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On Certain Identities Of Ramanujan

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In this paper we will start with one identity of Ramanujan about Lambert series related to modular relations of degree 7 to give some completely new proofs of several important theta functions identities proved by Berndt and Zhang [1994, *J. Number Theory* **48**, 224–242] via the theory of modular forms. This study also yields a simple proof of one important identity of Ramanujan connected with partitions modulo 7. We also derive some new theta-function identities. © 2000 Academic Press

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1. INTRODUCTION

Using the theory of modular forms, Evans [5] has developed a powerful method for proving theta-function identities, and his method has been employed in proving several of Ramanujan's most difficult theta-function identities [1]. Furthermore, Berndt and Zhang [2] have used some of Evans' ideas to prove some difficult identities found in the unorganized pages of Ramanujan's notebooks [9], and they also hope that others can find more natural proofs, not dependent upon the theory of modular forms, so that we might be led to a better understanding of not only Ramanujan's work on theta functions but also of his discoveries on mock theta functions as well. In this paper we shall start from the following well-known identity. (A proof of this identity without using the theory of modular forms is given in [1, pp. 467–472].)

THEOREM 1.

$$\left(1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}\right)^3 = \frac{(q;q)_\infty^7}{(q^7;q^7)_\infty} + 13q(q;q)_\infty^3\,(q^7;q^7)_\infty^3 + 49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}. \tag{1.1}$$

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We will combine it with the residue theorem of elliptic functions to prove all of the Ramanujan identities proved in [2]. In a famous manuscript on the partition $p(n)$ and the tau function $\tau(n)$, published with the lost notebook, Ramanujan [10, p. 145] recorded the following important identity:

THEOREM 2.

$$\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n(1+q^n)}{(1-q^n)^3}=q(q;q)_\infty^3\,(q^7;q^7)_\infty^3+8q^2\frac{(q^7;q^7)_\infty^7}{(q:q)_\infty}, \tag{1.2}$$

where $(n/7)$ *is the Legendre symbol, and*

$$(a;q)_\infty=\prod_{n=0}^{\infty}(1-aq^n).$$

Starting from (1.2), Ramanujan proves in [10, p. 145] that

THEOREM 3.

$$\sum_{n=0}^{\infty}p(7n+5)\,q^n=7\frac{(q^7;q^7)_\infty^3}{(q:q)_\infty^3}+49q\frac{(q^7;q^7)_\infty^7}{(q:q)_\infty^8}. \tag{1.3}$$

From the above equation one can easily deduces the famous Ramanujan's congruence $p(7n+5)\equiv 0 \pmod 7$. Raghavan [8] proves (1.2) by using the theory of modular forms, and Chan [3] proves that (1.2) is equivalent to the following identity of Ramanujan [10, p. 145]:

THEOREM 4.

$$8-7\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{n^2q^n}{1-q^n}=49q(q;q)_\infty^3\,(q^7;q^7)_\infty^3+8\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}. \tag{1.4}$$

In this paper we shall prove (1.2) by our method which does not require the use of the theory of modular forms. In this paper we will also prove the following interesting result:

THEOREM 5.

$$1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}$$
$$=(q^7;q^7)_\infty^3\times\left(q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)}-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)}+\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}\right), \tag{1.5}$$

where $f(a, b)$ is Ramanujan's general theta function defined by (2.7) below.

Note that in this paper we call the following fundamental theorem of elliptic functions (see, for example, [4, p. 22, Theorem 2]) the residue theorem of elliptic functions.

THEOREM 6. *The sum of all the residues of an elliptic function at the poles inside a period-parallelogram is zero.*

The above fact plays a pivotal role in our derivations of the classical theta-function identities. The idea is to use the classical theta functions to construct an elliptic function whose poles are known and then to compute the residues of the elliptic function at these poles, and finally by setting the sum of the residues equal to zero we obtain the desired theta-function identity.

2. SOME BASIC FACTS ABOUT THE JACOBI THETA FUNCTION $\theta_1(z\,|\,q)$

Let $q = e^{2\pi i\tau}$, $\operatorname{Im}\tau > 0$, and define the classical theta function [13, p. 464]

$$\theta_1(z\,|\,q) = -iq^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{(1/2)\,n(n+1)}e^{(2n+1)\,iz}$$

$$= 2q^{1/8}\sum_{n=-\infty}^{\infty}(-1)^n q^{(1/2)\,n(n+1)}\sin(2n+1)z. \tag{2.1}$$

By the Jacobi triple product formula we have [13, p. 470]

$$\theta_1(z\,|\,q) = -iq^{1/8}e^{iz}(q;q)_\infty\,(qe^{2iz};q)_\infty\,(e^{-2iz};q)_\infty. \tag{2.2}$$

The above equation can be rewritten as

$$\theta_1(z\,|\,q) = 2q^{1/8}\sin z(q;q)_\infty\,(qe^{2iz};q)_\infty\,(qe^{-2iz};q)_\infty. \tag{2.3}$$

