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Amanda Folsom

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# Perspectives on mock modular forms

Amanda Folsom

## Abstract

Mock modular forms have played many prominent roles in number theory and other areas of mathematics over the course of the last 15 years. While the term “mock modular form” was not formally defined in the literature until 2007, we now know in hindsight that evidence of this young subject appears much earlier, and that mock modular forms are intimately related to ordinary modular and Maass forms, and Ramanujan’s mock theta functions. In this expository article, we offer several different perspectives on mock modular forms – some of which are number theoretic and some of which are not – which together exhibit the strength and scope of their developing theory. They are: combinatorics,  $q$ -series and mock theta functions, mathematical physics, number theory, and Moonshine. We also describe some essential results of Bruinier and Funke, and Zagier, both of which have made tremendous impacts on the development of the theory of mock modular forms. We hope that this article is of interest to both number theorists and enthusiasts – to any reader who is interested in or curious about the history, development, and applications of the subject of mock modular forms, as well as some amount of the mathematical details that go along with them.

## 1 Preamble

Modular forms have played central and far-reaching roles in number theory and mathematics over the last two centuries. Their footprints can be seen in the elliptic functions of the early 1800s, and their prominence has only risen since then. The theory and importance of their younger descendants, mock modular forms, has evolved in many analogous ways. Among the many papers written in the subject of mock modular forms are three excellent articles which are accessible to the non-specialist, and which we recommend to the readers of this article: Duke’s 2014 article [54] in the *the Notices of the AMS*, Ono’s 2010 article [120] in the *Notices of the AMS*, and Zagier’s 2007 *Séminaire Bourbaki* article [138]. These articles highlight different aspects of the history, theory, and applications of mock modular forms, including discussions of Ramanujan’s mock theta functions from 1920. The general term mock modular form was not defined in the literature until 2007 [138], and we now know in hindsight that Ramanujan’s mock theta functions are among the oldest examples.

In this expository article, we begin by offering a few concrete examples of mock modular forms in Section 2, which can be viewed as companions to some familiar ordinary modular forms. We offer these examples as previews to the more formal definitions and basic properties given in Section 3. In all of the following sections, we offer several perspectives on mock modular forms from the standpoint of different areas in which mock modular forms

have played a role – some are areas within number theory, some are not. They are: combinatorics (Section 2),  $q$ -series and mock theta functions (Section 4), mathematical physics (Section 6), number theory (i.e., elliptic curves and traces of singular moduli, Section 7), and Moonshine (Section 8). We also devote two sections to describing some of the essential results of Bruinier and Funke (Section 3), and Zagier (Section 5), both of which have been highly influential, and have made tremendous impacts on the development of the theory of mock modular forms over the last 15 years.

These diverse perspectives exhibit the strength and scope of the theory of mock modular forms. In the remainder of this section, we give a glimpse of some of the results which illustrate these aspects of the theory of mock modular forms, and which are discussed in more detail in this article.

From the perspective of combinatorics (Section 2), we can now show that various combinatorial generating functions are also mock modular forms. Armed with mock modularity, we have been able to leverage new information about the underlying combinatorial functions. This is very much parallel to the well-known interplay between ordinary modular forms and combinatorics. For example, a long-standing conjecture of Andrews and Dragonette from the 1960s on ranks of integer partitions has been resolved using the theory of mock modular forms. The conjecture, now theorem (Theorem 2.1), gives a beautiful exact formula for a partition rank function analogous to the revered Hardy-Ramanujan-Rademacher exact formula for the partition function. We are also sometimes able to exploit mock modularity to establish combinatorial congruence (divisibility) properties, asymptotic properties, and more. Conversely, we can also sometimes use combinatorial theory to shed new light on mock modular forms.

From the perspective of the mock theta functions (Section 4), the theory of mock modular forms has played a monumental role. Since the time of their inception in 1920, it was not well-understood how exactly Ramanujan’s curious “modular-like” mock theta functions fit into the theory of modular forms. Almost a century later, marking a major breakthrough, this question was resolved using the theory of mock modular forms (see Theorem 4.3). Moreover, this new realization surrounding the mock theta functions played a key role in catapulting the development of the theory of mock modular forms and the overarching theory of harmonic Maass forms (see Section 3 and Section 5).

From the perspective of mathematical physics (Section 6), mock modular forms have also played prominent roles in recent years. For example, the surrounding theory has led to a proof of a conjecture of Vafa and Witten in support of  $S$ -duality. The conjecture, now theorem (Theorem 6.1), rests in the setting of topologically twisted gauge theory, and establishes the mock modularity of associated Euler number generating functions. Exploiting this mock modularity subsequently led to exact analytic Hardy-Ramanujan-Rademacher-like formulas for the Euler numbers. Also within mathematical physics, the theory of mock modular forms has led to proofs of conjectures of Moore and Witten on Donaldson invariants (Conjecture 6.2), certain correlation functions for a supersymmetric topological gauge theory. In another direction, within string theory, the theory of mock modular forms has shed light on the quantum theory of black holes. In particular, we now understand the degeneracies of single centered black holes as Fourier coefficients of a mixed mock Jacobi form (see Section 6.3). While strongly motivated from a physical standpoint, this has also led to further developments in the theory of mock modular (Jacobi) forms.

From the perspective of number theory (Section 7), there are numerous applications of the theory of mock modular forms. For example, we now have a mock modular extension of Zagier’s foundational work relating traces of singular moduli arising from quadratic forms with negative discriminants to Fourier coefficients of modular forms. Theorem 7.1 beautifully completes the picture to include positive discriminants, illuminating the roles played by mock modular forms and cycle integrals, and also the interplay between mock modular forms and Zagier’s original modular theory. Also within number theory, we now see how elliptic curves may be used to construct mock modular forms (Theorem 7.2), and how associated mock modular Fourier coefficients encode information about the vanishing of  $L$ -values and derivatives (Theorem 7.3). By known results towards the Birch and Swinnerton-Dyer Conjecture, a major area of research in number theory, this gives results on the ranks of quadratic twist elliptic curves over  $\mathbb{Q}$ .

Finally, from the perspective of Moonshine (Section 8), mock modular forms have been used to establish, and prove, new Moonshine theories beyond the original monster group, the largest of the finite sporadic simple groups. This is quite remarkable, as we now know decades after Borcherds’ celebrated resolution of Monstrous Moonshine, that coefficients of mock modular forms, as opposed to ordinary modular forms, are graded traces arising from other types of groups (see Conjecture 8.1, which is now a theorem). This connection between Moonshine and mock modular forms has since opened the door to further explorations on the algebraic side of the story as well.

We hope that this article is of interest to both number theorists and enthusiasts – to any reader who is interested in or curious about the history, development, and applications of the subject of mock modular forms, as well as some amount of the mathematical details that go along with them. The several perspectives offered in this article are by no means meant to be an exhaustive list. In addition to the articles [54, 120, 138] mentioned above, there are many other accessible articles in the subject which we recommend, including [9, 10, 83, 116, 119]. This article is also a more recent, more detailed, cousin to [67], and a less comprehensive and less detailed predecessor to [23].

## 2 Combinatorics and first examples

Modular forms often serve as “combinatorial” generating functions, in the sense that their Fourier coefficients may enumerate quantities of interest. For example, with  $q = e^{2\pi i\tau}$ ,  $\tau \in \mathbb{H} := \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ , we have

$$(2.1) \quad q^{\frac{1}{24}}\eta^{-1}(\tau) = \sum_{n=0}^{\infty} p(n)q^n, \quad E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \quad \theta^4(\tau) = \sum_{n=0}^{\infty} r_4(n)q^n.$$

On the left-hand sides of the equalities in (2.1) we see the weight  $1/2$  modular forms

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{and} \quad \theta(\tau) := \sum_{n=-\infty}^{\infty} q^{n^2},$$

and the weight  $2k$  modular Eisenstein series  $E_{2k}$ , where  $k$  is any integer at least equal to 2. On the other hand, on the right-hand sides in (2.1), we have the combinatorial functions

$p(n) := \#\{\text{partitions of } n\}$  (the partition function),  $\sigma_m(n) := \sum_{d|n, d>0} d^m$  (the  $m$ th power divisor function), and  $r_4(n)$ , which counts the number of representations of  $n$  as a sum of 4 squares. A *partition* of  $n$  is any non-increasing sequence of positive integers whose sum is  $n$  (e.g.  $\{3\}, \{2, 1\}, \{1, 1, 1\}$  are the three partitions of  $n = 3$ , so  $p(3) = 3$ ). See for example [11, 137] for further background on many of the things discussed in this section.

Using modular properties of functions like those in (2.1), one can often leverage information about the combinatorial functions which they encode. A notable example of this is found in the work of Hardy and Ramanujan, who developed the Circle Method in analytic number theory in order to determine the asymptotic behavior of the partition function  $p(n)$  as  $n \rightarrow \infty$ . Their work heavily relies upon the modular properties satisfied by Dedekind's  $\eta$ -function. Rademacher later extended the work of Hardy and Ramanujan and established the beautiful exact formula

$$(2.2) \quad p(n) = \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{m=1}^{\infty} \frac{A_m(n)}{m} I_{\frac{3}{2}} \left( \frac{\pi\sqrt{24n-1}}{6m} \right),$$

where  $I_{\frac{3}{2}}$  is the modified Bessel function of the 1st kind, and  $A_m$  is a Kloosterman-like exponential sum. It is remarkable that the infinite sum of complex numbers on the right-hand side of (2.2) converges to a positive integer, and doubly remarkable that this positive integer is a fundamental combinatorial function which counts integer partitions. Using another modular property, namely the finite dimensionality of relevant vector spaces of modular forms, leads to non-trivial combinatorial identities for divisor functions such as

$$2640 \sum_{r=1}^{n-1} \sigma_3(r)\sigma_9(n-r) = \sigma_{13}(n) - 11\sigma_9(n) + 10\sigma_3(n),$$

as well as a simple exact formula for  $r_4(n)$  as 8 multiplied by the sum of the positive divisors of  $n$  which are not divisible by 4. Conversely, combinatorial functions which appear as modular Fourier coefficients can sometimes be used to reveal new information about the modular forms which encode them. For example, counting partitions according to the sizes of their *Durfee squares* shows that the partition generating function in (2.1), and hence the reciprocal of the modular  $\eta$ -function (up to multiplication by  $q^{\frac{1}{24}}$ ), can be expressed as

$$(2.3) \quad \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n^2},$$

a *q-hypergeometric series*, also called a *basic hypergeometric series*, for which there is a rich theory in and of itself [14, 66, 80]. Here and throughout, the  $q$ -Pochhammer symbol is defined by  $(a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$ , for  $n \in \mathbb{N}$ , and  $(a; q)_0 := 1$ . For more on  $q$ -hypergeometric series and mock modular forms, see Sections 4-5.

In the remainder of this section, we give some concrete examples of functions which may be viewed as mock modular counterparts to the three combinatorial modular forms from (2.1). We offer these example functions before giving the formal definition of a mock modular form in Section 3, with the intention that these functions act as a preview to the formalisms established there. We begin with the holomorphic weight 2 Eisenstein series

$E_2$ , defined using (2.1) with  $k = 1$ . While  $E_2$  is not modular, it is well-known to have a non-holomorphic *completion* (a term which we discuss further in Section 4)

$$(2.4) \quad E_2^*(\tau) := E_2(\tau) - \frac{3}{\pi v},$$

where here and throughout we let  $\tau = u + iv \in \mathbb{H}$ , with  $u \in \mathbb{R}, v \in \mathbb{R}^+$ . The non-holomorphic function  $E_2^*$  satisfies the following weight 2 transformation properties under the generators of the modular group  $\mathrm{SL}_2(\mathbb{Z})$ :

$$E_2^*(\tau + 1) = E_2^*(\tau), \quad E_2^*(-1/\tau) = \tau^2 E_2^*(\tau).$$

The function  $E_2^*$  additionally satisfies a moderate growth condition, so its *holomorphic part*  $E_2$  may be viewed as a prototype of a *mock modular form*. That is, the holomorphic Eisenstein series  $E_2$  does not exhibit a proper transformation law with respect to  $\mathrm{SL}_2(\mathbb{Z})$ , but when a suitable *non-holomorphic part* (namely  $-3/\pi v$ ) is added to  $E_2(\tau)$ , the resulting non-holomorphic function transforms correctly. This situation is typical of a general mock modular form, as we formally explain in Section 3. As an aside, we remark that  $E_2$  (resp.  $E_2^*$ ) is also an example of a *quasimodular form* (resp. *almost holomorphic modular form*) (see [102] by Kaneko and Zagier).

