Journal of Number Theory 202 (2019) 347–387

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

General Section

On a class of quartic Thue equations with three parameters

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ARTICLE INFO

Article history:
Received 28 November 2017
Received in revised form 19 January 2019
Accepted 21 January 2019
Available online 15 February 2019
Communicated by L. Smajlovic

MSC:
primary: 11D59, 11D09, 11D75
secondary: 11A55

Keywords:
Thue equations
Method of Tzanakis
System of pellian equations
Linear forms in logarithms

ABSTRACT

Let k, m and n be integers. In this paper, for a fixed integer $\mu \neq 0$, we show that the family of Thue equation

$$x^4 - kmnx^3y + (km^2 - kn^2 + 2)x^2y^2 + kmnxy^3 + y^4 = \mu,$$

is reducible by Tzanakis's method into a system of pellian equations

$$kV^2 - (km^2 + 4)U^2 = -4\mu, \qquad kZ^2 - (kn^2 - 4)U^2 = 4\mu,$$

with any triple of integers (k, m, n) such that $k > 0$, $|n| \geq 2$, $|m| \geq 2$. We consider this system for any even integer $k \neq 2\square$, $\mu = 1$ and we prove that for all integers $|n| \geq 2$ and $|m| \geq 2$ that are sufficiently large and have sufficiently large common divisor this system has only the trivial solutions $(U, V, Z,) = (\pm 1, \pm m, \pm n)$. We also show that if $k \neq 2\square$ is even, then the system has in general at most 8 solutions in positive integers.
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https://doi.org/10.1016/j.jnt.2019.01.002
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1. Introduction

Let $f \in \mathbb{Z}[X,Y]$ be an irreducible binary form of degree $d \geq 3$ and μ a non-zero rational integer. Then Diophantine equation of the form

$$f(x,y) = k, \quad (x,y) \in \mathbb{Z} \times \mathbb{Z}$$

is called a Thue equation. The name is given in honor of the Norwegian mathematician A. Thue [19] who proved in 1909 that the equation has only finitely many solutions. His proof was not effective. For a nice summary of the history on Thue equations, one can refer to the introduction of V. Ziegler [21].

In recent years, general powerful methods have been developed for explicit solution of Thue equations (see [16,20]). It was in 1990 that E. Thomas [18] has solved the first parameterized family of Thue equations

$$x^3 - (n-1)x^2y - (n+2)xy^2 - y^3 = 1,$$

where n is an integer. Since then, several families of parameterized Thue equations have been studied (see [10] for references).

In 1993, N. Tzanakis [20] considered Thue equations of the form $f(x,y) = \mu$, where f is a homogeneous quartic form that corresponds to a Galois non cyclic field $\mathbb{K}$. If $\mathbb{K}$ is not totally complex, the assumption $\mathbb{K}$ is Galois non cyclic is equivalent to $\mathbb{K} = \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$, where d_1 and d_2 are distinct squarefree positive integers (see [14], [4]). These conditions hold if the cubic resolvent of $f(x,1) = 0$ has three distincts rationals roots.

Tzanakis proved that solving the equation $f(x,y) = \mu$ consists in solving a system of Pell equations. This method was successfully applied by several authors not only for families of quartic Thue equations but also to families of quartic Thue inequalities. One can cite for examples [4–6,8,9,11–13,21].

We notify here that all these studied equations have one parameter. It is B. Jadrijević [12], who in 2005 studied for the first time the following family with two parameters

$$x^4 - 2mnx^3y + 2(m^2 - n^2 + 1)x^2y^2 + 2mnxy^3 + y^4 = 1.$$

In 2006, V. Ziegler [21] studied for the first time the following family with three parameters

$$x^4 - 4sx^3y - (2ab + 4s(a+b))x^2y^2 - 4absxy^3 + a^2b^2y^4 = \pm 1.$$

The aim of this paper is to study the following families of quartic Thue equations with three integral parameters k, m and n

$$x^4 \mp kmnx^3y + (k(m^2 - n^2) + 2)x^2y^2 \pm kmnxy^3 + y^4 = \mu, \tag{1}$$

where $\mu = 1$ and k is even and not a double of perfect square ($k \neq 2\square$).

The fundamental reason which leads us to study these families with three integral parameters is that they extend the family with two parameters studied by B. Jadrijević [12] and unify many others families with one parameter:

- For $(k,m,n) = (2\square, m, n)$, one obtains the family with two parameters studied by B. Jadrijević [12];
- For $(k,m,n) = (2c, 2, 1)$, $|c| \geq 2$, one obtains the family with one parameter studied by B. Jadrijević and A. Dujella [4];
- For $(k,m,n) = (2c, 1, 1)$, $|c| \geq 3$, one obtains the family with one parameter studied by B. Ibrahimpašić [11].

But how did we get these families? We consider the quartic Thue equation with two integral parameters a and b

$$x^4 - ax^3y + bx^2y^2 + axy^3 + y^4 = \mu \tag{2}$$

and we look its reduction by Tzanakis's method. We obtain the following results:

Theorem 1. *The Thue equation (2) is reducible in a system of pellian equations by Tzanakis's method if and only if*

$$a = \pm kmn \quad \text{and} \quad b = km^2 - kn^2 + 2, \tag{3}$$

where $k \in \mathbb{Z} \setminus \{0\}$ and m, n are coprime positive integers with $m > n$ or $(m,n) = (1,1)$ or $(m,n) = (1,0)$ and the triple (k,m,n) does not satisfy any of the following conditions:

(1) $(k,m,n) = (k,1,0)$, $k \geq -4$;
(2) $(k,m,n) = (k,1,1)$, $|k| \leq 4$;
(3) $(k,m,n) = (k,m,1)$, $1 \leq k \leq 4$, $m \geq 2$;
(4) $(k,m,n) = (1,m,2)$, $m \geq 3$;
(5) $(k,m,n) = (-1,2,1)$.

Theorem 2. *Let (c,m,n) be a triple of integers with $c > 0$, $c \neq \square$, $|n| \geq 2$ and $|m| \geq 2$. For any $0.5 < \varepsilon < 1$, there exists an effectively computable constant $C(\varepsilon)$ such that if $\max\{|m|,|n|\} \geq C(\varepsilon)$ and $\gcd(m,n) \geq \max\{|m|^\varepsilon, |n|^\varepsilon\}$, then the system of pellian equations*

$$cV^2 - (cm^2+2)U^2 = -2, \qquad cZ^2 - (cn^2-2)U^2 = 2 \tag{4}$$

has only the trivial solutions $(U,V,Z) = (\pm 1, \pm m, \pm n)$. In particular, we take $C(0.999) = 10^{23}$, $C(0.995) = 10^{26}$, $C(0.99) = 10^{28}$, $C(0.95) = 10^{38}$, $C(0.9) = 10^{48}$, $C(0.8) = 10^{71}$, $C(0.7) = 10^{116}$, $C(0.6) = 10^{256}$, $C(0.55) = 10^{551}$.

Corollary 3. *Let $m \geq 2$ and $n \geq 2$ be a coprime positive integers and let $\mu = 1$ and $k = \pm 2c$ with $c = tg^2 \neq \square$, where t and g are positive integers. For any $0.5 < \varepsilon < 1$, there exists an effectively computable constant $C(\varepsilon)$ such that if $\max\{mg, ng\} \geq C(\varepsilon)$ and $g^{1-\varepsilon} \geq \max\{m^{\varepsilon}, n^{\varepsilon}\}$, then the only solutions of Thue equation (1) are $(x, y) = (0, \pm 1), (\pm 1, 0)$.*

Theorem 4. *For any (c, m, n) triple of integers with $c \neq \square$, $|n| \geq 2$ and $|m| \geq 2$, the system of pellian equations (4) has at most 8 solutions in positive integers.*

Theorem 5. *Let $g = \gcd(m, n)$.*

(1) *If $|n| > |m| \geq 2$ and the system of pellian equations (4) has no solution for $\frac{g^2\sqrt{c}\log|m|}{|n|\log|n|} \leq 5$ then it has at most 2 solutions in positive integers $(U, V, Z) = (U_{i_1}, V_{i_1}, Z_{j_1}), (U_{i_2}, V_{i_2}, Z_{j_2})$ which satisfy $2i_2 + 1 = q(2i_1 + 1)$ with integer $q \leq 40$.*
(2) *If $|m| \geq |n| \geq 2$ and the system of pellian equations (4) has no solution for $\frac{g^2\sqrt{c}\log|n|}{|m|\log|m|} \leq 13.244$ then it has at most 2 solutions in positive integers $(U, V, Z) = (U_{i_1}, V_{i_1}, Z_{j_1}), (U_{i_2}, V_{i_2}, Z_{j_2})$ which satisfy $2i_2 + 1 = q(2i_1 + 1)$ with integer $q \leq 46$.*

Here is the organization of this paper. In the next section, we recall the method of Tzanakis for solving a class of quartic equations. We use it in Section 3 to prove Theorem 1, where we obtain the system of pellian equations (4). In Section 4 and Section 5, after defining the recurrences sequences related to the pellian equations, we will give a relation between them and Lehmer sequences. We set a linear form in logarithms from theses sequences and obtain an upper bound of this linear form for each case. We also use the congruence method introduced in [7] with a result due to Baker and Wüstholz to complete the proof of Theorem 2. In Section 6, we study some special cases of the equation (13) and prove Theorem 4 using theorem of Bennett [1] on simultaneous approximations of algebraic numbers We finish this paper with the proof of our last theorem in Section 7.

2. The method of Tzanakis

In this section, we will describe the method of Tzanakis for solving a quartic Thue equation, corresponding to a quartic field $\mathbb{K}$ that has the properties described below. So let us consider a Thue equation

$$f(x, y) = \mu, \tag{5}$$

where

$$f(x, y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4 \in \mathbb{Z}[x, y],\ a_0 > 0.$$

We assign to this equation the cubic equation

$$4\rho^3 - g_2\rho - g_3 = 0, \tag{6}$$

where

$$g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \in \frac{1}{12}\mathbb{Z}, \quad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbb{Z}.$$

In 1993, Tzanakis [20] proved that a necessary condition for applying his method is that equation (6) has three distinct rational roots ρ_1, ρ_2, ρ_3 that satisfy

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}. \tag{7}$$

Let $H(x, y)$ and $G(x, y)$ be the quartic and sextic covariants of $f(x, y)$ respectively, i.e.

$$H(x,y) = -\frac{1}{144}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} \in \frac{1}{48}\mathbb{Z}[x,y],$$

$$G(x,y) = -\frac{1}{8}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{vmatrix} \in \frac{1}{96}\mathbb{Z}[x,y].$$

Then, we have $4H^3 - g_2Hf^2 - g_3f^3 = G^2$. Let us put

$$H = \frac{1}{48}H_0, G = \frac{1}{96}G_0, \rho_i = \frac{1}{12}r_i, i = 1, 2, 3,$$

then

$$H_0, G_0 \in \mathbb{Z}[x, y], r_i \in \mathbb{Z}, i = 1, 2, 3,$$

and

$$(H_0 - 4r_1f)(H_0 - 4r_2f)(H_0 - 4r_3f) = 3G_0^2.$$

There exist positive square-free integers k_1, k_2, k_3 and quadratic $G_1, G_2, G_3 \in \mathbb{Z}[x, y]$ such that

$$H_0 - 4r_if = k_iG_i^2, \quad i = 1, 2, 3$$

and $k_1k_2k_3(G_1G_2G_3)^2 = 3G_0^2$. If $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of (5), then we obtain

$$k_2G_2^2 - k_1G_1^2 = 4(r_1 - r_2)\mu, \tag{8}$$

$$k_3G_3^2 - k_1G_1^2 = 4(r_1 - r_3)\mu. \tag{9}$$

3. Proof of the Theorem 1

Now, we apply the Tzanakis method to equation (2). In this particular case, we obtain

$$a_0 = 1, \quad a_1 = -\frac{a}{4}, \quad a_2 = \frac{b}{6}, \quad a_3 = \frac{a}{4}, \quad a_4 = 1,$$

and a quick computation of invariants g_2 and g_3 give

$$g_2 = \frac{b^2 + 3a^2 + 12}{12} \quad \text{and} \quad g_3 = -\frac{2b^3 + (9a^2 - 72)b + 54a^2}{432}.$$

Then the cubic resolvent is

$$(12\rho - b - 6)[144\rho^2 + (12b + 72)\rho - 2b^2 + 12b - 9a^2] = 0. \tag{10}$$

The discriminant of the polynomial $144\rho^2 + (12b + 72)\rho - 2b^2 + 12b - 9a^2$ is

$$\Delta' = 18^2 \left[(b-2)^2 + (2a)^2\right] \geq 0.$$

Therefore

$$\Delta' = 0 \Leftrightarrow b = 2 \text{ and } a = 0.$$

To have two distincts rational roots ρ_2 and ρ_3 for

$$144\rho^2 + (12b + 72)\rho - 2b^2 + 12b - 9a^2,$$

it suffices that

$$(b-2)^2 + (2a)^2 = z^2 \text{ with } z \neq 0. \tag{11}$$

Lemma 6. *All solutions $(b, a, z) \in \mathbb{Z}^3$ of the equation (11) are given by:*

$$b = k(m^2 - n^2) + 2, \ a = \pm kmn, \ z = \pm k(m^2 + n^2),$$

where $k \in \mathbb{Z} \setminus \{0\}$ and m, n are coprime positive integers with $m > n$ or $(m, n) = (1, 1)$ or $(m, n) = (1, 0)$.

