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Level stripping for Siegel modular forms with reducible Galois representations

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In this paper we consider level stripping for genus 2 cuspidal Siegel eigenforms. In particular, we show that it is possible to strip primes from the level of Saito–Kurokawa lifts that arise as theta lifts and weak endoscopic lifts with a mild condition on the associated character. The main ingredients into our results are a level stripping result for elliptic modular forms and the explicit nature of the forms under consideration.

1. Introduction

We fix a prime $\ell \geqslant 3$ throughout the paper. Let υ be a place dividing ℓ in the field of algebraic numbers in $\mathbb{C}$ and let $\mathbb{F}$ denote the residue field of υ. Note that $\mathbb{F}$ is an algebraic closure of the field $\mathbb{F}_\ell$. Given a number field $E \subset \mathbb{C}$, the place υ induces a prime λ of E that divides ℓ along with a canonical inclusion $\mathbb{F}_\lambda \hookrightarrow \mathbb{F}$. Note that this allows us to consider our Galois representations defined over $\mathbb{F}_\lambda$ as taking values in $\mathbb{F}$.

Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F})$ be an odd, continuous, irreducible Galois representation. Serre's conjecture, now a theorem of Khare–Wintenberger, gives that ρ is the residual representation $\overline{\rho}_{f,\lambda}$ for some newform $f \in S_{k(\rho)}(\Gamma_1(N(\rho)))$ where $k(\rho)$ and $N(\rho)$ are the so-called Serre weight and level. An important early step in studying this conjecture was showing that given a ρ as above so that $\rho \simeq \overline{\rho}_{f,\lambda}$ for some normalized eigenform f of level $\Gamma_1(N\ell^\alpha)$, then in fact $\rho \simeq \overline{\rho}_{g,\lambda}$ for some normalized eigenform g of level $\Gamma_1(N)$. This result is given in [12, Theorem 2.1], which was well known by the time of its publication, for instance see [15]. It is natural to ask if there is an analogous Serre-type conjecture in the case of odd, continuous, Galois representations $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathbb{F})$ and to what

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extent the results known for GL(2) can be carried over to GL(4). One can see [6] for a statement of a Serre-type conjecture in this context. Our interest is in the special case that such a ρ is given as $\overline{\rho}_{F,\lambda}$ for some eigenform $F \in S_k(\Gamma_0^{(2)}(N\ell^\alpha), \chi)$. In this paper we consider the case where $\rho_{F,\lambda}$ is reducible. This happens precisely when F is a CAP form (cuspidal associated to parabolic) or a weak endoscopic lift.

In this paper we work classically, so we frame the question in this context. Let $k \geqslant 2$ and $M \geqslant 1$ be integers. Let χ be a primitive even Dirichlet character modulo M. Write $M = N\ell^\alpha$ with $\alpha \in \mathbb{Z}_{\geqslant 1}$ and $\ell \nmid N$. Define the level M congruence subgroup of $\mathrm{Sp}_4(\mathbb{Z})$ by

$$\Gamma_0^{(2)}(M) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) \colon C \equiv 0_2 \pmod{M} \right\}.$$

Let $F \in S_k(\Gamma_0^{(2)}(M), \chi)$ be a Siegel eigenform with associated character χ. We note here that we will only be interested in the eigenvalues of F away from $N\ell$. As such, the Hecke algebra of interest to us is generated by $T_S(p)$ and $T_S(p^2)$ for $p \nmid N\ell$ where one can see [2] for a definition of $T_S(p)$ and $T_S(p^2)$. In light of this, by "stripping primes from the level of F" we mean finding an integer k' and a Dirichlet character ψ modulo N such that there exists a Siegel eigenform $G \in S_{k'}(\Gamma_0^{(2)}(N), \psi)$ satisfying

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\upsilon},$$
$$\lambda_F\left(p^2\right) \equiv \lambda_G\left(p^2\right) \pmod{\upsilon},$$

for primes $p \nmid N\ell$ with $\lambda_F(p)$ and $\lambda_F(p^2)$ the eigenvalues of F with respect to the Hecke operators $T_S(p)$ and $T_S(p^2)$, respectively. The main results of this paper are that with a mild restriction on χ one can strip primes from the level in the case that F is a CAP form that arises as a theta lift or F is a weak endoscopic lift. These results rely heavily on Ribet's result mentioned above (see Theorem 4 below) and a careful analysis of the proof of this result.

2. The Saito–Kurokawa lifting

In order to frame our result properly, we begin with the following well-known result.

