $E(z, s; k, \chi, N)$ of weight $k = (n + 3)/2$, level $N > 1$ and a Dirichlet character χ modulo N such that $\chi^2 = 1$. Shimura proved that $E(z, k/2; k, \chi, N)$ is a nearly holomorphic function. We prove that $E(z, k/2; k, \chi, N)$ generates an indecomposable reducible $(\mathfrak{g}, K)$-module of length 2. These are new examples of indecomposable reducible $(\mathfrak{g}, K)$-modules generated by nearly holomorphic modular forms.

1. Introduction

Let G and $\mathfrak{H}_n$ be the real symplectic group of degree n and the Siegel upper half space of degree n, respectively. Let K be the maximal compact subgroup of G which stabilizes $\mathfrak{i} = i \cdot 1_n \in \mathfrak{H}_n$. We denote by K^c the complexification of K. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexification of the Lie algebra of G and K, respectively. We then have the well-known decomposition

$$\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_- \tag{1.1}$$

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where $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) is corresponding to the holomorphic tangent space (resp. anti-holomorphic tangent space) at $\mathfrak{i} \in \mathfrak{H}_n$. For an element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $a, d \in \mathrm{Mat}_n(\mathbb{R})$, we let $a_g = a$, $b_g = b$, $c_g = c$, and $d_g = d$. For a finite-dimensional representation (ρ, V) of K^c and a V-valued C^∞ function f, we define a V-valued function f^ρ on G by

$$f^\rho(g) = \rho(c_g\mathfrak{i} + d_g)^{-1} f(g(\mathfrak{i})), \qquad g \in G. \tag{1.2}$$

Then, by [12], a function f is nearly holomorphic if and only if f^ρ is $\mathfrak{p}_-$-finite under the right translation. We then call a C^∞ function φ on G nearly holomorphic type if the function φ is $\mathfrak{p}_-$-finite. Fix a congruence subgroup Γ of $\mathrm{Sp}_{2n}(\mathbb{Q})$. We define the space of nearly holomorphic automorphic forms $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ on G with respect to Γ by the space of scalar valued C^∞ functions φ which satisfy the following conditions:

- φ is nearly holomorphic type.
- φ is left Γ invariant.
- φ is right K-finite.
- φ is right $\mathcal{Z}$-finite.
- φ is slowly increasing.

Here, the algebra $\mathcal{Z}$ is the center of the universal enveloping algebra of $\mathfrak{g}$. Then the space $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ is a $(\mathfrak{g}, K)$-module by the right translation. Pitale-Saha-Schmidt proved the structure theorem of $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ for $n = 1, 2$ in [7] and [8]. For a dominant weight λ, let $N(\lambda)$ and $N(\lambda)^\vee$ be a parabolic Verma module of highest weight λ with respect to a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_- + \mathfrak{k}$ and its contragredient module, respectively. Then the module $N(\lambda)$ has a unique irreducible quotient $L(\lambda)$.

Theorem 1.1 *([7]). If $n = 1$, as a $(\mathfrak{g}, K)$-module, we have a decomposition*

$$\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}} \cong \bigoplus_{k \in \mathbb{Z}_{>0}} n_k L(k) \oplus N(0)^\vee.$$

Here the multiplicity n_k is the dimension of holomorphic modular forms of weight k with respect to Γ. Moreover the weight 2 Eisenstein series E_2 generates $N(0)^\vee$.

Theorem 1.2 *([8]). If $n = 2$, as a $(\mathfrak{g}, K)$-module, we have a decomposition*

$$\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}} \cong \mathbb{C} \oplus \bigoplus_{(i,j) \in \mathbb{Z}^2, i \geq j \geq 1} n_{i,j} L(i, j) \oplus \bigoplus_{i \in \mathbb{Z}_{\geq 0}} m_i N(i + 3, 1)^\vee.$$

Here the multiplicities $n_{i,j}$ and m_i are the dimension of the suitable subspace of nearly holomorphic modular forms with respect to Γ. Moreover, if Γ is the full modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$, the multiplicities m_i are zero for all i.

Unfortunately, no examples of modular forms, except E_2, which generate indecomposable reducible modules are known. In this paper, we give new examples of such modular forms.

For an element $\alpha \in G$ and $z \in \mathfrak{H}_n$, we define a factor of automorphy j by $j(\alpha, z) = \det(c_\alpha z + d_\alpha)$. Let P be the Siegel parabolic subgroup of G. Then we consider the Eisenstein series

$$E(z,s) = E(z,s;k,\chi,N) \qquad = \sum_{\alpha \in (P\cap\Gamma)\backslash\Gamma} \chi(\det(d_\alpha))j(\alpha,z)^{-k}|j(\alpha,z)|^{-s+k/2}.$$

Here $z \in \mathfrak{H}_n$, $s \in \mathbb{C}$, $k \in 2^{-1}\mathbb{Z}$, $N \in \mathbb{Z}_{>0}$, χ is a Dirichlet character modulo N and Γ is a congruence subgroup of $\mathrm{Sp}_{2n}(\mathbb{Q})$ depending on k and N. Suppose $n > 1$, $k = (n+3)/2$, $\chi^2 = 1$, and $N > 1$. Then the Eisenstein series $E(z, k/2)$ is not a holomorphic function but a nearly holomorphic function. Note that when $n = 1$, we let $\chi = 1$ and $N = 1$ and then the Eisenstein series is equal to E_2.

Let $E^*(z,s) = E^*(z,s;k,\chi,N)$ be the Eisenstein series defined by Shimura. Here E^* is given by the right translation of certain Siegel Eisenstein series E by the suitable element at finite places, i.e., there exists an element $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$ such that we have $E^*(z,s) = (E|_k\gamma)(z,s)$. We suppose that $n > 1$, $k = (n+3)/2$, $\chi^2 = 1$, and $N > 1$. Then the Eisenstein series $E^*(z, k/2)$ is a nearly holomorphic modular form. We now state the main theorem of this paper. For simplicity, we let

$$\underline{k} = (k, \dots, k) \in \mathbb{Q}^n, \qquad \underline{k-2} = (k-2, \dots, k-2) \in \mathbb{Q}^n.$$

Theorem 1.3. *Under the above assumptions, the Eisenstein series $E^*(z, k/2)$ generates $N(\underline{k-2})^\vee$ as a $(\mathfrak{g}, K)$-module.*

Note that there exists a unique non-split exact sequence

$$0 \longrightarrow L(\underline{k-2}) \longrightarrow N(\underline{k-2})^\vee \longrightarrow L(\underline{k}) \longrightarrow 0.$$

In particular the module $N(\underline{k-2})^\vee$ has length 2. These are new examples of indecomposable reducible modules generated by nearly holomorphic modular forms. The Fourier coefficients and the constant term of $E^*(z, k/2)$, calculated by Shimura, play the key roll of our proof.

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2. Notation

1. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ have the usual meaning. The symbol $\mathbb{A}$ be the adele ring of $\mathbb{Q}$.

2. For any commutative ring R and a positive integer n, $\mathrm{Mat}_n(R)$ is the ring of $n \times n$ matrices with entries in R. If $A \in \mathrm{Mat}_n(R)$, we let ${}^tA$ be its transpose. Let $\mathrm{Sym}_n(R)$ be the set of symmetric matrices in $\mathrm{Mat}_n(R)$. For a Hermitian matrix M, we say $M > 0$ if M is positive definite. For $1 \leq i,\, j \leq n$, let

$$e_{i,j} = (\delta_{i,k}\delta_{j,l})_{k,l}\mathrm{Mat}_n(R).$$

Here δ is the Kronecker's delta function.

3. We denote by GL_n and Sp_{2n} the algebraic groups defined by

$$\mathrm{GL}_n(R) = \{g \in \mathrm{Mat}_n(R) \mid \det g \in R^\times\},$$
$$\mathrm{Sp}_{2n}(R) = \{g \in \mathrm{GL}_{2n}(R) \mid {}^tgJ_ng = J_n\},$$

where R is a commutative ring and $J_n = \left(\begin{smallmatrix} 0 & -1_n \\ 1_n & 0 \end{smallmatrix}\right)$, respectively. For any element in $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Mat}_{2n}(R)$ with $a, d \in \mathrm{Mat}_n(R)$, write $a = a_g$, $b = b_g$, $c = c_g$ and $d = d_g$. We define a maximal compact subgroup K of $\mathrm{Sp}_{2n}(\mathbb{R})$ by

$$K = \{g \in \mathrm{Sp}_{2n}(\mathbb{R}) \mid a_g = d_g, b_g = -c_g\}.$$

Let K^c be the complexification of K.