Proof. All solutions (b, a, z) of the equation (11) are given by

$$b = k(m^2 - n^2) + 2, \ a = \pm kmn, \ z = \pm k(m^2 + n^2),$$

where:

- $k \in \mathbb{Z} \setminus \{0\}$ and m, n are positive integers with $m > n$, and with m and n coprime and not both odd ($b \neq 2$ and $a \neq 0$);

- $k \in \mathbb{Z} \setminus \{0\}$ and $m = n = 1$ ($b = 2$ and $a \neq 0$);
- $k \in \mathbb{Z} \setminus \{0\}$ and $m = n = 1$ ($b \neq 2$ and $a = 0$);

or by

$$b = 2kmn + 2,\ a = \pm k(m^2 - n^2),\ z = \pm k(m^2 + n^2),$$

where $k \in \mathbb{Z} \setminus \{0\}$ is even and m, n are positive integers with $m > n$, and with m and n coprime and not both odd.

If m and n are coprime and both odd, then there exist m_1, n_1 and with $m_1 > n_1$, m_1 and n_1 coprime and not both odd such that

$$\begin{aligned} b =& k(m^2 - n^2) + 2 = 4km_1n_1 + 2, \\ a =& \pm\ kmn = \pm 2k(m_1^2 - n_1^2), \\ z =& \pm\ k(m^2 + n^2) = \pm 2k(m_1^2 + n_1^2). \end{aligned}$$

This completes the proof of Lemma 6. □

For all triples (k, m, n) given by Lemma 6, the equation (10) has two distinct rational roots $\rho_2 = \frac{km^2+2kn^2-4}{12}$ and $\rho_3 = -\frac{2km^2+kn^2+4}{12}$, and the third root is $\rho_1 = \frac{km^2-kn^2+8}{12}$. So, we have to find all possible (k, m, n) for which

$$\rho_1 \neq \rho_2,\ \rho_1 \neq \rho_3$$

and

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\}.$$

Define

$$\begin{aligned} A_1 &= \frac{a_1^2}{a_0} - a_2 - \rho_1 = \frac{1}{16}(km^2 + 4)(kn^2 - 4), \\ A_2 &= \frac{a_1^2}{a_0} - a_2 - \rho_2 = \frac{1}{16}km^2(kn^2 - 4), \\ A_3 &= \frac{a_1^2}{a_0} - a_2 - \rho_3 = \frac{1}{16}kn^2(km^2 + 4). \end{aligned}$$

We have

$$\begin{aligned} A_i - A_j = 0 &\Leftrightarrow \rho_i = \rho_j, \\ A_i - A_j > 0 &\Leftrightarrow \rho_i < \rho_j, \end{aligned}$$

$$\frac{a_1^2}{a_0} - a_2 \geq \max\{\rho_1, \rho_2, \rho_3\} \Leftrightarrow A_i \geq 0 \text{ for all } i = 1, 2, 3.$$

This implies

$$\rho_1 = \rho_2 \Leftrightarrow A_1 - A_2 = \frac{kn^2 - 4}{4} = 0 \Leftrightarrow (k, n) = (1, 2), (4, 1),$$

$$\rho_1 = \rho_3 \Leftrightarrow A_1 - A_3 = \frac{km^2 + 4}{4} = 0 \Leftrightarrow (k, m) = (-1, 2), (-4, 1).$$

Precisely,

$\rho_1 = \rho_2 \Leftrightarrow (k, m, n) = (1, m, 2)$, for all odd $m \geq 3$ and $(k, m, n) = (4, m, 1)$, for all $m \geq 1$,

$\rho_1 = \rho_3 \Leftrightarrow (k, m, n) = (-1, 2, 1), (-4, 1, 1), (-4, 1, 0)$.

Let $k > 0$. Then

$$\begin{aligned} A_1 &= \frac{1}{16}(km^2 + 4)(kn^2 - 4) \geq 0, && \text{if } kn^2 \geq 4, \\ A_2 &= \frac{1}{16}km^2(kn^2 - 4) \geq 0, && \text{if } kn^2 \geq 4, \\ A_3 &= \frac{1}{16}kn^2(km^2 + 4) \geq 0. \end{aligned}$$

Therefore, for $k > 0$

$$\frac{a_1^2}{a_0} - a_2 < \max\{\rho_1, \rho_2, \rho_3\} \iff \begin{cases} (k, m, n) = (1, m, 1), (2, m, 1), (3, m, 1), \text{ for all } m \geq 1 \\ \text{or} \\ (k, m, n) = (k, 1, 0) \text{ for all } k \geq 1. \end{cases}$$

Let $k < 0$. Then

$$\begin{aligned} A_1 &= \frac{1}{16}(km^2 + 4)(kn^2 - 4) \geq 0, && \text{if } km^2 \leq -4, \\ A_2 &= \frac{1}{16}km^2(kn^2 - 4) \geq 0, \\ A_3 &= \frac{1}{16}kn^2(km^2 + 4) \geq 0, && \text{if } km^2 \leq -4. \end{aligned}$$

Thus, we get

$$\frac{a_1^2}{a_0} - a_2 < \max\{\rho_1, \rho_2, \rho_3\} \iff$$

$$(k, m, n) = (-1, 1, 1), (-2, 1, 1), (-3, 1, 1), (-2, 1, 0), (-3, 1, 0), (-1, 1, 0).$$

Hence, as all triples (k, m, n) given by Lemma 6 don't satisfy any of the following conditions:

- $(k,m,n)=(k,1,0),\ k\geq -4$;
- $(k,m,n)=(k,1,1),\ |k|\leq 4$;
- $(k,m,n)=(k,m,1),\ 1\leq k\leq 4,\ m\geq 2$;
- $(k,m,n)=(1,m,2),\ m\geq 3$;
- $(k,m,n)=(-1,2,1)$,

one can apply the reduction method to a quartic Thue equation (1). We have

$$\begin{aligned}H_0(x,y)-4r_1f(x,y)&=3(km^2+4)(kn^2-4)(x^2+y^2)^2\\ H_0(x,y)-4r_2f(x,y)&=3k(kn^2-4)(nx^2\pm 2mxy-ny^2)^2\\ H_0(x,y)-4r_3f(x,y)&=3k(km^2+4)(nx^2\mp 2mxy-ny^2)^2.\end{aligned}$$

Put

$$U=x^2+y^2,\quad V=nx^2\pm 2mxy-ny^2,\quad Z=nx^2\mp 2mxy-ny^2. \tag{12}$$

One obtains the system of pellian equations

$$kV^2-(km^2+4)U^2=-4\mu,\quad kZ^2-(kn^2-4)U^2=4\mu.$$

This completes the proof of Theorem 1.

Let us consider ours two parametric families of Thue equations (1) with three parametric integers k, m and n and put

$$F^{\pm}_{(k,m,n)}(x,y)=x^4\mp kmnx^3y+(k(m^2-n^2)+2)x^2y^2\pm kmnxy^3+y^4.$$

Without loss of generality, we can assume that m and n are coprime. Namely, if $\gcd(m,n)=g$, $m=m_1g$, $n=n_1g$ and if we put $k'=kg^2$, then

$$F^{\pm}_{(k,m,n)}(x,y)=F^{\pm}_{(k',m_1,n_1)}(x,y).$$

Also, we have

$$F^{\pm}_{(0,m,n)}(x,y)=x^4+2x^2y^2+y^4=(x-y)^2(x+y)^2.$$

Hence $F^{\pm}_{(0,m,n)}(x,y)=\mu$ is not a Thue equation, so we assume $k\neq 0$. Further, we have

$$\begin{aligned}&F^{+}_{(k,m,n)}(x,y)=F^{-}_{(k,m,n)}(y,x),\\ &F^{+}_{(k,m,-n)}(x,y)=F^{+}_{(k,-m,n)}(x,y)=F^{-}_{(k,m,n)}(x,y)=F^{+}_{(k,m,n)}(y,x),\\ &F^{+}_{(k,-m,-n)}(x,y)=F^{+}_{(k,m,n)}(x,y),\\ &F^{+}_{(k,m,n)}(x,y)=F^{-}_{(-k,n,m)}(x,y)=F^{+}_{(-k,n,m)}(y,x),\end{aligned}$$

and

$$F^+_{(k,m,0)}(x,y) = F^+_{(km^2,1,0)}(x,y) = F^+_{(-km^2,0,1)}(y,x),$$
$$F^+_{(k,0,n)}(x,y) = F^+_{(kn^2,0,1)}(x,y) = F^+_{(-kn^2,1,0)}(y,x),$$
$$F^+_{(k,m,m)}(x,y) = F^+_{(km^2,1,1)}(y,x).$$

Therefore, if we consider equation (1), it's enough to consider

$$F^+_{(k,m,n)}(x,y) = x^4 - kmnx^3y + (k(m^2-n^2)+2)x^2y^2 + kmnxy^3 + y^4 = \mu, \tag{13}$$

where $k > 0$ and m and n are coprime distinct positive integers or $(m,n) = (1,0)$ or $(m,n) = (0,1)$ or $(m,n) = (1,1)$. According Theorem 1, if the triple (k,m,n) does not satisfy any of the following conditions: $(k,m,n) = (k,1,0),\ k \geq 1$; $(k,m,n) = (k,0,1),\ 1 \leq k \leq 4$; $(k,m,n) = (k,m,1),\ 1 \leq k \leq 4,\ m \geq 1$; $(k,m,n) = (1,m,2),\ m \geq 3$; $(k,m,n) = (1,1,2)$, then solving the Thue equation (13) reduces to solving the system of pellian equations

$$kV^2 - (km^2+4)U^2 = -4\mu, \quad kZ^2 - (kn^2-4)U^2 = 4\mu, \tag{14}$$

where U, V and Z are given by (12).

4. Some properties of the solutions of system (4)

Let us consider the system of pellian equations (14). For our study, we take $\mu = 1$, $k > 0$ and m, n are integers (not necessary coprime) with $|n| \geq 2$ and $|m| \geq 2$. We also assume that $k = 2c$ with $c \neq \square$, since for $k = 2\square$ and $\mu = 1$ the system (14) was already studied in [12]. Then the system (14) becomes

$$cV^2 - \left(cm^2+2\right)U^2 = -2 \tag{15}$$
$$cZ^2 - \left(cn^2-2\right)U^2 = 2. \tag{16}$$

This is equivalent to

$$v^2 - \left(c^2m^2+2c\right)U^2 = -2c \tag{17}$$
$$z^2 - \left(c^2n^2-2c\right)U^2 = 2c, \tag{18}$$

where $v = cV,\ z = cZ$.

Without loss of generality, we also assume that $n \geq 2$, and $m \geq 2$. Therefore, we consider system of pellian equations (4) with $k = 2c,\ c \neq \square,\ c \geq 2$ and $n \geq 2$ and $m \geq 2$.

4.1. Linear recurrent sequences as solutions of a system of Pell equation

We have the following result.

Proposition 7. *(See [22]) Let a, b be integers and let c_1 be a non negative integer such that $c_1 \leq ab$. If c_1 divide ab and $h = \frac{ab}{c_1}$, then we have*

(1) $\sqrt{a^2b^2 + 2c_1} = \left[ab, \overline{h, 2ab}\right]$,
(2) $\sqrt{a^2b^2 - 2c_1} = \left[ab - 1, \overline{1, h-2, 1, 2ab-2}\right]$, $h > 2$;

$$\sqrt{a^2b^2 - 2c_1} = \left[2c_1 - 1, \overline{2, 4c_1 - 2}\right] \quad \text{if } h = 2;$$
$$\sqrt{a^2b^2 - 2c_1} = \left[c_1 - 2, \overline{1, 2c_1 - 4}\right] \quad \text{if } h = 1.$$

From Proposition 7, we obtain the following continued fractions

$$\sqrt{c^2m^2 + 2c} = \left[cm, \overline{m, 2cm}\right], \ m \geq 1;$$
$$\sqrt{c^2n^2 - 2c} = \left[cn - 1, \overline{1, n-2, 1, 2cn-2}\right], \quad n \geq 3.$$

For $n = 2$, we have

$$\sqrt{4c^2 - 2c} = \left[2c - 1, \overline{2, 4c - 2}\right].$$

One can easily see that the fundamental solution of the equation

$$v^2 - \left(c^2m^2 + 2c\right)U^2 = 1$$

is $cm^2 + 1 + m\sqrt{c^2m^2 + 2c}$ and that of the equation

$$z^2 - \left(c^2n^2 - 2c\right)U^2 = 1$$

is $cn^2 - 1 + n\sqrt{c^2n^2 - 2c}$, for any $n \geq 2$. Furthermore, by [15, Theorem 108a], the possible fundamental solutions of (17) are

$$v + U\sqrt{c^2m^2 + 2c} = cm + \sqrt{c^2m^2 + 2c} \ \text{ and } \ v' + U'\sqrt{c^2m^2 + 2c} = -cm + \sqrt{c^2m^2 + 2c}$$

which belong to the same class of solutions since $\frac{vv' - UU'(c^2m^2+2c)}{-2c}$ and $\frac{Uv' - vU'}{-2c}$ are integers (see [15, page 205]). Therefore, we have only one fundamental solution which is

$$cm + \sqrt{c^2m^2 + 2c}.$$

In a similar way, we find that (18) has only one fundamental solution which is

$$cn + \sqrt{c^2n^2 - 2c}.$$

Hence, all the positives solutions of (17) are determined by

$$v_i + U_i\sqrt{c^2m^2+2c} = \left(cm+\sqrt{c^2m^2+2c}\right)\left(cm^2+1+m\sqrt{c^2m^2+2c}\right)^i,$$

where the sequences $(v_i)_{i\geq 0}$ and $(U_i)_{i\geq 0}$ are given by the recurrent relations

$$v_0 = c\,m, \quad v_1 = 2c^2m^3+3c\,m, \quad v_{i+2} = 2(c\,m^2+1)v_{i+1} - v_i,\ i\geq 0;$$
$$U_0 = 1, \quad U_1 = 2c\,m^2+1, \quad U_{i+2} = 2(c\,m^2+1)U_{i+1} - U_i,\ i\geq 0.$$

All the positives solutions of (18) are

$$z_j + T_j\sqrt{c^2n^2-2c} = \left(cn+\sqrt{c^2n^2-2c}\right)\left(cn^2-1+n\sqrt{c^2n^2-2c}\right)^j,$$

where the sequences $(z_j)_{j\geq 0}$ and $(T_j)_{j\geq 0}$ are given by the recurrent relations

$$z_0 = c\,n, \quad z_1 = 2c^2n^3-3c\,n, \quad z_{j+2} = 2(c\,n^2-1)z_{j+1} - z_j,\ j\geq 0;$$
$$T_0 = 1, \quad T_1 = 2c\,n^2-1, \quad T_{j+2} = 2(c\,n^2-1)T_{j+1} - T_j,\ j\geq 0.$$

We summarize all the above by the following lemma.