Theorem 1. *(See [21, Theorem I], [17, Theorem 3.1.3].) Let Π be a unitary irreducible cuspidal automorphic representation of $\mathrm{GSp}_4(\mathbb{A})$ for which Π_∞ belongs to the discrete series of weight (k_1, k_2). Let S be the set of places of ramification of Π. There exists a number field E so that for any prime number ℓ and any extension λ of ℓ to E, there is a continuous semi-simple Galois representation*

$$\rho_{\Pi,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\overline{E}_\lambda)$$

that is unramified away from $S_\ell := S \cup \{\ell\}$ and satisfies

$$L_p(X, \Pi, \mathrm{spin}) = \det\left(X1_4 - \rho_{\Pi,\lambda}(\mathrm{Frob}_p)\right)$$

for all $p \notin S_\ell$ where $L_p(s, \Pi, \mathrm{spin})$ denotes the p-th Euler factor of the spinor L-function of Π.

While these Galois representations are not always irreducible when attached to a cusp form as is in the GL(2)-case, one does have a characterization of when they are reducible. We deal with the first case in this and the subsequent section. Namely, one has the following theorem.

Theorem 2. *(See [21, Theorem II].) The Galois representation $\rho_{\Pi,\lambda}$ has a one-dimensional invariant subspace if and only if Π is a CAP representation of Saito–Kurokawa type.*

For a detailed discussion of automorphic representations of “Saito–Kurokawa type” one can see [11,13,14]. One knows that if a holomorphic cuspidal Siegel modular F arises as a CAP form, i.e., the automorphic form associated to F generates a CAP automorphic representation, then it must be CAP with respect to the Siegel parabolic subgroup of GSp(4). These CAP forms are characterized by Piatetski-Shapiro in [11]. He shows that such representations arise either as the representations generated by theta lifts from the metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbb{A})$ of $\mathrm{SL}_2(\mathbb{A})$ or as a twist of such a representation by a one-dimensional representation of $\mathrm{GSp}_4(\mathbb{A})$. Let $F \in S_k(\Gamma_0^{(2)}(M), \chi)$ be a CAP form and suppose that it arises as a theta lift. This case corresponds to the classical Saito–Kurokawa lifting. As we will work classically, we now recall the Saito–Kurokawa lifting in this setting in more detail. The details of the Saito–Kurokawa lift can be found in [7,9,16]. For an explicit combination and discussion of the results from these papers which are relevant to our setting, the reader is referred to [1, §6]. We present a slightly more general result below to allow for non-trivial characters.

We now fix k to be an even integer and χ an even character. Let $J_{k,1}^{\mathrm{cusp}}(M, \chi)$ denote the space of Jacobi cusp forms of weight k, index 1, level M, and character χ. There is a linear Hecke-equivariant map from $S_{2k-2}(\Gamma_0(M), \chi^2)$ into $J_{k,1}^{\mathrm{cusp}}(M, \chi)$. The precise definition of this map will not be important for our purposes, but one can consult [1, §6] for the details.

Let $f_1, \ldots, f_r$ be a basis of eigenforms of $S_{2k-2}(\Gamma_0(M), \chi^2)$ away from M. For each i, set

$$J_{k,1}^{\mathrm{cusp}}(M, \chi; f_i) = \left\{\phi \in J_{k,1}^{\mathrm{cusp}}(M, \chi)\colon\ T_J(p)\phi = \lambda_{f_i}(p)\phi,\ p \nmid M\right\}$$

where $T_J(p)$ denotes the pth Hecke operator on the space of Jacobi forms. We know that $J_{k,1}^{\mathrm{cusp}}(M, \chi; f_i)$ has dimension at least one for each i, and in general the dimension can be strictly greater than 1.

It remains to connect the Jacobi forms to Siegel modular forms. Let V_m denote index raising operator

$$V_m : J_{k,1}^{\mathrm{cusp}}(M, \chi) \to J_{k,m}^{\mathrm{cusp}}(M, \chi).$$

One can see [7] for the precise definition as it will not be needed here.