4. For $z \in \mathrm{Mat}_n(\mathbb{C})$, we let $\bar{z}$ its complex conjugate. We also let

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - \sqrt{-1}\,\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + \sqrt{-1}\,\frac{\partial}{\partial y}\Big),$$

where $z = x + \sqrt{-1}\,y \in \mathrm{Mat}_n(\mathbb{C})$ and $x, y \in \mathrm{Mat}_n(\mathbb{R})$.

5. The Siegel upper half space of degree n is defined by

$$\mathfrak{H}_n = \{z \in \mathrm{Mat}_n(\mathbb{C}) \mid {}^tz = z, \sqrt{-1}\,(\bar{z} - z) > 0\}.$$

6. Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$, i.e., we have $\mathfrak{g} = \{X \in \mathrm{Mat}_n(\mathbb{C}) \mid {}^tXJ_n + J_nX = 0\}$. For a Lie algebra $\mathfrak{a}$, we let $\mathcal{U}(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$.
7. For manifolds M and N, we denote by $C^\infty(M, N)$ the space of C^∞ functions from M to N.

3. The category $\mathcal{O}^{\mathfrak{p}}$

In this section, we will review the theory of the category $\mathcal{O}^{\mathfrak{p}}$. A more complete theory may be found in [3]. It is well-known that there exists a decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ as in (1.1).

Here Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{p}_\pm$ are described as follows

$$\mathfrak{g} = \{x \in \mathrm{Mat}_{2n}(\mathbb{C}) \mid {}^txJ_n + J_nx = 0\}, \qquad \mathfrak{k} = \{x \in \mathfrak{g} \mid a_x = d_x, b_x = -c_x\},$$

$$\mathfrak{p}_+ = \{x \in \mathfrak{g} \mid a_x = -\sqrt{-1}\,b_x = -\sqrt{-1}\,c_x = -d_x\},$$
$$\mathfrak{p}_- = \{x \in \mathfrak{g} \mid a_x = \sqrt{-1}\,b_x = \sqrt{-1}\,c_x = -d_x\}.$$

It is well-known that the Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ have the same Cartan subalgebra. The root system of $\mathfrak{g}$ is

$$\Phi = \{\ \pm(e_i + e_j),\ \pm(e_k - e_l) \mid 1 \le i \le j \le n, 1 \le k < l \le n\ \}.$$

We declare the set

$$\Phi^+ = \{\ -(e_i + e_j),\ e_k - e_l \mid 1 \le i \le j \le n, 1 \le k < l \le n\ \}$$

to be a positive root system.

Let ρ be half the sum of positive roots. We let $\Lambda = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\ge 0},\ i = 1, \dots, n-1\}$. We regard e_i as a vector $(0, \dots, 0, 1, 0, \dots, 0) \in \Lambda$ canonically. We say that a weight $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ is dominant if $\lambda_1 \ge \dots \ge \lambda_n$ holds. Note that a dominant weight is dominant with respect to the positive root system of $\mathfrak{k}$ and is not dominant with respect to $\mathfrak{g}$ in general. Let Λ^+ be the set of dominant weights. For a dominant weight $\lambda \in \Lambda^+$, we denote by ρ_λ an irreducible $\mathcal{U}(\mathfrak{k})$-module of highest weight λ. Let V_λ be any model of ρ_λ. We consider V_λ a module for $\mathfrak{p} = \mathfrak{p}_- + \mathfrak{k}$ by letting $\mathfrak{p}_-$ act trivially. Let

$$N(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V_\lambda.$$

The modules $N(\lambda)$ are often called the parabolic Verma module of highest weight λ with respect to $\mathfrak{p}$. It is well-known that the module $N(\lambda)$ has a unique irreducible quotient $L(\lambda)$. We denote by χ_λ an infinitesimal character of $L(\lambda)$. By Harish-Chandra's Theorem, an infinitesimal character χ_λ is equal to χ_μ if and only if $\lambda = w \cdot \mu$ holds for some $w \in W$, where W is the Weyl group of $\mathfrak{g}$ and $w \cdot \lambda = w(\lambda + \rho) - \rho$.

The category $\mathcal{O}^\mathfrak{p}$ is defined to be the full subcategory of the category of $\mathcal{U}(\mathfrak{g})$-modules whose objects M satisfy the conditions ($\mathcal{O}^\mathfrak{p}1$), ($\mathcal{O}^\mathfrak{p}2$), and ($\mathcal{O}^\mathfrak{p}3$).

($\mathcal{O}^\mathfrak{p}1$) M is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.
($\mathcal{O}^\mathfrak{p}2$) Viewed as a $\mathcal{U}(\mathfrak{k})$-module, M is a direct sum of finite-dimensional irreducible modules.
($\mathcal{O}^\mathfrak{p}3$) M is locally $\mathfrak{p}_-$-finite.

In this section, for simplicity, let

$$k = \frac{n+3}{2}, \qquad \underline{k-2} = (k-2, \dots, k-2), \qquad \underline{k} = (k, \dots, k),$$
$$\lambda = \Big(\frac{n+3}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2}\Big), \qquad \mu = \Big(\frac{n+1}{2}, \dots, \frac{n+1}{2}, \frac{n-1}{2}\Big).$$

Here, we have $\underline{k-2}, \underline{k}, \lambda$ and $\mu \in \mathbb{Q}^n$.

We will classify indecomposable $\mathcal{U}(\mathfrak{g})$-modules in $\mathcal{O}^{\mathfrak{p}}$ with an infinitesimal character $\chi_{\underline{k}}$.

Lemma 3.1. *Let $\mathcal{X}$ be the set of dominant weights ω such that $\omega = w \cdot \underline{k}$ for some $w \in W$. We then have*

$$\mathcal{X} = \{\underline{k-2}, \quad \underline{k}, \quad \mu, \quad \lambda\}.$$

Proof. For each $w \in W \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, there exist elements $\sigma \in \mathfrak{S}_n$ and $\tau \in (\mathbb{Z}/2\mathbb{Z})^n$ such that $w = \sigma\tau$. For a subset $I \subset \{1, \dots, n\}$, we define $\tau_I \in (\mathbb{Z}/2\mathbb{Z})^n$ by

$$\tau_I((\omega_1, \dots, \omega_n)) = (\varepsilon_I(1)\omega_1, \dots, \varepsilon_I(n)\omega_n),$$

where we let

$$\varepsilon_I(j) = \begin{cases} 1 & \text{if } j \notin I \\ -1 & \text{if } j \in I. \end{cases}$$

Note that for any $\tau \in (\mathbb{Z}/2\mathbb{Z})^n$ there exists a unique subset $I \subset \{1, \dots, n\}$ such that $\tau = \tau_I$. Given $\tau \in (\mathbb{Z}/2\mathbb{Z})^n$, it is easy to see that the weight $(\sigma\tau) \cdot \underline{k}$ is dominant for some $\sigma \in \mathfrak{S}_n$ only if the weight $(\omega_1, \dots, \omega_n) = \tau(\underline{k} + \rho)$ satisfies $\omega_i \neq \omega_j$ for every $i \neq j$. Therefore we have $(\sigma\tau_I) \cdot \underline{k}$ is dominant for some $\sigma \in \mathfrak{S}_n$ if and only if we have

$$n + 3 - j \in I \text{ for any } j \in I. \tag{3.1}$$

It is easy to see that $\tau_I \cdot \underline{k}$ is equal to one of the following weights up to the action of $\sigma \in \mathfrak{S}_n$

$$\underline{k-2}, \qquad \underline{k}, \qquad \mu, \qquad \lambda$$

for any I which satisfy (3.1). Indeed, up to the action of $\sigma \in \mathfrak{S}_n$, weights $\tau_I \cdot \underline{k}$ and $(\tau_{I \cup \{j, n+3-j\}}) \cdot \underline{k}$ are same for any $3 \leq j \leq n$. This completes the proof. □

We note that a highest weight of a parabolic Verma module with an infinitesimal character $\chi_{\underline{k}}$ are equal to one of the elements in $\mathcal{X}$.