Another foundational example of a mock modular form, which, like  $E_2$  was studied before the term mock modular form was defined in the literature, arises from the weight  $3/2$  modular form  $\theta^3(\tau)$ . If we define its Fourier coefficients by  $\theta^3(\tau) =: \sum_{n \geq 0} r_3(n)q^n$ , then it was proved by Gauss that

$$r_3(4n + 1) = 12H(16n + 4), \quad r_3(4n + 2) = 12H(16n + 8), \quad r_3(8n + 3) = 24H(8n + 3),$$

where the values  $H(n)$  are the Hurwitz class numbers for discriminants  $-n$ . Hurwitz class numbers count the number of equivalence classes of binary quadratic forms of a given discriminant, where each class  $\mathcal{C}$  is counted with multiplicity  $1/\mathrm{Aut}(\mathcal{C})$ ; they are also related to class numbers of rings of integers in imaginary quadratic fields. Zagier studied the generating function for the  $H(n)$  in [135], and his results can be reinterpreted using the more recent terminology of mock modular forms as saying that the following non-holomorphic completion of the generating function for Hurwitz class numbers

$$(2.5) \quad -\frac{1}{12} + \sum_{n=1}^{\infty} H(n)q^n + \frac{1}{4\sqrt{\pi}} \sum_{n=-\infty}^{\infty} n \Gamma\left(-\frac{1}{2}, 4\pi n^2 v\right) q^{-n^2} + \frac{1}{8\pi\sqrt{2v}}$$

is a weight  $\frac{3}{2}$  *harmonic Maass form of moderate growth* on  $\Gamma_0(4)$ . The non-holomorphic part in (2.5) is defined using the incomplete gamma function (initially defined for  $\alpha > 0$  and  $w \in \mathbb{C}$ , or  $\alpha \in \mathbb{C}$  and  $w \in \mathbb{H}$ , and which can be analytically continued),

$$(2.6) \quad \Gamma(\alpha, w) := \int_w^{\infty} e^{-t} t^{\alpha-1} dt.$$

The holomorphic part in (2.5), namely the generating function for Hurwitz class numbers (including the constant term), can be regarded as a mock modular form.<sup>1</sup>

<sup>1</sup>Throughout this article, for ease of notation, we may slightly abuse terminology and refer to a function as a modular form, mock modular form, harmonic Maass form, etc. when in fact the function must be multiplied by a power of  $q$  and/or renormalized by  $\tau \mapsto \kappa\tau$ , in order to exhibit suitable transformation properties.

Lastly, we consider a refined partition statistic called the *rank* of a partition, which was originally defined by Dyson [62] to be the largest part of the partition minus the number of parts of the partition (e.g. the partition  $\{3, 3, 3, 1\}$  of 10 has rank  $3 - 4 = -1$ ). Compared to integer partitions, which are a very natural construct, their ranks may seem artificial without further context. In fact, Dyson defined ranks in order to combinatorially explain two of Ramanujan's celebrated partition congruences,  $p(5n + 4) \equiv 0 \pmod{5}$  and  $p(7n + 5) \equiv 0 \pmod{7}$ , which hold for every non-negative integer  $n$ . That is, if we let  $N(m, t; n)$  count the number of partitions of  $n$  whose rank is equal to  $m \pmod{t}$ , Dyson conjectured, and Atkin and Swinnerton-Dyer proved [18], that for any fixed positive integer of the form  $5n + 4$  (resp.  $7n + 5$ ), each set  $N(m, 5; 5n + 4)$  (resp.  $N(m, 7; 7n + 5)$ ) is equal in size, where  $m$  runs through a set of representatives for the integers modulo 5 (resp. 7). A combinatorial argument using Durfee squares similar to the one used to establish (2.3) leads to the well-known identity for the two-variable generating function for the partition rank function  $N(m, n) := p(n \mid \text{rank } m)$

$$(2.7) \quad \mathcal{R}(\zeta; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m, n) \zeta^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(\zeta q; q)_n (\zeta^{-1} q; q)_n},$$

which reduces to the ordinary partition generating function when  $\zeta = 1$ . In light of (2.1), it is natural to ask about the modular properties, if any, of  $\mathcal{R}(\zeta; q)$  at other values of  $\zeta$ . Perhaps the next simplest case to consider is the case  $\zeta = -1$ , which gives

$$(2.8) \quad \mathcal{R}(-1; q) = \sum_{n=0}^{\infty} \alpha_f(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2} =: f(q),$$

where  $\alpha_f(n) := N(0, 2; n) - N(1, 2; n)$  is a difference of two rank functions. The  $q$ -hypergeometric series  $f(q)$  on the right-hand side of (2.8) is not recognizable as a modular form, as is the case (in (2.7)) when  $\zeta = 1$ . However,  $f(q)$  is one of Ramanujan's original *mock theta functions*, curious  $q$ -series defined in his last letter to Hardy (see Section 4). For the remainder of this paragraph we put the cart before the horse and elaborate a bit on the mock modular properties of  $f(q)$ , and more generally  $\mathcal{R}(\zeta; q)$ , to foreshadow some of the results in Sections 3-5. Thanks to work of Zwegers, to which we devote Section 5 and part of Section 4, we now realize that the combinatorial generating function in (2.8) is also a mock modular form, as are all of Ramanujan's mock theta functions, resolving the decades-long question on the "modularity" of the mock theta functions. After Zwegers, in their paper [34], Bringmann and Ono considered the more general question of understanding the modular properties of  $\mathcal{R}(\zeta; q)$  as a function of  $\tau$  (or  $q$ ) for fixed  $\zeta$ . Indeed, by adopting results and methods of Zwegers [142] and Gordon-McIntosh [82], they proved that when  $\zeta = e^{2\pi i \frac{a}{b}}$  is any fixed root of unity not equal to 1, Dyson's rank function can be completed to the following non-holomorphic modular form

$$(2.9) \quad q^{-\ell_b/24} \mathcal{R}(e^{2\pi i \frac{a}{b}}; q^{\ell_b}) + i(3^{-1} \ell_b)^{\frac{1}{2}} \sin\left(\frac{\pi a}{b}\right) \int_{-\bar{\tau}}^{i\infty} \frac{\Theta\left(\frac{a}{b}; \ell_b z\right)}{\sqrt{-i(z + \tau)}} dz.$$

Here,  $\Theta$  is a certain weight  $3/2$  ordinary modular theta function, and  $\ell_b \in \mathbb{N}$ . In particular, results from [34] show that the function in (2.9) is an example of a *harmonic Maass form* of weight  $1/2$  and level 144, and its holomorphic part  $q^{-\ell_b/24} \mathcal{R}(e^{2\pi i \frac{a}{b}}; q^{\ell_b})$ , essentially Dyson's

rank function, is among the first examples of a mock modular form (see also Section 3). Zagier [138] later simplified some of the results from [34] by re-writing the rank generating function using an identity of Gordon-McIntosh [82] as follows<sup>2</sup> (with  $q = e^{2\pi i\tau}$ ,  $\zeta = e^{2\pi iz}$ ,  $z \notin \mathbb{Z} + \mathbb{Z}\tau$ ):

$$(2.10) \quad \mathcal{R}(\zeta; q) = -2 \sin(\pi z) \left( \frac{q^{\frac{1}{24}} \eta(3\tau)^3}{\eta(\tau) \vartheta(3z; 3\tau)} - q^{-\frac{1}{8}} \zeta^{-1} \mu(3z, -\tau; 3\tau) + q^{-\frac{1}{8}} \zeta \mu(3z, \tau; 3\tau) \right).$$

Here, the Jacobi  $\vartheta$ -function, a two-variable function which specializes to a multiple of  $\theta$  above, is given in Definition 5.1, and Zwegers' function  $\mu$  is given in Definition 5.2. Upon suitable specialization of its parameters, the function  $\mu$  becomes a mock modular form, a fact we explain in Section 5. Zagier's expression (2.10), combined with transformation properties for  $\eta$ ,  $\vartheta$ , and  $\mu$ , can be used to establish another proof of the mock modularity of the two-variable partition rank generating function  $\mathcal{R}$  when  $\zeta$  is any root of unity not equal to 1. Garvan also studied Dyson's rank function in [79], in which he extends and strengthens some related results from [34, 35], and also [2] by Ahlgren and Treener, including the transformation properties for  $\mathcal{R}$ .

Using the mock modularity of  $\mathcal{R}(-1; q) = f(q)$ , Bringmann and Ono [33] proved a conjecture of Andrews and Dragonette [8], and established the following exact formula for the combinatorial coefficients  $\alpha_f(n) = N(0, 2; n) - N(1, 2; n)$  of  $f(q)$ , analogous to the Hardy-Ramanujan-Rademacher formula (2.2).

**Theorem 2.1.** *If  $n \in \mathbb{N}$ , then*

$$(2.11) \quad \alpha_f(n) = \pi(24n - 1)^{-\frac{1}{4}} \sum_{m=1}^{\infty} \frac{(-1)^{\lfloor \frac{m+1}{2} \rfloor} A_{2m} \left( n - \frac{m(1+(-1)^m)}{4} \right)}{m} I_{\frac{1}{2}} \left( \frac{\pi \sqrt{24n - 1}}{12m} \right).$$

In addition to  $f$ , other of Ramanujan's mock theta functions encode combinatorial data, for example, the coefficients of his mock theta function

$$(2.12) \quad \omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2}$$

count partitions whose summands, except the last, form pairs of consecutive non-negative integers (see [12] for details). Garthwaite established an exact formula similar to (2.2) and (2.11) for these combinatorial coefficients in [77]. We discuss the mock modular properties of Ramanujan's mock theta functions in general, including  $f$  and  $\omega$ , in Section 4, but first turn to the formal definitions of harmonic Maass forms and mock modular forms in Section 3. In addition to the references already mentioned in this section, see for example [1, 17, 15, 28, 29, 30, 31, 68, 78, 130] by Ahlgren, Andrews, Bringmann, Dixit, Garvan, Holroyd, Lovejoy, Mahlburg, Ono, Rhoades, Vlasenko, Waldherr, Yee, and Zwegers, for some additional recent results relating mock modular forms and combinatorial functions. In particular some of these works (and others mentioned in this section) establish divisibility properties (i.e. congruence properties) of combinatorial mock modular coefficients by exploiting the mock modularity of associated generating functions. This is another natural and important area of study at the interface of combinatorics and (mock) modular forms.

<sup>2</sup>In (2.10), we have corrected a minor typographical error in [138].

### 3 Definitions and basic properties

The previews of some mock modular forms given in the previous section show that they share various characteristics. The functions  $E_2$ , the generating function for Hurwitz class numbers, and Dyson's rank function  $\mathcal{R}(\zeta; q)$  are all holomorphic functions which fail to exhibit ordinary modular transformation properties, however the ways in which they fail to do so can be corrected. That is, they can be completed to form the non-holomorphic functions in (2.4), (2.5), and (2.9), each of which does transform like an ordinary modular form. As mentioned, each of these non-holomorphic modular forms satisfies the definition of a harmonic Maass form, however, the growth conditions in the cusps satisfied by (2.4) and (2.5) slightly differ from those satisfied by (2.9). As a result, these functions would be classified as slightly different types of harmonic Maass forms as we explain below. In this section, we formalize and generalize some of the observations from the previous section surrounding the three examples discussed there. In particular, we formally define harmonic Maass forms and mock modular forms, and explain some relationships to ordinary modular forms.

As in the previous section, we let  $\tau = u + iv \in \mathbb{H}$ , where  $u \in \mathbb{R}$ , and  $v \in \mathbb{R}^+$ , and let  $q = e^{2\pi i\tau}$ . In words, harmonic Maass forms are real-analytic functions which satisfy three properties: i) they transform like ordinary modular forms on a suitable subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ , ii) they are annihilated by a Laplacian operator, and iii) they satisfy certain growth conditions at cusps. These functions were originally defined by Bruinier and Funke in their 2004 work on theta lifts [42]. The operator which annihilates a harmonic Maass form (of weight  $k$ ) is the *weight  $k$  Laplacian operator*, defined by

$$(3.1) \quad \Delta_k := -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) = -4v^2 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \bar{\tau}} + 2ikv \frac{\partial}{\partial \bar{\tau}}.$$

In what follows, we will consider forms for which  $k \in \frac{1}{2}\mathbb{Z}$ . As is the case with ordinary modular forms, the transformation laws satisfied by harmonic Maass forms differ depending on whether or not  $k \in \frac{1}{2} + \mathbb{Z}$ . To introduce the formal definition of a harmonic Maass form, we let  $(\cdot)$  denote the Kronecker symbol,  $\sqrt{\cdot}$  denote the principal branch of the holomorphic square root, and for odd integers  $d$ , we let  $\varepsilon_d$  equal 1 or  $i$ , depending on whether  $d \equiv 1 \pmod{4}$  or  $d \equiv -1 \pmod{4}$ . Following Bruinier and Funke [42], we have the following definitions.

**Definition 3.1.** Let  $k \in \frac{1}{2}\mathbb{Z}$ , and let  $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$  be a subgroup of the full modular group, where  $\Gamma \subseteq \Gamma_0(4)$  if  $k \in \frac{1}{2} + \mathbb{Z}$ . A **weight  $k$  harmonic Maass form on  $\Gamma$**  is any smooth function  $M: \mathbb{H} \rightarrow \mathbb{C}$  satisfying the following properties.

- i) For all  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$  and all  $\tau \in \mathbb{H}$ , we have that

$$M\left(\frac{a\tau + b}{c\tau + d}\right) = \begin{cases} (c\tau + d)^k M(\tau) & \text{if } k \in \mathbb{Z}, \\ \left(\frac{c}{d}\right)^{2k} \varepsilon_d^{-2k} (c\tau + d)^k M(\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

- ii) We have that  $\Delta_k(M) = 0$ .