Dividing both sides of the above equation by z and then letting $z \to 0$ we find that

$$\theta_1'(q) := \theta_1'(0\,|\,q) = 2q^{1/8}(q;q)_\infty^3, \tag{2.4}$$

where $\theta_1'(z\,|\,q)$ means the partial derivative with respect to the variable z. Replacing q by q^7 in (2.2) and then taking $z = \pi\tau$, $2\pi\tau$, and $3\pi\tau$ respectively we find that

$$\theta_1(\pi\tau\,|\,q^7) = iq^{3/8}(q;q^7)_\infty\,(q^6;q^7)_\infty\,(q^7;q^7)_\infty,$$
$$\theta_1(\pi\tau\,|\,q^7) = iq^{-1/8}(q^2;q^7)_\infty\,(q^5;q^7)_\infty\,(q^7;q^7)_\infty, \tag{2.5}$$
$$\theta_1(\pi\tau\,|\,q^7) = iq^{-3/8}(q^3;q^7)_\infty\,(q^4;q^7)_\infty\,(q^7;q^7)_\infty.$$

Multiplying the above equations together we find that

$$\theta_1(\pi\tau\,|\,q^7)\,\theta_1(2\pi\tau\,|\,q^7)\,\theta_1(3\pi\tau\,|\,q^7) = -iq^{-3/8}(q;q)_\infty\,(q^7;q^7)_\infty^2. \tag{2.6}$$

We now give Ramanujan's definition of his general theta function. For $|ab|<1$, define

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.7}$$

By the Jacobi triple product formula we have the following product expansion for $f(a,b)$ (see, for example [1, p. 35, Entry 16]):

$$f(a,b) = (-a,ab)_\infty\,(-b,ab)_\infty\,(ab,ab)_\infty. \tag{2.8}$$

It is easily verified that

$$\theta_1(z\,|\,q) = iq^{1/8}e^{-iz}f(-e^{2iz},\,-qe^{-2iz}). \tag{2.9}$$

From the above equation we find that

$$\theta_1(\pi\tau\,|\,q^7) = iq^{3/8}f(-q;\,-q^6),$$
$$\theta_1(\pi\tau\,|\,q^7) = iq^{-1/8}f(-q^2;\,-q^5), \tag{2.10}$$
$$\theta_1(\pi\tau\,|\,q^7) = iq^{-5/8}f(-q^3;\,-q^4).$$

From (2.6) and (2.9) we find that

$$f(-q,\,-q^7)\,f(-q^2,\,-q^5)\,f(-q^3,\,-q^4) = (q;q)_\infty\,(q^7;q^7)_\infty^2. \tag{2.11}$$

From (2.2) by logarithmic differentiation we find that

$$\frac{\theta_1'}{\theta_1}(z\,|\,q) = i - 2i\sum_{n=1}^{\infty}\frac{q^n e^{2iz}}{1-q^n e^{2iz}} + 2i\sum_{n=0}^{\infty}\frac{q^n e^{-2iz}}{1-q^n e^{-2iz}}, \tag{2.12}$$

and we also have the well-known fact [13, p. 489]

$$\frac{\theta_1'}{\theta_1}(z\,|\,q) = \cot z + 4\sum_{n=1}^{\infty}\frac{q^n}{1-q^n}\sin 2nz. \tag{2.13}$$

The following fact is well-known:

$$\cot z = \frac{1}{z} - \frac{1}{3} z - \frac{1}{45} z^3 + \cdots.$$

From the two eqautions above we easily find that

$$\lim_{z \to 0} \left(\frac{2}{z} - \frac{\theta_1'}{\theta_1}(z \mid q) \right) = 0. \tag{2.14}$$

From the definition of $\theta_1(z \mid q)$ the following functional equations are easily verified

$$\theta_1(z + n\pi \mid q) = (-1)^n \theta_1(z \mid q), \quad \theta_1(z + n\pi\tau \mid q) = (-1)^n q^{-n^2/2} e^{-2niz} \theta_1(z \mid q), \tag{2.15}$$

where n is any integer. Differentiating the second equation in (2.15) with respect to z and then setting $z = 0$ we have

$$\theta_1'(0 \mid n\pi\tau \mid q) = (-1)^n q^{-n^2/2} \theta_1'(q). \tag{2.16}$$

3. IDENTITIES OF THETA-FUNCTIONS

In this section we will prove all of the Ramanujan identities proved in [2]. By denoting

$$\begin{aligned} a &:= q^{-2/7} \frac{f(-q^2, -q^5)}{f(-q, -q^6)}, \\ b &:= -q^{-1/7} \frac{f(-q^3, -q^4)}{f(-q^2, -q^5)}, \\ c &:= q^{3/7} \frac{f(-q, -q^6)}{f(-q^3, -q^4)} \end{aligned} \tag{3.1}$$

we have

THEOREM 7.

$$ab^2 - a^2 + c = 0, \tag{3.2}$$

$$bc^2 - b^2 + a = 0, \tag{3.3}$$

$$ca^2 - c^2 + b = 0, \tag{3.4}$$

$$abc = -1. \tag{3.5}$$

Proof. By directly multiplying the three equations in (3.1) together we immediately obtain (3.5). By multiplying both sides of (3.2) by a^{-1} and c respectively and then using (3.5) in the resulting equations we deduce (3.3) and (3.4).