Lemma 3.2. *The following non-split exact sequences exist:*

$$0 \longrightarrow N(\underline{k}) \longrightarrow N(\underline{k-2}) \longrightarrow L(\underline{k-2}) \longrightarrow 0,$$

$$0 \longrightarrow N(\lambda) \longrightarrow N(\mu) \longrightarrow L(\mu) \longrightarrow 0.$$

Proof. By the calculation of first reduction points in the sense of [1], the modules $N(\underline{k})$ and $N(\lambda)$ are irreducible (see the proof of Proposition 4.2). Moreover, we can prove that the modules $N(\underline{k-2})$ and $N(\mu)$ are reducible. Indeed, the weights $\underline{k-2}$ and μ are exactly same as the first reduction points. Let $\Lambda^{++} = \{(\omega_1, \dots, \omega_n) \in \Lambda^+ \cap \mathbb{Z}^n \mid \omega_n \geq 0\}$. Since the polynomial algebra $\mathcal{U}(\mathfrak{p}_+)$ is isomorphic to $\bigoplus_{\omega \in 2\Lambda^{++}} V_\omega$ as a $\mathcal{U}(\mathfrak{k})$-module, the modules $N(\underline{k-2})$ and $N(\mu)$ are isomorphic to the following modules

$$N(\underline{k-2}) = \left(\bigoplus_{\omega \in 2\Lambda^{++}} V_\omega \right) \otimes V_{\underline{k-2}}, \quad N(\mu) = \left(\bigoplus_{\omega \in 2\Lambda^{++}} V_\omega \right) \otimes V_\mu,$$

as $\mathcal{U}(\mathfrak{k})$-modules, respectively. Therefore, the modules $N(\underline{k-2})$ and $N(\mu)$ are multiplicity-free as $\mathcal{U}(\mathfrak{k})$-modules. By the same method, the modules $N(\underline{k})$ and $N(\mu)$ are multiplicity-free as $\mathcal{U}(\mathfrak{k})$-modules. Since $N(\underline{k})$ and $N(\lambda)$ are reducible, they have proper submodules. Since $\mathcal{U}(\mathfrak{k})$-modules $\rho_{\underline{k}}$ and ρ_λ do not occur in $N(\mu)$ and $N(\underline{k-2})$, respectively, the following exact sequences exist:

$$0 \longrightarrow N(\underline{k}) \longrightarrow N(\underline{k-2}), \qquad 0 \longrightarrow N(\lambda) \longrightarrow N(\mu).$$

Neither $N(\underline{k-2})/N(\underline{k})$ nor $N(\mu)/N(\lambda)$ contain the modules $N(\underline{k}) = L(\underline{k})$ and $N(\lambda) = L(\lambda)$ as $\mathcal{U}(\mathfrak{k})$-modules. Therefore the quotient modules $N(\underline{k-2})/N(\underline{k})$ and $N(\mu)/N(\lambda)$ must be irreducible. We then obtain the desired exact sequences

$$0 \longrightarrow N(\underline{k}) \longrightarrow N(\underline{k-2}) \longrightarrow L(\underline{k-2}) \longrightarrow 0, \tag{3.2}$$

$$0 \longrightarrow N(\lambda) \longrightarrow N(\mu) \longrightarrow L(\mu) \longrightarrow 0. \tag{3.3}$$

Since a Verma module has a unique irreducible quotient, the exact sequences (3.2) and (3.3) are non-split. $\square$

Lemma 3.3. *Suppose that weights x and y belong to $\mathcal{X}$. Then the following assertions hold.*

$$\dim_{\mathbb{C}} \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), L(y)) = \begin{cases} 1 & (x,y) = (\underline{k}, \underline{k-2}), (\underline{k-2}, \underline{k}), (\lambda, \mu), (\mu, \lambda) \\ 0 & \textit{otherwise}. \end{cases} \tag{1}$$

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), N(\underline{k-2})^\vee) = 0, \qquad \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), N(\underline{k-2})) = 0, \tag{2}$$

for all $x \in \{\underline{k-2}, \lambda, \mu\}$.

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), N(\mu)^\vee) = 0, \qquad \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), N(\mu)) = 0, \tag{3}$$

for all $x \in \{\underline{k-2}, \underline{k}, \mu\}$.

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(x), N(y)) = 0, \qquad \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(x)^\vee, N(y)^\vee) = 0, \tag{4}$$

for all $x, y \in \{\underline{k-2}, \mu\}$.

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(x), N(y)^{\vee}) = 0, \qquad \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(x)^{\vee}, N(y)) = 0, \tag{5}$$

for all $x, y \in \{\underline{k-2}, \mu\}$.

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(\underline{k}), N(\underline{k-2})) = 0, \qquad \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(\lambda), N(\mu)) = 0. \tag{6}$$

Proof. We first prove (1). By [3, Proposition 3.1 (d)], the case $x = y$ is clear. Since we have $\lambda \not< \underline{k}$ and $N(\lambda) = L(\lambda)$, we have

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k}), L(\lambda)) \cong \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\lambda), L(\underline{k})) = 0$$

by [3, Proposition 3.1 (a) and Theorem 3.2 (c) and (e)]. Since $L(\underline{k})$ and $L(\lambda)$ are the maximal submodules in $N(\underline{k-2})$ and $N(\mu)$, respectively, we have

$$\mathbb{C} \cong \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k}), L(\underline{k})) \cong \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k-2}), L(\underline{k})),$$
$$\mathbb{C} \cong \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}(L(\lambda), L(\lambda)) \cong \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\mu), L(\lambda)),$$

by [3, Proposition (c)]. Consider the dual modules, then this proves the case $(x, y) = (\underline{k}, \underline{k-2}), (\underline{k-2}, \underline{k}), (\lambda, \mu), (\mu, \lambda)$. Let $x = \underline{k-2}$. We then consider the following exact sequence

$$0 \longrightarrow L(y) \longrightarrow M \longrightarrow L(\underline{k-2}) \longrightarrow 0$$

for $y = \lambda, \mu$ and a module M. We may assume that the exact sequence is non-split. Let v be a non-zero vector of weight $\underline{k-2}$ in M. Then the vector v generates M and hence M is a quotient of $N(\underline{k-2})$. By Lemma 3.2, we have $M \cong N(\underline{k-2})$. This contradicts to the condition on y. Therefore we have

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k-2}), L(\lambda))) \cong \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k-2}), L(\mu))) \cong 0.$$

This proves the case that x or y is equal to $\underline{k-2}$. Similarly we proved the case that x or y is equal to μ. This completes the proof of (1).

Next, we will prove (2). By Lemma 3.2, we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}(L(x), L(\underline{k-2})) \longrightarrow \mathrm{Hom}(L(x), N(\underline{k-2})^{\vee}) \longrightarrow \mathrm{Hom}(L(x), L(\underline{k})) \\ &\longrightarrow \mathrm{Ext}(L(x), L(\underline{k-2})) \longrightarrow \mathrm{Ext}(L(x), N(\underline{k-2})^{\vee}) \longrightarrow \mathrm{Ext}(L(x), L(\underline{k})) \longrightarrow \cdots \end{aligned}$$

Here we set $\mathrm{Hom} = \mathrm{Hom}_{\mathcal{O}^{\mathfrak{p}}}$ and $\mathrm{Ext} = \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}$ If $x = \lambda$ or μ, it is easy to see that $\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(x), N(\underline{k-2})^{\vee}) = 0$ by (1) and by computing the long exact sequence. If $x = \underline{k}$ or $\underline{k-2}$, we have

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k}), N(\underline{k-2})^{\vee}) = \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(\underline{k-2}), L(\underline{k})) = 0,$$

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k-2}), N(\underline{k-2})^{\vee}) = \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N(\underline{k-2}), L(\underline{k-2})) = 0$$

by Proposition 3.1 and Proposition 3.12 of [3] and $\underline{k-2} > \underline{k}$. We can prove (3) similarly.
Calculating long exact sequences, we obtain (4), (5) and (6). We omit the details. □

In order to give the complete classification, we recall properties of indecomposable projectives. For a dominant weight $\omega \in \Lambda^+$, let $P(\omega)$ be the projective cover of $L(\omega)$, i.e., the surjective map $P(\omega) \longrightarrow L(\omega)$ is essential (cf. [3, section 3.9]). Then the projective cover $P(\omega)$ is indecomposable. For an object M in $\mathcal{O}^{\mathfrak{p}}$, we say that the module M has a standard filtration if there exists a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ such that each quotient module M_{i+1}/M_i is isomorphic to some parabolic Verma module with respect to $\mathfrak{p}$. If a module M has a standard filtration, let $(M \colon N(\omega))$ be the multiplicity of $N(\omega)$ in the standard filtration. By the universality of Verma modules, the multiplicity $(M \colon N(\omega))$ is well-defined. We also let $[M : L(\lambda)]$ be the multiplicity of $L(\lambda)$ in the Jordan-Hölder sequence.