Lemma 8. *Let (U,V,Z) be a triple of positive integers and a solution of the system of pellian equations (15) and (16). Then, there exist i and j non negative integers such that*

$$U = U_i = T_j,$$

where the sequences (U_i) and (T_j) are given by

$$U_0 = 1, \quad U_1 = 2c\,m^2+1, \quad U_{i+2} = 2(c\,m^2+1)U_{i+1} - U_i, \quad \text{for } i\geq 0 \tag{19}$$

and

$$T_0 = 1, \quad T_1 = 2c\,n^2-1, \quad T_{j+2} = 2(c\,n^2-1)T_{j+1} - T_j, \quad \text{for } j\geq 0. \tag{20}$$

Note that for $i=j=0$, we have $U = U_0 = T_0 = 1$, which implies that $(U,V,Z) = (\pm 1, \pm m, \pm n)$ are (trivial) solutions of system of pellian equations (15) and (16).

Also note that (U_i) and (T_j) are strictly increasing sequences of positive integers.

- If $2\leq n\leq m$, then $U_1 = 4cm^2+1 > T_1 = 2cn^2-1$. Assume that $U_i > T_i$ holds for some positive integer i. We have

$$\begin{aligned} U_{i+1} - T_{i+1} &= 2(c\,m^2+1)U_i - U_{i-1} - 2(c\,n^2-1)T_i + T_{i-1} \\ &= 2\left(cm^2U_i - cn^2T_i\right) + 2U_i + 2T_i + T_{i-1} - U_{i-1} \\ &> 2\left(cm^2U_i - cn^2T_i\right) + U_i - U_{i-1}, \\ &= 2c(m^2-n^2)U_i + cn^2(U_i - T_i) + U_i - U_{i-1} > 0, \end{aligned}$$

since $2 \le n \le m$, $(U_i)_{i\ge 0}$ is an increasing sequence and by hypothesis. Thus we prove that if $2 \le n \le m$, then $U_i > T_i$, $\forall\, i \ge 1$. Therefore, if $U_i = T_j$ and $2 \le n \le m$, then necessary we have $j > i$.

- If $2 \le m < n$, then $U_1 = 2cm^2 + 1 < T_1 = 2cn^2 - 1$ and $U_2 = (2cm^2+1)^2 + 2cm^2 < T_2 = (2cn^2-1)^2 - 2cn^2$ since $n^2 \ge m^2 + 2m + 1$. The condition $2 \le m < n$ implies also that $m + 1 \le n$. Assume that $U_i < T_i$ holds some positive integer i. Then

$$\begin{aligned} T_{i+1} - U_{i+1} &= 2(c\,n^2-1)T_i - T_{i-1} - 2(c\,m^2+1)U_i + U_{i-1} \\ &= 2\left((cn^2-1)T_i - (cm^2+1)U_i\right) - T_{i-1} + U_{i-1} \\ &> 2(cm^2+1)(T_i - U_i) + (2cm + 2c - 5)T_i + (T_i - T_{i-1}), \\ &> (4cm + 2c - 5)T_i > 0, \end{aligned}$$

since $m \ge 2$, $c \ge 2$, $(T_i)_{i\ge 0}$ is an increasing sequence, and by hypothesis. Therefore if $2 \le m < n$, then $U_i < T_i$, $\forall\, i \ge 1$. We conclude that if $U_i = T_j$ and $2 \le m < n$, then necessary we have $i > j$.

Thus we have proved the following lemma.

Lemma 9. *Let $(U_i)_{i\ge 0}$ and $(T_j)_{j\ge 0}$ be sequences defined by (19) and (20), respectively. Assuming that $U_i = T_j$, we have*

(1) *if $2 \le n \le m$ then $i < j$.*
(2) *if $2 \le m < n$ then $j < i$.*

4.2. Lehmer sequences properties of the solutions

We start this subsection with the definition of Lehmer sequences.

Definition 10. *Let $p > 0$ and q be relative prime integers such that $p - 4q > 0$.*

We call Lehmer sequences with parameters (p, q), the sequence $\{L_n(p,q)\}_{n\ge 0}$ defined by

$$L_n(p,q) := \begin{cases} \frac{\gamma_1^n - \gamma_2^n}{\gamma_1 - \gamma_2} & \text{if } n \text{ is odd} \\ \frac{\gamma_1^n - \gamma_2^n}{\gamma_1^2 - \gamma_2^2} & \text{if } n \text{ is even,} \end{cases}$$

where $\gamma_1 = \frac{\sqrt{p}+\sqrt{p-4q}}{2}$ and $\gamma_2 = \frac{\sqrt{p}-\sqrt{p-4q}}{2}$.

Next, we prove the following result.

Proposition 11. *Let $a > 0$ be an integer and let b be nonzero integer such that $d^{\pm} = ab^2 \pm 2$ are positive integers. The solutions in positive integers of the pellian equations;*

(1) $ax^2 - d^+y^2 = -2$ *are* $(x, y) = \left(bL_{2i+1}(2d^+, 1); L_{2i+1}(2d^+ - 4, -1)\right)$, *for* $i \geq 0$;
(2) $ax^2 - d^-y^2 = 2$ *are* $(x, y) = \left(bL_{2j+1}(2d^-, -1); L_{2j+1}(2d^- + 4, 1)\right)$, *for* $j \geq 0$.

Proof. (1) Equation $ax^2 - d^+y^2 = -2$ is equivalent to $X^2 - ad^+y^2 = -2a$ with $X = ax$. From Proposition 7, we obtain $\sqrt{ad^+} = \left[ab, \overline{b, 2ab}\right]$. Then, the fundamental solution of $X^2 - ad^+y^2 = 1$ is

$$ab^2 + 1 + b\sqrt{ad^+}.$$

From [15, Theorem 108a], the possible fundamental solutions of $X^2 - ad^+y^2 = -2a$ are

$$\pm ab + \sqrt{ad^+},$$

which belong to the same class. Therefore, the fundamental solution is

$$ab + \sqrt{ad^+}.$$

Hence, all solutions in positive integers of $X^2 - ad^+y^2 = -2a$ are defined by $(X, y) = (X_i, y_i)$, where

$$X_i + y_i\sqrt{ad^+} = (ab + \sqrt{ad^+})\left(ab^2 + 1 + b\sqrt{ad^+}\right)^i, \ i \geq 0$$

which implies that all solutions in positive integers of $ax^2 - d^{\pm}y^2 = -2$ are given by $(x, y) = (x_i, y_i)$ where

$$x_i\sqrt{a} + y_i\sqrt{d^+} = (b\sqrt{a} + \sqrt{d^+})\left(ab^2 + 1 + b\sqrt{ad^+}\right)^i, \ i \geq 0.$$

This implies that

$$x_i = \frac{(b\sqrt{a} + \sqrt{d^+})\left(ab^2 + 1 + b\sqrt{ad^+}\right)^i + (b\sqrt{a} - \sqrt{d^+})\left(ab^2 + 1 - b\sqrt{ad^+}\right)^i}{2\sqrt{a}}$$

and

$$y_i = \frac{(b\sqrt{a} + \sqrt{d^+})\left(ab^2 + 1 + b\sqrt{ad^+}\right)^i - (b\sqrt{a} - \sqrt{d^+})\left(ab^2 + 1 - b\sqrt{ad^+}\right)^i}{2\sqrt{d^+}}.$$

As

$$ab^2 + 1 \pm b\sqrt{ad^+} = \left(\frac{b\sqrt{a} \pm \sqrt{d^+}}{\sqrt{2}}\right)^2$$

and putting

$$\gamma_1 = \frac{b\sqrt{a} + \sqrt{d^+}}{\sqrt{2}} = \frac{b\sqrt{2a} + \sqrt{2d^+}}{2}, \quad \gamma_2 = \frac{b\sqrt{a} - \sqrt{d^+}}{\sqrt{2}} = \frac{b\sqrt{2a} - \sqrt{2d^+}}{2},$$

we have

$$x_i = \frac{\gamma_1^{2i+1} + \gamma_2^{2i+1}}{\sqrt{2a}} = \frac{\gamma_1^{2i+1} - (-\gamma_2)^{2i+1}}{\sqrt{2a}} \text{ and } y_i = \frac{\gamma_1^{2i+1} - \gamma_2^{2i+1}}{\sqrt{2d^+}}.$$

Therefore, we obtain

$$x_i = b \cdot L_{2i+1}(2ab^2 + 4, 1) = b \cdot L_{2i+1}(2d^+, 1) \quad \text{since } \gamma_1 - (-\gamma_2) = b\sqrt{2a}$$

and

$$y_i = L_{2i+1}(2ab^2, -1) \quad \text{as } \gamma_1 - \gamma_2 = \sqrt{2d^+}.$$

(2) The second part of the proposition can be similarly proved. □

Lemma 12. *Let (U, V, Z) be a triple of non negative integers and a solution of the system of pellian equations (15) and (16). Then, there exist positive integers i and j such that*

$$U = L_{2i+1}(km^2, -1) = L_{2j+1}(kn^2, 1), \tag{21}$$

where $L_i(p, q)$ denotes the Lehmer sequence of parameter (p, q).

Proof. This comes from Proposition 11 with $d^+ = cm^2 + 2$, $d^- = cn^2 - 2$ and the fact that $k = 2c$. □

4.3. Linear form in logarithms of algebraic numbers

We put

$$\alpha = \frac{m\sqrt{k} + \sqrt{km^2 + 4}}{2}, \quad \beta = \frac{n\sqrt{k} + \sqrt{kn^2 - 4}}{2}$$

and

$$\Lambda = (2j + 1)\log\beta - (2i + 1)\log\alpha + \log\sqrt{\frac{km^2 + 4}{kn^2 - 4}}.$$

Lemma 13. *If $m \geq n \geq 2$, $L_{2i+1}(km^2, -1) = L_{2j+1}(kn^2, 1)$ and $i \neq 0$, then we have*

(1) $0 < \Lambda < 0.149 \cdot \alpha^{-4i}$;
(2) $i < j < 1.204 \cdot i \cdot \frac{\log m}{\log n}$.

Proof. (1) The equality $L_{2i+1}(km^2, -1) = L_{2j+1}(kn^2, 1)$ is equivalent to

$$\frac{\alpha^{2i+1} - (-\alpha^{-1})^{2i+1}}{\sqrt{km^2+4}} = \frac{\alpha^{2i+1} + \alpha^{-2i-1}}{\sqrt{km^2+4}} = \frac{\beta^{2j+1} - \beta^{-2j-1}}{\sqrt{kn^2-4}}.$$

Put

$$P = \frac{\alpha^{2i+1}}{\sqrt{km^2+4}}, \quad Q = \frac{\beta^{2j+1}}{\sqrt{kn^2-4}}, \text{ and } A = \sqrt{\frac{km^2+4}{kn^2-4}}.$$

Then, we obtain

$$P + \frac{1}{km^2+4}P^{-1} = Q - \frac{1}{kn^2-4}Q^{-1}$$

i.e.

$$Q - P = \frac{1}{km^2+4}P^{-1} + \frac{1}{kn^2-4}Q^{-1} > 0$$

and

$$0 < \Lambda = \log\left(\frac{Q}{P}\right) = -\log\left(1 - \frac{Q-P}{Q}\right).$$

Moreover, we have

$$\begin{aligned}
\frac{Q-P}{Q} &= \left(\frac{1}{km^2+4}P^{-1} + \frac{1}{kn^2-4}Q^{-1}\right)Q^{-1} \\
&= \frac{1}{km^2+4}\left(P^{-1} + A^2Q^{-1}\right)Q^{-1} \\
&< \frac{1}{km^2+4}(1+A^2)P^{-2} = \left(1+A^2\right)\alpha^{-4i-2}, \text{ since } Q^{-1} < P^{-1} \text{ as } Q-P>0 \\
&< \left(1+A^2\right)\alpha^{-6}, \qquad \text{since } i \geq 1 \\
&< \left(1+A^2\right)\left(m\sqrt{k}\right)^{-6} = \frac{1+A^2}{k^3m^6}, \qquad \text{since } \alpha > m\sqrt{k} \\
&= \frac{m^2+n^2}{k^2m^6(kn^2-4)} \\
&< \frac{2^2+2^2}{4^2\cdot 2^6(4\cdot 2^2-4)} = \frac{1}{3\times 2^9} < 6.52\cdot 10^{-4},
\end{aligned}$$

as the function $(k,m,n)\mapsto \frac{m^2+n^2}{k^2m^6(kn^2-4)}$ is decreasing with respect to each variable and $k\geq 4,\ m\geq n\geq 2$. Hence, we get

$$0<\Lambda<\frac{Q-P}{Q}+\left(\frac{Q-P}{Q}\right)^2=\left(1+\frac{Q-P}{Q}\right)\frac{Q-P}{Q}$$

and

$$\begin{aligned}0<\Lambda&<\left(1+6.52\cdot 10^{-4}\right)\frac{Q-P}{Q}\\&<\left(1+6.52\cdot 10^{-4}\right)\left(1+A^2\right)\alpha^{-4i-2}\\&<\left(1+6.52\cdot 10^{-4}\right)\left(\frac{1+A^2}{\alpha^2}\right)\alpha^{-4i}\\&<\left(1+6.52\cdot 10^{-4}\right)\frac{4k(m^2+n^2)}{(kn^2-4)(m\sqrt{k}+\sqrt{km^2+4})^2}\cdot\alpha^{-4i}\\&<\left(1+6.52\cdot 10^{-4}\right)\frac{4\cdot 4(2^2+2^2)}{(4\cdot 2^2-4)(2\sqrt{4}+\sqrt{4\cdot 2^2+4})^2}\cdot\alpha^{-4i}\\&<0.149\cdot\alpha^{-4i}.\end{aligned}$$