Given $\phi \in J_{k,1}^{\mathrm{cusp}}(M, \chi)$, define

$$F_\phi\big(\tau, z, \tau'\big) = \sum_{m \geqslant 0} V_m\phi(\tau, z)e\big(m\tau'\big).$$

It is known that $F_\phi \in S_k(\Gamma_0^{(2)}(M), \chi)$ and that the map $\phi \mapsto F_\phi$ gives an injective linear map of $J_{k,1}^{\mathrm{cusp}}(M, \chi)$ into $S_k(\Gamma_0^{(2)}(M), \chi)$ for nearly all cases; the only unknown case is dealt with below. In particular, details of this map for full level and trivial character is given in [5] and the more general case of level $\Gamma_0^{(2)}(M)$ can be found in [7]. However, the setting of non-trivial character and arbitrary level is not dealt with in any reference we know of and so we give a short proof of the result in this case. One should note the Saito–Kurokawa correspondence for non-trivial character and square-free level is given in [10] using representation theoretic arguments that do not directly translate to this setting.

Lemma 3. *Let $\phi(\tau, z) \in J_{k,1}^{\mathrm{cusp}}(M, \chi)$. Then $F_\phi \in S_k(\Gamma_0^{(2)}(M), \chi)$.*

Proof. First, we note that the generators of $\Gamma_0^{(2)}(M)$ are given by the following matrices,

$$X := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Y := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad Z := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M)$ and $(\lambda, \mu) \in \mathbb{Z}^2$. We need only check that F transforms appropriately with respect to these matrices to see the image is in $S_k(\Gamma_0^{(2)}(M), \chi)$.

The action of each matrix on (τ, z, τ') is given by:

$$X : (\tau, z, \tau') \mapsto \left(\frac{a\tau + b}{c\tau + d}, \frac{z^2}{c\tau + d}, \tau' - \frac{cz^2}{c\tau + d}\right),$$
$$Y : (\tau, z, \tau') \mapsto (\tau, z + \lambda\tau + \mu, \tau' + 2\lambda z + \lambda^2\tau),$$
$$Z : (\tau, z, \tau') \mapsto (\tau', z, \tau).$$

Given the action of X and Y, checking the transformation property with respect to these two matrices is a routine calculation. To see that F transforms appropriately with respect to Z we first write the Fourier expansion of F as

$$F(\tau, z, \tau') = \sum_{\substack{n,m,t \in \mathbb{Z} \\ n,m,4nm - t^2 \geqslant 0}} A(n, t, m) e(n\tau) e(tz) e(m\tau').$$

From [5, Theorem 4.2] we have that each Fourier coefficient $A(n, t, m)$ is symmetric in n and m. Hence,

$$\begin{aligned} F(\tau', z, \tau) &= \sum_{\substack{n,m,t \in \mathbb{Z} \\ n,m,4nm - t^2 \geqslant 0}} A(n, t, m) e(n\tau) e(tz) e(m\tau') \\ &= \sum_{\substack{n,m,t \in \mathbb{Z} \\ n,m,4nm - t^2 \geqslant 0}} A(m, t, n) e(m\tau) e(tz) e(n\tau') \\ &= F(\tau, z, \tau'). \end{aligned}$$

Since, k and χ are even we see that,

$$Z : F(\tau, z, \tau') \mapsto F(\tau, z, \tau') = \chi(-1) F(\tau, z, \tau').$$

Therefore, F transforms appropriately with respect to X, Y, Z, which completes the proof. $\square$

It now follows as in the known cases that the map $\phi \mapsto F_\phi$ gives an injective linear map from $J_{k,1}^{\text{cusp}}(M, \chi)$ to $S_k(\Gamma_0^{(2)}(M), \chi)$. In fact, this map is Hecke-equivariant in that

$$\lambda_{F_\phi}(p) = \lambda_\phi(p) + \chi(p) p^{k-2}(p + 1),$$
$$\lambda_{F_\phi}(p^2) = \chi(p)^2 p^{2k-6}(p^2 - 1) + \chi(p) \lambda_\phi(p) p^{k-3}(p + 1)$$

for all $p \nmid M$.

Recalling the basis of eigenforms $f_1, \ldots, f_r$ of $S_{2k-2}(\Gamma_0(M), \chi^2)$ given above, we define

$$S_k^{\text{SK}}(\Gamma_0^{(2)}(M), \chi; f_i) = \{F_\phi \colon \phi \in J_{k,1}^{\text{cusp}}(M, \chi; f_i)\}$$

and

$$S_k^{\mathrm{SK}}\big(\Gamma_0^{(2)}(M), \chi\big) = \bigoplus_{i=1}^{r} S_k^{\mathrm{SK}}\big(\Gamma_0^{(2)}(M), \chi; f_i\big).$$

The Maass Spezialschar $S_k^*(\Gamma_0^{(2)}(M), \chi)$ is a subspace of $S_k(\Gamma_0^{(2)}(M), \chi)$ consisting of those F whose Fourier coefficients satisfy

$$A(n,t,m) = \sum_{\substack{d \mid \gcd(n,t,m) \\ \gcd(d,M)=1}} d^{k-1} A\left(\frac{nm}{d^2}, \frac{t}{d}, 1\right).$$

Note that in general we have $S_k^{\mathrm{SK}}(\Gamma_0^{(2)}(M), \chi) \subset S_k^*(\Gamma_0^{(2)}(M), \chi)$. In the case of full level this is known to be an equality, but in the more general case we only know containment. We will not use the space of Maass Spezialschar in this paper, so we do not discuss it further here.