Theorem 3.4 *([3] Chapter 9). The following statements hold:*

(1) *The category $\mathcal{O}^{\mathfrak{p}}$ has enough projectives.*
(2) *The projective cover $P(\lambda)$ has a standard filtration.*
(3) *If $x, y \in \Lambda^+$, we have*

$$(P(x) : N(y)) = [N(y) \colon L(x)].$$

We then get the following Lemma.

Lemma 3.5. *For $x \in \mathcal{X}$, we have the following assertions:*

(1) *For $x = \underline{k-2}, \mu$, the projective cover $P(x)$ is isomorphic to $N(x)$.*
(2) *The projective cover $P(\underline{k})$ has a filtration $0 \subset P_1 \subset P(\underline{k})$ such that we have*

$$P_1 \cong N(\underline{k-2}), \qquad P/P_1 \cong N(\underline{k}) \cong L(\underline{k}).$$

Moreover the projective module $P(\underline{k})$ is self-dual, i.e., $P(\underline{k}) \cong P(\underline{k})^{\vee}$.
(3) *The projective cover $P(\lambda)$ has a filtration $0 \subset P_1 \subset P(\lambda)$ such that we have*

$$P_1 \cong N(\lambda), \qquad P/P_1 \cong N(\lambda) \cong L(\lambda).$$

Moreover the projective module $P(\lambda)$ is self-dual, i.e., $P(\lambda) \cong P(\lambda)^{\vee}$.

Proof. By Theorem 3.4, it is sufficient to prove the self-duality of projective covers. By some computations of long exact sequences, we have

$$\dim_{\mathbb{C}} \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k}), N(\underline{k-2})) \leq 1, \qquad \dim_{\mathbb{C}} \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\lambda), N(\mu)) \leq 1.$$

Since projective covers $P(\lambda)$ and $P(\underline{k})$ are indecomposable, we have

$$\dim_{\mathbb{C}} \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\underline{k}), N(\underline{k-2})) = 1, \qquad \dim_{\mathbb{C}} \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(L(\lambda), N(\mu)) = 1.$$

By taking the dual, we have

$$0 \longrightarrow L(\underline{k}) \longrightarrow P(\underline{k})^{\vee} \longrightarrow N(\underline{k})^{\vee} \longrightarrow 0$$

Let v be a non-zero vector of weight $\underline{k-2}$ in $P(\underline{k})$. Then, the vector v generates $L(\underline{k-2})$ or $N(\underline{k-2})$. If the vector v generates $L(\underline{k-2})$, the quotient $P(\underline{k})^{\vee}/L(\underline{k-2})$ is isomorphic to $L(\underline{k}) \oplus L(\underline{k})$ by Lemme 3.3. Since the socle of $P(\underline{k})$ is $L(\underline{k})$ and the module $P(\underline{k})$ has a unique quotient $L(\underline{k})$, it is contradiction. Hence the vector v generates $N(\underline{k-2})$. Therefore we have a non-split exact sequence

$$0 \longrightarrow N(\underline{k}) \longrightarrow P(\underline{k})^{\vee} \longrightarrow L(\underline{k}) \longrightarrow 0.$$

Hence the projective cover $P(\underline{k})$ is self-dual. Similarly, the projective cover $P(\lambda)$ is self-dual. This completes the proof. $\square$

Let $\mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},1}$ and $\mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},2}$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ whose an object is a direct sum of

$$L(\underline{k-2}), \qquad N(\underline{k-2}), \qquad N(\underline{k-2})^{\vee}, \qquad L(\underline{k}), \qquad P(\underline{k}),$$

and

$$L(\mu), \qquad N(\mu), \qquad N(\mu)^{\vee}, \qquad L(\lambda), \qquad P(\lambda),$$

respectively.

Lemma 3.3 and 3.5 imply the following two corollaries:

Corollary 3.6. *Let M be an indecomposable $\mathcal{U}(\mathfrak{g})$-module with an infinitesimal character $\chi_{\underline{k}}$. Then the module M is isomorphic to one of the following modules*

$$L(\underline{k}), \quad L(\underline{k-2}), \quad N(\underline{k-2})^{\vee}, \quad N(\underline{k-2}), \quad P(\underline{k}),$$
$$L(\mu), \quad L(\lambda), \quad N(\mu)^{\vee}, \quad N(\mu), \quad P(\lambda).$$

Corollary 3.7. *The categories $\mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},1}$ and $\mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},2}$ are closed under extension and we have*

$$\mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N_1, N_2) = \mathrm{Ext}_{\mathcal{O}^{\mathfrak{p}}}(N_2, N_1) = 0$$

for any $N_1 \in \mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},1}$ and $N_2 \in \mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},2}$.

By the calculation of K-types and Lemma 3.2, we have

Corollary 3.8. *Let $\rho_{\underline{k}}$ be an irreducible $\mathcal{U}(\mathfrak{k})$-module of highest weight $\underline{k}$. Then the K-type $\rho_{\underline{k}}$ does not occur in modules $L(\underline{k-2})$ and M for any object M in $\mathcal{O}^{\mathfrak{p}}_{\chi_{\underline{k}},2}$.*

4. Modular forms and differential operators

We define the functions $r_{i,j}$ on $\mathfrak{H}_n$ by $\mathrm{Im}(z)^{-1} = (r_{i,j}(z))_{i,j}$ for $z \in \mathfrak{H}_n$. For a polynomial P in $n(n+1)/2$ variables with coefficients in $\mathbb{C}$, we let $r_P = P((r_{i,j})_{1\le i\le j\le n})$. Given a representation of (ρ, V) of K^c, we call a V-valued C^∞ function f nearly holomorphic if there exist finite number of polynomials P and V-valued holomorphic functions f_P such that we have

$$f(z) = \sum_P r_P(z) f_P(z), \qquad z \in \mathfrak{H}_n.$$

For a congruence subgroup Γ and a representation (ρ, V) of K^c, we say that a V-valued C^∞ function f is a nearly holomorphic modular form of K-representation ρ with respect to Γ if f satisfies the following conditions (NH1), (NH2) and (NH3).

(NH1) f is a nearly holomorphic function.
(NH2) $f(\gamma(z)) = \rho(c_\gamma z + d_\gamma) f(z)$ for all $\gamma \in \Gamma$ and $z \in \mathfrak{H}_n$.
(NH3) f satisfies the cusp condition.

The cusp condition means that for any nearly holomorphic function f which satisfies the conditions (NH1) and (NH2) with Fourier expansion

$$f(z) = \sum_{h \in \mathrm{Sym}_n(\mathbb{Q})} c(h, y) \exp(2\pi i \,\mathrm{tr}(hz)),$$

we have $c(h, y) = 0$ for any non-semipositive definite matrix h. We denote by $N_\rho(\Gamma)$ the space of nearly holomorphic function of K-representation ρ with respect to Γ. By Koecher principle, we can remove the condition (NH3) if $n > 1$. For simplicity, if $\rho = \det^k$, we say that a modular form which is of K-representation $\det^k$ is a modular form of weight k.

For the convenience, we prove the Koecher principle.

Proposition 4.1 *(Koecher principle). Let $f \colon \mathfrak{H}_n \longrightarrow V$ be a nearly holomorphic function of K-representation (ρ, V) which satisfies the conditions (NH1) and (NH2). We denote Fourier expansion of f by*

$$f(z) = \sum_{h \in \mathrm{Sym}_n(\mathbb{Q})} c(h, y) \exp(2\pi i \,\mathrm{tr}(hz)).$$

If $n > 1$, the condition (NH3) is automatically satisfied.

Proof. Let N be a level of f. Take a non-semipositive definite matrix $h = (h_{i,j})$. Then there exists a vector $v = (v_1, \dots, v_n)$ such that vh^tv is negative. We may assume that integers v_i are divisible by N for $i \geq 2$ and the greatest common divisor $v_1, \dots, v_n$ is 1. Let α be an element in $\mathrm{GL}_n(\mathbb{Z})$ such that the first row is v and $\alpha \equiv 1_n \pmod{N}$ holds. Then, the matrix $\alpha h^t\alpha = (w_{i,j})$ satisfies $w_{11} < 0$.