The Proof of (3.2). Let

$$f(z)=\frac{\theta_1^3(z\,|\,q^7)\,\theta_1(2z\,|\,q^7)}{\theta_1(z\,|\,q)}.$$

By using (2.15) we can verify that $f(z)$ is an elliptic function with periods π and $7\pi\tau$. It has six simple poles $\pi\tau$, $2\pi\tau$, $3\pi\tau$, $4\pi\tau$, $5\pi\tau$, and $6\pi\tau$. Let $\mathrm{Res}(f;x)$ denote the residue of $f(z)$ at x. Let us now compute the residues.

$$\begin{aligned}\mathrm{Res}(f;\pi\tau)&=\lim_{z\to\pi\tau}(z-\pi\tau)\,f(z)\\&=\lim_{z\to\pi\tau}\theta_1^3(z\,|\,q^7)\,\theta_1(2z\,|\,q^7)\lim_{z\to\pi\tau}\frac{(z-\pi\tau)}{\theta_1(z\,|\,q)}.\end{aligned}$$

It is plain that

$$\lim_{z\to\pi\tau}\theta_1^3(z\,|\,q^7)\,\theta_1(2z\,|\,q^7)=\theta_1^3(\pi\tau\,|\,q^7)\,\theta_1(2\pi\tau\,|\,q^7).$$

By first applying L'Hôpital's rule and then using (2.16) we have

$$\lim_{z\to\pi\tau}\frac{(z-\pi\tau)}{\theta_1(z\,|\,q)}=\frac{1}{\theta_1'(\pi\tau\,|\,q)}=-\frac{q^{1/2}}{\theta_1'(q)}.$$

Hence we find that

$$\mathrm{Res}(f;\pi\tau)=-q^{1/2}\,\frac{\theta_1^3(\pi\tau\,|\,q^7)\,\theta_1(2\pi\tau\,|\,q^7)}{\theta_1'(q)}.$$

By a direct calculation we find that

$$\mathrm{Res}(f;6\pi\tau)=\mathrm{Res}(f;\pi\tau).\tag{3.6}$$

Similarly, we find that

$$\mathrm{Res}(f;2\pi\tau)=\mathrm{Res}(f;5\pi\tau)=q^{3/2}\,\frac{\theta_1^3(2\pi\tau\,|\,q^7)\,\theta_1(3\pi\tau\,|\,q^7)}{\theta_1'(q)},\tag{3.7}$$

$$\mathrm{Res}(f;\pi\tau)=\mathrm{Res}(f;6\pi\tau)=-q^2\,\frac{\theta_1^3(3\pi\tau\,|\,q^7)\,\theta_1(\pi\tau\,|\,q^7)}{\theta_1'(q)}.\tag{3.8}$$

The residue theorem of elliptic functions, Theorem 6, gives the identity

$$\sum_{k=1}^{6} \operatorname{Res}(f; k\pi\tau) = 0.$$

Substituting (3.6), (3.7), and (3.8) into the above equation we obtain the following identity:

$$q^{-1/2} \frac{\theta_1^2(\pi\tau \mid q^7)}{\theta_1(3\pi\tau \mid q^7)} - q^{1/2} \frac{\theta_1^2(2\pi\tau \mid q^7)}{\theta_1(\pi\tau \mid q^7)} + q \frac{\theta_1^2(3\pi\tau \mid q^7)}{\theta_1(2\pi\tau \mid q^7)} = 0.$$

By applying (2.10) into the above equation we find the following identity [2, Eq. (1.2)]:

$$\frac{f^2(-q^3, -q^4)}{f(-q^2, -q^5)} - \frac{f^2(-q^2, -q^5)}{f(-q, -q^6)} + q \frac{f^2(-q, -q^6)}{f(-q^3, -q^4)} = 0. \tag{3.9}$$

The above equation is equivalent to (3.2).