- iii) There exists a polynomial  $P_M(\tau) \in \mathbb{C}[q^{-1}]$ , called **the principal part of  $M$  at  $\infty$** , such that

$$M(\tau) - P_M(\tau) = O(e^{-\epsilon v})$$

as  $v \rightarrow \infty$  for some  $\epsilon > 0$ . Analogous conditions are required at all cusps.

We denote the space of all such forms by  $H_k(\Gamma)$ . One may also consider harmonic Maass forms on  $\Gamma_0(N)$  for some  $N \in \mathbb{N}$ , which transform with a (mod  $N$ ) Dirichlet character  $\chi$ . In this case, the right-hand side of the transformation law in part i) of Definition 3.1 is multiplied by  $\chi(d)$ .

If we replace the growth condition in part iii) of Definition 3.1 by  $M(\tau) = O(e^{\epsilon v})$  as  $v \rightarrow \infty$  for some  $\epsilon > 0$ , then we call  $M$  a *weight  $k$  harmonic Maass form of moderate growth on  $\Gamma$* . The non-holomorphic weight 2 Eisenstein series (2.4) and Zagier's non-holomorphic completion of the Hurwitz class number generating function (2.5) are examples of harmonic Maass forms of moderate growth, while the non-holomorphic completion of Dyson's rank function (2.9) is an example of a harmonic Maass form according to Definition 3.1.

The word "harmonic" is used to describe the Maass forms in Definition 3.1 due to the fact that they are annihilated by  $\Delta_k$ . We remark that it has also been a natural question in the literature to study eigenfunctions of  $\Delta_k$  which satisfy i) and iii) in Definition 3.1, but which have eigenvalues that are not necessarily equal to 0. Harmonic Maass forms are close relatives to the classical Maass forms introduced by Maass in 1949, which are weight 0 eigenfunctions of  $\Delta_0$ , and which have a less relaxed growth condition [45, 81, 100]. In general, it is a difficult problem to construct Maass cusp forms, rendering their study particularly interesting. Maass constructed such forms for some congruence subgroups, and for  $\mathrm{SL}_2(\mathbb{Z})$  in particular, Selberg proved existence. The problem of determining (non) existence of Maass cusp forms on non-congruence subgroups remains a major open problem today [123]. Like ordinary modular and Maass forms, harmonic Maass forms have Fourier expansions. The following lemma is established in [42].

**Lemma 3.2.** *If  $M \in H_k(\Gamma)$ , with  $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ , and  $\Gamma \in \{\Gamma_0(N), \Gamma_1(N)\}$  for some  $N \in \mathbb{N}$ , then  $M$  has Fourier expansion*

$$M(\tau) = \sum_{n=r_f}^{\infty} c_M^+(n)q^n + \sum_{n=-\infty}^{-1} c_M^-(n)\Gamma(1-k, 4\pi|n|v)q^n,$$

for some integer  $r_f$ .

Using Lemma 3.2, we see that the Fourier expansions of harmonic Maass forms split naturally into two parts. To this end, we have the following definitions.

**Definition 3.3.** Assume the notation and hypotheses as in Lemma 3.2. We define the **holomorphic part** of a harmonic Maass form  $M \in H_k(\Gamma)$  by

$$M^+(\tau) := \sum_{n=r_f}^{\infty} c_M^+(n)q^n,$$

and the **non-holomorphic part** of  $M$  by

$$M^-(\tau) := \sum_{n=-\infty}^{-1} c_M^-(n) \Gamma(1-k, 4\pi|n|v) q^n.$$

We remark that similar expansions as in Lemma 3.2 hold for forms of moderate growth, and in the case of weight 1. For more on weight  $k = 1$ , see the recent work of Duke and Li, which shows that Fourier coefficients of weight 1 harmonic Maass forms are related to complex Galois representations associated to weight 1 newforms. We also have analogous notions of holomorphic and non-holomorphic parts when in slightly different settings. For example,  $-\frac{1}{12} + \sum_{n=1}^{\infty} H(n)q^n$  is the holomorphic part of the weight  $3/2$  harmonic Maass form in (2.5), and the remaining sum involving the incomplete gamma function plus the term involving  $1/\sqrt{v}$  is its non-holomorphic part.

One immediate relationship between harmonic Maass forms and ordinary modular forms is the following. If  $M \in H_k(\Gamma)$  is a harmonic Maass form for which  $M^- = 0$ , then we have that  $M \in M_k^!(\Gamma)$ , the space of weight  $k$  weakly holomorphic modular forms on  $\Gamma$ . By *weakly holomorphic*, we mean holomorphic on  $\mathbb{H}$  with possible poles in the cusps. In other words, harmonic Maass forms with trivial non-holomorphic parts are weakly holomorphic modular forms. The reverse containment holds as well: a weakly holomorphic modular form is a harmonic Maass form with  $M^- = 0$ . A less immediate relationship between harmonic Maass forms and ordinary modular forms can be seen by applying the differential operator

$$\xi_k := 2iv^k \frac{\partial}{\partial \bar{\tau}}.$$

It is often useful to note that the Laplacian operator  $\Delta_k$  from (3.1) may be factored in terms of the  $\xi$ -operator as follows:

$$\Delta_k = -\xi_{2-k}\xi_k.$$

The following theorem due to Bruinier and Funke [42] shows how the  $\xi$ -operator maps harmonic Maass forms to cusps forms. Moreover, it reveals how the Fourier expansions of these ordinary modular forms in the image of  $\xi$  are built from the Fourier coefficients of the non-holomorphic parts of harmonic Maass forms. Below, we let  $S_k(\Gamma)$  denote the space of modular cusp forms of weight  $k$  on  $\Gamma$ .

**Theorem 3.4.** *Assume the notation and hypotheses as in Lemma 3.2, and let  $k \leq 0$ . We have that*

$$\xi_k : H_k(\Gamma_0(N)) \rightarrow S_{2-k}(\Gamma_0(N)).$$

*Explicitly, for  $M \in H_k(\Gamma_0(N))$ , we have that*

$$\xi_k(M(\tau)) = \xi_k(M^-(\tau)) = -(4\pi)^{1-k} \sum_{n=1}^{\infty} \overline{c_M^-(-n)} n^{1-k} q^n.$$

Slightly modifying Zagier's original definition, [138], the cusp forms in Theorem 3.4, as well as the holomorphic parts of harmonic Maass forms, have special names.

**Definition 3.5.**

- i) A **mock modular form** of weight  $k$  is the holomorphic part  $M^+$  of a harmonic Maass form of weight  $k$ .
- ii) If  $M \in H_k(\Gamma_0(N))$ , we refer to the cusp form  $\xi_k(M(\tau))$  in Theorem 3.4 as the **shadow** of the mock modular form  $M^+$ .

We point out that it is also natural to restrict the definition of a mock modular form to encompass harmonic Maass forms whose non-holomorphic parts are non-trivial. “Shadow” can also be attached to more general forms (see [23]).

The term *mock modular form* in Definition 3.5 reflects the fact that that Ramanujan’s mock theta functions (e.g.  $f(q)$  from Section 2) were among the first examples of mock modular forms, thanks to work of Zwegers [142]. In Sections 4-5, we discuss Ramanujan’s mock theta functions and Zwegers’ results in the context of harmonic Maass forms in more detail.

Having established the terminology in Definition 3.5, we state a last result in this section, which shows that the non-holomorphic parts of mock modular forms may be expressed in a different, perhaps simpler, way than as in Lemma 3.2. That is, the non-holomorphic part of a harmonic Maass form is a *period integral* of the (*complement* of the) corresponding mock modular form’s shadow.

**Lemma 3.6.** *Assume the notation and hypotheses as above. Let  $f \in H_k(\Gamma_0(N))$ , and suppose the mock modular form  $M^+$  has shadow  $g \in S_{2-k}(\Gamma_0(N))$ . Then the non-holomorphic part  $M^-$  satisfies*

$$M^-(\tau) = 2^{k-1}i \int_{-\bar{\tau}}^{i\infty} \frac{\overline{g(-\bar{w})}}{(-i(w + \tau))^k} dw.$$

We have already seen a non-holomorphic part of this shape in (2.9), and will see more in the following sections (e.g. Sections 4-6).

As a final remark, we mention that the kernel of the map in Theorem 3.4 is  $\ker(\xi_k) = M_k^!(\Gamma_0(N))$ . This shows that given a particular cusp form, there are in fact infinitely many harmonic Maass forms with that shadow. In general, it is of interest to construct a given lift of a cusp form which is in some way canonical (see Section 7 and references therein for more). In the next section, we discuss Ramanujan’s mock theta functions as mock modular forms.

## 4 Ramanujan’s mock theta functions

Ramanujan’s mock theta functions are a collection of 17  $q$ -hypergeometric series originally defined in Ramanujan’s last letter to Hardy from 1920 [19]; two such functions are the functions  $f$  and  $\omega$  defined in (2.8) and (2.12), respectively. All others may be found, for example, in [83], which provides a comprehensive treatment of numerous aspects of the mock theta functions and their generalizations. See also the works of Andrews [9, 10, 13], which have had a great impact on the study of mock theta functions.

Ramanujan called his functions “mock theta functions” due to the fact that they resemble ordinary modular forms in certain ways. In particular, he studied their asymptotic relationship to ordinary modular theta functions. Ramanujan did not phrase his characterization of mock theta functions exactly as in Definition 4.1 below, but he nearly did. In light of this, we attribute the following “definition” to him.

**Definition 4.1** (Ramanujan). A **mock theta function**  $m$  is a function defined on  $\mathbb{H}$  satisfying the following properties.

- i) There are infinitely many roots of unity  $\zeta$  for which  $m(\tau)$  grows exponentially as  $q = e^{2\pi i\tau}$  approaches  $\zeta$  radially from inside the unit disk.
- ii) For every root of unity  $\zeta$ , there exists a (weakly holomorphic) modular form  $B_\zeta$  and a rational number  $\alpha_\zeta$  such that

$$m(\tau) - q^{\alpha_\zeta} B_\zeta(\tau)$$

is bounded as  $q \rightarrow \zeta$  radially from within the unit disk.

- iii) There does not exist a single (weakly holomorphic) modular form  $B$  that satisfies ii).

We compare this definition to the modern definition of a mock modular form from Section 3.1 below, but first illustrate Ramanujan’s definition with an explicit example given in his last letter to Hardy. Ramanujan’s example involves his mock theta function  $f$  from (2.8), and the function  $b$  he defined by

$$b(q) := (1 - q) (1 - q^2) (1 - q^3) (1 - q^5) \cdots (1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \cdots).$$

In more modern language, Ramanujan’s function  $b$  may be expressed as  $b(q) = q^{\frac{1}{24}} \eta^3(\tau) / \eta^2(2\tau)$  ( $q = e^{2\pi i\tau}$ ,  $\tau \in \mathbb{H}$ ), which is a weight  $1/2$  modular form up to the multiplicative factor  $q^{\frac{1}{24}}$ . It is not difficult to show that  $f$  converges at odd order roots of unity, and has singularities at even order roots of unity. Ramanujan claimed that the two modular forms  $\pm b$  cut out these singularities of the mock theta function  $f$ . That is, Ramanujan claimed that as  $q$  approaches an even order  $2k$  primitive root of unity  $\zeta$ , we have (see [19, 131])

$$(4.1) \quad \lim_{q \rightarrow \zeta} (f(q) - (-1)^k b(q)) = O(1).$$

Following Ramanujan’s death, it remained a question of interest to further understand Ramanujan’s claim, especially in recent years with the development of a theory of mock modular forms. Using some elements from the theory of mock modular forms, and also the theory of  $q$ -hypergeometric series, Ramanujan’s claim was refined, proved, and generalized in [69, 70]. In particular, an exact formula for the implied  $O(1)$  constants from Ramanujan’s claim in (4.1) is given in [69]. As shown in (4.2) these constants are explicit polynomials in  $\mathbb{Z}[\zeta]$ , which at first glance may appear to be rather peculiar. That is, from [69], we have that

$$(4.2) \quad \lim_{q \rightarrow \zeta} (f(q) - (-1)^k b(q)) = -4 \sum_{n=0}^{k-1} (1 + \zeta)^2 (1 + \zeta^2)^2 \cdots (1 + \zeta^n)^2 \zeta^{n+1}.$$

It turns out that these polynomials in  $\zeta$  on the right-hand side of (4.1) are special values of modular-like objects originally defined by Zagier called *quantum modular forms* [139]. Quantum modular forms exhibit a modular-like transformation law not on  $\mathbb{H}$ , but on  $\mathbb{Q}$ , or perhaps  $\mathbb{Q} \setminus S$  for some appropriate set  $S$ , up to the addition of suitably continuous or analytic error functions in  $\mathbb{R}$ . If one treats the right-hand side of (4.2) as a function of  $x \in \mathbb{Q}$  by replacing  $\zeta = e^{2\pi ix}$ , and extends the upper limit of summation from  $k - 1$  to  $\infty$ , the resulting function turns out to be an example of a quantum modular form [44, 139]. Upon replacing  $x = 1/2k$ , it is not difficult to show that the infinite sum becomes the finite one shown in (4.2).