Since the function f has a level N, we have

$$f(\beta z \cdot {}^t\beta) = \rho({}^t\beta^{-1})f(z), \qquad \beta \in \mathrm{GL}_n(\mathbb{Z}), \beta \equiv 1_n \,(\mathrm{mod.}\ N).$$

Hence the equality

$$\rho({}^t\beta)c({}^t\beta^{-1}h\beta^{-1}, \beta y^t\beta) = c(h, y)$$

holds. In particular, we have

$$\rho(\alpha^{-1})c(\alpha h^t\alpha, {}^t\alpha^{-1}y\alpha^{-1}) = c(h, y).$$

Hence we may assume $h_{1,1} < 0$.

Fix an imaginary part y. By the definition of nearly holomorphy, there exists a polynomial P_h, depending on h, such that $P_h((r_{i,j}(z))) = c(h, y)$. For a positive integer ℓ, let

$$a_\ell = \begin{pmatrix} 1 & \ell N & \\ & 1 & \\ & & 1_{n-2} \end{pmatrix}.$$

Since h and y are fixed, there exists a rational polynomial $Q(\ell)$ with the variable ℓ such that the inequality

$$|c(h, y)| \leq |Q(\ell)||c(h, a_\ell y \cdot {}^ta_p)|$$

holds as a function of ℓ where $|\cdot|$ is some norm on V.

Combine the above formulas, the following inequality holds:

$$|c(h, y)| \leq |Q(\ell)||c(h, a_\ell y^t a_\ell)| = |Q(\ell)||\rho({}^ta_\ell^{-1})c({}^ta_\ell h a_\ell, y)|.$$

Therefore, for a certain norm for matrices, we have

$$|c(h, y)| \leq |Q(\ell)||\rho({}^ta_\ell^{-1})||c({}^ta_\ell h a_\ell, y)|.$$

The Fourier coefficient $c(h, y)$ can be expressed by

$$c(h, y) = \left(\int_{\mathrm{Sym}_n(\mathbb{R})/L} f(x + iy) \exp(2\pi i \operatorname{tr}(hx))\, dx \right) \times \exp(2\pi \operatorname{tr}(hy)),$$

where L and dx is a lattice of $\mathrm{Sym}_n(\mathbb{R})$ and a normalized measure, respectively. By taking the absolute value, we have

$$|c(h,y)| \le |M(y)| \exp(2\pi \operatorname{tr}(hy)),$$

where $M(y)$ is a constant depending only on the fixed imaginary part y. To sum it up, there exists a polynomial $R(\ell)$ such that we have

$$|c(h,y)| \le |R(\ell)||M(y)| \exp(2\pi \operatorname{tr}({}^t a_\ell h a_\ell y)).$$

Then $\operatorname{tr}({}^t a_\ell h a_\ell y)$ is equal to $h_{1,1} y_{2,2} \ell^2 + O(\ell)$. Since we assume $h_{11} < 0$ and $y_{2,2} > 0$, the right hand side is an exponential decay function in ℓ. Take a limit $\ell \longrightarrow \infty$, we have $|c(h,y)| = 0$. This completes the proof. □

Let σ be a representation of $K^c \cong \mathrm{GL}_n(\mathbb{C})$ on the dual space of $\mathrm{Sym}_n(\mathbb{C})$ defined by

$$(\sigma(k)h)(x) = h(k^{-1}x \cdot {}^t k^{-1}), \qquad h \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}_n(\mathbb{C}), \mathbb{C}), k \in K^c.$$

For a finite-dimensional representation (ρ, V) of K^c, we regard the representation $\rho \otimes \sigma$ as the representation on $\mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}_n(\mathbb{C}), V)$ defined by

$$((\rho \otimes \sigma)(k)h)(x) = \rho(k)h(k^{-1}x \cdot {}^t k^{-1}), \qquad h \in \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}_n(\mathbb{C}), V), k \in \mathrm{GL}_n(\mathbb{C}).$$

Let $\epsilon_{i,j} = (e_{i,j} + e_{j,i})$ be basis of $\mathrm{Sym}_n(\mathbb{C})$. For $u \in \mathrm{Sym}_n(\mathbb{C})$ define $u = \sum_{i,j} u_{i,j}\epsilon_{i,j}$. We also put $z = \sum_{i,j} z_{i,j}\epsilon_{i,j}$ with $z_{i,j}$ for the variable $z \in \mathfrak{H}_n \subset \mathrm{Sym}_n(\mathbb{C})$. Then, for any function $f \in C^\infty(\mathfrak{H}_n, V)$, we define $\overline{D}f$ and $Ef \in C^\infty(\mathfrak{H}_n, \mathrm{Hom}_{\mathbb{C}}(\mathrm{Sym}_n(\mathbb{C}), V))$ by

$$\overline{D}f(u) = \sum_{i,j} u_{i,j} \partial f / \partial \overline{z}_{i,j}, \qquad Ef(u)(z) = \overline{D}f(\mathrm{Im}(z) u \mathrm{Im}(z))(z)$$

for $u \in \mathrm{Sym}_n(\mathbb{C})$ and $z \in \mathfrak{H}_n$. Then a C^∞ function f is a nearly holomorphic function if and only if we have $E^m f = 0$ for some m (cf. [13].)

Given a representation of (ρ, V) of K, we denote by $C^\infty(G, \rho)$ the set of all functions f in $C^\infty(G, V)$ such that $f(gk) = \rho(k^{-1})f(g)$ for every $g \in G$ and $k \in K$. We denote by ρ^c the holomorphic representation of K^c corresponding to ρ. For the sake of simplicity, let us say ρ^c to ρ. For such a (ρ, V) and $f \in C^\infty(\mathfrak{H}_n, V)$, we define $f^\rho \in C^\infty(G, \rho)$ by (1.2). Then a map $f \longmapsto f^\rho$ is a $\mathbb{C}$-linear isomorphism of $C^\infty(\mathfrak{H}_n, V)$ onto $C^\infty(G, \rho)$. Now we have

$$\iota(u) g^\rho = (Eg)^{\rho \otimes \sigma}(u), \qquad g \in C^\infty(\mathfrak{H}_n, V), u \in \mathrm{Sym}_n(\mathbb{C}) \tag{4.1}$$

where $\iota \colon \mathrm{Sym}_n(\mathbb{C}) \xrightarrow{\sim} \mathfrak{p}_-$ defined by

$$\iota(u) = \frac{\sqrt{-1}}{4} \begin{pmatrix} {}^t u & -\sqrt{-1}\,{}^t u \\ -\sqrt{-1}\,{}^t u & -{}^t u \end{pmatrix}$$

by [12, section 7]. Hence a C^∞ function f is nearly holomorphic if and only if f^ρ is $\mathfrak{p}_-$-finite. More complete theory of correspondences (4.1) can be found in [12] and [13].

Let Γ be a congruence subgroup of G. Let $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ be the space of a scalar valued C^∞ functions φ which satisfy the following conditions (NH'1), (NH'2), (NH'3), (NH'4) and (NH'5).

(NH'1) φ is left Γ invariant.
(NH'2) φ is right $\mathcal{U}(\mathfrak{k})$ finite.
(NH'3) φ is right $\mathcal{Z}$ finite.
(NH'4) φ is slowly increasing.
(NH'5) φ is $\mathfrak{p}_-$-finite.

Here, the algebra $\mathcal{Z}$ is the center of $\mathcal{U}(\mathfrak{g})$. Then the space $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ is a $(\mathfrak{g}, K)$-module by the right translation. For $f \in C^\infty(\mathfrak{H}_n, V)$ and $v^* \in V^*$, we have a scalar valued function $\varphi_{f,v^*}(g) = \langle f^\rho(g), v^*\rangle$ on G. Then, if f is a nearly holomorphic modular form, we have $\varphi_{f,v^*} \in \mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$ by [9] and, moreover, a map $f \otimes v^* \longmapsto \varphi_{f,v^*}$ is a $\mathbb{C}$-linear injective map from $N_\rho \otimes V^*$ to $\mathcal{A}(\Gamma)_{\mathfrak{p}_-\text{-fin}}$. The $(\mathfrak{g}, K)$-module M generated by φ_{f,v^*} is independent of the choice of v^* if ρ is irreducible. Indeed, let v_1^* be a highest weight vector in V^*. There exists an element $X \in \mathcal{U}(\mathfrak{k})$ such that $X \cdot v^* = v_1^*$. Then, we have $-X \cdot \varphi_{f,v^*} = \varphi_{f,v_1^*}$. Conversely, there exists an element $Y \in \mathcal{U}(\mathfrak{k})$ such that $-Y \cdot \varphi_{f,v_1^*} = \varphi_{f,v^*}$. Hence, the module M is independent of the choice of v^*. Let $M_f = \mathcal{U}(\mathfrak{g})\varphi_{f,v^*}$ for $v^* \neq 0$. We denote by M_f the $(\mathfrak{g}, K)$-module generated by f.