(2) From above, we have

$$(2j+1)\log\beta-(2i+1)\log\alpha<0.149\cdot\alpha^{-4i}-\frac{1}{2}\log\left(\frac{km^2+4}{kn^2-4}\right)$$

and, since $i\geq 1,\ \alpha+\alpha^{-1}=\sqrt{km^2+4}$ and $\beta+\beta^{-1}=\sqrt{kn^2-4}$, we obtain

$$\begin{aligned}2\left(j\log\beta-i\log\alpha\right)&<0.149\cdot\alpha^{-4}-\frac{1}{2}\log\left(\frac{km^2+4}{kn^2-4}\right)+\log\alpha-\log\beta\\&=0.149\cdot\alpha^{-4}+\log\frac{\alpha}{\beta}-\log\frac{\alpha+\alpha^{-1}}{\beta+\beta^{-1}}\\&=0.149\cdot\alpha^{-4}-\log\left(1+\frac{\beta^{-2}+\alpha^{-2}}{1-\beta^{-2}}\right)\\&\leq 0.149\cdot\alpha^{-4}-\frac{\beta^{-2}+\alpha^{-2}}{2},\\&\qquad\text{since }\log(1+x)\geq\frac{x}{1+x},\text{ for all }x>-1\\&\leq 0.149\cdot\alpha^{-4}-\alpha^{-2},\quad\text{since }\alpha^{-2}<\beta^{-2}\text{ for all }m\geq n\\&<0.\end{aligned}$$

Thus, one can see that

$$\begin{aligned}\frac{j}{i} \le \frac{\log\alpha}{\log\beta} &= \frac{\log\frac{\sqrt{km^2}}{2} + \log\left(1+\sqrt{1+\frac{4}{km^2}}\right)}{\log\frac{\sqrt{kn^2}}{2} + \log\left(1+\sqrt{1-\frac{4}{kn^2}}\right)} \\ &\le \frac{\log\frac{\sqrt{k}}{2} + \log m + \log\left(1+\sqrt{1+\frac{1}{4}}\right)}{\log\frac{\sqrt{k}}{2} + \log n + \log\left(1+\sqrt{1-\frac{1}{4}}\right)} \\ &\le \frac{\log m + \log\left(1+\frac{\sqrt{5}}{2}\right)}{\log n + \log\left(1+\frac{\sqrt{3}}{2}\right)} \le \frac{\log\left(1+\frac{\sqrt{5}}{2}\right)}{\log\left(1+\frac{\sqrt{3}}{2}\right)} \cdot \frac{\log m}{\log n} \\ &< 1.204 \cdot \frac{\log m}{\log n}.\end{aligned}$$

Moreover, from Lemma 9 we have $j > i$ since $2 \le n \le m$. □

Lemma 14. *If $n > m \ge 2$, $L_{2i+1}(km^2,-1) = L_{2j+1}(kn^2,1)$ and $j \ne 0$, then one has*

(1) $0 < \Lambda < 0.102 \cdot \beta^{-4j}$;
(2) $j < i < 1.5 \cdot j \cdot \frac{\log n}{\log m}$.

Proof. (1) We use the same notations as in the above lemma. As $Q - P > 0$ and $P > 1$, then $Q^{-1} < P^{-1} < 1$, and we obtain the following sequence of inequalities

$$\begin{aligned} Q - \left(\frac{1}{km^2+4} + \frac{1}{kn^2-4}\right) P^{-1} &< Q - \frac{1}{kn^2-4}Q^{-1} - \frac{1}{km^2+4}P^{-1} = P \\ Q - \left(\frac{1}{km^2+4} + \frac{1}{kn^2-4}\right) &< P \\ Q - \frac{1}{km^2+4}\left(1+A^2\right) &< P. \end{aligned}$$

Moreover, $A^2 < 1$ since $n > m \ge 2$. This implies

$$\frac{1}{km^2+4}\left(1+A^2\right) < \frac{2}{km^2+4} < \frac{1}{10}.$$

Therefore, we get $P > Q - \frac{1}{10}$ and then $P^{-1} < \left(Q - \frac{1}{10}\right)^{-1} = Q^{-1}\left(1 - \frac{Q^{-1}}{10}\right)^{-1}$. We deduce that

$$\begin{aligned} Q - P &= \frac{1}{km^2+4}P^{-1} + \frac{1}{kn^2-4}Q^{-1} \\ &< \frac{1}{km^2+4}Q^{-1}\left(1-\frac{Q^{-1}}{10}\right)^{-1} + \frac{1}{kn^2-4}Q^{-1} \\ &< \frac{1}{km^2+4}Q^{-1}\left(1-\frac{Q^{-1}}{10}\right)^{-1} + \frac{1}{km^2+4}Q^{-1}, \quad \text{since } kn^2-4 > km^2+4 \end{aligned}$$

$$< \frac{2.003}{km^2+4}Q^{-1}, \quad \text{since } 1-\frac{Q^{-1}}{10} > \frac{349}{350} \text{ (or since } Q > 35).$$

Then, we see that

$$\begin{aligned}\frac{Q-P}{Q} &< \frac{2.003}{km^2+4}Q^{-2} = 2.003 \cdot A^{-2}\beta^{-4j-2} \\ &< 2.003 \cdot A^{-2}\beta^{-6} && \text{since } j \geq 1 \\ &< 2.003 \cdot A^{-2}\left(\sqrt{kn^2-4}\right)^{-6} && \text{since } \beta > \sqrt{kn^2-4} \\ &= \frac{2.003}{(km^2+4)(kn^2-4)^2} < 0.000098,\end{aligned}$$

as the function $g(k,m,n) = \frac{2.003}{(km^2+4)(kn^2-4)^2}$ is decreasing with respect to each variable and then less than or equal to $g(4,2,3)$ (since $n > m$). We also have

$$\begin{aligned}\frac{Q-P}{Q} &< \frac{2.003}{km^2+4}Q^{-2} = 2.003 \cdot A^{-2}\beta^{-4j-2} \\ &= 2.003 \cdot A^{-2}\beta^{-2}\beta^{-4j} \\ &< \frac{2.003}{km^2+4}\beta^{-4j} && \text{since } A^2\beta^2 > km^2+4 \\ &< 0.1002 \cdot \beta^{-4j},\end{aligned}$$

because the function $g_1(k,m) = \frac{2.003}{km^2+4}$ is decreasing with respect to each variable and then less than or equal to $g_1(4,2)$. It follows that

$$\begin{aligned}\Lambda &< (1+0.000098) \times 0.1002\beta^{-4j} \\ &< 0.10021\beta^{-4j}.\end{aligned}$$

(2) From Lemma 9 one gets $j < i$, as $2 \leq m < n$. We have

$$0 < (2j+1)\log\beta - (2i+1)\log\alpha + \frac{1}{2}\log\frac{km^2+4}{kn^2-4}.$$

Then, since $km^2+4 < kn^2-4$ and $\alpha > 4$, we obtain

$$0 < (j+0.5)\log\beta - i\log\alpha \leq 1.5j\log\beta - i\log\alpha$$

and then

$$\frac{i}{j} \leq 1.5\frac{\log\beta}{\log\alpha} = 1.5 \cdot \frac{\log\frac{\sqrt{k}}{2} + \log n + \log\left(1+\sqrt{1-\frac{4}{kn^2}}\right)}{\log\frac{\sqrt{k}}{2} + \log m + \log\left(1+\sqrt{1+\frac{4}{km^2}}\right)}$$

$$< 1.5 \cdot \frac{\log \frac{\sqrt{k}}{2} + \log n + \log 2}{\log \frac{\sqrt{k}}{2} + \log m + \log 2} < 1.5 \cdot \frac{\log n}{\log m}. \quad \square$$

4.4. Use of congruences

We put $g = \gcd(m, n)$ and $m = m_1 g$, $n = n_1 g$.

Lemma 15. *Let $(U_i)_{i\geq 0}$ and $(T_j)_{j\geq 0}$ be the sequences defined in (19) and (20) respectively. For all $i, j \geq 0$, we have*

$$U_i \equiv \frac{i(i+1)}{2} km^2 + 1 \pmod{kg^4}, \tag{22}$$

$$T_j \equiv (-1)^{j+1} \left[\frac{j(j+1)}{2} kn^2 - 1 \right] \pmod{kg^4}. \tag{23}$$

Proof. The proof can easily done by induction. $\square$

$U_i = T_j$ implies that $U_i \equiv T_j \pmod{kg^4}$. Hence, from Lemma 15 we have $1 \equiv (-1)^{j+2} \pmod{kg^2}$ then j is even since $k > 2$. From Lemma 15, we deduce that

$$\frac{i(i+1)}{2} km^2 + 1 \equiv -\frac{j(j+1)}{2} kn^2 + 1 \pmod{kg^4}$$

and so

$$\frac{i(i+1)}{2} m_1^2 \equiv -\frac{j(j+1)}{2} n_1^2 \pmod{kg^4}. \tag{24}$$

Put $X = \frac{i(i+1)}{2} m_1^2 + \frac{j(j+1)}{2} n_1^2$, and assume $(i, j) \neq (0, 0)$.

(1) Case $2 \leq n \leq m$.
Assume $j \leq \sqrt{\frac{k}{2}} \frac{g}{m_1}$. Then, we have from Lemma 13 (2), $i < j$ and because $n_1 \leq m_1$ and $j \geq 2$, we get

$$0 < \frac{j(j+1)}{2} n_1^2 \leq \frac{j(j+1)}{2} m_1^2 < j^2 m_1^2 \leq \frac{kg^2}{2}$$

and

$$0 < \frac{i(i+1)}{2} m_1^2 < \frac{j(j+1)}{2} m_1^2 \leq \frac{kg^2}{2}.$$

This implies that $0 < X < kg^2$ contradicting (24). Therefore, $j > \sqrt{\frac{k}{2}} \frac{g}{m_1}$.

(2) Case $2 \leq m < n$. We assume here that $i \leq \sqrt{\frac{k}{2}}\frac{g}{n_1}$. Thus, from Lemma 14 (2) we have $i > j$ and $i \geq 3$ (we have $i > j > 0$ and j is even) and because $n_1 > m_1$ we also get

$$0 < \frac{i(i+1)}{2}m_1^2 < \frac{i(i+1)}{2}n_1^2 < i^2 n_1^2 \leq \frac{kg^2}{2}$$

and

$$0 < \frac{j(j+1)}{2}n_1^2 < \frac{i(i+1)}{2}n_1^2 \leq \frac{kg^2}{2}.$$

This implies that $0 < X < kg^2$, which contradicts (24). Hence, we deduce that $i > \sqrt{\frac{k}{2}}\frac{g}{n_1}$.

Therefore, we have just proved the following result.

Proposition 16. *If $U_i = T_j$ with $(i,j) \neq (0,0)$, then*

(1) *if $2 \leq n \leq m$, we have $j > \sqrt{\frac{k}{2}}\frac{g^2}{m}$*
(2) *if $2 \leq m < n$, we have $i > \sqrt{\frac{k}{2}}\frac{g^2}{n}$.*

5. Proof of Theorem 2

In this section, we will completely prove Theorem 2 similarly as in [12, Section 7]. We begin by recalling of the following result due to Baker and Wüstholz [12, Theorem 3].

Theorem 17. *For a linear form $\Lambda \neq 0$ in logarithms of s algebraic numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ with rational integer coefficients $b_1, b_2, \cdots, b_s$ we have*

$$\log \Lambda \geq -18(s+1)!s^{s+1}(32d)^{s+2}h'(\alpha_1)\cdots h'(\alpha_s)\log(2sd)\log G$$

where $G = \max\{|b_1|, \cdots, |b_s|\}$ and where d is the degree of number fields generated by $\alpha_1, \alpha_2, \cdots, \alpha_s$.

Here

$$h'(\chi) = \frac{1}{d}\max\{h(\chi), |\log \chi|, 1\}$$

and $h(\chi)$ denotes the standard logarithmic Weil height of χ.