It is interesting to see mock modular, ordinary modular, and quantum modular forms related by the single expression in (4.2). This result, it turns out, is not limited to the mock theta function  $f$ , and has been generalized to other mock modular forms, including the two-variable rank generating function  $\mathcal{R}$ , in [36, 69, 101]. We do not give a full treatment of the developing subject of quantum modular forms here, but note that there has been much recent progress in the area; in addition to the references mentioned above, the interested reader may also wish to consult, for example, [24, 26, 37, 94, 97, 105, 125, 140].

One sees via Ramanujan's Definition 4.1, and in (4.1) and (4.2), how mock theta functions asymptotically "resemble" modular theta functions. It is also natural to ask how the mock theta functions transform under  $\mathrm{SL}_2(\mathbb{Z})$ , if at all. Using Poisson summation, classically used to give modular transformations for ordinary theta functions, Watson [131] established the following modular-like transformation property relating  $f$  and  $\omega$

$$(4.3) \quad q^{-\frac{1}{24}} f(q) = 2\sqrt{\frac{2\pi}{\alpha-1}} q_1^{\frac{4}{3}} \omega(q_1^2) + 4\sqrt{\frac{3\alpha}{2\pi}} \int_0^\infty \frac{\sinh(\alpha t)}{\sinh\left(\frac{3\alpha t}{2}\right)} e^{-\frac{3\alpha t^2}{2}} dt,$$

where  $q := e^{-\alpha}$ ,  $\beta := \pi^2/\alpha$ ,  $q_1 := e^{-\beta}$  ( $\alpha, \beta \in \mathbb{C}$  with  $\mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0$ ). If we adopt more modern notation and let  $\alpha = -2\pi i\tau$  for some  $\tau \in \mathbb{H}$ , then (4.3) shows a modular-like transformation under  $\tau \mapsto -1/(2\tau)$ . The relationship shown in (4.3) suggests a vector-valued transformation, up to the error integral involving  $\sinh$ . Decades later, Zwegers [141] made a major breakthrough in understanding the modularity of the mock theta functions. For the mock theta functions  $f$  and  $\omega$  in particular, Zwegers packaged them together in a vector, and nicely completed this vector by subtracting from it a suitable non-holomorphic vector-valued function. This non-holomorphic subtraction compensates for the error to modularity suggested by (4.3).

Before stating Zwegers' result, we digress for a moment to discuss the term *completion*, which we have mentioned a number of times in the narrative thus far. The term can be explicitly defined with respect to a given example, but is more implicitly described in general. Completed functions, in the sense we have been using the term here, typically result from adding another (non-holomorphic) function to a starting function so that the resulting sum transforms appropriately on a subgroup of the modular group; moreover, this should be done in a non-trivial way. In general, the function added should perhaps be simpler in some way than the starting function. For example, the completion of  $E_2(\tau)$  in (2.4) results from adding the simple function  $-3/(\pi v)$ , and the completions of the generating function for Hurwitz class numbers in (2.5), and Dyson's rank function in (2.9), result from adding period integrals of ordinary modular forms, which we understand the modular properties of very well.

Returning to the description of Zwegers' work, define the vector-valued function  $F$  by

$$F(\tau) = (F_0(\tau), F_1(\tau), F_2(\tau))^T := \left( q^{-\frac{1}{24}} f(q), 2q^{\frac{1}{3}} \omega \left( q^{\frac{1}{2}} \right), 2q^{\frac{1}{3}} \omega \left( -q^{\frac{1}{2}} \right) \right)^T$$

(where  $\mathbf{v}^T$  denotes the transpose of a vector  $\mathbf{v}$ ), and the vector-valued non-holomorphic function  $G$  by

(4.4)

$$G(\tau) := 2i\sqrt{3} \left( \int_{-\bar{\tau}}^{i\infty} \frac{\mathcal{G}_1(w)}{\sqrt{-i(w+\tau)}} dw, \int_{-\bar{\tau}}^{i\infty} \frac{\mathcal{G}_0(w)}{\sqrt{-i(w+\tau)}} dw, \int_{-\bar{\tau}}^{i\infty} \frac{-\mathcal{G}_2(w)}{\sqrt{-i(w+\tau)}} dw \right)^T.$$

Here, the functions  $\mathcal{G}_0, \mathcal{G}_1$  and  $\mathcal{G}_2$  are weight  $3/2$  theta functions defined by

$$\mathcal{G}_j(\tau) := \sum_{n \in \mathbb{Z}} (-1)^{\binom{j+1}{2}n} \left( n + \frac{1}{3} \right) e^{3\pi i \left( n + \frac{1}{3} \right)^2 \tau} \text{ if } j \in \{0, 2\}, \text{ and } \mathcal{G}_1(\tau) := - \sum_{n \in \mathbb{Z}} \left( n + \frac{1}{6} \right) e^{3\pi i \left( n + \frac{1}{6} \right)^2 \tau}.$$

Using these functions, Zwegers defined the vector-valued function  $H$  by

$$H(\tau) := F(\tau) - G(\tau),$$

and he proved the following theorem in [141], which we have slightly rephrased here to incorporate the terminology of harmonic Maass forms.

**Theorem 4.2.** *The function  $H$  is a vector-valued real-analytic modular form of weight  $1/2$  that satisfies the following transformation laws*

$$H(\tau + 1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} H(\tau), \quad H(-1/\tau) = \sqrt{-i\tau} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} H(\tau).$$

*In particular,  $H$  is a vector-valued weight  $1/2$  harmonic Maass form whose shadow is a vector-valued weight  $3/2$  cuspidal unary theta function.*

Zwegers' result also yields the mock modular properties of Ramanujan's original mock theta functions  $f$  and  $\omega$  individually. For example, the first component of  $H$  is a harmonic Maass form of weight  $1/2$  on  $\Gamma_0(144)$  with Nebentypus  $\left(\frac{12}{\cdot}\right)$ . More generally, Zwegers' work from [141, 142] implies the following important result.

**Theorem 4.3.** *Ramanujan's mock theta functions are (up to multiplication by a power of  $q$ ) weight  $1/2$  mock modular forms. More precisely, if  $m$  is one of Ramanujan's mock theta functions, then*

$$m(\tau) = q^\alpha M^+(\tau),$$

*for some  $\alpha \in \mathbb{Q}$ , where  $M^+$  is the holomorphic part of a weight  $1/2$  harmonic Maass form whose shadow is a weight  $3/2$  unary theta function.*

The above theorem shows that Ramanujan's mock theta functions satisfy (up to multiplication by a power of  $q$ ) the modern definition of a mock modular form from Section 3. However, it was not confirmed until recently that Ramanujan's mock theta functions actually satisfy his own definition (Definition 4.1) of a mock theta function. Using the theory of mock modular forms, Griffin, Ono and Rolin proved that this is indeed the case in [89].

**Theorem 4.4.** *Suppose that  $M = M^+ + M^- \in H_k(\Gamma_1(N))$ , where  $k \in \frac{1}{2}\mathbb{Z}$ . If  $M^-$  is non-trivial and  $g$  is a weight  $k$  weakly holomorphic modular form on  $\Gamma_1(N')$  for some  $N' \in \mathbb{N}$ , then there are infinitely many roots of unity  $\zeta$  for which  $M^+ - g$  has exponential growth as  $q$  approaches  $\zeta$  radially from within the unit disk.*

As a corollary (see Corollary 1.2 of [89]), recalling Theorem 4.3, it follows that for any of Ramanujan's mock theta functions, there can not be a single (weakly holomorphic) modular form as in Definition 4.1 ii) and iii) which carves out all of its singularities. Rhoades [124] also recently showed that Ramanujan's Definition 4.1 of mock theta function is not equivalent to the modern definition of a mock modular form (for appropriate weights) from Section 3.1, by showing that there are two explicit functions  $V_1$  and  $V_2$  such that either  $V_1$  satisfies the modern definition but not Ramanujan's, or  $V_2$  satisfies Ramanujan's definition but not the modern.

## 5 Zwegers' $\mu$ -function

Generalizing the results described for Ramanujan's mock theta functions from Section 4, Zwegers defined a certain multivariable function he named  $\mu$  which has since been used to construct a strikingly large number of mock modular forms, and to prove mock modularity of many functions of interest, including some of the functions discussed in Section 2, Section 4, Section 6 and Section 8. In particular, upon suitable specialization of parameters, Zwegers'  $\mu$ -functions become mock modular forms (see Corollary 5.6). To define Zwegers'  $\mu$ -functions, we first define the Jacobi theta function. For further details on Zwegers' results discussed in this section, see [142]. In this section, we let  $\zeta = e^{2\pi iz}$ , and  $q = e^{2\pi i\tau}$ .

**Definition 5.1.** Let  $z \in \mathbb{C}$  and  $\tau \in \mathbb{H}$ . The **Jacobi theta function**  $\vartheta(z)$  is defined by

$$(5.1) \quad \vartheta(z; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n(z + \frac{1}{2})} = -iq^{\frac{1}{8}} \zeta^{-\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n) (1 - \zeta q^{n-1}) (1 - \zeta^{-1} q^n).$$

The infinite product expansion given on the right-hand side of (5.1) is known as the Jacobi triple product identity. The  $\vartheta$ -function is an example of a weight  $1/2$  holomorphic Jacobi form [65], a two-variable function which satisfies both elliptic and modular transformation properties, namely

$$(5.2) \quad \begin{aligned} \vartheta(z+1; \tau) &= -\vartheta(z; \tau), & \vartheta(z+\tau; \tau) &= -e^{-\pi i\tau - 2\pi iz} \vartheta(z; \tau), \\ \vartheta(z; \tau+1) &= e^{\frac{\pi i}{4}} \vartheta(z; \tau), & \vartheta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) &= -i\sqrt{-i\tau} e^{\frac{\pi iz^2}{\tau}} \vartheta(z; \tau). \end{aligned}$$

Using the Jacobi form  $\vartheta$ , Zwegers defined his Appell-Lerch sums  $\mu$  as follows.

**Definition 5.2.** For  $\tau \in \mathbb{H}$  and  $z_1, z_2 \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$ , **Zwegers'  $\mu$ -function** is defined by

$$\mu(z_1, z_2; \tau) := \frac{\rho_1^{\frac{1}{2}}}{\vartheta(z_2; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n \rho_2^n q^{\frac{n(n+1)}{2}}}{1 - \rho_1 q^n},$$

where  $\rho_j := e^{2\pi i z_j}$  ( $j \in \{1, 2\}$ ).

The function  $\mu$  does not quite transform like a true Jacobi form, but does exhibit some mock-like behavior. In particular, Zwegers showed that there is a non-holomorphic correction term which can be added to make it transform like a Jacobi form.

**Definition 5.3.** For  $z_1, z_2 \in \mathbb{C}$  and  $\tau \in \mathbb{H}$ , we define the **completed  $\mu$ -function**

$$\widehat{\mu}(z_1, z_2; \tau) := \mu(z_1, z_2; \tau) + \frac{i}{2} R(z_1 - z_2; \tau),$$

where  $(\tau = u + iv, z = x + iy)$ ,

$$(5.3) \quad R(z; \tau) := \sum_{n \in \frac{1}{2} + \mathbb{Z}} \left( \operatorname{sgn}(n) - 2 \int_0^{(n + \frac{y}{v})\sqrt{2v}} e^{-\pi t^2} dt \right) (-1)^{n - \frac{1}{2}} \zeta^{-n} q^{-\frac{n^2}{2}}.$$

We remark, especially in light of Corollary 5.6 below, that it is sometimes more convenient to regard the non-holomorphic function  $R$  from (5.3) as a period integral as in Lemma 3.6. When  $z$  is a suitable linear function in  $\tau$ , this can be done using the following weight  $3/2$  modular theta functions ( $a, b \in \mathbb{R}$ )

$$(5.4) \quad g_{a,b}(\tau) := \sum_{n \in a + \mathbb{Z}} n e^{2\pi i n b} q^{\frac{n^2}{2}}.$$

To this end, Zwegers established the following result.

**Theorem 5.4.** For  $\tau \in \mathbb{H}$ ,  $a \in (-\frac{1}{2}, \frac{1}{2})$ , and  $b \in \mathbb{R}$ , we have that

$$\int_{-\bar{\tau}}^{\infty} \frac{g_{a+\frac{1}{2}, b+\frac{1}{2}}(w)}{\sqrt{-i(w+\tau)}} dw = -e^{-\pi a^2 \tau + 2\pi i a(b+\frac{1}{2})} R(a\tau - b; \tau).$$

A main result from [142], stated as Theorem 5.5 below, shows that Zwegers' completed functions  $\widehat{\mu}$  transform like non-holomorphic Jacobi forms of weight  $1/2$ . For this reason, Zwegers'  $\mu$ -function is referred to as a *mock Jacobi form*. Ordinary Jacobi forms transform on  $\mathbb{C} \times \mathbb{H}$ , while Zwegers'  $\mu$ -functions are defined in  $\mathbb{C}^2 \times \mathbb{H}$ , so this statement may seem somewhat misleading. However, as suggested by Definition 5.3, the completed  $\widehat{\mu}$ -functions can almost be viewed as functions of  $z_1 - z_2$ , rather than of  $z_1$  and  $z_2$  separately, up to the addition of a Jacobi form. See [142, Theorem 1.11 (4)] and the footnote on p986-07 of [138].