We complete the proof of the above theorem. Let

$$\begin{aligned} y_1 &:= a^3 b = -q^{-1}(q; q)_\infty (q^7; q^7)_\infty^2 \frac{f(-q^2, -q^5)}{f^4(-q, -q^6)}, \\ y_2 &:= b^3 c = -(q; q)_\infty (q^7; q^7)_\infty^2 \frac{f(-q^3, -q^4)}{f^4(-q^2, -q^5)}, \\ y_3 &:= c^3 a = q(q; q)_\infty (q^7; q^7)_\infty^2 \frac{f(-q, -q^6)}{f^4(-q^3, -q^4)}. \end{aligned} \tag{3.10}$$

From (2.5)–(2.9) and the above equations by some direct calculations we derive the following relations:

$$y_1 y_2 = -y_1 - 1, \tag{3.11}$$

$$y_2 y_3 = -y_2 - 1, \tag{3.12}$$

$$y_3 y_1 = -y_3 - 1, \tag{3.13}$$

$$y_1 y_2 y_3 = 1. \tag{3.14}$$

The above relations about a, b, c and y_1, y_2, y_3 are also obtained by other authors by using differential methods for other goals [6, 7, 12].

THEOREM 8.

$$\left(q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)}-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)}+q\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}\right)^3$$
$$=q^3\frac{f^3(-q^2,-q^5)}{f^6(-q^3,-q^4)}-q^3\frac{f^3(-q,-q^6)}{f^6(-q^2,-q^5)}+q\frac{f^3(-q^3,-q^4)}{f^6(-q,-q^6)}$$
$$+3q^2(q;q)_\infty^{-1}(q^7;q^7)_\infty^{-2}. \tag{3.15}$$

Proof. Denoting

$$\alpha:=q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)},\qquad \beta:=-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)},$$
$$\gamma:=\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}, \tag{3.16}$$

(3.9) can be rewritten as

$$\alpha^{-1}+\beta^{-1}+\gamma^{-1}=0. \tag{3.17}$$

Hence we find that

$$\alpha\beta+\beta\gamma+\gamma\alpha=\alpha\beta\gamma(\alpha^{-1}+\beta^{-1}+\gamma^{-1})=0. \tag{3.18}$$

By using the above equation and some elementary calculations we find that

$$\left(q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)}-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)}+\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}\right)^3$$
$$=(\alpha+\beta+\gamma)^3$$
$$=\alpha^3+\beta^3+\gamma^3+3(\alpha+\beta)(\alpha\beta+\alpha\gamma+\beta\gamma+\gamma^2)$$
$$=\alpha^3+\beta^3+\gamma^3+3(\alpha\gamma+\beta\gamma)\,\gamma$$
$$=\alpha^3+\beta^3+\gamma^3-3\alpha\beta\gamma$$
$$=q^3\frac{f^3(-q^2,-q^5)}{f^6(-q^3,-q^4)}-q^3\frac{f^3(-q,-q^6)}{f^6(-q^2,-q^5)}+\frac{f^3(-q^3,-q^4)}{f^6(-q,-q^6)}$$
$$+3q^2(q;q)_\infty^{-1}(q^7;q^7)_\infty^{-2}.$$

We complete the proof of the above theorem.

The proof of Theorem 6 goes as follows: We construct the elliptic function

$$f_1(z) = e^{-2\pi iz} \frac{\theta_1^2(z-3\pi\tau\,|\,q^7)\,\theta_1(z-2\pi\tau\,|\,q^7)}{\theta_1^2(z\,|\,q^7)\,\theta_1(z-\pi\tau\,|\,q^7)}.$$

We can verify that $f_1(z)$ is an elliptic function with periods π and $7\pi\tau$. $\pi\tau$ is its simple pole and 0 is its pole of order 2.

$$\begin{aligned}
\operatorname{Res}(f_1;\pi\tau) &= \lim_{z\to\pi\tau}(z-\pi\tau)\,f_1(z)\\
&= \lim_{z\to\pi\tau} e^{-2\pi iz}\theta_1^2(z-3\pi\tau\,|\,q^7)\,\frac{\theta_1(z-2\pi\tau\,|\,q^7)}{\theta_1^2(z\,|\,q^7)}\lim_{z\to\pi\tau}\frac{(z-\pi\tau)}{\theta_1(z-\pi\tau\,|\,q^)}\\
&= \frac{q^{-1}\theta_1^2(2\pi\tau\,|\,q^7)}{\theta_1'(q)\,\theta_1(\pi\tau\,|\,q^7)}. \qquad (3.19)\\
\operatorname{Res}(f_1;0) &= \lim_{z\to 0}(z^2 f_1(z))'\\
&= \lim_{z\to 0} z^2 f_1(z)\left(\frac{2}{z}+\frac{f_1'(z)}{f_1(z)}\right)\\
&= \lim_{z\to 0} z^2 f_1(z)\lim_{z\to 0}\Bigg(-2\pi i+\frac{2}{z}-2\frac{\theta_1'}{\theta_1}(z\,|\,q^7)-\frac{\theta_1'}{\theta_1}(z-\pi\tau\,|\,q^7)\\
&\quad +\frac{\theta_1'}{\theta_1}(z-2\pi\tau\,|\,q^7)+2\frac{\theta_1'}{\theta_1}(z-3\pi\tau\,|\,q^7)\Bigg).
\end{aligned}$$