In summary, we have that if $f \in S_{2k-2}(\Gamma_0(M), \chi^2)$ is a normalized eigenform, then there exists (at least one) $F_f \in S_k^{\mathrm{SK}}(\Gamma_0^{(2)}(M), \chi)$ so that

$$\lambda_{F_f}(p) = \lambda_f(p) + \chi(p)p^{k-2}(p+1),$$
$$\lambda_{F_f}\big(p^2\big) = \chi(p)^2 p^{2k-6}\big(p^2-1\big) + \chi(p)\lambda_f(p)p^{k-3}(p+1)$$

for all $p \nmid M$.

3. Level stripping for Saito–Kurokawa forms

In this section we show that given a Saito–Kurokawa lift

$$F_f \in S_k^{\mathrm{SK}}\big(\Gamma_0^{(2)}\big(N\ell^\alpha\big), \chi\big),$$

we can strip the prime ℓ from the level of F_f as long as χ satisfies a minor technical condition given below. In particular, the case that F_f has trivial character will be a special case of our result. In fact, we show that there is a k', a Dirichlet character ψ, and a normalized eigenform $g \in S_{2k'-2}(\Gamma_0(N), \psi^2)$ so that $\lambda_{F_f}(p) \equiv \lambda_{F_g}(p) \pmod{\upsilon}$ and $\lambda_{F_f}(p^2) \equiv \lambda_{F_g}(p^2) \pmod{\upsilon}$ for all $p \nmid M$.

Using that χ is a Dirichlet character modulo M and that $(\mathbb{Z}/M\mathbb{Z})^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \times (\mathbb{Z}/\ell^\alpha\mathbb{Z})^\times$, we see that we can decompose χ as $\epsilon\tau$ where ϵ has conductor dividing N and τ has conductor dividing ℓ^α. We now use that the order of $(\mathbb{Z}/\ell^\alpha\mathbb{Z})^\times$ is $\ell^{\alpha-1}(\ell-1)$ to decompose τ into $\eta\omega^i$ where η has conductor a power of ℓ and has order a power of ℓ, ω is the Teichmüller character of conductor ℓ and order $\ell-1$, and i is an integer modulo ℓ. In light of this, we see that if $i=0$ then χ is trivial on the tame inertia at ℓ. In this case we say that χ has trivial tame ramification. We require that our χ satisfy this condition for the level stripping result. We explain exactly where this is needed below.

We now state the result that will form the basis of our proof.

Theorem 4. *(See [12, Theorem 2.1].) Let $M = \ell^\alpha N$ where $\ell \nmid N$. Let $f \in S_k(\Gamma_0(M), \varphi)$ be a normalized eigenform. Then there exists a normalized eigenform $g \in S_{k'}(\Gamma_0(N), \varphi')$ for some integer k' such that $\lambda_f(p) \equiv \lambda_g(p) \pmod{\upsilon}$ for all $p \nmid N\ell$.*

Before we can apply this theorem to our $f \in S_{2k-2}(\Gamma_0(M), \chi^2)$, we must verify that the weight k' of g is even and that we can express the character φ' as ψ^2 for some Dirichlet character ψ modulo N.

The fact that φ' can be expressed as ψ^2 can be found in [4]. In fact, it is shown there that $\psi(p) \equiv \chi(p) \pmod{\upsilon}$ for all $p \nmid N\ell$.