**Theorem 5.5.** With notation and hypotheses as above, we have that

$$\text{i) } \widehat{\mu}(z_1 + k\tau + \ell, z_2\tau + m\tau + n; \tau) = (-1)^{k+\ell+m+n} q^{\frac{1}{2}(k-m)^2} (\rho_1 \rho_2^{-1})^{k-m} \widehat{\mu}(z_1, z_2; \tau),$$

for  $k, \ell, m, n \in \mathbb{Z}$ ,

$$\text{ii) } \widehat{\mu}\left(\frac{z_1}{c\tau+d}, \frac{z_2}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = \nu_\eta^{-3}(\gamma)(c\tau+d)^{\frac{1}{2}} e^{-\frac{\pi ic}{c\tau+d}(z_1-z_2)^2} \widehat{\mu}(z_1, z_2; \tau),$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ , where  $\nu_\eta$  is the multiplier of  $\eta$ ,

$$\text{iii) } \widehat{\mu}(-z_1, -z_2; \tau) = \widehat{\mu}(z_1, z_2; \tau) = \widehat{\mu}(z_2, z_1; \tau).$$

Upon suitable specialization of the parameters  $z_1$  and  $z_2$  as linear functions in  $\tau$ , Zwegers' work reveals that the completed  $\mu$ -functions give rise to harmonic Maass forms. Corollary 5.6 is parallel to a result from the theory of ordinary holomorphic Jacobi forms due to Eichler and Zagier [65], which shows that specializations of holomorphic Jacobi forms give rise to ordinary modular forms.

**Corollary 5.6.** *Let  $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Q}$  such that  $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \notin \mathbb{Z}^2$ . Then as a function of  $\tau \in \mathbb{H}$ ,*

$$(5.5) \quad e^{-\pi i(\alpha_1 - \alpha_2)^2 \tau} \widehat{\mu}(\alpha_1 \tau + \beta_1, \alpha_2 \tau + \beta_2; \tau)$$

*is a harmonic Maass form (for some congruence subgroup) of weight  $1/2$ .*

Using (2.10) for example, with  $\zeta = -1$ , ( $z = \frac{1}{2}$ ), we see how the mock theta function  $f$  can be written in terms of Zwegers'  $\mu$ -function:

$$f(q) = -2 \left( \frac{q^{\frac{1}{24}} \eta(3\tau)^3}{\eta(\tau) \vartheta(\frac{3}{2}; 3\tau)} + q^{-\frac{1}{8}} \mu\left(\frac{3}{2}, -\tau; 3\tau\right) - q^{-\frac{1}{8}} \mu\left(\frac{3}{2}, \tau; 3\tau\right) \right).$$

Similarly, all of Ramanujan's mock theta functions can be expressed in terms of  $\mu$  and ordinary modular forms; one can quickly write down exact expressions using work of Gordon and McIntosh on their *universal mock theta functions* [83], and work of Kang [103], which expresses Zwegers'  $\mu$ -functions in terms of  $q$ -hypergeometric functions of Gordon and McIntosh.

As alluded to here and in Section 2, in general, there is an intimate connection between  $q$ -hypergeometric series and mock modular forms, and understanding this connection more precisely remains a topic of current research interest. A series of papers by Lovejoy and Osburn [106, 107, 108] offers a valuable approach to this topic, as does the recent work [96] by Hickerson and Mortenson. In the former, the authors show how to produce  $q$ -hypergeometric mock modular forms using elements from the theory of  $q$ -hypergeometric series; Bailey pairs play particularly prominent roles. In the latter, the authors study Appell-Lerch sums and Hecke-type double sums in order to study mock theta functions. Zwegers' thesis [142] too extends beyond the results described in this section; he also studies indefinite theta series and Hecke-type double sums, and meromorphic Jacobi forms, in relation to mock modular forms. All of these topics play prominent roles in the theory and applications of mock modular forms. In addition to the references already mentioned in this section, the interested reader may also wish to consult, for example, [7, 16, 95, 144] by Andersen, Andrews, Dyson, Hickerson, and Zwegers. In the next section, we describe some applications to mathematical physics, all of which make use of work of Zwegers.

## 6 Mathematical physics

In this section, we stray from the foundations of mock modular forms established in the previous sections, and turn to some applications. In recent years, mock modular forms have been shown to play important roles in mathematical physics. Here, we discuss certain applications to  $S$ -duality, topological gauge theory, and string theory.

### 6.1 A conjecture of Vafa and Witten

Certain topological invariant generating functions associated to moduli spaces of coherent sheaves on complex surfaces  $S$  have been of great interest in physics, due to a property called  $S$ -duality.  $S$ -duality was conjectured in the late 1970s by Montonen and Olive [114], as a duality of gauge theory under the group  $\mathrm{SL}_2(\mathbb{Z})$ . The setup is to consider a moduli space  $\mathcal{M}$  of semi-stable sheaves, and study the Poincaré polynomials  $p$

$$(6.1) \quad p(\mathcal{M}, s) := \sum_{j=0}^{2 \dim_{\mathbb{C}} \mathcal{M}} b_j(\mathcal{M}) s^j,$$

and the Euler numbers  $\chi = \chi(\mathcal{M}) := p(\mathcal{M}, -1)$ , where  $b_j(\mathcal{M}) := \dim H_j(\mathcal{M}, \mathbb{Z})$  denotes the  $j$ th Betti number. Vafa and Witten [129] tested  $S$ -duality in the mid-1990s in the setting of topologically twisted gauge theory with  $\mathcal{N} = 4$  symmetry, and their results led them to make a conjecture regarding the modular properties of the associated Euler number generating functions when  $\mathcal{M} \in \{\mathcal{M}(2, -1, n), \mathcal{M}(2, 0, n)\}$ . Here,  $\mathcal{M}(r, c_1, c_2)$  denotes the moduli space of semi-stable sheaves of rank  $r$  and  $j$ th Chern class  $c_j$ . Their conjectured modular properties strongly supported the  $S$ -duality conjecture. The Vafa-Witten conjecture is precisely described in terms of Hurwitz class number  $H(n)$  generating functions for discriminants  $-n$  ( $j \in \{0, 1\}$ ) (see also Section 2)

$$h_j(\tau) := \sum_{n=0}^{\infty} H(4n + 3j) q^{n + \frac{3j}{4}}.$$

Using the theory of mock modular forms, Bringmann and Manschot [32] proved the Vafa-Witten conjecture, which we state as Theorem 6.1.

**Theorem 6.1.** *With notation as above, we have that*

$$q^{-\frac{1}{2}} \sum_{n=1}^{\infty} \chi(\mathcal{M}(2, -1, n)) q^n = \frac{3h_1(\tau)}{\eta^6(\tau)}, \quad q^{-\frac{1}{4}} \sum_{n=2}^{\infty} \chi(\mathcal{M}(2, 0, n)) q^n = \frac{3h_0(\tau)}{\eta^6(\tau)} + \frac{1}{4\eta^3(2\tau)}.$$

As explained in Section 2, due to Zagier, the generating function for Hurwitz class numbers (e.g. the holomorphic part of the function in (2.5)) is a mock modular form; similarly, the restricted generating functions  $h_j$  ( $j \in \{0, 1\}$ ) are mock modular forms. In particular, the following functions are harmonic Maass forms of weight  $3/2$  (see [32])

$$\widehat{h}_j(\tau) := h_j(\tau) + \frac{(1+i)}{8\pi} \int_{-\bar{\tau}}^{i\infty} \frac{\sum_{n=-\infty}^{\infty} e^{\pi i w(2n+j)^2/2}}{(\tau+w)^{\frac{3}{2}}} dw.$$

This fact, combined with Theorem 6.1, and the ordinary modular transformation properties of the  $\eta$ -function, show that the Vafa-Witten generating functions for Euler numbers are weight  $3/2$  *mixed mock modular forms* [53], that is, they lie in the tensor space of mock modular forms and modular forms.

A first step in the proof of Theorem 6.1 given in [32] is deducing closed expressions for the generating functions for Poincaré polynomials  $p(\mathcal{M}, s)$  from (6.1) when  $\mathcal{M}$  is in the set  $\{\mathcal{M}(2, -1, n), \mathcal{M}(2, 0, n)\}$  in terms of Zwegers'  $\mu$ -function (see Definition 5.2). The authors of [32] do so by making use of prior work of Yoshioka [134]. For example, with  $\zeta = e^{2\pi iz}$ ,

$$(6.2) \quad q^{-\frac{1}{2}} \sum_{n=1}^{\infty} p(\mathcal{M}(2, -1, n), \zeta^{\frac{1}{2}}) (q\zeta^{-2})^n = -\frac{(1-\zeta)}{\zeta^{\frac{5}{2}} \vartheta^2(z; \tau)} \mu(2z - \tau, \frac{1}{2} - \tau - z; 2\tau).$$

To specialize the Poincaré polynomial generating functions to the Euler polynomial generating functions  $\chi(\mathcal{M})$  from the Vafa-Witten conjecture requires one to take certain derivatives of the  $\mu$ -function in (6.2) in the variable  $\zeta$  (or equivalently, in  $z$ ). Using (6.2) and its counterpart for the numbers  $\mathcal{M}(2, 0, n)$  yields exact expressions for the Vafa-Witten Euler number generating functions as (mixed) mock modular forms. To prove they are the same (mixed) mock modular forms as those given by the functions on the right-hand sides of the two expressions in Theorem 6.1, the authors of [32] use work of Kronecker, Mordell, and Watson to explicitly rewrite the Hurwitz class number generating functions  $h_j$  ( $j \in \{0, 1\}$ ).

An exact formula for the Fourier coefficients of the mixed mock modular forms  $h_j(\tau)/\eta^6(\tau)$  is also given in [32], similar to the Hardy-Ramanujan-Rademacher exact formula for the partition numbers (2.2). For  $j = 1$ , by Theorem 6.1, these coefficients are (up to a constant multiple) the Euler numbers  $\chi(\mathcal{M}(2, -1, n))$ . We also mention another interesting recent related paper, due to Alim, Haghighat, Hecht, Klemm, Rauch, and Wotschke [5], which uses the theory of mock modular forms to derive an anomaly equation for two M5-branes wrapping a rigid divisor inside a Calabi-Yau manifold. Among other things, their results make use of work of Göttsche and Zagier [84] on indefinite theta functions (see also the discussion at the end of Section 5).

## 6.2 A conjecture of Moore and Witten

In addition to the topological invariant Poincaré polynomials and Euler numbers, mathematical physics has also led to a study of Donaldson invariants and Seiberg-Witten invariants. In physics, these invariants are the correlation functions for a supersymmetric topological gauge theory with gauge groups  $SU(2)$  and  $SO(3)$ . The Donaldson invariants are graded homogeneous polynomials on the homology  $H_0(\mathbb{C}P^2) \oplus H_2(\mathbb{C}P^2)$ . Witten [133] explained that the correlation functions should be able to be computed in a “low energy effective field theory.” This theory is parameterized by the “ $u$ -plane,” a rational elliptic surface which should be equal to the modular curve  $\mathbb{H}/\Gamma_0(4)$ , together with a meromorphic one-form [126]. Moore and Witten went on to find that the correlation functions could be expressed as a regularized integral over the  $u$ -plane, where the regularization involves constant term contributions from the cusps  $\{0, 2, \infty\}$ . They showed that the contributions at 0 and 2 vanish, and made the following conjecture [115].