By an elementary calculation we have

$$\lim_{z\to 0} z^2 f_1(z) = -\frac{\theta_1(2\pi\tau\,|\,q^7)\,\theta_1^2(3\pi\tau\,|\,q^7)}{\theta_1'(q^7)^2\,\theta_1(\pi\tau\,|\,q^7)},$$

and by using (2.14) we immediately have

$$\begin{aligned}
&\lim_{z\to 0}\Bigg(-2\pi i+\frac{2}{z}-2\frac{\theta_1'}{\theta_1}(z\,|\,q^7)-\frac{\theta_1'}{\theta_1}(z-\pi\tau\,|\,q^7)\\
&\qquad +\frac{\theta_1'}{\theta_1}(z-2\pi\tau\,|\,q^7)+2\frac{\theta_1'}{\theta_1}(z-3\pi\tau\,|\,q^7)\Bigg)\\
&\quad = -2\pi i+\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^7)-\frac{\theta_1'}{\theta_1}(2\pi\tau\,|\,q^7)+2\frac{\theta_1'}{\theta_1}(3\pi\tau\,|\,q^7).
\end{aligned}$$

Hence

$$\operatorname{Res}(f_1;0) = -\frac{\theta_1(2\pi\tau\,|\,q^7)\,\theta_1^2(3\pi\tau\,|\,q^7)}{\theta_1'(q^7)^2\,\theta_1(\pi\tau\,|\,q^7)} \times\left(-2\pi i+\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^7)-\frac{\theta_1'}{\theta_1}(2\pi\tau\,|\,q^7)+2\frac{\theta_1'}{\theta_1}(3\pi\tau\,|\,q^7)\right). \quad (3.20)$$

The residue theorem of elliptic functions, Theorem 6, tells us that

$$\operatorname{Res}(f_1;\pi\tau)+\operatorname{Res}(f_1;0)=0.$$

By substituting (3.19) and (3.20) we obtain the identity

$$-2i+\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^7)-\frac{\theta_1'}{\theta_1}(2\pi\tau\,|\,q^7)-2\frac{\theta_1'}{\theta_1}(3\pi\tau\,|\,q^7) = q^{-1}\theta_1'(q^7)\frac{\theta_1(2\pi\tau\,|\,q^7)}{\theta_1^2(3\pi\tau\,|\,q^7)}. \quad (3.21)$$

Similarly, if we apply the residue theorem of elliptic functions to the elliptic functions

$$f_2(z)=\frac{\theta_1^2(z-2\pi\tau\,|\,q^7)\,\theta_1(z+\pi\tau\,|\,q^7)}{\theta_1^2(z\,|\,q^7)\,\theta_1(z-3\pi\tau\,|\,q^7)},$$

$$f_3(z)=e^{2\pi iz}\frac{\theta_1^2(z+\pi\tau\,|\,q^7)\,\theta_1(z+2\pi\tau\,|\,q^7)}{\theta_1^2(z\,|\,q^7)\,\theta_1(z-3\pi\tau\,|\,q^7)}.$$

we obtain the following facts respectively

$$\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^7)-2\frac{\theta_1'}{\theta_1}(2\pi\tau\,|\,q^7)+\frac{\theta_1'}{\theta_1}(3\pi\tau\,|\,q^7)=q^{-1/2}\theta_1'(q^7)\frac{\theta_1(\pi\tau\,|\,q^7)}{\theta_1^2(2\pi\tau\,|\,q^7)}, \quad (3.22)$$

$$2i+\frac{\theta_1'}{\theta_1}(\pi\tau\,|\,q^7)+\frac{\theta_1'}{\theta_1}(2\pi\tau\,|\,q^7)+\frac{\theta_1'}{\theta_1}(3\pi\tau\,|\,q^7)=q^{1/2}\theta_1'(q^7)\frac{\theta_1(3\pi\tau\,|\,q^7)}{\theta_1^2(\pi\tau\,|\,q^7)}. \quad (3.23)$$

Adding (3.21) and (3.22) and sustracting (3.23) we find that

$$2\frac{\theta_1'}{\theta}(\pi\tau\,|\,q^7)+2\frac{\theta_1'}{\theta}(2\pi\tau\,|\,q^7)-2\frac{\theta_1'}{\theta}(3\pi\tau\,|\,q^7) = \theta_1'(q^7)\left(q^{-1}\frac{\theta_1(2\pi\tau\,|\,q^7)}{\theta_1^2(3\pi\tau\,|\,q^7)}-q^{-1/2}\frac{\theta_1(\pi\tau\,|\,q^7)}{\theta_1^2(2\pi\tau\,|\,q^7)}+q^{1/2}\frac{\theta_1(3\pi\tau\,|\,q^7)}{\theta_1^2(\pi\tau\,|\,q^7)}\right). \quad (3.24)$$