We also recall that if $x \geq x_0 > 1$ and $y \geq y_0 > 1$ are reals numbers, then the following inequality holds

$$\log(x\,y) \le \frac{\log(x_0\,y_0)}{\log(x_0)\cdot\log(y_0)}\cdot\log(x)\cdot\log(y). \tag{25}$$

Case 1: Assume $m \ge n \ge 2$. We apply the above theorem with

$$s=3,\ \ d=4,\ \ G=j,\ \ \alpha_1=\alpha,\ \ \alpha_2=\beta,\ \ \alpha_3=\sqrt{\frac{km^2+4}{kn^2-4}}.$$

Then, $h(\alpha_1)=\frac{1}{2}\log\alpha$ implies that

$$\begin{aligned} h'(\alpha_1)=\frac{1}{4}\log\alpha &= \frac{1}{4}\log(m\sqrt{k})+\frac{1}{4}\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{km^2}}\right)\\ &< \frac{1.2}{4}\log(m\sqrt{k}) = 0.3\log(m_1 g\sqrt{k}) \end{aligned}$$

As $k=2c$, $g\sqrt{c}\ge\sqrt{2}$, $m_1\sqrt{2}\ge\sqrt{2}$ and using (25), we obtain

$$h'(\alpha_1) < 0.3\cdot\frac{\log 2}{\log\sqrt{2}\cdot\log\sqrt{2}}\cdot\log(g\sqrt{c}) < 1.7313\cdot\log(m_1\sqrt{2})\cdot\log(g\sqrt{c}).$$

The facts $n\sqrt{k}\ge\beta$ and $h(\alpha_2)=\frac{1}{2}\log\beta$ imply that

$$\begin{aligned} h'(\alpha_2)=\frac{1}{4}\log\beta &\le \frac{1}{4}\log(n\sqrt{k})\\ &\le \frac{1}{4}\log(m\sqrt{k}) = 0.25\log(m_1 g\sqrt{k}) \le 1.44275\cdot\log(m_1\sqrt{2})\cdot\log(g\sqrt{c}). \end{aligned}$$

We know that $h(\alpha_3)=\frac{1}{2}\log(kn^2-4)+\log\alpha_3=\frac{1}{2}\log(km^2+4)$ and the minimal polynomial of α_3 over $\mathbb{Q}$ is $(kn^2-4)X^2-(km^2+4)$, then

$$\begin{aligned} h'(\alpha_3)=\frac{1}{8}\log(km^2+4) &= \frac{1}{8}\log(km^2)+\frac{1}{8}\log\left(1+\frac{4}{km^2}\right)\\ &< \frac{1}{4}\log(m\sqrt{k})+\frac{1}{8}\log\left(\frac{5}{4}\right)\\ &< \frac{1.2}{4}\log(m\sqrt{k}) = 0.3\cdot\log(m\sqrt{k}) < 1.7313\cdot\log(m_1\sqrt{2})\cdot\log(g\sqrt{c}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \log\Lambda &\ge -18\cdot 4!\cdot 3^4\cdot(32\cdot 4)^5\cdot 1.7313^2\cdot 1.44275\cdot\log 24\cdot\log^3(m_1\sqrt{2})\cdot\log^3(g\sqrt{c})\cdot\log j\\ &\ge -1.65241\cdot 10^{16}\cdot\log^3(m_1\sqrt{2})\cdot\log^3(g\sqrt{c})\cdot\log j. \end{aligned}$$

Moreover, using Lemma 13 (1) and $\alpha_1 > m\sqrt{k} > g\sqrt{c}$, we get

$$\log\Lambda < -4i\log\alpha_1+\log(0.149) < -4i\log\alpha_1 < -4i\log(g\sqrt{c}).$$

Thus, we obtain

$$i < 4.1311 \cdot 10^{15} \cdot \log^3(m_1\sqrt{2}) \cdot \log^2(g\sqrt{c}) \cdot \log j.$$

Furthermore, from Lemma 13 (2), we have

$$\begin{aligned} j <& 1.204 \cdot i \cdot \frac{\log m}{\log n} < 1.737 \cdot i \cdot \log(m\sqrt{k}), \quad \text{since} \quad \log n \geq \log 2; \\ <& 10.0243 \cdot i \cdot \log(m_1\sqrt{2}) \cdot \log(g\sqrt{c}), \quad \text{since} \quad (25). \end{aligned}$$

Hence, since $j \geq 2$, we obtain

$$\frac{j}{\log j} < 4.14114 \cdot 10^{16} \cdot \log^4(m_1\sqrt{2}) \cdot \log^3(g\sqrt{c}).$$

By Proposition 16 (1), we have $g\sqrt{c} < jm_1 < jm_1\sqrt{2}$. Since $\log(jm_1\sqrt{2}) < 4.3281 \cdot \log(m_1\sqrt{2}) \cdot \log j$, this implies that

$$\begin{aligned} \frac{j}{\log j} <& 4.14114 \cdot 10^{16} \cdot \log^4(m_1\sqrt{2}) \cdot \log^3(jm_1\sqrt{2}) \\ <& 3.35747 \cdot 10^{18} \cdot \log^7(m_1\sqrt{2}) \cdot \log^3 j, \end{aligned}$$

i.e.

$$j < 3.35747 \cdot 10^{18} \cdot \log^7(m_1\sqrt{2}) \cdot \log^4 j,$$

which has the form

$$j < K \cdot \log^4 j, \tag{26}$$

where

$$K = K(m_1) = 3.35747 \cdot 10^{18} \cdot \log^7(m_1\sqrt{2}) > 2.016 \cdot 10^{15},$$

for $m_1 \geq 1$. As

$$K < K \cdot \log^4 K,$$

and if (26) implies $j < j_0$ then $K < j_0$. Therefore, $j_0 = K^{1+t_0}$, where $t_0 > 0$. We implicitly define a positive function t_{min} by

$$K \log^4(K^{1+t_{min}}) = K^{1+t_{min}}.$$

We find that $t_{min}(K)$ is a continuous and decreasing function. So we have $t_{min}(K) < t_{min}(2.016 \cdot 10^{15}) < 0.4463$. We take $t_0 = \frac{5}{11}$ and then

$$j_0 = K^{16/11} < 8.8497 \cdot 10^{26} \log^{10.182}(m_1\sqrt{2}),$$

which implies that

$$g < 8.8497 \cdot 10^{26} \cdot m_1 \log^{10.182}(m_1\sqrt{2}).$$

We assume that $g = m^{\varepsilon},\ 0 \le \varepsilon \le 1$. Thus, we obtain

$$m^{\varepsilon} < G_0 \cdot m^{1-\varepsilon} \log^{10.182}(m^{1-\varepsilon}\sqrt{2}), \tag{27}$$

where $G_0 = 8.8497 \cdot 10^{26}$. Also we implicitly define a positive function $M_0(\varepsilon)$ by

$$M_0^{\varepsilon} = G_0 \cdot M_0^{1-\varepsilon} \log^{10.182}(M_0^{1-\varepsilon}\sqrt{2}). \tag{28}$$

We see that $M_0(\varepsilon)$ is defined for any $\varepsilon \in (0.5, 1]$ and $M_0(\varepsilon)$ is strictly decreasing. Hence, $M_0(\varepsilon) \ge M_0(1) > 1.824 \cdot 10^{22}$.

If $\varepsilon \in [0, 0.5]$, then (27) is satisfied for every m. We leave this case that is not interesting.

If $\varepsilon = 1$, then one has $m = n = g$. This case will be discussed in Subsection 6.1.

Let us suppose that $\varepsilon_0 \in (0.5, 1)$ and $M_0(\varepsilon_0) = m_0$. Then, the inequality

$$m^{\varepsilon_0} < 8.8497 \cdot 10^{26} \cdot m^{1-\varepsilon_0} \log^{10.182}(\sqrt{2} \cdot m^{1-\varepsilon_0})$$

implies that $m < m_0$. So for $m \ge m_0$ i.e. $g \ge m^{\varepsilon_0} \ge m_0^{\varepsilon_0}$, we only have the trivial solutions $(U, V, Z) = (\pm 1, \pm m, \pm n)$ for system (4).

Let us discuss about some special cases:

- If $\varepsilon_0 = 0.999$, then $m_0 < 8.593 \cdot 10^{22}$, so for $m \ge 8.593 \cdot 10^{22}$, and $\varepsilon \ge 0.999$, i.e.

$$g \ge m^{0.999} = \max\{m^{0.999}, n^{0.999}\} \ge 8.15 \cdot 10^{22} > m_0^{0.999},$$

  we have only the trivial solutions.
- Similarly, for $g \ge m^{0.995} \ge 1.1936 \cdot 10^{25}$, $g \ge m^{0.99} \ge 1.31 \cdot 10^{27}$, $g \ge m^{0.95} \ge 4.2895 \cdot 10^{35}$, $g \ge m^{0.9} \ge 2.1013 \cdot 10^{42}$, $g \ge m^{0.8} \ge 3.3411 \cdot 10^{56}$, $g \ge m^{0.7} \ge 1.2712 \cdot 10^{81}$, $g \ge m^{0.6} \ge 2.021 \cdot 10^{153}$, $g \ge m^{0.55} \ge 3.5836 \cdot 10^{302}$, we obtain the trivial solutions.

Case 2: Let $2 \le m < n$. We apply Theorem 17 with

$$s = 3,\ \ d = 4,\ \ G = i,\ \ \alpha_1 = \alpha,\ \ \alpha_2 = \beta,\ \ \alpha_3 = \sqrt{\frac{km^2 + 4}{kn^2 - 4}}.$$

Similarly, we get

$$h'(\alpha_1) < 1.7313 \cdot \log(n_1\sqrt{2}) \cdot \log(g\sqrt{c}),$$
$$h'(\alpha_2) < 1.44275 \cdot \log(n_1\sqrt{2}) \cdot \log(g\sqrt{c}),$$
$$h'(\alpha_3) < 1.44275 \cdot \log(n_1\sqrt{2}) \cdot \log(g\sqrt{c}),$$

and

$$\log \Lambda \geq -1.377 \cdot 10^{16} \cdot \log^3(n_1\sqrt{2}) \cdot \log^3(g\sqrt{c}) \cdot \log i.$$

Moreover, from Lemma 14 (1) and since $\alpha_2 > \sqrt{kn^2-4} > g\sqrt{c}$, we have $\log \Lambda < -4j \log(g\sqrt{c})$ and then

$$j < 3.4425 \cdot 10^{15} \cdot \log^3(n_1\sqrt{2}) \cdot \log^2(g\sqrt{c}) \cdot \log i.$$

Furthermore, from Lemma 14 (2) and (25) we deduce that

$$i < 1.5 \cdot j \cdot \frac{\log n}{\log m} < 2.165 \cdot j \cdot \log(n\sqrt{k}) < 8.33 \cdot j \cdot \log(n_1\sqrt{2}) \cdot \log(g\sqrt{c}),$$

since $n_1\sqrt{2} \geq 2\sqrt{2}$ and $g\sqrt{c} \geq \sqrt{2}$. Hence, for $i \geq 2$, we obtain

$$\frac{i}{\log i} < 2.8677 \cdot 10^{16} \cdot \log^4(n_1\sqrt{2}) \cdot \log^3(g\sqrt{c}).$$

By Proposition 16 (2), we have $g\sqrt{c} < i\, n_1 < i\, n_1\sqrt{2}$. This implies that

$$\frac{i}{\log i} < 3.9867 \cdot 10^{17} \cdot \log^7(n_1\sqrt{2}) \cdot \log^3 i.$$

Since $\log(i\, n_1\sqrt{2}) < 2.4045 \cdot \log(n_1\sqrt{2}) \cdot \log i$. Thus, we get

$$i < 3.9867 \cdot 10^{17} \cdot \log^7(n_1\sqrt{2}) \cdot \log^4 i,$$

which has the form

$$i < K' \cdot \log^4 i, \tag{29}$$

where $K' = K'(n_1) = 3.9867 \cdot 10^{17} \cdot \log^7(n_1\sqrt{2}) > 5.2364 \cdot 10^{17}$, for all $n_1 \geq 2$. It is obvious that

$$K' < K' \cdot \log^4 K'.$$

If (29) implies $i < i_0$ then $K' < i_0$. Therefore, $i_0 = {K'}^{1+t'_0}$, where $t'_0 > 0$. We implicitly define a positive function t'_{min} by

$$K' \log^4({K'}^{1+t_{min}}) = {K'}^{1+t'_{min}}.$$

The function $t'_{min}(K')$ is continuous and decreasing. Thus, we have $t'_{min}(K') < t'_{min}(5.2364 \cdot 10^{17}) < 0.396330$. We take $t'_0 = \frac{2}{5}$ and then

$$i_0 = K'^{7/5} < 4.37381 \cdot 10^{24} \log^{9.8}(n_1\sqrt{2}),$$

which implies that

$$g < 4.37381 \cdot 10^{24} \cdot n_1 \log^{9.8}(n_1\sqrt{2}).$$

We assume that $g = n^{\varepsilon},\ 0 \le \varepsilon < 1$. Then we obtain

$$n^{\varepsilon} < G'_0 \cdot n^{1-\varepsilon} \log^{9.8}(n^{1-\varepsilon}\sqrt{2}), \tag{30}$$

where $G'_0 = 4.37381 \cdot 10^{24}$. Similarly, we implicitly define a positive function $N_0(\varepsilon)$ by

$$N_0^{\varepsilon} = G'_0 \cdot N_0^{1-\varepsilon} \log^{9.8}(N_0^{1-\varepsilon}\sqrt{2}). \tag{31}$$

So $N_0(\varepsilon)$ is defined for all $\varepsilon \in (0.5, 1]$ and $N_0(\varepsilon)$ is a strictly decreasing. Hence, $N_0(\varepsilon) \ge N_0(1) > 1.3516 \cdot 10^{20}$.

If $\varepsilon \in [0, 0.5]$ then (30) is satisfied for every n. We leave this non-interesting case.