To see that the weight k' is even we will need to make a minor restriction which becomes clear after working through the proof of Theorem 4 and keeping track of the weight at each step. We factor our character χ as above. Following the proof of Theorem 4 we have the following expression for the weight of g:

$$k' = \left\{ \begin{array}{ll} \ell^j(\ell-1) + \ell^\alpha(2k-2+\ell-1+i) & \text{if } \ell > 3, \\ \ell^j(\ell+1) + \ell^\alpha(2k-1+\ell+1+i) & \text{if } \ell = 3 \end{array} \right\},$$

where j is a sufficiently large integer. One can see [8] for a detailed rewrite of the proof of [12, Theorem 2.1] keeping track of the weights. Note, in both cases we have that k' is even if i is even and so we can take the Saito–Kurokawa lift of g in this situation. By doing so we obtain a Siegel eigenform $F_g \in S_m^{\mathrm{SK}}(\Gamma_0^{(2)}(N), \psi)$ where,

$$m := \left\{ \begin{array}{ll} \ell^j(\frac{\ell-1}{2}) + \ell^\alpha(k-1+\frac{\ell-1+i}{2}) + 1 & \text{if } \ell > 3, \\ \ell^j(\frac{\ell+1}{2}) + \ell^\alpha(k-1+\frac{\ell+1+i}{2}) + 1 & \text{if } \ell = 3 \end{array} \right\}.$$

As F_g is the Saito–Kurokawa lift of g we have the following relationships between the eigenvalues of g and G,

$$\lambda_{F_g}(p) = \lambda_g(p) + \psi(p)p^{m-2}(p+1), \tag{1}$$

$$\lambda_{F_g}\big(p^2\big) = \psi(p)^2 p^{2m-6}\big(p^2-1\big) + \psi(p)\lambda_g(p)p^{m-3}(p+1), \tag{2}$$

where $p \nmid N\ell$ is a prime. For what follows we will need to further restrict to the case when $i = 0$, i.e., when χ has trivial tame ramification. Combining Eq. (1), Theorem 4, and making heavy use of the fact that $p^\ell \equiv p \pmod{\ell}$, we have the following for $\ell > 3$ and primes $p \nmid N\ell$:

$$\begin{aligned} \lambda_{F_g}(p) &= \lambda_g(p) + \psi(p)p^{m-2}(p+1) \\ &= \lambda_g(p) + \psi(p)p^{\ell^j(\frac{\ell-1}{2})+\ell^\alpha(k-1+\frac{\ell-1}{2})-1}(p+1) \\ &\equiv \lambda_g(p) + \psi(p)p^{\frac{\ell-1}{2}+k+\frac{\ell-1}{2}-2}(p+1) \pmod{\upsilon} \\ &\equiv \lambda_f(p) + \chi(p)p^{k-2}(p+1) \pmod{\upsilon} \\ &= \lambda_{F_f}(p). \end{aligned}$$

Note that we require $i = 0$ in order to deduce the first congruence given. From Eq. (2) we have

$$\begin{aligned} \lambda_{F_g}\big(p^2\big) &= \psi(p)^2 p^{2m-6}\big(p^2-1\big) + \psi(p)\lambda_g(p)p^{m-3}(p+1) \\ &= \psi(p)^2 p^{2\ell^j(\frac{\ell-1}{2})+2\ell^\alpha(k-1+\frac{\ell-1}{2})-4}\big(p^2-1\big) \\ &\quad + \psi(p)\lambda_g(p)p^{\ell^j(\frac{\ell-1}{2})+\ell^\alpha(k-1+\frac{\ell-1}{2})-2}(p+1). \end{aligned}$$

Reducing modulo ν we obtain

$$\begin{aligned} \lambda_{F_g}\big(p^2\big) &\equiv \psi(p)^2 p^{2(\ell-1)+2k-6}\big(p^2-1\big) + \psi(p)\lambda_g(p)p^{\ell-1+k-3}(p+1) \pmod{\upsilon} \\ &\equiv \chi(p)^2 p^{2k-6}\big(p^2-1\big) + \chi(p)\lambda_f(p)p^{k-3}(p+1) \pmod{\upsilon} \\ &= \lambda_{F_f}\big(p^2\big). \end{aligned}$$

Note, a similar argument works for $\ell = 3$. Thus, we have shown the following theorem.

Theorem 5. *Let $F \in S_k^{\mathrm{SK}}(\Gamma_0^{(2)}(N\ell^\alpha), \chi)$ be a Siegel eigenform, with k even, α a positive integer, $\ell \nmid N$ an odd prime, and χ an even Dirichlet character modulo $N\ell^\alpha$ having trivial tamely ramified part. Then there exists a Siegel eigenform $G \in S_{k'}^{\mathrm{SK}}(\Gamma_0^{(2)}(N), \psi)$, with k' a positive even integer and ψ an even Dirichlet character modulo N such that,*

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\upsilon},$$
$$\lambda_F\left(p^2\right) \equiv \lambda_G\left(p^2\right) \pmod{\upsilon},$$

for every prime $p \nmid N\ell$.