Proposition 4.2. *Let f be a holomorphic modular form. Then the $(\mathfrak{g}, K)$-module M_f is semisimple.*

Proof. We may assume that f is a holomorphic modular form of an irreducible K-representation ρ_λ. Here, the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a highest weight of ρ_λ. Then, there exists a canonical exact sequence

$$N(\lambda) \longrightarrow M_f \longrightarrow 0.$$

Hence, if the Verma module $N(\lambda)$ is irreducible, the module M_f is irreducible. Let $p = \#\{i \mid \lambda_i = \lambda_n\}$ and $q = \#\{i \mid \lambda_i = \lambda_n + 1\}$. By the calculation of first reduction point as in [1], we may assume that $\lambda_n \leq n - (p + q + 1)/2$. Then, by the square-integrability theorem of Weissauer [14, Satz 3], the holomorphic modular form f is square-integrable if

$$p/2 \leq n - \lambda_n.$$

Therefore, the holomorphic modular form f is square-integrable and, moreover, the module M_f is unitarizable. Since $N(\lambda)$ has the unique irreducible quotient, we have $M_f \cong L(\lambda)$. This completes the proof. $\square$

5. Eisenstein series

5.1. Degenerate principal series representation

In this section, we review briefly the degenerate principal series representation of the metaplectic groups. For the details, see [5] and [6]. For any real symplectic group $G = \mathrm{Sp}_{2n}(\mathbb{R})$, we denote by $\widetilde{G}$ its metaplectic two fold cover. Let $\mathrm{pr}\colon \widetilde{G} \longrightarrow G$ be the canonical projection. We let $\widetilde{K} = \mathrm{pr}^{-1}(K)$. For the sake of simplicity, we denote $\widetilde{K}$ by K. We shall identify $\widetilde{G}$ as a set with

$$G \times \mathbb{Z}/2\mathbb{Z} = \{(g, \epsilon) \mid g \in G, \epsilon = \pm 1\}.$$

The multiplicative relation is described by

$$(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1 g_2, \epsilon_1 \epsilon_2 c(g_1, g_2))$$

where c is the Rao's 2-cocycle of G as in [10]. For $a \in \mathrm{GL}_n(\mathbb{R})$ and $b \in \mathrm{Sym}_n(\mathbb{R})$, we define $l(a), n(b) \in G$ by

$$l(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad n(b) = \begin{pmatrix} 1_n & b \\ 0_n & 1_n \end{pmatrix}.$$

Let

$$L = \{(l(a), \epsilon) \mid a \in \mathrm{GL}_n(\mathbb{R}), \epsilon = \pm 1\}$$

and

$$N = \{(n(b), 1) \mid b \in \mathrm{Sym}_n(\mathbb{R})\}.$$

Then $P = LN$ is a maximal parabolic subgroup of $\widetilde{G}$, called the Siegel parabolic subgroup.

Let $\chi\colon L \longrightarrow \mathbb{C}^\times$ be given by

$$\chi((l(a), \epsilon)) = \epsilon \cdot \begin{cases} i & \text{if } \det a < 0 \\ 1 & \text{if } \det a > 0. \end{cases}$$

This is a character of L of order 4. For $s \in \mathbb{C}$ and $\alpha \in \{1, 2, 3, 4\}$, let χ_s^α be the character of P given by

$$\chi_s^\alpha((l(a), \epsilon) \cdot (n(b), 1)) = |\det a|^s \chi((m(a), \epsilon))^\alpha.$$

For $\alpha = 0, 1, 2,$ and 3, let $I^\alpha(s)$ be the normalized induced representation

$$I^\alpha(s) = \mathrm{Ind}_P^{\widetilde{G}} \chi_s^\alpha.$$

We have multiplicity-free decomposition

$$I^{\alpha}(s)|_K = \bigoplus_{\lambda\in\Lambda^{++}} \rho_{2\lambda+\frac{\alpha}{2}}$$

as a K-module. Fix $v_{2\lambda+\frac{\alpha}{2}}$ to be the unique (up to constant) K-highest weight vector in $\rho_{2\lambda+\frac{\alpha}{2}}$. We then consider the K-map given by

$$m\colon (\mathfrak{p}_+ + \mathfrak{p}_-)\otimes \rho_{2\lambda+\frac{\alpha}{2}} \longrightarrow I^{\alpha}(s)|_K$$
$$m(p\otimes v) = p\cdot v.$$

Since $\mathfrak{p}_+ + \mathfrak{p}_- \cong \rho_{(2,0,\dots,0)} \oplus \rho_{(0,\dots,0,-2)}$, highest weights μ in $(\mathfrak{p}_+ + \mathfrak{p}_-)\otimes\rho_\lambda$ are of the form

$$\lambda \pm e_i \pm e_j, \qquad 1\le i\le j\le n$$

for a dominant weight $\lambda = (\lambda_1,\dots,\lambda_n)$. For each $1\le j\le n$, there exists an element X_j in $\mathcal{U}(\mathfrak{g})$ such that $X_j\cdot v_{2\lambda+\frac{\alpha}{2}}$ is a constant multiple of $v_{2\lambda+\frac{\alpha}{2}\pm 2e_j}$. Then we have the coefficients $c_{\lambda,j,\pm}\in\mathbb{C}$ such that $X_j\cdot v_{2\lambda+\frac{\alpha}{2}} = c_{\lambda,j,\pm}\cdot v_{2\lambda+\frac{\alpha}{2}\pm 2e_j}$. Note that the coefficients $c_{\lambda,j,\pm}$ is depending only on the choice of the highest weight vectors v_λ and elements X_j. For suitable choices of v_λ and X_j in [5] and [6], we have

$$c_{\lambda,j,\pm} = -s-1\pm\Big(\frac{n+1}{2}-\frac{\alpha}{2}+j-\lambda_j\Big).$$

Let $\mathrm{Ad}\colon \widetilde{G}\longrightarrow \mathrm{Aut}(\mathfrak{g})$ be the adjoint representation of $\widetilde{G}$. Then the algebra $\mathcal{U}(\mathfrak{p}_\pm)$ is stable under $\mathrm{Ad}(k)$ for $k\in K$. The algebra $\mathcal{U}(\mathfrak{p}_+)$ (resp. $\mathcal{U}(\mathfrak{p}_-)$) decompose into

$$\bigoplus_{\lambda}\rho_\lambda$$

where λ runs through weights in $\Lambda^+\cap 2\mathbb{Z}^n_{\ge 0}$ (resp. λ runs through weights in $\Lambda^+\cap 2\mathbb{Z}^n_{\le 0}$) as a representation of K.

Lemma 5.1. *Let $\alpha\in\{0,1,2,3\}$ and $k=(n+3)/2$. Suppose $2k\equiv\alpha$ (mod. 4). Let π be an irreducible K-subrepresentation in $I^{\alpha}(-1)$ of highest weight $\underline{k}$. Then the representation π generates $N(\underline{k-2})^\vee$ as a $(\mathfrak{g},K)$-module.*

Proof. Let M be the $(\mathfrak{g},K)$-module generated by π. By calculations of $c_{\lambda,j,\pm}$, there exists a non-split exact sequence

$$0\longrightarrow L(\underline{k-2})\longrightarrow M\longrightarrow L(\underline{k})\longrightarrow 0.$$

For details, see [5, section 5]. By Lemma 3.2, the module M is isomorphic to $N(\underline{k-2})^\vee$. This completes the proof. $\square$

5.2. Eisenstein series and Fourier coefficients

In this section, we consider the metaplectic group Mp_{2n} as a non-trivial central extension of Sp_{2n} by the circle S^1. We denote by $\widetilde{G}$ the central extension by $\mathbb{Z}/2\mathbb{Z}$ as in the previous section. We consider the group $\widetilde{G}$ as a subgroup of Mp_{2n}. The map $\widetilde{G} \longrightarrow \mathrm{Mp}_{2n}$ may found in Kudla's note [4, Chapter 1] and [10].