By using (2.12) we easily find that

$$\frac{\theta_1'}{\theta}(\pi\tau\,|\,q^7)+\frac{\theta_1'}{\theta}(2\pi\tau\,|\,q^7)-\frac{\theta_1'}{\theta}(3\pi\tau\,|\,q^7)=-i\left(1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}\right).$$

Subistituting the above equation into the left side of (3.24) and then using (2.4) and (2.10) to the right side of (3.24) we obtain the following important result:

$$1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-q^n}$$
$$=(q^7;q^7)_\infty^3\left(q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)}-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)}+\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}\right).$$

We complete the proof of [5]. Substituting (1.1) into the above equation we find that

THEOREM 9.

$$(\alpha+\beta+\gamma)^3=\left(q\frac{f(-q^2,-q^5)}{f^2(-q^3,-q^4)}-q\frac{f(-q,-q^6)}{f^2(-q^2,-q^5)}+\frac{f(-q^3,-q^4)}{f^2(-q,-q^6)}\right)^3$$
$$=(q^7;q^7)_\infty^{-9}\left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}+13q(q;q)_\infty^3(q^7;q^7)_\infty^3\right.$$
$$\left.+49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}\right). \tag{3.25}$$

The above equation is equivalent to [2, Eq. (1.3)].

THEOREM 10.

$$(\alpha^{-2}+\beta^{-2}+\gamma^{-2})^3$$
$$=\left(q^{-2}\frac{f^4(-q^3,-q^4)}{f^2(-q^2,-q^5)}+q^{-2}\frac{f^4(-q^2,-q^5)}{f^2(-q,-q^6)}+\frac{f^4(-q,-q^6)}{f^2(-q^3,-q^4)}\right)^3$$
$$=q^{-6}(q;q)_\infty^3(q^7;q^7)_\infty^{-3}$$
$$\times\left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}+13q(q;q)_\infty^3(q^7;q^7)_\infty^3+49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}\right), \tag{3.26}$$

$$(\alpha^2+\beta^2+\gamma^2)^3$$
$$=\left(q^2\frac{f^2(-q^2,-q^5)}{f^4(-q^3,-q^4)}+q^2\frac{f^2(-q,-q^6)}{f^4(-q^2,-q^5)}+\frac{f^2(-q^3,-q^4)}{f^4(-q,-q^6)}\right)^3$$
$$=(q^7;q^7)_\infty^{-18}\left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}+13q(q;q)_\infty^3(q^7;q^7)_\infty^3+49q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}\right). \tag{3.27}$$

Equation (3.26) is equivalent to [2, Eq. (4.24)] and (3.27) is equivalent to [2, Eq. (4.25)].

Proof. From (3.17) we easily find that

$$(\alpha^{-1}+\beta^{-1}+\gamma^{-1})^2=\alpha^{-2}+\beta^{-2}+\gamma^{-2}+2(\alpha\beta\gamma)^{-1}(\alpha+\beta+\gamma)=0.$$

Hence we obtain

$$\alpha^{-2}+\beta^{-2}+\gamma^{-2}=-2(\alpha\beta\gamma)^{-1}(\alpha+\beta+\gamma).$$

Hence

$$(\alpha^{-2}+\beta^{-2}+\gamma^{-2})^3=-8(\alpha\beta\gamma)^{-3}(\alpha+\beta+\gamma)^3. \tag{3.28}$$

By using (3.19) we find that

$$(\alpha+\beta+\gamma)^2=\alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+\gamma\alpha$$
$$=\alpha^2+\beta^2+\gamma^2.$$

Hence we have

$$(\alpha^2+\beta^2+\gamma^2)^3=(\alpha+\beta+\gamma)^6. \tag{3.29}$$

Noting the definitions of α, β, and γ we know that (3.28) is the same as (3.27) and that (3.29) is the same as (3.26). Combining (3.25) with (3.15) we immediately obtain

THEOREM 11.

$$\alpha^3+\beta^3+\gamma^3=q^3\frac{f^3(-q^2,-q^5)}{f^6(-q^3,-q^4)}-q^3\frac{f^3(-q,-q^6)}{f^6(-q^2,-q^5)}+q\frac{f^3(-q^3,-q^4)}{f^6(-q,-q^6)}$$
$$=(q^7;q^7)_\infty^{-9}\left(\frac{(q;q)_\infty^7}{(q^7;q^7)_\infty}+13q(q;q)_\infty^3(q^7;q^7)_\infty^3\right.$$
$$\left.+46q^2\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty}\right). \tag{3.30}$$