If we restrict to the case that F is a CAP form arising from a theta lift with trivial central character, Theorem 5 gives a complete solution to the problem of stripping odd primes from the level. In the case of F giving rise to a general CAP form it remains to deal with the case that F is a CAP form arising from the twist of a theta lift. Unfortunately, the situation in this case is not so clear. Suppose that the automorphic form associated to F generates an automorphic representation of the desired form, namely, Π_F is given by $\sigma \times \pi$ where π is a theta lift. The obvious thing to try would be to strip the desired prime from the level of the Saito–Kurokawa form generating π using Theorem 5. Let π' denote the automorphic representation generated by the form we obtain upon applying Theorem 5. One would then naturally consider $\sigma \times \pi'$. However, it is not known when such a representation gives rise to a holomorphic cuspidal Siegel modular form of the desired level $\Gamma_0^{(2)}(N)$. For instance, in the case of level $\mathrm{Sp}_4(\mathbb{Z})$ or $\Gamma_0^{(2)}(M)$ with M square-free, it is known that there are no holomorphic cuspidal Siegel eigenforms F generating a representation of the form $\sigma \times \pi$. In the case that M is not square-free, the representation theory necessary to determine when such forms exist is not currently known. The situation of a non-trivial character is also unknown. Thus, with the current state of knowledge we can say that Theorem 5 completely solves the problem for general CAP forms only in the case of trivial character and square-free level.

4. Level stripping for weak endoscopic lifts

The previous two sections dealt with the case that $\rho_{F,\lambda}$ has a one-dimensional invariant subspace. If $\rho_{F,\lambda}$ is reducible the only other possibility is that $\rho_{F,\lambda}$ has no one-dimensional invariant subspace, but has a two-dimensional invariant subspace. In this case Π_F is a weak endoscopic lift (see below for a definition). One can see [17, Theoreme 3.2.1] for a proof of the facts about the reducibility of $\rho_{F,\lambda}$.

To this point we have restricted ourselves to scalar-valued Siegel modular forms. However, to deal with endoscopic lifts we require vector-valued Siegel modular forms. For a more thorough introduction to vector-valued Siegel modular forms one can consult [3] or [18]. Let ρ be a finite dimensional representation of $\mathrm{GL}_2(\mathbb{C})$ with representation space V_ρ. A holomorphic function $F : \mathfrak{h}^2 \to V_\rho$ is called a V_ρ-valued Siegel modular form of weight ρ and level $\Gamma_0^{(2)}(M)$ if $F(\gamma Z) = \rho(CZ + D)F(Z)$ for all $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0^{(2)}(M)$. We denote the space of such forms by $M_\rho(\Gamma_0^{(2)}(M), V_\rho)$. As in the scalar-valued case, there is a Siegel operator that one can use to define the subspace of cusp forms, denoted $S_\rho(\Gamma_0^{(2)}(M), V_\rho)$. We now give the ρ that are of interest to us.

Let x, y be indeterminates, $j \in \mathbb{Z}$ with $j \geqslant 0$, and set $V = \mathbb{C}x \oplus \mathbb{C}y$. We can identify $\mathrm{Sym}^j(V)$, the j-th symmetric tensor product, with the space of homogeneous polynomials of degree j in $\mathbb{C}[x, y]$. Let $k_1, k_2 \in \mathbb{Z}$ with $k_1 \geqslant k_2 \geqslant 3$. We define a representation $\rho_{k_1,k_2} = \mathrm{Sym}^{k_1-k_2} \otimes \det^{k_2}$ of $\mathrm{GL}_2(\mathbb{C})$ on $\mathrm{Sym}^{k_1-k_2}(V)$ by setting

$$\rho_{k_1,k_2}(g)\bigl(v(x, y)\bigr) = \bigl(\det(g)^{k_2}\bigr)v\bigl((x, y)g\bigr)$$

for $g \in \mathrm{GL}_2(\mathbb{C})$ and $v(x, y) \in \mathrm{Sym}^{k_1-k_2}(V)$. Note here we have $V_{\rho_{k_1,k_2}} = \mathrm{Sym}^{k_1-k_2}(V)$ where $V = \mathbb{C}x \oplus \mathbb{C}y$. One knows that a vector-valued Siegel cusp form F of weight ρ_{k_1,k_2} corresponds to an irreducible cuspidal automorphic representation Π_F of $\mathrm{GSp}_4(\mathbb{A})$ whose archimedean component belongs to the discrete series of weight (k_1, k_2) and whose non-archimedean components are determined by the level of F. We generally write F has weight (k_1, k_2) to shorten the notation. Note that if $k = k_1 = k_2$, then $\rho_{k,k} = \det^k$, $V_{\rho_{k,k}} = \mathbb{C}$ and we recover the classical scalar-valued Siegel modular forms.