**Conjecture 6.2.** *The contribution from the cusp  $\infty$  to the regularized  $u$ -plane integral is the generating function for the Donaldson invariants of  $\mathbb{CP}^2$ .*

Moore and Witten [115] computed the first 40 invariants in the  $SU(2)$  case as evidence towards their conjecture, and also showed relationships between Hurwitz class numbers (see Section 2 and Section 6.1), an early instance of a relationship to mock modular forms. The conjecture was proved in the case of  $SO(3)$ , where less was known, by Malmendier and Ono in [111], and later in the case of  $SU(2)$  by Griffin, Malmendier, and Ono in [87]. As was the case with the functions in the previous section (see (6.2)), the strategies used to prove the Moore-Witten conjectures involve reformulating the relevant functions in terms of Zwegers'  $\mu$ -functions and studying their derivatives. The methods used in [111] for  $SO(3)$  and in [87] for  $SU(2)$  are similar, so here we limit our discussion to the case of  $SO(3)$ . As a key step in their proof of the Moore-Witten conjecture, the authors of [111] compute the regularized  $u$ -plane integral, and from their computations, deduce that the Moore-Witten conjecture is equivalent to the vanishing of the constant terms for every pair  $(m, n) \in \mathbb{N}_0^2$  of a series of the form

$$(6.3) \quad \theta_{m,n}(\tau) \sum_{k=0}^n \sum_{j=0}^k a_{j,k,n} E_2^{k-j}(\tau) \cdot \left( b_{k,n} \theta_j(\tau) F_{2(n-k)}(\tau) + c_{j,k,n} \theta^*(\tau) \theta_{n-k}(\tau) \left( q \frac{d}{dq} \right)^j \mathcal{Q}^+(\tau) \right).$$

All of the functions in (6.3) are explicitly defined in [111]:  $a_{j,k,n}$ ,  $b_{k,n}$  and  $c_{j,k,n}$  are constants,  $E_2$  is the Eisenstein series discussed in Section 2, and the functions  $\theta_{m,n}$ ,  $\theta_j$ , and  $\theta^*$  are ordinary modular forms. The function  $\mathcal{Q}^+$  is the holomorphic part of a certain harmonic Maass form of weight  $1/2$ , which the authors re-write in terms of Zwegers'  $\mu$ -functions (see Definition 5.2) as follows:

$$\mathcal{Q}^+(\tau) = 2iq^{-\frac{1}{8}} \mu(-2\tau, -\tau - \frac{1}{2}; 4\tau) - 2iq^{-\frac{1}{8}} \mu(-2\tau, -3\tau - \frac{1}{2}; 4\tau) + g(\tau),$$

where  $g$  is an explicit weakly holomorphic ordinary modular form. Similarly, the functions  $F_t$  in (6.3) may be expressed as a product of the  $t$ -th derivative of a Zwegers  $\mu$ -function, multiplied by an ordinary modular theta function. When  $n = 0$ , the sum in (6.3) collapses. One is left with a difference of two harmonic Maass forms of weight  $1/2$  with the same non-holomorphic parts, which is therefore an ordinary weakly holomorphic modular form. The proof in this case follows by an explicit calculation. For  $n > 0$  one encounters derivatives of the  $\mu$ -function from the functions  $F_t$ , and this gives rise to a similar to but more complicated proof than the case  $n = 0$  just described.

Malmendier continued the work from [111] in his paper [109]. Namely, he computed the regularized  $u$ -plane integral on  $\mathbb{CP}^1 \times \mathbb{CP}^1$ , and determined the explicit formulas for the  $SU(2)$  and  $SO(3)$  Donaldson invariants of  $\mathbb{CP}^1 \times \mathbb{CP}^1$  in terms of mock modular forms. Malmendier and Ono [110] also made a connection between a certain ‘‘Moonshine’’ mock modular form (see (8.3)) originally studied by Eguchi, Ooguri, and Tachikawa [64] which we discuss in Section 8, and the  $SO(3)$  Donaldson invariants for  $\mathbb{CP}^2$ .

### 6.3 Quantum black holes

The quantum theory of black holes is of interest within the context of string theory. It is expected that certain counting functions for quantum degeneracies of black hole horizons are modular, and from a physical perspective, modular symmetry properties are essential for understanding the quantum entropy of these black holes. One obtains such information from a Hardy-Ramanujan-Rademacher-type expansion for the degeneracies, similar to the expression for  $p(n)$  given in (2.2). When there is a “wall-crossing” phenomenon, something we elaborate upon more below, the situation is more complicated from the points of view of both physics and number theory. Dabholkar, Murthy, and Zagier address this in [53], and we describe some of their findings here. One of their main results applies to a certain meromorphic Jacobi form which counts quarter-BPS states in  $\mathcal{N} = 4$  string theories. As described in Section 5, Jacobi forms are two-variable functions which exhibit both modular and elliptic transformation properties. (See [65] for a formal definition.) For example, the function  $\vartheta$  in (5.1) is a holomorphic Jacobi form, and some of its transformation properties are shown in (5.2). The authors of [53] study the meromorphic Jacobi forms  $\psi_m(z; \tau)$  which arise as the Fourier-Jacobi coefficients of a meromorphic Siegel modular form. That is,

$$(6.4) \quad \frac{1}{\Phi_{10}(\Omega)} = \sum_{m=-1}^{\infty} \psi_m(z; \tau) e^{2\pi i \sigma m},$$

where  $\Phi_{10}$  is the Igusa cusp form of weight 10. Here,  $\Omega$  lies in the Siegel upper half-space of degree 2, defined by  $\mathbb{H}_2 := \{\Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix} \in M_2(\mathbb{C}) \mid \text{Im}(\tau), \text{Im}(\sigma), \det(\text{Im}(\Omega)) > 0\}$ . We omit the formal definition of (degree 2) Siegel modular forms here, but note that they are analogous to ordinary modular forms in that they satisfy a symmetry property  $F(g \circ \Omega) = \det(C\Omega + D)^k F(\Omega)$  with respect to the action of elements  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  of a symplectic group on  $\mathbb{H}_2$ . See for example [65] for more on Siegel modular and Jacobi forms.

The fact that the Jacobi forms  $\psi_m$  from (6.4) are meromorphic, as opposed to holomorphic, is intertwined with wall-crossing, and turns out to dictate a relationship to mock modular forms, as opposed to ordinary modular forms. A main application of the more general results proved in [53] show that the Fourier-Jacobi coefficients  $\psi_m$  ( $m \geq 1$ ) naturally decompose as a sum of two parts

$$\psi_m = \psi_m^P + \psi_m^F.$$

This decomposition should not be confused with the decomposition of a harmonic Maass form as a sum of a holomorphic part and a non-holomorphic part (see Section 3). Here, the two parts  $\psi_m^P$  and  $\psi_m^F$  are referred to as the *polar part* and the *finite part*, respectively. The polar part  $\psi_m^P$  is completely determined by the poles of  $\psi_m$ . The finite part  $\psi_m^F$  is a finite linear combination of (mixed) mock modular forms  $f_{m,\ell}^*$  multiplied by Jacobi theta functions  $\vartheta_{m,\ell}$

$$\psi_m^F(z; \tau) = \sum_{\ell \pmod{2m}} f_{m,\ell}^*(\tau) \vartheta_{m,\ell}(z; \tau).$$

In this way, we view the finite part  $\psi^F$  as a (mixed) *mock Jacobi form*. From the point of view of physics, this result from [53] shows that the degeneracies  $d^*(\ell, m, n)$  ( $\text{Im}(\sigma) = 2n/\varepsilon, \text{Im}(\tau) = 2m/\varepsilon, \text{Im}(z) = -\ell/\varepsilon$ ) of single centered black holes with magnetic charge

invariant ( $M^2/2 = m$ ) are Fourier coefficients of a (mixed) mock Jacobi form of index  $m$ . That is,

$$d^*(n, \ell, m) = \int e^{\pi i \ell^2 \tau / 2m} f_{m, \ell}^*(\tau) d\tau,$$

where the integral is over an interval of length one ( $\text{Im}(\tau)$  is fixed). From the point of view of both number theory and physics, the wall-crossing behavior exhibited by the degeneracies is the same as that of the Fourier coefficients of the meromorphic Jacobi forms, and this is the origin of their connection. The wall-crossing behavior comes from the polar part  $\psi_m^P$ , which is shown to equal the Appell-Lerch sum

$$(6.5) \quad \mathcal{A}_{2,m}(z; \tau) := \sum_{s=-\infty}^{\infty} \frac{q^{ms^2+s} \zeta^{2ms+1}}{(1 - q^s \zeta)^2}$$

divided by a constant multiple of the modular  $\Delta$ -function ( $\Delta := \eta^{24}$ ). The wall crossing phenomenon is seen in the fact that  $\mathcal{A}_{2,m}$  has different Fourier expansions depending on which interval  $n < \text{Im}(z)/\text{Im}(\tau) < n + 1$  (i.e., for which integer  $n$ ) the ratio  $\text{Im}(z)/\text{Im}(\tau)$  lies in. From a physical perspective, wall crossing may be interpreted in terms of counting two-centered black holes.

Dabholkar, Murthy and Zagier generalize their findings above for the forms  $\psi_m$  in [53], which we summarize in the following theorem. The number  $m \in \mathbb{N}$  seen in what follows is the *index* of the Jacobi form  $\varphi$  (see [53, 65]).

**Theorem 6.3.** *Let  $\varphi$  be a meromorphic Jacobi form with simple poles at  $z = z_s = \alpha\tau + \beta$ ,  $s = (\alpha, \beta) \in S \subseteq \mathbb{Q}^2$ , and with Fourier coefficients  $h_\ell^{(-\ell\tau/2m)}(\tau)$ , where for  $z_0 \in \mathbb{C}$ ,*

$$h_\ell^{(z_0)}(\tau) = q^{-\ell^2/4m} \int_{z_0}^{z_0+1} \varphi(z; \tau) e^{-2\pi i \ell z} dz.$$

Then  $\varphi$  has the decomposition

$$\varphi(z; \tau) = \varphi^F(z; \tau) + \varphi^P(z; \tau)$$

into a “finite part”  $\varphi^F$  and a “polar part”  $\varphi^P$ . Each  $h_\ell^{(-\ell\tau/2m)}$  is a mixed mock modular form, and  $\varphi^F$  is a mixed mock Jacobi form.

We reiterate that the two “parts” shown in Theorem 6.3 should not be confused with the holomorphic and non-holomorphic parts of a harmonic Maass form. In [53], the polar parts  $\varphi^P$  are expressed in terms of the residues of  $\varphi$  and *universal Appell-Lerch sums*, which are essentially due to Zwegers [142, Definition 3.2]. In general, these functions resemble (6.5), and are meromorphic mock Jacobi forms of weight 1. The finite parts  $\varphi^F$  are expressed in terms of the  $h_\ell^{(z_0)}$  and holomorphic Jacobi theta functions as follows:

$$(6.6) \quad \varphi^F(z; \tau) = \sum_{\ell \pmod{2m}} h_\ell^{(-\ell\tau/2m)}(\tau) \vartheta_{m, \ell}(z; \tau).$$

This expression for  $\varphi^F$  nicely extends what is known for holomorphic Jacobi forms. Namely, holomorphic Jacobi forms admit a *theta decomposition* (see equation (5) of Chapter II §5 in

[65]), which is nearly identical to the meromorphic Jacobi finite part  $\varphi^F$  in (6.6), save for the fact that the analogues to the functions  $h_\ell^{(z_0)}$  which arise in the holomorphic setting are ordinary modular forms, while in the meromorphic setting they are mixed mock modular forms. Since there are no poles in the holomorphic setting, the holomorphic analogue of the meromorphic polar part  $\varphi^P$  is identically zero. For an extension of Theorem 6.3 for meromorphic Jacobi forms with higher order poles, and an application to representation theory, see [25]; more recent related works than [25] include [118] by Olivetto, and [143], by Zagiers. The work of Dabholkar, Murthy and Zagier in [53] (some of which extends work of Zagiers from [142]), sheds much light on the theory of meromorphic Jacobi forms and mock modular forms; while strongly motivated by its applications to theoretical physics, it is also of independent number theoretic interest.

## 7 Number theory

In this section, we offer some number theoretic applications of mock modular forms. In Section 7.1, we discuss some recent work of Duke, Imamoglu, and Tóth on cycle integrals and mock modular forms, in Section 7.2 we discuss recent work of Alfes, Griffin, Guerzhoy, Ono, and Rolin on elliptic curves and mock modular forms, and in Section 7.3, we discuss work of Bruinier and Ono on Borcherds products and mock modular forms. For additional number theoretic consequences of the development of the theory of mock modular forms not mentioned in this article, we refer the reader to the works [23, 54, 119, 138] mentioned in Section 1.