Let $k = (n+3)/2$. We let N be a positive integer greater than 1 if k is an integer. We also let $N = 4$ if k is not an integer. Define a congruence subgroup Γ of $\mathrm{Sp}_{2n}(\mathbb{Q})$ by

$$\Gamma = \begin{cases} \{g \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid c_g \equiv 0 \ (\mathrm{mod.}\ N)\} & (k \in \mathbb{Z}), \\ \{g \in \mathrm{Sp}_{2n}(\mathbb{Z}) \mid b_g \equiv c_g \equiv 0 \ (\mathrm{mod.}\ 2)\} & (k \notin \mathbb{Z}). \end{cases}$$

If a weight k is not an integer, the congruence subgroup Γ is a subgroup of the theta subgroup (cf. [13]). Fix a Dirichlet character χ modulo N of order 2. Let $j(g,z)$ be a factor of automorphy on $G \times \mathfrak{H}_n$ defined by

$$j(g,z) = \det(c_g z + d_g), \qquad (g,z) \in G \times \mathfrak{H}_n.$$

Let h be a factor of automorphy of weight $1/2$ defined in [13, Appendix 2]. Then the factor of automorphy h satisfies

$$h((g,\epsilon),\mathfrak{i})^2 = t \cdot j(c_g\mathfrak{i} + d_g), \qquad (g,\epsilon) \in \mathrm{Mp}_{2n}(\mathbb{R}),$$

with some $t \in S^1$. Let j^k be a factor of automorphy of weight k defined by

$$j^k = \begin{cases} j^k & \text{if } k \text{ is an integer} \\ j^{k-1/2}h & \text{if } k \text{ is not an integer.} \end{cases}$$

For every $s \in \mathbb{C}$ and $\alpha \in \{0,1,2,3\}$, we take an element δ_s which belongs to $I^\alpha(2s-k)$ defined by

$$\delta_s(g) = j(g,\mathfrak{i})^k \det(\mathrm{Im}(g(\mathfrak{i})))^{s-k/2}.$$

We then define the Eisenstein series $E(g,s)$ on the metaplectic group by

$$E(g,s) = E(g,s;k,\chi,N) = \sum_{\gamma \in \Gamma\cap P\backslash\Gamma} \chi(\det d_\gamma)\delta_s(\gamma g), \quad g \in \mathrm{Mp}_{2n}(\mathbb{R}).$$

The Eisenstein series $E(g,s)$ is absolutely convergent for $\mathrm{Re}(s) \geq (n+1)/2$. Due to Langlands' theory for Eisenstein series, $E(g,s)$ is meromorphically continued to whole s-plane. Note that if k is an integer, $E(g,s)$ can be defined on G via the canonical projection $\mathrm{pr}\colon \mathrm{Mp}_{2n} \longrightarrow \mathrm{Sp}_{2n}$. For $z \in \mathfrak{H}_n$, we define the function $E(z,s)$ on $\mathfrak{H}_n$ by

$$E(z,s) = h(g,i)^{2k} E(g,s), \qquad g \in \widetilde{G} \text{ such that } g(\mathfrak{i}) = z.$$

This is a well-defined function. In order to compute the Fourier coefficients of Eisenstein series, we will twist the Eisenstein series at finite places. Let $K_{\mathbb{A}}$ be a subgroup of $\mathrm{Sp}_{2n}(\mathbb{A})$, the adele valued points of G, defined by

$$K_{\mathbb{A}} = K_{\mathrm{fin}} \times K,$$
$$K_{\mathrm{fin}} = \{g \in \mathrm{Sp}_{2n}(\mathbb{A}_{\mathrm{fin}}) \mid a_g, d_g \in \mathrm{Mat}_n(Z), b_g \in \mathrm{Mat}_n(b^{-1}\mathbb{Z}), c_g \in \mathrm{Mat}_n(bN\mathbb{Z})\},$$

where $b = 1$ and N is a positive integer if k is an integer and $b = 1/2$ and $N = 4$ if k is not an integer. Note that the open compact subgroup K_{fin} is the closure of the congruence subgroup Γ in $\mathrm{Sp}_{2n}(\mathbb{A}_{\mathrm{fin}})$. By the strong approximation in $\mathrm{Sp}_{2n}(\mathbb{A})$, for every $g \in \mathrm{Mp}_{2n}(\mathbb{A})$, there exist $\gamma \in \mathrm{Sp}_{2n}(\mathbb{Q})$, $g_\infty \in \mathrm{Mp}_{2n}(\mathbb{R})$ and $k \in K_{\mathbb{A}}$ such that $g = \gamma g_\infty k$. Then we define the Eisenstein series $E_{\mathbb{A}}(g, s)$ on $\mathrm{Sp}_{2n}(\mathbb{A})$ or $\mathrm{Mp}_{2n}(\mathbb{A})$ by

$$E_{\mathbb{A}}(g, s) = j(k, \mathfrak{i})^k E(g_\infty, s).$$

Define an element $\zeta \in \mathrm{Sp}_{2n}(\mathbb{A})$ by

$$\zeta_\infty = 1_{2n}, \qquad \zeta_p = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}.$$

We also define an element $\widetilde{\zeta}$ of $\mathrm{Mp}_{2n}(\mathbb{A})$ by

$$\mathrm{pr}(\widetilde{\zeta}) = \zeta, \qquad h(z, \widetilde{\zeta}) = 1.$$

Define a function $E^*_{\mathbb{A}}(g, s)$ by

$$E^*_{\mathbb{A}}(g, s) = \begin{cases} E_{\mathbb{A}}(g\zeta, s) & (g \in \mathrm{Sp}_{2n}(\mathbb{A}), k \in \mathbb{Z}) \\ E_{\mathbb{A}}(g\widetilde{\zeta}, s) & (g \in \mathrm{Mp}_{2n}(\mathbb{A}), k \notin \mathbb{Z}). \end{cases}$$

We also define the function $E^*(z, s)$ on $\mathfrak{H}_n$, similarly. Eisenstein series $E^*(z, s)$ have the Fourier expansion of the form

$$E^*(z, s) = \sum_{h \in \mathrm{Sym}_n(\mathbb{Q})_{\geq 0}} c_h(y, s) \exp(2\pi\sqrt{-1}\,\mathrm{tr}(hz)), \quad z = x + \sqrt{-1}\,y \in \mathfrak{H}_n.$$

The Fourier coefficients $c_h(y, s)$ are already calculated by Shimura. For the details, see [11] and [13]. In order to obtain the formula, we first put

$$\xi(g, h; s, s') = \int_{\mathrm{Sym}_n(\mathbb{R})} \exp(-2\pi\sqrt{-1}\,\mathrm{tr}(hx)) \det(x + ig)^{-s} \det(x - ig)^{-s'} dx,$$

where s, $s' \in \mathbb{C}$, $0 < g \in \mathrm{Sym}_n(\mathbb{R})$ and $h \in \mathrm{Sym}_n(\mathbb{R})$. We also put, for a half integral matrix $\tau \in \mathrm{Mat}_n(\mathbb{Q}_p)$,

$$\alpha_N^0(\tau,s,\chi)=\prod_{p\nmid N}\sum_{\sigma\in\mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)}\exp_p(-\mathrm{tr}(\tau\sigma))\chi^*(\nu_0(\sigma))\nu(\sigma)^{-s}$$

$$\alpha_N^1(\tau,s,\chi)=\prod_{p\nmid N}\sum_{\sigma\in\mathrm{Sym}_n(\mathbb{Q}_p)/\mathrm{Sym}_n(\mathbb{Z}_p)}\exp_p(-\mathrm{tr}(\tau\sigma))\chi^*(\nu_0(\sigma))\omega(\sigma)\nu(\sigma)^{-s}.$$

Here, for $x\in\mathbb{Q}_p$, $\exp_p(x)=\exp(-2\pi\sqrt{-1}\,y)$ with $y\in\cup_{m=1}^{\infty}p^{-m}\mathbb{Z}$ and $x-y\in\mathbb{Z}_p$, χ^* is the ideal character associated to χ, $\nu_0(\sigma)$ is the denominator ideal, $\nu(\sigma)$ is the norm of $\nu_0(\sigma)$, and ω is described as follows. For $a\in\mathrm{Sym}_n(\mathbb{A})$, we put

$$\gamma(a)=\prod_p\gamma_p(a),\qquad \gamma_p=\int_{\mathbb{Z}_p^n}\exp_p({}^tx\cdot ax/2)dx,\qquad \omega(a)=\gamma(a)/|\gamma(a)|,$$

where p runs finite places, the measure dx is the Haar measure of $\mathbb{Z}_p^n$ such that $\int_{\mathbb{Z}_p^n}dx=1$ and we assume that $\gamma(a)\neq 0$. The following Proposition is due to Shimura.