We say a unitary cuspidal automorphic representation Π of $\mathrm{GSp}_4(\mathbb{A})$ is a weak endoscopic lift if there are unitary irreducible cuspidal automorphic representations π_1 and π_2 of $\mathrm{GL}_2(\mathbb{A})$ with central characters $\omega_{\pi_1} = \omega_{\pi_2}$ so that

$$L_p(s, \Pi, \mathrm{spin}) = L_p(s, \pi_1) L_p(s, \pi_2) \tag{3}$$

for almost all places p. We will call a Siegel modular form F a weak endoscopic lift if Π_F is a weak endoscopic lift.

Let Π be a weak endoscopic lift so that Π_∞ belongs to the discrete series of weight (k_1, k_2). Let π_1 and π_2 be the unitary irreducible cuspidal automorphic representations giving rise to Π. Then $\pi_{i,\infty}$ belongs to the discrete series of weight r_i where after a suitable ordering we have $r_1 = k_1 + k_2 - 2$ and $r_2 = k_1 - k_2 + 2$. One also has the converse statement if one begins with two unitary irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$, namely, one can associate a weak endoscopic lift to π_1 and π_2 with the weights given as above. One can see [21] for a statement of these facts and [19,20] for proofs. One should note here that if F is a weak endoscopic lift and F is a scalar-valued Siegel modular form, then F must have weight (k, k) and so the eigenforms f_1, f_2 associated to π_1, π_2 must have weights given by $r_1 = 2k - 2$ and $r_2 = 2$.

As we again want to work classically for our level stripping, we begin by translating equation (3) into a statement about the L-functions associated to the elliptic modular forms f_1 and f_2 along with the spinor L-function associated to the vector-valued Siegel modular form. Let $f_i \in S_{r_i}(\Gamma_0(M), \tau)$ and π_{f_i} the associated automorphic representations. It is well known that the L-functions of f_i and π_{f_i} are related by

$$L_p(s, \pi_{f_i}) = L_p\big(s + (r_i - 1)/2, f_i\big)$$

for all $p \nmid M$. Let F be a vector-valued Siegel modular form of weight (k_1, k_2), level $\Gamma_0^{(2)}(M)$, and character χ with associated automorphic representation Π_F. The relation between the spinor L-functions of F and Π_F is given by

$$L_p(s, \Pi_F, \mathrm{spin}) = L_p\big(s + (k_1 + k_2 - 3)/2, F, \mathrm{spin}\big)$$

for all $p \nmid M$ where the p-th Euler factor of the spinor L-function of a vector-valued Siegel modular form is given by

$$\begin{aligned} L_p(s, F, \mathrm{spin}) = 1 - \lambda_F(p) p^{-s} + \big(\lambda_F(p)^2 - \lambda_F\big(p^2\big) - \chi\big(p^2\big) p^{\mu - 1}\big) p^{-2s} \\ - \chi\big(p^2\big) \lambda_F(p) p^{\mu - 3s} + \chi\big(p^4\big) p^{2\mu - 4s} \end{aligned}$$

where $\mu = k_1 + k_2 - 3$. One can immediately check this Euler factor reduces to the familiar one in the case that $k_1 = k_2 = k$. Combining all of this, if F is a weak endoscopic lift of weight (k_1, k_2), level $\Gamma_0^{(2)}(M)$, and character χ associated to $f_1 \in S_{r_1}(\Gamma_0(M), \tau)$ and $f_2 \in S_{r_2}(\Gamma_0(M), \tau)$ for $r_1 > r_2$, then the L-functions are related by

$$L_p(s, F, \mathrm{spin}) = L_p(s, f_1) L_p\big(s + (r_2 - r_1)/2, f_2\big),$$

for all $p \nmid N\ell$ and we must have $\tau = \chi^2$. Using this factorization of the L-function for F we obtain the following relationships between the eigenvalues,

$$\lambda_F(p) = \lambda_{f_1}(p) + p^{\frac{r_1-r_2}{2}}\lambda_{f_2}(p),$$
$$\lambda_F\bigl(p^2\bigr) = \lambda_{f_1}(p)^2 + p^{r_1-r_2}\lambda_{f_2}(p)^2 + p^{\frac{r_1-r_2}{2}}\lambda_{f_1}(p)\lambda_{f_2}(p) - \chi\bigl(p^2\bigr)p^{r_1-2}(2p+1),$$

for every prime $p \nmid N\ell$.