### 7.1 Cycle integrals

A lauded result of Zagier [136] relates traces of singular moduli arising from quadratic forms with negative discriminants to Fourier coefficients of modular forms. To describe this, let  $j_m(\tau)$  ( $m \in \mathbb{N}_0$ ) denote the unique basis element for  $\mathbb{C}[j]$  satisfying  $j_m(\tau) = q^{-m} + O(q)$ . Here,  $j = j(\tau) = q^{-1} + 744 + 196884q + \dots$  is the modular invariant function. Then we have that  $j_0 = 1, j_1 = j - 744, j_2 = j^2 - 1488j + 159768$ , and so on. The *twisted traces* of these functions are twisted, weighted sums evaluated at CM points in  $\mathbb{H}$ . Precisely, for negative discriminants  $d$  and fundamental positive discriminants  $D$ ,

$$(7.1) \quad \mathrm{Tr}_{d,D}(j_m) := D^{-\frac{1}{2}} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) |\Gamma_Q|^{-1} j_m(\tau_Q),$$

where  $\mathcal{Q}_\Delta$  is the set of discriminant  $\Delta$  positive definite integral binary quadratic forms  $Q = Q(x, y) = ax^2 + bxy + cy^2$  ( $a, b, c \in \mathbb{Z}$ ), and  $\tau_Q$  is the unique number in  $\mathbb{H}$  that is a root of  $Q(x, 1)$ . Here,  $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ , a group which naturally acts on  $\mathcal{Q}_{dD}$ , and  $\Gamma_Q := \mathrm{Aut}(Q)$ . The sum (7.1) (including the character  $\chi$ ) is well-defined on  $\Gamma \backslash \mathcal{Q}_{dD}$ . See [90] for more details on these ingredients used to form (7.1). Zagier showed that the twisted traces may be expressed in terms of Fourier coefficients of modular forms. That is,

$$\mathrm{Tr}_{d,D}(j_m) = \sum_{n|m} \left( \frac{D}{m/n} \right) na(n^2 D, d),$$

where  $\{g_d\}_{d>0}$ ,  $g_d(\tau) = -q^{-d} + \sum_{n \leq 0} a(d, n)q^{|n|}$ , are a basis of weakly holomorphic modular forms of weight  $3/2$  and level 4. Equivalently, the forms  $\{f_d\}_{d \leq 0}$ , with Fourier coefficients  $a(n, d)$  defined by

$$(7.2) \quad f_d(\tau) = q^d + \sum_{n=1}^{\infty} a(n, d)q^n,$$

constructed with the dual coefficients (with  $d$  and  $n$  interchanged) turn out to coincide with Borcherds' basis for weakly holomorphic modular forms of the dual weight  $2 - 3/2 = 1/2$  and level 4. The first terms in the expansions of the first few such functions are

$$(7.3) \quad f_0(\tau) = 1 + 2q + 2q^4 + 2q^9 + \dots$$

$$(7.4) \quad f_{-3}(\tau) = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \dots$$

$$f_{-4}(\tau) = q^{-4} + 492q + 143376q^4 + 565760q^5 + 18473000q^8 + \dots$$

$$f_{-7}(\tau) = q^{-7} - 4119q + 8288256q^4 - 52756480q^5 + \dots$$

It is natural to ask if there are analogues to Zagier's results for positive discriminants  $dD$ . One first needs an appropriate analogue of (7.1) in this setting, and then a modular family which encodes the "traces" in their Fourier coefficients. Duke, Imamoglu, and Tóth beautifully answer this question in [56]. They define an extension of the function in (7.1) to positive discriminants using *cycle integrals* as follows:

$$(7.5) \quad \widetilde{\text{Tr}}_{d,D}(j_m) := \begin{cases} \text{Tr}_{d,D}(j_m), & d < 0, \\ (2\pi)^{-1} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \int_{C_Q} j_m(\tau) (Q(\tau, 1))^{-1} d\tau, & d > 0, \end{cases}$$

where the cycle integral of  $j_m$  above is taken over any smooth curve  $C_Q$  from a point  $z \in \mathbb{H}$  to the point  $\gamma z$ , where  $\gamma = \gamma_Q$  is a certain generator of  $\Gamma_Q$ . Some of the main results from [56] show how these "traces" for positive coefficients are related to Fourier coefficients of a basis of mock modular forms. We summarize some results of Duke, Imamoglu, and Tóth from [56] in the following theorem.

**Theorem 7.1.** *With notation as above, the following are true.*

i) *For each positive discriminant  $d$  there is a unique mock modular form  $f_d$  with shadow  $g_d$ , with Fourier expansion of the form*

$$(7.6) \quad f_d(\tau) = \sum_{n=1}^{\infty} a(n, d)q^n.$$

*Moreover, the set  $\{f_d\}_{d>0}$  forms a basis for the space of mock modular forms of weight  $1/2$  and level 4 (in the plus space).*

ii) *For any integer  $d \equiv 0, 1 \pmod{4}$ , and fundamental discriminant  $D > 0$  with  $dD$  not a square,*

$$\widetilde{\text{Tr}}_{d,D}(j_m) = \sum_{n|m} \left( \frac{D}{m/n} \right) na(n^2 D, d),$$

*where for  $d < 0$ , the coefficients  $a(n, d)$  are as defined in (7.2), and for  $d > 0$ , the coefficients  $a(n, d)$  are as defined in (7.6).*

Rephrasing this another way, Theorem 7.1 gives a certain basis of mock modular forms of weight  $1/2$  with shadows given by Zagier's weakly holomorphic modular forms of weight  $3/2$ , and whose Fourier coefficients are given in terms of cycle integrals of the modular  $j$ -function. The authors prove Theorem 7.1 in [56] using the delicate theory of Poincaré series, and Kloosterman sums. The work in [56] has inspired and is related to a number of additional interesting recent works, including [6, 27, 39, 55, 57, 112] by Andersen, Bringmann, Bruinier, Duke, Funke, Guerzhoy, Imamoglu, Kane, Masri, and Tóth. Moreover, the results from [56] are not limited to Theorem 7.1. For example, another result from [56] is a lifting result; the authors study certain holomorphic modular integrals of weight 2 which can be viewed as Shimura-type lifts of mock modular forms of weight  $1/2$ , and which give quadratic analogues to Borcherds products. In Section 7.3 and Section 8, we also mention lifts and Borcherds products with respect to mock modular forms and Moonshine.

## 7.2 Elliptic curves

Classically, we have that an elliptic curve  $E \cong \mathbb{C}/\Lambda_E$ , where  $\Lambda_E$  is a two-dimensional lattice in  $\mathbb{C}$ , is parameterized by the Weierstrass  $\wp$ -function via the mapping  $z \mapsto (\wp(\Lambda_E; z), \wp'(\Lambda_E; z))$ . The Weierstrass  $\zeta$ -function

$$\zeta(\Lambda_E; z) := \frac{1}{z} + \sum_{w \in \Lambda_E \setminus \{0\}} \left( \frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right)$$

also plays a role in the theory of elliptic curves. For example, it has an addition law  $\zeta(\Lambda_E; z_1 + z_2) = \zeta(\Lambda_E; z_1) + \zeta(\Lambda_E; z_2) + P(\Lambda_E; z_1, z_2)$  (where  $P$  is a certain rational function in  $\wp$  and  $\wp'$ ) which can be interpreted in terms of the group law of  $E$ . The Weierstrass  $\zeta$ -function is not lattice invariant, but Eisenstein [132] observed that it could be corrected in a similar way to how mock modular forms can be completed to form a (non-holomorphic) modular form by adding a suitable non-holomorphic function (see Sections 3-5). Eisenstein's corrected function is defined by

$$\mathfrak{z}_E(z) := \zeta(\Lambda_E; z) - S(\Lambda_E)z - \frac{\pi}{a(\Lambda_E)}\bar{z},$$

where  $a(\Lambda_E)$  is the area of a fundamental parallelogram for  $\Lambda_E$ , and

$$S(\Lambda_E) := \lim_{s \rightarrow 0^+} \sum_{w \in \Lambda_E \setminus \{0\}} w^{-2} |w|^{-2s}.$$

In his study of a differential equation due to Kaneko and Zagier, Guerzhoy [92, 93] first showed how the Weierstrass  $\zeta$ -function could be used to construct a weight 0 harmonic Maass form associated to an elliptic curve over  $\mathbb{Q}$ . Alfes, Griffin, Ono and Rolin extended Guerzhoy's work in [4], which we now describe. The idea, stemming from Guerzhoy's construction in [92], is that Eisenstein's correction  $\mathfrak{z}_E$  to the Weierstrass  $\zeta$ -function gives rise to a weight 0 harmonic Maass form. The Maass form is obtained by specializing  $\mathfrak{z}_E$  at  $z \mapsto \mathcal{E}_E(\tau)$  ( $\tau \in \mathbb{H}$ ), where  $\mathcal{E}_E$  is an *Eichler integral* associated to  $E$ , and then subtracting a canonical modular function. More precisely, for an elliptic curve  $E$  over  $\mathbb{Q}$ , its Eichler

integral is defined in terms of its associated weight 2 newform  $F_E(\tau) = \sum_{n=1}^{\infty} a_E(n)q^n$  as follows

$$\mathcal{E}_E(\tau) = -2\pi i \int_{\tau}^{i\infty} F_E(w)dw = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n.$$

The authors of [4] define the function  $\widehat{\mathfrak{z}}_E(\tau) := \mathfrak{z}_E(\mathcal{E}_E(\tau))$  (for  $\tau \in \mathbb{H}$ ) to be the specialization of Eisenstein's  $\mathfrak{z}_E$ -function at the Eichler integral  $\mathcal{E}_E(\tau)$ . The function  $\widehat{\mathfrak{z}}_E(\tau)$  decomposes into a sum of a holomorphic part plus a non-holomorphic part, parts which we denote by  $\widehat{\mathfrak{z}}_E^{\pm}(\tau)$ . We summarize a main theorem from [4] in Theorem 7.2.

**Theorem 7.2.** *With notation as above, the following are true.*

- i) The poles of  $\widehat{\mathfrak{z}}_E^+(\tau)$  are precisely those  $\tau$  for which  $\mathcal{E}_E(\tau) \in \Lambda_E$ .
- ii) If  $\widehat{\mathfrak{z}}_E^+(\tau)$  has poles in  $\mathbb{H}$ , then there is a canonical modular function  $B_E(\tau)$  with algebraic coefficients for which  $\widehat{\mathfrak{z}}_E^+(\tau) - B_E(\tau)$  is holomorphic on  $\mathbb{H}$ .
- iii) The function  $\widehat{\mathfrak{z}}_E(\tau) - B_E(\tau)$  is a weight 0 harmonic Maass form with level equal to the conductor of  $E$ . In particular,  $\widehat{\mathfrak{z}}_E^+(\tau)$  is a weight 0 mock modular form.

In addition to giving rise to a harmonic Maass form, it turns out that the function  $\widehat{\mathfrak{z}}_E(\tau)$  encodes information about the vanishing and non-vanishing of central  $L$ -values and derivatives. Such a connection between harmonic Maass forms (of weight 1/2) and  $L$ -values and derivatives was first made by Bruinier and Ono in [38], however computing their Maass forms poses some non-trivial difficulties. (See also [41] by Bruinier and Stromberg for more on computing harmonic Maass forms.) Using a Siegel theta function originally studied by Hövel [98], the authors of [4] define a twisted theta lift, which lifts their weight 0 harmonic Maass forms to weight 1/2 harmonic Maass forms  $f_E$ . These weight 1/2 theta lifts have Fourier expansions as in Lemma 3.2 (with  $k = 1/2$ ), and we let  $c_E^{\pm}(n)$  denote their Fourier coefficients. We have the following result from [4].

**Theorem 7.3.** *Let  $E$  be an elliptic curve over  $\mathbb{Q}$  with prime conductor  $p$  and sign of functional equation  $-1$ . Let  $d$  be any fundamental discriminant such that  $\left(\frac{d}{p}\right) = 1$ , and let  $E_d$  be the quadratic twist of  $E$ . The following are true.*

- i) If  $d < 0$ , then  $L(E_d, 1) = 0$  if and only if  $c_E^-(d) = 0$ .
- ii) If  $d > 0$ , then  $L'(E_d, 1) = 0$  if and only if  $c_E^+(d)$  is in  $\mathbb{Q}$ .

As remarked in [4], by known results towards the Birch and Swinnerton-Dyer Conjecture, this shows for fundamental discriminants with  $\left(\frac{d}{p}\right) = 1$  such that  $d < 0$  and  $c_E^-(d) \neq 0$ , we have that the rank of  $E_d(\mathbb{Q})$  is 0; if  $d > 0$  and  $c_E^+(d)$  is transcendental, then the rank of  $E_d(\mathbb{Q})$  is 1.

### 7.3 Borcherds products

A celebrated result of Borcherds shows that the space of weight 1/2 modular forms in Kohnen's plus space (e.g. with integer coefficient Fourier expansions supported on certain residue classes (mod 4)) is in natural bijection with the set  $\mathcal{M}_H$  of integer weight meromorphic modular forms of level 1 with integer coefficients, leading coefficient 1, and with

*Heegner divisor*, meaning their divisors are supported at  $\infty$  and CM points. See [21, 136] for further details on some of the results described in this section. Here are two examples to illustrate Borcherds' result. Consider the weight  $1/2$  forms  $\tilde{f}_0 := 12f_0$  and  $\tilde{f}_{-3} := 3f_{-3}$ , where  $f_0$  and  $f_{-3}$  are as defined in (7.3) and (7.4). If we define the coefficient of  $q^r$  in the Fourier expansion of  $\tilde{f}_j$  ( $j \in \{0, -3\}$ ) by  $c_j(r)$ , then Borcherds' theorem implies that the functions

$$\Psi(\tilde{f}_0) := q \prod_{n=1}^{\infty} (1 - q^n)^{c_0(n^2)} = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 + \dots$$

$$\Psi(\tilde{f}_{-3}) := q^{-1} \prod_{n=1}^{\infty} (1 - q^n)^{c_{-3}(n^2)} = q^{-1} (1 - q)^{-744} (1 - q^2)^{80256} \dots = q^{-1} + 744 + 196884q + \dots$$

are in  $\mathcal{M}_H$ . Borcherds' product defining  $\Psi(\tilde{f}_0)$  is visibly equal to the modular form  $\Delta = \eta^{24}$  of weight 12. Moreover, by considering the divisor of the resulting function, it follows that  $\Psi(\tilde{f}_{-3}) = j(\tau)$ , the modular invariant function, as suggested by the first few terms in the Fourier expansion given above. These are examples of Borcherds' more general theorem, which we now summarize. To describe this, we let  $\mathcal{H}^+(\tau)$  denote the holomorphic part of Zagier's Eisenstein series from (2.5). For a weight  $1/2$  form  $g$  in Kohnen's plus space, we let  $b_g$  denote the constant term in the Fourier expansion of  $g(\tau)\mathcal{H}^+(\tau)$ .