Proposition 5.2 *([13]). Let $q\in\mathrm{GL}_n(\mathbb{R})$. Suppose that $N>1$ and $\det q>0$. Let $y={}^tqq$. Then $c_h(y,s)\neq 0$ only if $h\in\mathrm{Sym}_n(b^{-1}N^{-1}\mathbb{Z}_p)$ for every finite places p, in which case*

$$\begin{aligned}&c_h(y,s)\exp(-2\pi\,\mathrm{tr}(hy))\\&=C\cdot(bN)^{-n(n+1)/2}\det(y)^{s-k/2}\xi(y,h;s+k/2,s-k/2)\alpha_N^e({}^tqhq,2s,\chi),\end{aligned}$$

where $C=1$ and $e=0$ if k is an integer and $C=\exp(\pi\sqrt{-1}\,n/4)$ and $e=1$ if k is not an integer.

Let $b_0(x)=1$, $b_j(x)=\prod_{m=0}^{j-1}(x+(m/2))$ if $j>0$. For an indeterminate T, we define

$$\det(T1_n-X)=\sum_{j=0}^{n}(-1)^r\phi_r(X)T^{n-r},\quad X\in\mathrm{Mat}_n(\mathbb{C}).$$

By the explicit formula of confluent hypergeometric functions and Siegel series, we have the following Lemma.

Lemma 5.3. *Suppose $n>1$ and $k=(n+3)/2$, $\chi^2=1$ and $N>1$. Then the Fourier coefficient $c_h(y,k/2)$ is described as follows: If $h=0$, we have*

$$c_0(y,k/2)=c\,\det y^{-1}\quad\text{with}\quad c\in\mathbb{C}.$$

If $h>0$, the Fourier coefficient $c_h(y,k/2)$ is a constant independent of y. If $h\geq 0$ and $0<\mathrm{rank}(h)<n$, we have

$$c_h(y,k/2)=c\,\det y^{-1}\sum_{j=0}^{\mathrm{rank}(h)}b_j((n-r)/2)\phi_{r-j}(4\pi hy),\quad c\in\mathbb{C}.$$

Moreover, the Fourier coefficients $c_0(y,k)$ and $c_h(y,k)$ are non-zero for some $h>0$.

Proof. It is sufficient to prove that Fourier coefficients $c_0(y,s)$ and $c_h(y,s)$ are non-zero at $s=k/2$ for some $h>0$ by [13]. By the explicit formula of Siegel series, $c_0(y,s)$ is described as follows. Define $\Lambda(s)$ and $\Lambda_0(s)$ by

$$\Lambda(s)=\begin{cases} L(2s,\chi)\displaystyle\prod_{i=1}^{(n-1)/2} L(4s-2i,\chi^2) & n\in 2\mathbb{Z}+1 \\ \displaystyle\prod_{i=1}^{n/2} L(4s-2i+1,\chi^2) & n\in 2\mathbb{Z} \end{cases}$$

$$\Lambda_0(s)=\begin{cases} L(2s-n,\chi)\displaystyle\prod_{i=1}^{(n-1)/2} L(4s-2n+2i-1,\chi^2) & n\in 2\mathbb{Z}+1 \\ \displaystyle\prod_{i=1}^{n/2} L(4s-2n+2i-2,\chi^2) & n\in 2\mathbb{Z} \end{cases}$$

where $L(s,\chi)$ is the Dirichlet L function. Then, up to bad local factor, we have

$$c_0(y,s)=(\Lambda(2s)/\Lambda_0(2s))\cdot\det(y)^{-1}.$$

Then it is easy to see that $c_0(y,k/2)\neq 0$. By some computation of L-factors as in [13, Proposition 16.10], it is clear that $c_h(y,k/2)\neq 0$ for some $h>0$. This completes the proof. □

5.3. Main theorem

We define a function ϕ on G or $\widetilde{G}$ by

$$\phi(g)=h(g,\mathfrak{i})^{-(n+3)}\det(\operatorname{Im} g(\mathfrak{i}))^{-1}.$$

It is what is often called the constant term of E^* along the Siegel parabolic subgroup.

Lemma 5.4. *Let $k=(n+3)/2$. The constant term ϕ generates $N(\underline{k-2})^\vee$ as a $(\mathfrak{g},K)$-module. In particular, the constant term ϕ has an infinitesimal character $\chi_{\underline{k}}$.*

Proof. It is easy to see that ϕ belongs to $I^\alpha(-1)$ for $\alpha\equiv 2k \pmod{4}$. By Lemma 5.1, ϕ generates $N(\underline{k-2})^\vee$. □

Then we can prove the main theorem.

Theorem 5.5. *With the same assumption as in Lemma 5.3, let M be the $(\mathfrak{g}, K)$-module generated by $E^*(g, k/2)$. We then have*

$$M \cong N(\underline{k-2})^\vee.$$

Proof. In this proof, we follow the notation as in section 2. By the definition of E^*, the Eisenstein series E^* has the same infinitesimal character as the Siegel Eisenstein series E. Note that Eisenstein series E has the same infinitesimal character as its constant term ϕ'. Since the constant terms ϕ and ϕ' are different only in finite places, they have the same infinitesimal character. Hence the action of $\mathcal{Z}$ on M is equal to the character $\chi_{\underline{k}}$. By Corollary 3.6 and Corollary 3.8, the module M is a direct sum of following modules:

$$L(\underline{k}), \quad N(\underline{k-2})^\vee, \quad N(\underline{k-2}), \quad P(\underline{k}).$$

Let M' be the submodule of M generated by the functions $X \cdot E^*$ for $X \in \mathfrak{p}_-$. Since $E^*(g, k/2)$ is non-holomorphic, the submodule M' is non-zero. By (4.1) and Lemma 5.3, for a non-constant vector $X \in \mathcal{U}(\mathfrak{p}_-)$, the Fourier coefficient $c(X, h, y)$ of $X \cdot E^*$ at a positive definite matrices h is 0. Therefore, the submodule M' is a non-zero proper submodule of M. Let $L(\omega) = L((\omega_1, \dots, \omega_n))$ and v be an irreducible submodule of M' and its highest weight vector, respectively. Let f be the holomorphic modular form corresponding to v. Since we have $c(X, h, y) = 0$ for a non-constant $X \in \mathcal{U}(\mathfrak{p}_-)$ and a positive definite matrix $h > 0$, the modular form f is a singular form. By [2] and [14], we have $\omega_n < n/2$. Therefore we have $\omega = \underline{k-2}$. Since the module M/M' is a non-zero module of highest weight $\underline{k}$ and the module M is generated by only one element of weight $\underline{k}$, we have the following exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow L(\underline{k}) \longrightarrow 0. \tag{5.1}$$

Since the socle of $N(\underline{k-2})$ and $P(\underline{k})$ are $L(\underline{k})$, there exist integers a and b such that we have

$$M \cong aL(\underline{k}) \oplus bN(\underline{k-2})^\vee.$$

Then, by definition of M', we have $M' \cong bL(\underline{k-2})$ and hence the multiplicity b is non-zero. By the exact sequence (5.1), we have $a + b = 1$. Hence, we have $a = 0$ and $b = 1$. This completes the proof. □

Remark 5.6. We define a complex number c_h for a semi-positive matrices h by

$$E^*(z, k/2) = \sum_{h \geq 0, h \not> 0} c_h(\det y^{-1} + f_h(\mathrm{Im}(z)^{-1})) \exp(2\pi i \operatorname{tr}(hz)) + \sum_{h > 0} c_h \exp(2\pi i \operatorname{tr}(hz)).$$

Here, the function f_h is a polynomial with $n(n+1)/2$ variables of degree less than n (cf. section 4). We can compute c_h by Lemma 5.3. Then, by the computation of the differential operator $\mathcal{D} = c_n \det(\partial/\partial r_{i,j})$ with some normalization factor c_n, the singular form

$$\mathcal{D}E^*(z) = \sum_{h \geq 0, h \not> 0} c_h \exp(2\pi i \operatorname{tr}(hz))$$

generates $L(\underline{k-2})$. Note that, the singular form $\mathcal{D}E^*$ is a residue of some Eisenstein series (see [13, section 17]).

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