Let us suppose that $\varepsilon_0 \in (0.5, 1)$ and $N_0(\varepsilon_0) = n_0$. Then, (30) implies that $n < n_0$. So for $n \ge n_0$ i.e. $g \ge n^{\varepsilon_0} \ge n_0^{\varepsilon_0}$ we have only the trivial solutions $(U, V, Z) = (\pm 1, \pm m, \pm n)$ for system (4). Let us give some special cases:

- If $\varepsilon_0 = 0.999$, thus $n_0 < 5.26326 \cdot 10^{20}$, this implies that for $n \ge 5.26326 \cdot 10^{20}$, and $\varepsilon \ge 0.999$, i.e.

$$g \ge n^{0.999} = \max\{m^{0.999}, n^{0.999}\} \ge 5.0181 \cdot 10^{20} > n_0^{0.999}.$$

  We obtain only the trivial solutions.
- Similarly, for $g \ge n^{0.995} \ge 4.3722 \cdot 10^{22}$, $g \ge n^{0.99} \ge 3.3041 \cdot 10^{24}$, $g \ge n^{0.95} \ge 3.5815 \cdot 10^{32}$, $g \ge n^{0.9} \ge 7.6775 \cdot 10^{38}$, $g \ge n^{0.8} \ge 1.7117 \cdot 10^{52}$, $g \ge n^{0.7} \ge 1.7311 \cdot 10^{75}$, $g \ge n^{0.6} \ge 5.7580 \cdot 10^{142}$, $g \ge n^{0.55} \ge 2.9767 \cdot 10^{282}$, we have the trivial solutions.

For any $\varepsilon_0 \in (0.5, 1)$, one notices that $M_0(\varepsilon_0) > N_0(\varepsilon_0)$. Thus, if $\max\{m, n\} \ge M_0(\varepsilon_0)$ and $g \ge \max\{m^{\varepsilon}, n^{\varepsilon}\} \ge M_0(\varepsilon_0)$, for any $\varepsilon > \varepsilon_0$, we have only the trivial solutions of equation (13). This completes the proof of Theorem 2 and Corollary 3.

6. Some special cases and the proof of Theorem 4

6.1. Some special cases

Here we consider equation (13) when $(k, m, n) \in \{(2c, 1, 1), (2c, 2, 1), (2cn^2, 0, 1)\}$.

Case: $(k, m, n) = (2c, 1, 1)$.

This case leads to the equation

$$x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4 = 1, \tag{32}$$

which was solved by Ibrahimpasić for all $c \geq 0$. He also showed that corresponding system

$$cV^2 - (4c+2)U^2 = -2, \quad cZ^2 - (4c-2)U^2 = 2, \text{ for } c \geq 3$$

has only trivial solutions $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$.

The case $m = n$ in the equation (13) when $k = 2c$ and $\mu = 1$ is a special case of (32) since for $m = n = g$, we have $F^{(+)}_{(k,m,n)}(x, y) = F^{(+)}_{(kg^2,1,1)}(x, y)$.

Case: $(k, m, n) = (2c, 2, 1)$.

In this case, equation (13) becomes

$$x^4 - 4cx^3y + (6c+2)x^2y^2 + 4cxy^3 + y^4 = 1, \tag{33}$$

which was solved by Dujella and Jadrijević for all $c \geq 0$. They also showed that corresponding system

$$cV^2 - (4c+2)U^2 = -2, \quad cZ^2 - (c-2)U^2 = 2, \text{ for } c \geq 3$$

has only trivial solutions $(U, V, Z) = (\pm 1, \pm 2, \pm 1)$. The case $m = 2n$ in equation (13) with $k = 2c$ and $\mu = 1$ is a special case of (33) since $F^{(+)}_{(k,m,n)}(x, y) = F^{(+)}_{(kn^2,2,1)}(x, y)$.

Case: $(k, m, n) = (2cn^2, 0, 1)$.

The equation is

$$x^4 - 2(cn^2 - 1)x^2y^2 + y^4 = 1, \tag{34}$$

which can be solved by using a result of Walsh (see [8, Theorem 4.2]).

- If $2(cn^2 - 1)$ is not a perfect square, then the only solutions are $(x, y) = (\pm 1, 0), (0, \pm 1)$. This implies that $(U, V, Z) = (\pm 1, 0, \pm n)$ are solutions of the corresponding system

$$cV^2 - 2U^2 = -2, \quad cZ^2 - (cn^2 - 2)U^2 = 2.$$

- If $2(cn^2 - 1)$ is a perfect square, then the only solutions are $(x, y) = (\pm 1, 0)$, $(0, \pm 1)$, $(\pm\sqrt{2(cn^2-1)}, \pm 1)$, $(\pm 1, \pm\sqrt{2(cn^2-1)})$. Then $(U, V, Z) = (\pm 1, 0, \pm n)$, $(\pm(2cn^2 - 1), \pm 2n\sqrt{2(cn^2-1)}, \pm n(2cn^2 - 3))$ are the solutions of the corresponding system.
- If $2(cn^2 - 1) = 338$, then we get $n = \pm 1$, $c = 170$. We ignore this case as $|n| \geq 2$.

6.2. Proof of Theorem 4

Jadrijević [12] proved that when $c = \square$ then system (4) has at most 7 solutions in positive integers. We follow the same method to prove Theorem 4. As in Section 5, we consider the case when $n \geq 2,\ m \geq 2$ and $2 \leq c \neq \square$. Let $(U, V, Z) = (U_{i_l} = T_{j_l}, V_{i_l}, Z_{j_l}),\ l = 1, 2, 3$ be three solutions in positive integers of system (4) with $U_{i_1} < U_{i_2} < U_{i_3}$. As system (4) is equivalent to the system of equations (17) and (18) with $v = cV$ and $z = cZ$, we define $v_{i_l} = cV_{i_l},\ z_{j_l} = cZ_{j_l},\ l = 1, 2, 3$ and we consider

$$\Delta = \begin{vmatrix} U_{i_1} & z_{j_1} & v_{i_1} \\ U_{i_2} & z_{j_2} & v_{i_2} \\ U_{i_3} & z_{j_3} & v_{i_3} \end{vmatrix} = \begin{vmatrix} U_{i_1} & z_{j_1} - \sqrt{c^2n^2 - 2c}T_{j_1} & v_{i_1} - \sqrt{c^2m^2 + 2c}U_{i_1} \\ U_{i_2} & z_{j_2} - \sqrt{c^2n^2 - 2c}T_{j_2} & v_{i_2} - \sqrt{c^2m^2 + 2c}U_{i_2} \\ U_{i_3} & z_{j_3} - \sqrt{c^2n^2 - 2c}T_{j_3} & v_{i_3} - \sqrt{c^2m^2 + 2c}U_{i_3} \end{vmatrix}.$$

Put

$$\alpha_l = v_{i_l} + \sqrt{c^2m^2 + 2c}U_{i_l}, \quad \beta_l = z_{j_l} + \sqrt{c^2n^2 - 2c}T_{j_l}.$$

We have

$$(v_{i_l} - \sqrt{c^2m^2 + 2c}U_{i_l})\alpha_l = -2c, \quad (z_{j_l} - \sqrt{c^2n^2 - 2c}T_{j_l})\beta_l = 2c.$$

Using

$$\begin{aligned} v_{i_l} - \sqrt{c^2m^2 + 2c}U_{i_l} &= \frac{1}{2\sqrt{c^2m^2 + 2c}\,U_{i_l}} \left[-2c - (v_{i_l} - \sqrt{c^2m^2 + 2c}U_{i_l})^2\right] \\ &= \frac{\sqrt{c}}{\sqrt{cm^2 + 2}\,U_{i_l}} \left(-1 - 2c\alpha_l^{-2}\right) \end{aligned}$$

and

$$\begin{aligned} z_{j_l} - \sqrt{c^2n^2 - 2c}T_{j_l} &= \frac{1}{2\sqrt{c^2n^2 - 2c}\,T_{j_l}} \left[2c - (z_{j_l} - \sqrt{c^2n^2 - 2c}T_{j_l})^2\right] \\ &= \frac{\sqrt{c}}{\sqrt{cn^2 - 2}\,T_{j_l}} \left(1 - 2c\beta_l^{-2}\right), \end{aligned}$$

we obtain

$$\Delta = \frac{c}{\sqrt{(cm^2 + 2)(cn^2 - 2)}U_{i_1}U_{i_2}U_{i_3}} \sum_{1 \leq l \leq 3} U_{i_l}^2 \delta_l,$$

where

$$\begin{aligned} \delta_1 &= 2c(\beta_2^{-2} - \beta_3^{-2}) + 2c(\alpha_2^{-2} - \alpha_3^{-2}) + 4c^2(\beta_2^{-2}\alpha_3^{-2} - \beta_3^{-2}\alpha_2^{-2}) \\ \delta_2 &= 2c(\beta_1^{-2} - \beta_3^{-2}) + 2c(\alpha_1^{-2} - \alpha_3^{-2}) + 4c^2(\beta_1^{-2}\alpha_3^{-2} - \beta_3^{-2}\alpha_1^{-2}) \\ \delta_3 &= 2c(\beta_1^{-2} - \beta_2^{-2}) + 2c(\alpha_1^{-2} - \alpha_2^{-2}) + 4c^2(\beta_1^{-2}\alpha_2^{-2} - \beta_2^{-2}\alpha_1^{-2}). \end{aligned}$$

One can see that

$$\alpha_l^2 > (c^2m^2+2c)U_{i_1}^2,\ \ \beta_l^2 > (c^2n^2-2c)U_{i_1}^2,\ \ \alpha_l^2\beta_t^2 > (c^2m^2+2c)(c^2n^2-2c)U_{i_1}^4.$$

Then, since $U_{i_1}^4 > U_1^4 \geq 17^4$, we get

$$\begin{aligned}|\delta_l| &< \frac{4c}{(c^2m^2+2c)U_{i_1}^2} + \frac{4c}{(c^2n^2-2c)U_{i_1}^2} + \frac{8c^2}{(c^2n^2-2c)(c^2m^2+2c)U_{i_1}^4}\\ &= \frac{4}{(cm^2+2)U_{i_1}^2} + \frac{4}{(cn^2-2)U_{i_1}^2} + \frac{8}{(cn^2-2)(cm^2+2)U_{i_1}^4}\\ &< \frac{4}{10U_{i_1}^2} + \frac{4}{6U_{i_1}^2} + \frac{8}{60U_{i_1}^4} \leq \frac{16}{15U_{i_1}^2}.\end{aligned}$$

Hence, we obtain

$$|\Delta| < \frac{16c\,U_{i_3}}{5\sqrt{(cm^2+2)(cn^2-2)}U_{i_1}^3U_{i_2}}.$$

Moreover, Δ is a nonzero integer. Therefore, we get $\frac{5\sqrt{(cm^2+2)(cn^2-2)}}{16c}U_{i_1}^3U_{i_2} < U_{i_3}$, i.e.

$$\frac{5\sqrt{(m^2+2/c)(n^2-2/c)}}{16}U_{i_1}^3U_{i_2} < U_{i_3}.$$

If $m \geq n \geq 2$ and as $c > 1$, then one obtains

$$U_{i_3} > 1.21\,U_{i_1}^3U_{i_2}. \tag{35}$$

The sequences of positive solutions defined in (19) and (20) are increasing and

$$\begin{aligned}U_4 &= 16c^4m^8 + 56c^3m^8 + 60c^2m^4 + 20cm^2 + 1\\ &< 1.21(2cm^2+1)^3(4c^2m^4+6cm^2+1) = 1.21U_1^3U_2\\ U_5 &= 32c^5m^{10} + 144c^4m^8 + 224c^3m^6 + 140c^2m^4 + 30cm^2 + 1\\ &< 1.21(2cm^2+1)^3(4c^2m^4+6cm^2+1) = 1.21U_1^3U_2\end{aligned}$$

and

$$\begin{aligned}U_6 &= 64c^6m^{12} + 352c^5m^{10} + 720c^4m^8 + 672c^3m^6 + 280c^2m^4 + 42cm^2 + 1\\ &> 1.21U_1^3U_2.\end{aligned}$$

Then, we deduce that $i_3 \geq 6$.

If $2 \leq m < n$ and as $c > 1$, using $U_{i_l} = T_{j_l}$, one obtains

$$T_{j_3} > 1.97\,T_{j_1}^3T_{j_2}, \tag{36}$$

$$T_5 = 32c^5n^{10} - 144c^4n^8 + 224c^3n^6 - 140c^2n^4 + 30cn^2 - 1$$
$$< 1.97(2cn^2-1)^3(4c^2n^4 - 6cn^2 + 1) = 1.97T_1^3T_2$$

and

$$T_6 = 64c^6n^{12} - 352c^5n^{10} + 720c^4n^8 - 672c^3n^6 + 280c^2n^4 - 42cn^2 + 1$$
$$> 1.97T_1^3T_2.$$

Then, we get $j_3 \geq 6$.

Let $(U_{i_l}, V_{i_l}, Z_{j_l})$ and $(U_{i_t}, V_{i_t}, Z_{j_t})$ be two solutions in positive integers of system (4) such that $U_{i_l} < U_{i_t}$. We will apply the following theorem of Bennett (see [12, Theorem 3]).

Theorem 18. *If a_i, p_i, and N are integers, $0 \leq i \leq 2$, with $a_0 < a_1 < a_2$, $a_j = 0$, for some $j \in \{0,1,2\}$, q nonzero and $N > M^9$ where $M = \max_{0\leq i\leq 2}\{|a_i|\}$, then*

$$\max_{0\leq i\leq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(33N\gamma)}{\log\left(1.7N^2\prod_{0\leq i<j\leq 2}(a_i-a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} (a_2-a_0)^2(a_2-a_1)^2/(2a_2-a_0-a_1) & \text{if } a_2-a_1 \geq a_1-a_0 \\ (a_2-a_0)^2(a_1-a_0)^2/(a_1+a_2-2a_0) & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

Put

$$a_0 = -kn^2+4,\ \ a_1 = 0,\ \ a_2 = km^2+4,\ \ N = (cn^2-2)(cm^2+2)U_{i_l}^2,$$
$$q = (cn^2-2)(cm^2+2)U_{i_l}U_{i_t},\ \ p_0 = c(cn^2-2)V_{i_l}V_{i_t},\ \ p_2 = c(cm^2+2)Z_{j_l}Z_{j_t}.$$

Before applying the above theorem, let us prove the following lemma.