Adding (3.11), (3.12), and (3.13) together we find that

$$y_1 y_2 + y_2 y_3 + y_3 y_1 = -(y_1 + y_2 + y_3) - 3. \tag{3.31}$$

Hence we easily find that

$$\begin{aligned}
&(y_1 + y_2 + y_3)^2 + 3(y_1 + y_2 + y_3) + 6 \\
&\quad = y_1^2 + y_2^2 + y_3^2 + 2(y_1 y_2 + y_2 y_3 + y_3 y_1) + 3(y_1 + y_2 + y_3) + 6 \\
&\quad = y_1^2 + y_2^2 + y_3^2 + y_1 + y_2 + y_3 \\
&\quad = y_1(1 + y_1) + y_2(1 + y_2) + y_3(1 + y_3) \\
&\quad = -y_1^2 y_2 - y_2^2 y_3 - y_3^2 y_1 .
\end{aligned}$$

In the last step (3.11), (3.12), and (3.13) were used. By (3.10) we easily find that

$$\begin{aligned}
&-y_1^2 y_2 - y_2^2 y_3 - y_3^2 y_1 \\
&\quad = q^{-2}(q; q)_\infty (q^7; q^7)_\infty^2 \left(q^3 \frac{f^3(-q^2, -q^5)}{f^6(-q^3, -q^4)} \right. \\
&\qquad \left. - q^3 \frac{f^3(-q, -q^6)}{f^6(-q^2, -q^5)} + q \frac{f^3(-q^3, -q^4)}{f^6(-q, -q^6)} \right).
\end{aligned}$$

Substituting (3.30) into the above equation we find that

$$\begin{aligned}
&(y_1 + y_2 + y_3)^2 + 3(y_1 + y_2 + y_3) + 6 \\
&\quad = \frac{(q; q)_\infty^8}{q^2 (q^7; q^7)_\infty^8} + 13 \frac{(q; q)_\infty^4}{q (q^7; q^7)_\infty^4} + 46.
\end{aligned} \tag{3.32}$$

From (3.10) we know that

$$\begin{aligned}
y_1 + y_2 + y_3 &= (q; q)_\infty (q^7; q^7)_\infty^2 \left(q \frac{f(-q, -q^6)}{f^4(-q^3, -q^4)} \right. \\
&\quad \left. - \frac{f(-q^3, -q^4)}{f^4(-q^2, -q^5)} - q^{-1} \frac{f(-q^2, -q^5)}{f^4(-q, -q^6)} \right).
\end{aligned}$$

For q small and positive we easily find that $y_1 + y_2 + y_3 < 0$. Hence by solving Eq. (3.31) we obtain the following important result.

THEOREM 12.

$$y_1+y_2+y_3=(q;q)_\infty\,(q^7;q^7)_\infty^2 \times\left(q\,\frac{f(-q,-q^6)}{f^4(-q^3,-q^4)}-\frac{f(-q^3,-q^4)}{f^4(-q^2,-q^5)}-q^{-1}\,\frac{f(-q^2,-q^5)}{f^4(-q,-q^6)}\right) =-\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}-8. \tag{3.33}$$

The above equation is the same as [2, Eq. (4.22)]. By using (3.31) and the above equation we immediately have

THEOREM 13.

$$\begin{aligned} y_1\,y_2+y_2\,y_3+y_3\,y_1&=(q;q)_\infty^{-1}\,(q^7;q^7)_\infty^{-2}\left(q^{-1}\,\frac{f^4(-q^3,-q^4)}{f(-q,-q^6)}\right.\\ &\quad\left.-\frac{f^4(-q^2,-q^5)}{f(-q^3,-q^4)}-q\,\frac{f^4(-q,-q^6)}{f(-q^2,-q^5)}\right)\\ &=-(y_1+y_2+y_3)-3\\ &=\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}+5. \end{aligned} \tag{3.34}$$

The above equation is equivalent to [2, Eq. (4.23)].

THEOREM 14.

$$\begin{aligned} a^7+b^7+c^7&=q^{-2}\,\frac{f^7(-q^2,-q^5)}{f^7(-q,-q^6)}-q^{-1}\,\frac{f^7(-q^3,-q^4)}{f^7(-q^2,-q^5)}+q^3\,\frac{f^7(-q,-q^6)}{f^7(-q^3,-q^4)}\\ &=57+14\,\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}+\frac{(q;q)_\infty^8}{q^2(q^7;q^7)_\infty^8}\\ &\quad-(a^7b^7+b^7c^7+c^7a^7) \end{aligned} \tag{3.35}$$

$$=q^2\,\frac{f^7(-q,-q^6)}{f^7(-q^2,-q^5)}-q\,\frac{f^7(-q^2,-q^5)}{f^7(-q^3,-q^4)}+q^{-3}\,\frac{f^7(-q^3,-q^4)}{f^7(-q,-q^6)}$$

$$=289+126\,\frac{(q;q)_\infty^4}{q(q^7;q^7)_\infty^4}+19\,\frac{(q;q)_\infty^8}{q^2(q^7;q^7)_\infty^8}+\frac{(q;q)_\infty^{12}}{q^3(q^7;q^7)_\infty^{12}}. \tag{3.36}$$

Equation (3.35) is equivalent to [1, p. 305, Eq. (18.2); 5, p. 114, Eq. (5.20)] and (3.36) is equivalent to [1, p. 305, Eq. (18.2)].