We now return to the case of interest to us, namely, we assume $M = N\ell^\alpha$ with $\alpha \geqslant 0$ and $\ell \nmid N$. We apply Theorem 4 to f_1 and f_2 and obtain two eigenforms $g_1 \in S_{m_1}(\Gamma_0(N), \psi^2)$ and $g_2 \in S_{m_2}(\Gamma_0(N), \psi^2)$ such that the following congruences are satisfied for all primes $p \nmid N\ell$:

$$\lambda_{f_1}(p) \equiv \lambda_{g_1}(p) \pmod{\upsilon},$$
$$\lambda_{f_2}(p) \equiv \lambda_{g_2}(p) \pmod{\upsilon}.$$

Further, we have the following expressions for the weights m_1 and m_2,

$$m_1 := \left\{ \begin{array}{ll} \ell^{j_1}(\ell-1) + \ell^\alpha(r_1+\ell-1+i) & \text{if } \ell > 3, \\ \ell^{j_1}(\ell+1) + \ell^\alpha(r_1+\ell+1+i) & \text{if } \ell = 3 \end{array} \right\},$$
$$m_2 := \left\{ \begin{array}{ll} \ell^{j_2}(\ell-1) + \ell^\alpha(r_2+\ell-1+i) & \text{if } \ell > 3, \\ \ell^{j_2}(\ell+1) + \ell^\alpha(r_2+\ell+1+i) & \text{if } \ell = 3 \end{array} \right\}.$$

Note, it is again necessary to restrict to the case when $i = 0$. As j_1 and j_2 are both arbitrarily large, we are free to choose j_1 such that $m_1 \geqslant m_2$. Taking the endoscopic lift of g_1 and g_2, we obtain a Siegel eigenform G of weight (k_1', k_2'), level $\Gamma_0^{(2)}(N)$, and character ψ where $k_1' := \frac{m_1+m_2}{2}$ and $k_2' := \frac{m_1-m_2}{2} + 2$.

It only remains to show the eigenvalues of F are congruent to the eigenvalues of G for all $p \nmid N\ell$:

$$\begin{aligned} \lambda_G(p) &= \lambda_{g_1}(p) + p^{\frac{m_1-m_2}{2}}\lambda_{g_2}(p) \\ &\equiv \lambda_{f_1}(p) + p^{\frac{r_1-r_2}{2}}\lambda_{f_2}(p) \pmod{\upsilon} \\ &= \lambda_F(p), \end{aligned}$$
$$\begin{aligned} \lambda_G\bigl(p^2\bigr) &= \lambda_{g_1}(p)^2 + p^{m_1-m_2}\lambda_{g_2}(p)^2 + p^{\frac{m_1-m_2}{2}}\lambda_{g_1}(p)\lambda_{g_2}(p) - \psi\bigl(p^2\bigr)p^{m_1-2}(2p+1) \\ &\equiv \lambda_{f_1}(p)^2 + p^{r_1-r_2}\lambda_{f_2}(p)^2 + p^{\frac{r_1-r_2}{2}}\lambda_{f_1}(p)\lambda_{f_2}(p) - \chi\bigl(p^2\bigr)p^{r_1-2}(2p+1) \pmod{\upsilon} \\ &= \lambda_F\bigl(p^2\bigr). \end{aligned}$$

Note, a similar argument works for $\ell = 3$. Thus, we have shown the following theorem.

Theorem 6. *Let ℓ be an odd prime, and let N, α be positive integers such that $\ell \nmid N$. Let F be a weak endoscopic lift of weight (k_1, k_2), level $\Gamma_0^{(2)}(N\ell^\alpha)$, and character χ having trivial tame ramification. Then, there exists a weak endoscopic lift G of weight (k_1', k_2'), level $\Gamma_0^{(2)}(N)$, and character ψ, with k_1' and k_2' positive integers such that,*

$$\lambda_F(p) \equiv \lambda_G(p) \pmod{\upsilon},$$
$$\lambda_F\bigl(p^2\bigr) \equiv \lambda_G\bigl(p^2\bigr) \pmod{\upsilon},$$

for every prime $p \nmid N\ell$.

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