Lemma 19. *With above hypothesis on $(U_{i_l}, V_{i_l}, Z_{j_l})$ and $(U_{i_t}, V_{i_t}, Z_{j_t})$ and the choice of a_i, p_i, $0 \leq i \leq 2$ ($p_1 = q$) and N, we have*

$$\max_{0\leq i\leq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} < \frac{2}{U_{i_t}^2}. \tag{37}$$

Proof. One can see that

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|=\left|\sqrt{1+\frac{-2}{(cm^2+2)U_{i_l}^2}}-\frac{cV_{i_l}V_{i_t}}{(cm^2+2)U_{i_l}U_{i_t}}\right|.$$

Since $cV_{i_l}^2-(cm^2+2)U_{i_l}^2=-2$, we have

$$1+\frac{-2}{(cm^2+2)U_{i_l}^2}=\frac{c}{cm^2+2}\cdot\frac{V_{i_l}^2}{U_{i_l}^2} \tag{38}$$

and then

$$\begin{aligned}\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|=&\left|\frac{V_{i_l}}{U_{i_l}}\sqrt{\frac{c}{cm^2+2}}-\frac{cV_{i_l}V_{i_t}}{(cm^2+2)U_{i_l}U_{i_t}}\right|\\ =&\frac{c}{cm^2+2}\cdot\frac{V_{i_l}}{U_{i_l}}\left|\sqrt{\frac{cm^2+2}{c}}-\frac{V_{i_t}}{U_{i_t}}\right|.\end{aligned}$$

We multiply (38) by $\frac{c}{cm^2+2}$ to obtain

$$\frac{c^2}{(cm^2+2)^2}\cdot\frac{V_{i_l}^2}{U_{i_l}^2}=\frac{c}{cm^2+2}\left(1-\frac{2}{(cm^2+2)U_{i_l}^2}\right)<\frac{c}{cm^2+2}.$$

Hence, we get

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|<\sqrt{\frac{c}{cm^2+2}}\left|\sqrt{\frac{cm^2+2}{c}}-\frac{V_{i_t}}{U_{i_t}}\right|.$$

Additionally, from $cV_{i_t}^2-(cm^2+2)U_{i_t}^2=-2$, we have $cU_{i_t}^2\left(\frac{V_{i_t}^2}{U_{i_t}^2}-\frac{cm^2+2}{c}\right)=-2$ and

$$\left|\frac{V_{i_t}}{U_{i_t}}-\sqrt{\frac{cm^2+2}{c}}\right|\cdot\left|\frac{V_{i_t}}{U_{i_t}}+\sqrt{\frac{cm^2+2}{c}}\right|=\frac{2}{cU_{i_t}^2}.$$

We deduce that

$$\left|\frac{V_{i_t}}{U_{i_t}}-\sqrt{\frac{cm^2+2}{c}}\right|=\frac{2}{cU_{i_t}^2}\cdot\left(\frac{V_{i_t}}{U_{i_t}}+\sqrt{\frac{cm^2+2}{c}}\right)^{-1}<\frac{2}{cU_{i_t}^2},$$

since $\frac{V_{i_t}}{U_{i_t}}+\sqrt{\frac{cm^2+2}{c}}>1$. Therefore, we get

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|<\frac{2}{\sqrt{c(cm^2+2)}\,U_{i_t}^2}. \tag{39}$$

Similarly, we have

$$\left|\sqrt{1+\frac{a_0}{N}}-\frac{p_0}{q}\right|<\frac{2\sqrt{3}}{3}\frac{2}{\sqrt{c(cn^2-2)}\,U_{i_t}^2}. \tag{40}$$

From (39) and (40), we deduce inequality (37). □

By considering the choice of a_i, p_i, $0\le i\le 2$ ($p_1=q$) and N, we have $N\ge M\,U_{i_l}^2$. Then we must have $U_{i_l}>M^4$ in order to apply Theorem 18.

• We assume $m\ge n\ge 2$, then we see that $M=km^2+4$. In this case, $U_{i_l}>M^{i_l/2}$. This implies that we must have $i_l\ge 8$. We know that $\gamma=\frac{k^2(m^2+n^2)^2(km^2+4)^2}{2km^2+kn^2+4}$,

$$N\gamma=\frac{8c^2(cm^2+2)^3(cn^2-2)(m^2+n^2)^2U_{i_l}^2}{2cm^2+cn^2+2}=\frac{k^2(km^2+4)^3(kn^2-4)(m^2+n^2)^2U_{i_l}^2}{4(2km^2+kn^2+4)},$$

since $k^2(m^2+n^2)^2<4M^2$, $(km^2+4)^3(kn^2-4)<M^4$, and $2M<2km^2+kn^2+4\le 3M$, we have

$$N\gamma<0.5M^5U_{i_l}^2.$$

Moreover, we see that

$$\log\left(1.7N^2\prod_{0\le i<j\le 2}(a_i-a_j)^{-2}\right)=\log\left(\frac{1.7}{16}\frac{U_{j_l}^4}{k^2(m^2+n^2)^2}\right)\ge\log\left(\frac{1.7U_{j_l}^4}{16\times 4M^2}\right).$$

Hence, as $U_{i_l}>M^{i_l/2}$, we get

$$\begin{aligned}\lambda<&1+\frac{\log\left(16.5M^5U_{i_l}^2\right)}{\log\left(\frac{1.7U_{i_l}^4}{16\times 4M^2}\right)}<1+\frac{i_l\log(M)+5\log(M)+\log(16.5)}{2i_l\log(M)-2\log(M)-\log\left(\frac{64}{1.7}\right)}\\<&1+\frac{i_l+5+\log(16.5)/\log(M)}{2i_l-2-\log\left(\frac{64}{1.7}\right)/\log(M)}<1+\frac{i_l+5.936}{2i_l-3.212}=\frac{3i_l+2.724}{2i_l-3.212},\end{aligned}$$

since $M\ge 20$. If $i_l\ge 12$, then $\lambda<1.8629$. Therefore, one gets

$$\begin{aligned}\max_{0\le i\le 2}\left\{\left|\sqrt{1+\frac{a_i}{N}}-\frac{p_i}{q}\right|\right\}>&\frac{1}{65}M^{-5}U_{i_l}^{-2}\left[(cn^2-2)(cm^2+2)U_{i_l}U_{i_t}\right]^{-1.8629}\\>&\frac{4^{1.8629}}{65}M^{-8.6902}U_{i_l}^{-3.8451}U_{i_t}^{-1.8629}.\end{aligned}$$

Combining this inequality with (37), we get $U_{i_t}^{0.1371}<9.8258\,U_{i_l}^{6.01765}$, i.e.

$$U_{i_t}<1.731\cdot 10^7\,U_{i_l}^{43.893}<U_{i_l}^{45.284}<U_{i_l}^{46}, \tag{41}$$

since $U_{i_l} > M^4 \geq 20^4$. Hence, $U_{i_t} < U_{i_3}^{46}$, for all $t \geq 4$. Moreover, from inequality (35), we check that

$$U_{i_7} > 1.21\, U_{i_5}^3 U_{i_6} > U_{i_3}^{19}, \quad U_{i_8} > 1.21\, U_{i_6}^3 U_{i_7} > U_{i_3}^{40}$$

and

$$U_{i_9} > 1.21\, U_{i_7}^3 U_{i_8} > U_{i_3}^{97}.$$

This contradicts the fact that $U_{i_t} < U_{i_3}^{46}$, for any $t \geq 4$. Thus, U_{i_9} cannot exist.

• If $2 \leq m < n$, then we obtain $M = kn^2 - 4$ and $U_{i_l} = T_{j_l} > M^{j_l}$. This implies that we must have $j_l \geq 4$. We have

$$\gamma = \frac{k^2(m^2+n^2)^2(kn^2-4)^2}{2kn^2+km^2-4}$$

and

$$\lambda < \frac{6j_l + 2.76}{4j_l - 3.04}.$$

If $j_l \geq 6$, then $\lambda < 1.8493$. We obtain

$$T_{j_t} < 43837.09\, T_{j_l}^{39.974} < T_{j_l}^{40}. \tag{42}$$

Therefore, $T_{j_t} < T_{j_3}^{40}$, for any $t \geq 4$. From (36), we also check that

$$T_{j_7} > T_{j_5}^3 T_{j_6} > T_{j_3}^{19}, \quad T_{j_8} > T_{i_6}^3 T_{j_7} > T_{j_3}^{40},$$

and

$$T_{j_9} > T_{i_7}^3 T_{j_8} > T_{j_3}^{97}.$$

We get a contradiction to the condition $T_{j_t} < T_{j_3}^{46}$, for any $t \geq 4$. Hence, T_{j_9} cannot exist. This completes the proof of Theorem 4.

7. Proof of Theorem 5

In this last section, we will obviously prove the last theorem, i.e. Theorem 5. We will start by recalling a useful result. According to [17, Lemma 5], for any solution (U, V, Z) in positive integers of the system (4), there exist exponents $(2i+1, 2j+1)$ such that $(U, V, Z) = (U_i = T_j, V_i, Z_j)$, where

$$V_i\sqrt{k} + U_i\sqrt{km^2+4} = 2\alpha^{2i+1} \text{ and } Z_j\sqrt{k} + T_j\sqrt{kn^2-4} = 2\beta^{2j+1} \tag{43}$$

and

$$\alpha = \frac{m\sqrt{k} + \sqrt{km^2+4}}{2} \text{ and } \beta = \frac{n\sqrt{k} + \sqrt{kn^2-4}}{2}.$$

Lemma 20 *([3] and [17, Lemma 5]). Let $(2i_1+1, 2j_1+1)$ be exponents of the smallest solution in positive integers to system (4) and $(2i+1, 2j+1)$ exponents of a further solution in positive integers to system (4). Then, we have $(2i_1+1) \mid (2i+1)$ and $(2j_1+1) \mid (2j+1)$.*

Now, we prove the next result.

Lemma 21. *Let $(U, V, Z) = (U_{i_l} = T_{j_l}, V_{i_l}, Z_j)_l$, $l = 1, 2$ be two solutions in positive integers of the system (4) given by (43) where $i_1 < i_2$.*

(1) *If $2 \le m < n$ and $j_1 \ge 6$ then $i_2 < 40.82\, i_1$.*
(2) *If $2 \le n \le m$ and $i_1 \ge 12$ then $i_2 < 46.94\, i_1$.*

Proof. 1) Assume $2 \le m < n$. Then if $j_1 \ge 6$, one has $\lambda < 1.8493$. By Lemma 14 (2), $j_1 \ge 6$ implies $i_1 > 6$. From inequalities (42), we have

$$\log U_{i_2} < 40 \log U_{i_1}. \tag{44}$$

Put $i_2 = \sigma\, i_1$ with $\sigma > 1$. We have

$$\begin{aligned}
\frac{U_{i_2}}{U_{i_1}^{\sigma}} =& (km^2+4)^{\frac{\sigma-1}{2}} \left(\alpha^{2i_1+1} + \alpha^{-2i_1-1}\right)^{-\sigma} \left(\alpha^{2i_2+1} + \alpha^{-2i_2-1}\right) \\
=& (km^2+4)^{\frac{\sigma-1}{2}} \alpha^{1-\sigma} \left(1 + \alpha^{-4i_1-2}\right)^{-\sigma} \left(1 + \alpha^{-4i_2-2}\right) \\
>& \left(1 + \alpha^{-4i_1-2}\right)^{-\sigma} \left(1 + \alpha^{-4i_2-2}\right) \\
>& \left(1 - \alpha^{-4i_1-2}\right)^{\sigma} \left(1 + \alpha^{-4i_2-2}\right) \\
>& \left(1 - \alpha^{-4i_1-2}\right)^{\sigma} > \left(1 - 4^{-26}\right)^{\sigma},
\end{aligned}$$

since $\alpha \ge 4$ and $i_1 > 6$. Therefore, we get

$$\begin{aligned}
\log U_{i_2} >& \sigma \left(\log U_{i_1} + \log\left(1 - 4^{-26}\right)\right) \\
>& \sigma \left[\log(U_{i_1}^{0.98}) + \log(U_{i_1}^{0.02}) + \log\left(1 - 4^{-26}\right)\right] \\
>& 0.98 \cdot \sigma \log U_{i_1}
\end{aligned}$$

as

$$\begin{aligned}
\log\left((1 - 4^{-26}) U_{i_1}^{0.02}\right) >& \log\left((1 - 4^{-26})(M^6)^{0.02}\right) = \log\left((1 - 4^{-26})(kn^2-4)^{0.12}\right) \\
>& \log\left[(1 - 4^{-26})(4 \cdot 3^2 - 4)^{0.12}\right] > 0,
\end{aligned}$$

i.e.

$$\log U_{i_2} > 0.98 \cdot \sigma \log U_{i_1}. \tag{45}$$

Combining (44) and (45), we obtain

$$i_2 < 40.82\, i_1. \tag{46}$$

2) If $2 \le n \le m$, then for $i_1 \ge 12$, we get $\lambda < 1.8629$. From (41), we see that

$$\log U_{i_2} < 46 \cdot \log U_{i_1}. \tag{47}$$

Let $i_2 = \sigma i_1$ with $\sigma > 1$. By similar computation as above, we obtain

$$\log U_{i_2} > \sigma \log\left(\left(1 - 4^{-50}\right) \cdot U_{i_1}\right) = \sigma \log(U_{i_1}^{0.98}) + \sigma \log\left(\left(1 - 4^{-50}\right) \cdot U_{i_1}^{0.02}\right)$$

and since $U_{i_1} > M^{\frac{i_1}{2}} \ge M^6 = (km^2+4)^6 \ge (4.2^2+4)^6$, we have

$$\log\left(\left(1 - 4^{-50}\right) \cdot U_{i_1}^{0.02}\right) \ge \log\left(20^{0.12}\left(1 - 4^{-50}\right)\right) > 0.$$