Proof. Multiplying (3.2) by ab, (3.3) by bc, (3.4) by ac and noting (3.10) we easily find that

$$a^2b^3 = y_1 + 1, \qquad b^2c^3 = y_2 + 1, \qquad c^2a^3 = y_3 + 1. \tag{3.37}$$

Multiplying (3.2) by b^3, (3.3) by c^3, and (3.4) by a^3 and noting (3.10) we easily find that

$$ab^5 = a^2b^3 - y_2, \qquad bc^5 = b^2c^3 - y_3, \qquad ca^5 = a^3c^2 - y_1. \tag{3.38}$$

Combining (3.37) and (3.38) we find that

$$ab^5 = y_1 - y_2 + 1, \qquad bc^5 = y_2 - y_3 + 1, \qquad ca^5 = y_3 - y_1 + 1. \tag{3.39}$$

Multiplying (3.2) by a^5, (3.3) by b^5, and (3.4) by c^5 and noting (3.10) we easily find that

$$a^7 = a^5c + y_1^2, \qquad b^7 = b^5a + y_2^2, \qquad c^7 = c^5b + y_3^2. \tag{3.40}$$

From (3.39) and (3.40) we find the following important relations:

$$a^7 = y_1^2 - y_1 + y_3 + 1, \qquad b^7 = y_2^2 - y_2 + y_1 + 1, \qquad c^7 = y_3^2 - y_3 + y_2 + 1. \tag{3.41}$$

Using the above equations and (3.31) we easily find that

$$a^7 + b^7 + c^7 = (y_1 + y_2 + y_3)^2 + 2(y_1 + y_2 + y_3) + 9. \tag{3.42}$$

By using (3.41) and (3.11)–(3.14) we easily find that

$$a^7b^7 = y_1(1 + y_1)^2, \qquad b^7c^7 = y_2(1 + y_2)^2, \qquad c^7a^7 = y_3(1 + y_3)^2. \tag{3.43}$$

Using the above equations, (3.31), and (3.14) we can easily obtain

$$\begin{aligned} &a^7b^7 + b^7c^7 + c^7a^7 \\ &\quad = (y_1 + y_2 + y_3)^3 + 5(y_1 + y_2 + y_3)^2 + 14(y_1 + y_2 + y_3) + 15. \end{aligned} \tag{3.44}$$

Substituting (3.33) into (3.42) and (3.44) respectively we complete the proofs.

The following identity of theta functions is well known (see, for example, [11]):

$$\left(\frac{\theta_1'}{\theta_1}\right)'(x\,|\,q) - \left(\frac{\theta_1'}{\theta_1}\right)'(y\,|\,q) = \frac{\theta_1'(q)^2\,\theta_1(x-y\,|\,q)\,\theta_1(x+y\,|\,q)}{\theta_1^2(x\,|\,q)\,\theta_1^2(y\,|\,q)}.$$

Dividing both sides of the above equation by $x-y$ and then letting $y \to x$ we get

$$\left(\frac{\theta_1'}{\theta_1}\right)''(x|q) = \theta_1'(q)^3 \frac{\theta_1(2x|q)}{\theta_1^4(x|q)}. \tag{3.45}$$

By writing q for q^7 in the above equation and noting (2.4), (2.9), and (2.12) we find that

$$\sum_{n=-\infty}^{\infty} \frac{q^{7n}e^{2ix}(1+q^{7n}e^{2ix})}{(1-q^{7n}e^{2ix})^3} = e^{2ix}(q^7;q^7)_\infty^9 \frac{f(-e^{4ix}, -q^7e^{-4ix})}{f^4(-e^{2ix}, -q^7e^{-2ix})}.$$

By taking $x=\pi\tau, 2\pi\tau, 3\pi\tau$ respectively in the above equation and then combining the resulting equations and finally using (3.33) we obtain

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \left(\frac{q^{7n+1}(1+q^{7n+1})}{(1-q^{7n+1})^3} + \frac{q^{7n+2}(1+q^{7n+2})}{(1-q^{7n+2})^3} - \frac{q^{7n+3}(1+q^{7n+3})}{(1-q^{7n+3})^3} \right) \\
&\quad = \sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n(1+q^n)}{(1-q^n)^3} \\
&\quad = -q^2(q^7;q^7)_\infty^9 \left(\frac{qf(-q, -q^6)}{f^4(-q, -q^6)} - \frac{f(-q^3, -q^4)}{f^4(-q^2, -q^5)} - q^{-1} \frac{f(-q^2, -q^5)}{f^4(-q, -q^6)} \right) \\
&\quad = q(q;q)_\infty^3 (q^7;q^7)_\infty^3 + 8q^2 \frac{(q^7;q^7)_\infty^7}{(q:q)_\infty}.
\end{aligned} \tag{3.46}$$

This completes the proof of (1.2).

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