Then we get (45) again and we combine it with (47) to obtain

$$i_2 < 40.82\, i_1.$$

Therefore, the proof of the lemma is completed. □

Proposition 22. *If system (4) has three solutions in positive integers with exponents* $(2i_l+1, 2j_l+1)$, $l = 1, 2, 3$ *where* $i_1 < i_2 < i_3$ *and* $(2i_1+1, 2j_1+1)$ *are exponents of the smallest solution, then*

$$i_3 - i_2 > \frac{j_1 i_1 \log(kn^2-4)}{0.129}.$$

To prove Proposition 22, we need result similar to one given in the proof of [2, Lemma 2.2]. If $(2i_1+1, 2j_1+1)$ are exponents of the smallest solution in positive integers to system (4), then by Lemma 20, we have

$$2i_3 + 1 \equiv 2i_2 + 1 \equiv 0 \pmod{2i_1+1}, \quad 2j_3 + 1 \equiv 2j_2 + 1 \equiv 0 \pmod{2j_1+1}. \tag{48}$$

Also, we have

$$\begin{vmatrix} 2j_3+1-(2j_2+1) & 2j_2+1-(2j_1+1) \\ 2i_3+1-(2i_2+1) & 2i_2+1-(2i_1+1) \end{vmatrix} = 4 \begin{vmatrix} j_3-j_2 & j_2-j_1 \\ i_3-i_2 & i_2-i_1 \end{vmatrix} = 4\kappa$$

and

$$\frac{V_{i_l}\sqrt{k} + U_{i_l}\sqrt{km^2+4}}{2} = \alpha^{2i_l+1}, \quad \frac{Z_{j_l}\sqrt{k} + T_{j_l}\sqrt{kn^2-4}}{2} = \beta^{2j_l+1}, \tag{49}$$

$l = 1, 2, 3$. From (48) and (49), we obtain

$$\frac{V_{i_l}\sqrt{k} + U_{i_l}\sqrt{km^2+4}}{2} = \left(\frac{V_{i_1}\sqrt{k} + U_{i_1}\sqrt{km^2+4}}{2}\right)^{\frac{2i_l+1}{2i_1+1}}, \ l = 2, 3,$$

$$\frac{Z_{j_l}\sqrt{k} + T_{j_l}\sqrt{kn^2-4}}{2} = \left(\frac{Z_{j_1}\sqrt{k} + T_{j_1}\sqrt{kn^2-4}}{2}\right)^{\frac{2j_l+1}{2j_1+1}}, \ l = 2, 3.$$

We have $(2j_1+1)(2i_l+1) \equiv (2i_1+1)(2j_l+1) \pmod 2$, $l = 2, 3$. Then since (48), and $\gcd((2i_1+1)(2j_1+1), 2) = 1$, we obtain

$$\frac{2i_l+1}{2i_1+1} \equiv \frac{2j_l+1}{2j_1+1} \pmod 2, \ l = 2, 3.$$

Therefore, we get

$$\frac{4\kappa}{(2i_1+1)(2j_1+1)} = \begin{vmatrix} \frac{2j_3+1}{2j_1+1} - \frac{2j_2+1}{2j_1+1} & \frac{2j_2+1}{2j_1+1} - 1 \\ \frac{2i_3+1}{2i_1+1} - \frac{2i_2+1}{2i_1+1} & \frac{2i_2+1}{2i_1+1} - 1 \end{vmatrix} \tag{50}$$

is a positive even integer, i.e. $\frac{4\kappa}{(2i_1+1)(2j_1+1)} \geq 2$, which implies that

$$\kappa \geq \frac{(2i_1+1)(2j_1+1)}{2} > 2i_1j_1.$$

We summarize this as follows.

Lemma 23. *If system (4) has three solutions in positive integers with exponents $(2i_l + 1, 2j_l + 1)$, $l = 1, 2, 3$ where $i_1 < i_2 < i_3$ and $(2i_1 + 1, 2j_1 + 1)$ are exponents of the smallest solution, then*

$$\begin{vmatrix} j_3 - j_2 & j_2 - j_1 \\ i_3 - i_2 & i_2 - i_1 \end{vmatrix} > 2i_1j_1.$$

Now, we can prove Proposition 22.

Proof of Proposition 22. System (4) has solutions if and only if $U_i = T_j$ has solutions in positive integers (i, j). If $\alpha^{2i+1} = e^t$ and $\beta^{2j+1} = e^u$, then we obtain

$$\sinh u = \cosh t\sqrt{\frac{kn^2-4}{km^2+4}}. \tag{51}$$

We know that

$$e^u - e^t\sqrt{\frac{kn^2-4}{km^2+4}} = e^{-u} + e^{-t}\sqrt{\frac{kn^2-4}{km^2+4}} > 0.$$

This implies that

$$u > t + \frac{1}{2}\log\frac{kn^2-4}{km^2+4} > t + \overbrace{\log\frac{n}{m} + \frac{1}{2}\log\frac{3}{5}}^{:=\nu}.$$

If $i \geq 1$ and $j \geq 1$, then we have the following lower bounds

$$t = (2i+1)\log\alpha > (2i+1)\log\sqrt{km^2} > 3\log 4 > 4.15,$$
$$u = (2j+1)\log\beta > (2j+1)\log\sqrt{kn^2-4} > \frac{3}{2}\log 12 > 3.72.$$

From equation (51), we have

$$\frac{du}{dt} = \frac{\sinh t}{\cosh u}\sqrt{\frac{kn^2-4}{km^2+4}} = \tanh u \tanh t.$$

This implies that

$$0 < \frac{du}{dt} < 1.$$

Taking the logarithm of both sides of equation (51), we obtain

$$\log\frac{e^u - e^{-u}}{e^t + e^{-t}} = \frac{1}{2}\log\frac{kn^2-4}{km^2+4}.$$

It follows that

$$\begin{aligned} u - t + \log\frac{1-e^{-2u}}{1+e^{-2t}} &= \log\frac{n}{m} + \frac{1}{2}\log\frac{1-4/kn^2}{1+4/km^2} \\ &= \log\frac{n}{m} + \frac{1}{2}\log\left(1 - \frac{4/km^2 + 4/kn^2}{1+4/km^2}\right) < \log\frac{n}{m}. \end{aligned}$$

Thus, we deduce that

$$u - t - \log\frac{n}{m} < -\log\frac{1-e^{-2u}}{1+e^{-2t}} \leq 2\frac{e^{-2t}+e^{-2u}}{1+e^{-2t}},$$

since $0.99 < \frac{1-e^{-2u}}{1+e^{-2t}} < 1$. Therefore, we have

$$\begin{aligned} 0 < u - t - \nu \leq & -\frac{1}{2}\log\frac{3}{5} + 2\frac{e^{-2t}+e^{-2u}}{1+e^{-2t}} \\ \leq & \frac{1}{2}\log\frac{5}{3} + 2(e^{-2t}+e^{-2u}) \\ \leq & \frac{1}{2}\log\frac{5}{3} + 2(e^{-7.44}+e^{-8.30}) \leq 0.258, \end{aligned}$$

i.e.

$$0 < u - t - \nu \leq 0.258. \tag{52}$$

The second derivation of u with respect t gives

$$\frac{d^2u}{dt^2} = \left[1 - \left(\frac{du}{dt}\right)^2\right] \tanh u.$$

As $0 < \left(\frac{du}{dt}\right)^2 < 1$, we obtain

$$0 < \frac{d^2u}{dt^2} < \frac{1 - e^{-2u}}{1 + e^{-2u}} < \frac{1}{1 + e^{-2u}}.$$

Let $\alpha^{2i_l+1} = e^{t_l}$ and $\beta^{2j_l+1} = e^{u_l}$, $l = 1, 2, 3$. Then $u_1 < u_2 < u_3$ and $t_1 < t_2 < t_3$. Therefore, one has

$$0 < \frac{\frac{u_3-u_2}{t_3-t_2} - \frac{u_2-u_1}{t_2-t_1}}{t_3 - t_1}$$

and moreover

$$0 < \frac{u_3 - u_2}{t_3 - t_2} - \frac{u_2 - u_1}{t_2 - t_1}. \tag{53}$$

Also we have $\frac{u_3-u_2}{t_3-t_2} < 1$ and $\frac{u_2-u_1}{t_2-t_1} < 1$ since $\frac{du}{dt} < 1$. Thus, $u_2 - t_2 < u_1 - t_1$. From (52), we deduce that

$$0 < u_2 - t_2 - \nu < u_1 - t_1 - \mu \leq 0.258.$$

This leads to

$$0 < u_1 - u_2 + t_2 - t_1 < 0.258$$

and

$$0 < 1 - \frac{u_2 - u_1}{t_2 - t_1} < \frac{0.258}{t_2 - t_1}.$$

Hence, using (53) we obtain

$$0 < \frac{u_3 - u_2}{t_3 - t_2} - \frac{u_2 - u_1}{t_2 - t_1} < \frac{0.258}{t_2 - t_1}.$$

It follows that

$$0 < \frac{j_3 - j_2}{i_3 - i_2} - \frac{j_2 - j_1}{i_2 - i_1} < \frac{0.258}{2(i_2 - i_1)\log\beta}.$$

Put $\kappa = \begin{vmatrix} j_3 - j_2 & j_2 - j_1 \\ i_3 - i_2 & i_2 - i_1 \end{vmatrix}$. Hence, since $\log\beta > \log\sqrt{kn^2-4}$, we obtain

$$\frac{\kappa}{i_3 - i_2} < \frac{0.258}{2\log\beta} < \frac{0.258}{\log(kn^2-4)}.$$

Moreover, $\kappa > 2j_1 i_1$ (Lemma 23). Finally, we obtain

$$\frac{j_1 i_1}{i_3 - i_2} < \frac{0.129}{\log(kn^2-4)}.$$

The conclusion is easily obtained. □

Proposition 24.

(1) *Assume that $n > m \geq 2$. If there is no solution coming from $1 \leq j \leq 5$, then system (4) has at most two solutions in positive integers $(U_{i_1}, V_{i_1}, Z_{j_1})$, $(U_{i_2}, V_{i_2}, Z_{j_2})$ which satisfy $2i_2 + 1 = q(2i_1 + 1)$ with $q \leq 40$.*

(2) *Assume that $m \geq n \geq 2$. If there is no solution coming from $1 \leq i \leq 11$, then system (4) has at most two solutions in positive integers $(U_{i_1}, V_{i_1}, Z_{j_1})$, $(U_{i_2}, V_{i_2}, Z_{j_2})$ which satisfy $2i_2 + 1 = q(2i_1 + 1)$ with $q \leq 40$.*

Proof. If there are three solutions in positive integers $(U, V, Z) = (U_{i_l} = T_{j_l}, V_{i_l}, Z_{j_l})$, $l = 1, 2, 3$ where $i_1 < i_2 < i_3$ and $(2i_1 + 1, 2j_1 + 1)$ are exponents of the smallest solution, then from Lemma 21 and Lemma 20, we have:

1) If $n > m \geq 2$, then $2i_3 + 1 = q(2i_1 + 1)$ where $1 \leq q \leq 40$ is an integer. Moreover, from Proposition 22 we get

$$i_3 - i_2 > \frac{j_1 i_1 \log(kn^2-4)}{0.129}.$$

Thus, we have

$$2i_3 + 1 = q(2i_1 + 1) > \frac{6}{0.129} i_1 \log(kn^2-4) \geq 46.51\, i_1 \log(kn^2-4) \geq 161.19\, i_1$$

since $j_1 \geq 6$. Therefore, since $i_1 > j_1 \geq 6$, we obtain

$$i_3 > \frac{161.19}{2} i_1 - \frac{1}{2} > 80\, i_1.$$

This is a contradiction.

2) If $m \geq n \geq 2$, then $2i_3 + 1 = q(2i_1 + 1)$ where $1 \leq q \leq 46$ is an integer. Moreover, from Proposition 22 we obtain

$$2i_3 + 1 = q(2i_1 + 1) > \frac{12}{0.129} i_1 \log(kn^2-4) \geq 93.023\, i_1 \log(kn^2-4) \geq 231.146\, i_1,$$

since $j_1 > i_1 \geq 12$. We get

$$i_3 > \frac{231.146}{2} i_1 - \frac{1}{2} > 115\, i_1,$$

which is also a contradiction. This completes the proof of the proposition. □

Now, we are ready to prove Theorem 5.

1) Let $2 \leq m < n$. Then, from Lemma 14, (2) and Proposition 16 (2) we have

$$j > \frac{g^2 \sqrt{\frac{k}{2}} \log m}{1.5 \cdot n \log n}.$$

Therefore, if system (4) has no solution for $\frac{g^2 \sqrt{\frac{k}{2}} \log m}{n \log n} < 7.5$, then from Proposition 24 (1) it has at most two solutions in positive integers which exponents satisfy $2i_2{+}1 = q(2i_1{+}1)$ with $q \leq 40$ integer.

2) Let $2 \leq n \leq m$. Then, from Lemma 13 (2) and Proposition 16 (1), we have

$$i > \frac{g^2 \sqrt{\frac{k}{2}} \log n}{1.204 \cdot m \log m}.$$

Therefore, if for $\frac{g^2 \sqrt{\frac{k}{2}} \log n}{m \log m} < 13.244$, then system (4) has no solution. Thus, from Proposition 24 (2) it has at most two solutions in positive integers which exponents satisfy $2i_2 + 1 = q(2i_1 + 1)$ with $q \leq 46$ integer. This completes the proof of Theorem 5.

Acknowledgments

The authors express their gratitude to the anonymous referee for the constructive suggestions to improve an earlier draft of this paper. The first author was supported by Natural Science Foundation of China (Grant No. 11301363). The third author thanks Purdue University Northwest for the partial support.

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