**Theorem 7.4.** *Let  $g(\tau) = \sum_n c(n)q^n$  be in Kohnen's plus space  $M_{\frac{1}{2}}^+(\Gamma_0(4))$ , and define*

$$\Psi(g(\tau)) := q^{-b_g} \prod_{n=1}^{\infty} (1 - q^n)^{c(n^2)}.$$

*Then  $\Psi(g) \in \mathcal{M}_H$ , and has weight  $c(0)$ . Moreover, the map  $\Psi : M_{\frac{1}{2}}^+(\Gamma_0(4)) \rightarrow \mathcal{M}_H$  is an isomorphism.*

In fact, Borcherds' results do more than what is stated in Theorem 7.4. For example, he also gives the multiplicity of zeros at CM points.

Bruinier and Ono provide an analogous Borcherds-like lift for mock modular forms in [38]. We first illustrate their general result with an example involving Ramanujan's mock theta function  $\omega$  as defined in (2.12). Define the coefficients  $a(n)$  by

$$-2q^{\frac{1}{3}} \left( \omega(q^{\frac{1}{2}}) + \omega(-q^{\frac{1}{2}}) \right) =: \sum_{n \in \frac{1}{3} + \mathbb{Z}} a(n)q^n = -4q^{\frac{1}{3}} - 12q^{\frac{4}{3}} - 24q^{\frac{7}{3}} - 40q^{\frac{10}{3}} - \dots$$

In [38] it is shown that

$$\prod_{n=1}^{\infty} \left( \frac{1 + \sqrt{-2}q^n - q^{2n}}{1 - \sqrt{-2}q^n - q^{2n}} \right)^{\binom{n}{3} a(n^2/3)} = 1 - 8\sqrt{-2}q - (64 - 24\sqrt{-2})q^2 + (384 + 168\sqrt{-2})q^3 + \dots$$

is a meromorphic modular form of weight 0, whose divisor can be determined explicitly. Moreover, this function can be written down explicitly in terms of the holomorphic Eisenstein series  $E_4$ , and two functions on  $\Gamma_0^*(6)$ : the Hauptmodul, and the unique normalized cusp form of weight 4.

The general mock modular lifting theorem from [38] is stated for vector valued forms and certain Weil representations. Many technical ingredients are required to properly state it, and we refer the reader to [38] for explicit details. Briefly speaking, the result is a lifting result on weight  $k < 1$  harmonic Maass forms with respect to the metaplectic group  $\tilde{\Gamma} := \text{Mp}_2(\mathbb{Z})$ , and Weil representations  $\rho_L$  for certain even lattices  $L$ . These types of harmonic Maass forms are defined in an analogous way to ordinary harmonic Maass forms (see Definition 3.1): they are twice-continuously differentiable, satisfy a suitable transformation law, namely  $M(\gamma\tau) = \phi(\tau)^{2k} \rho_L(\gamma, \phi) M(\tau)$  for all  $(\gamma, \phi) \in \tilde{\Gamma}$ , are annihilated by  $\Delta_k$ , and satisfy appropriate growth conditions in the cusps. If  $L'$  denotes the dual of  $L$ , the lifting result from is stated for fundamental discriminants  $\Delta \equiv r^2 \pmod{4N}$ , and harmonic Maass forms of weight  $1/2$  with respect to  $\tilde{\Gamma}$  and  $\tilde{\rho}_L$ , where  $\tilde{\rho}_L$  is equal to  $\rho_L$  or  $\bar{\rho}_L$ , depending on whether or not  $\Delta > 0$ . This space of harmonic Maass forms is denoted by  $H_{1/2, \tilde{\rho}_L}$ . As we see in Theorem 7.5 below, the Borcherds-like products from [38] are taken over  $0 < \lambda \in K'$ , where  $K$  is a one dimensional lattice arising from  $L$  satisfying  $K'/K \cong L'/L$ . A main result from [38], generalizing the example above for the mock theta function  $\omega$ , is as follows.

**Theorem 7.5.** *Let  $M \in H_{1/2, \tilde{\rho}_L}$  be a harmonic Maass form with real coefficients  $c^+(m, h)$  for all  $m \in \mathbb{Q}$  and  $h \in L'/L$ , and  $c^+(n, h) \in \mathbb{Z}$  for all  $n \leq 0$ . Then the product*

$$\Psi_{\Delta, r}(\tau, M) = e((\rho_{M, \ell}, \tau)) \prod_{0 < \lambda \in K'} \prod_{b(\Delta)} (1 - e((\lambda, \tau) + b/\Delta))^{(\frac{\Delta}{b})^{c^+(\Delta|\lambda^2/2, r\lambda)}}$$

*is a meromorphic modular form on  $\Gamma_0(N)$  with weight  $c^+(0, 0)$  or 0, depending on whether or not  $\Delta$  is equal to 1.*

In addition to the references given above, there are a number of other related lifting results to which we refer the interested reader, such as works by Alfes, Bruinier, Choie, Duke, Ehlen, Funke, Jenkins, Kim, Li, and Lim [3, 43, 50, 58, 59, 104].

## 8 Moonshine

One of the most beautiful and well-known results relating representations of groups and ordinary modular forms is given by ‘‘Monstrous Moonshine’’. Throughout this section, we refer the reader to the articles by Gannon [75], Borcherds [22], and Duncan-Griffin-Ono [60] for more on the history of this topic. The adjective ‘‘monstrous’’ arises from the *monster group*  $\mathbb{M}$ , largest of the finite sporadic simple groups with over  $8 \times 10^{53}$  elements. In the 1970s, Conway and Norton [51] made a surprising conjecture, beginning with observations of McKay and Thompson, relating the dimensions of the irreducible representations of  $\mathbb{M}$  to the Fourier coefficients of the modular  $j$ -function. To describe this, let  $\rho_n$  denote the  $n$ -th smallest irreducible representation of  $\mathbb{M}$ , and let  $\delta_n$  denote its dimension. Moreover, let  $\beta(k)$  denote the coefficient of  $q^k$  in the Fourier expansion of the modular  $j$ -function  $j(\tau) = q^{-1} + 744 + 196884q + \dots$  (where  $q = e^{2\pi i\tau}$  as usual). Some of McKay’s and Thompson’s observations [128] are as follows:

$$196884 = \beta(1) = \delta_1 + \delta_2, \quad 21493760 = \beta(2) = \delta_1 + \delta_2 + \delta_3, \quad 864299970 = \beta(3) = 2\delta_1 + 2\delta_2 + \delta_3 + \delta_4,$$

where  $\delta_1 = 1, \delta_2 = 196883, \delta_3 = 21296876, \delta_4 = 842609326$ .

These observations of McKay and Thompson can be interpreted as evidence of a grading, in which case the dimensions are graded traces of the identity element  $e \in \mathbb{M}$ . That is, there should be an infinite-dimensional graded module  $V = \bigoplus_{n=0}^{\infty} V_n$  for the Monster at work, with subspaces  $V_0 = \rho_1, V_1 = \emptyset, V_2 = \rho_1 \oplus \rho_2, V_3 = \rho_1 \oplus \rho_2 \oplus \rho_3$ , etc. so that the module  $V$  should have graded dimension  $\dim_V(q) = q(j(\tau) - 744)$ . It was surmised that studying the graded traces of other elements  $g \in \mathbb{M}$  (not just  $g = e$ ) may also be of interest. The *McKay-Thompson series*  $T_g(q)$  for elements  $g \in \mathbb{M}$  are defined to have coefficients given by these graded traces, normalized so that  $T_e(q) + 744$  should equal the modular  $j$ -function. The Monstrous Moonshine Conjecture claimed that for any  $g \in \mathbb{M}$ , the McKay-Thompson series  $T_g$  is a Hauptmodul, that is, a generator of a genus zero modular function field. Atkin, Fong, and Smith [127], Griess [85, 86], and Frenkel, Lepowsky, and Meurman [71, 72], established key results towards the conjecture. In particular, Frenkel-Lepowsky-Meurman constructed a module  $V^\natural$ , a vertex operator algebra, whose graded dimension is given by  $q(j(\tau) - 744)$  and whose automorphism group equal to  $\mathbb{M}$ . Borcherds famously proved the Monstrous Moonshine Conjecture many years later [20], by reconciling the module  $V^\natural$  with that fact that all Hauptmoduls satisfy *replication formulas*, meaning that their coefficients satisfy certain recursions. With respect to the  $j$ -function, replication formulas can be captured by the following identity [136] ( $\zeta = e^{2\pi iz}$ ):

$$(8.1) \quad j(z) - j(\tau) = \zeta^{-1} \prod_{\substack{m>0 \\ n \in \mathbb{Z}}} (1 - \zeta^m q^n)^{\beta^*(mn)},$$

where  $\beta^*(k) := \beta(k)$  if  $k \neq 0$ , and  $\beta^*(0) := 0$ . For example, one recursion satisfied by the coefficients  $\beta(k)$  which is implied by (8.1) is

$$(8.2) \quad \beta(4n + 2) = \beta(2n + 2) + \sum_{k=1}^n \beta(k)\beta(2n - k + 1).$$

Conway and Norton conjectured analogous formulas for all McKay-Thompson series  $T_g$ , which would mean that each  $T_g$  is determined by only finitely many Fourier coefficients. For his proof, Borcherds defined a Lie algebra, which inherits the action of  $\mathbb{M}$  from  $V^\natural$ . Remarkably, it turned out that Borcherds' monster Lie algebra has associated denominator identity given by (8.1). A "twisting" procedure by the elements  $g \in \mathbb{M}$  led to the other conjectured replication identities for each Thompson series  $T_g$ .

Further generalizations and extensions of "classical" Monstrous Moonshine have since been explored (see [52] and [75], for example). Also since the time of Monstrous Moonshine, we have recently made contact with mock modular forms within a similar framework. The tale of mock modular Moonshine begins with Eguchi, Ooguri, and Tachikawa [64], who made similar observations to those of McKay and Thompson. In this case, the largest sporadic simple Mathieu group  $\mathbb{M}_{24}$  plays the role of the monster group  $\mathbb{M}$ ; they observed that dimensions of representations of  $\mathbb{M}_{24}$  are the multiplicities of superconformal algebra characters in the  $K3$  elliptic genus. An expansion of the elliptic genus led to the mock

modular form

$$(8.3) \quad -8i\left(\mu\left(\frac{1}{2}, \frac{1}{2}; \tau\right) + \mu\left(\frac{1+\tau}{2}, \frac{1+\tau}{2}; \tau\right) + \mu\left(\frac{\tau}{2}, \frac{\tau}{2}; \tau\right)\right) \\ = 2q^{-\frac{1}{8}} \left(-1 + 45q + 231q^2 + 770q^3 + 2277q^4 + \dots\right),$$

expressed in terms of Zwegers'  $\mu$ -functions (see Definition 5.2), which plays the role of the  $j$ -function in the original Moonshine. The first few  $q$ -coefficients shown in (8.3) are dimensions of irreducible representations of  $\mathbb{M}_{24}$ . This led to the conjectured existence of an infinite-dimensional graded  $\mathbb{M}_{24}$ -module, such that the graded dimensions are the coefficients of the mock modular form in (8.3). Moreover, the analogous McKay-Thompson series formed using the graded traces for general  $g \in \mathbb{M}_{24}$  should be mock modular forms. Beautiful work by Gannon [76] proved the existence of this module, following work of Cheng [46], Eguchi and Hikami [63], and Gaberdiel, Hohenegger, and Volpato [73, 74]. Cheng, Duncan, and Harvey [47, 48] later formulated the *Umbral Moonshine Conjectures*, which conjecture the existence of even more additional graded infinite-dimensional modules relating finite groups and mock modular forms, extending the result just described. These new modules arise from *Niemeyer lattices*. A result of Niemeier [117] shows that up to isomorphism, there are 24 even unimodular positive-definite lattices of rank 24, one of which is the Leech lattice, and the others of which have roots systems of full rank, called *Niemeyer root systems*. Attached to such a root system  $X$  is an *umbral group*  $G^X$ , and vector valued modular forms  $(H_{g,r}^X(\tau))_r$ ,  $g \in G^X$ . When  $X = A_1^{24}$ , then  $G^X \cong \mathbb{M}_{24}$ , and  $(H_{g,r}^X)_r$  is a four-dimensional vector indexed by  $r \in \mathbb{Z}/4\mathbb{Z}$ . For this  $X$ ,  $H_{g,r}^X = 0$  for  $r \equiv 0 \pmod{2}$ , and  $H_{g,r}^X = -H_{g,-r}^X$ , thus, the entire vector is determined by  $H_{g,1}^X$ . For each  $g \in \mathbb{M}_{24}$ , these are exactly the mock modular forms from [46, 63, 73, 74]. Generalizing work from Eguchi, Ooguri, and Tachikawa, we have the following ‘‘Umbral’’ conjecture of Cheng, Duncan, and Harvey [47, 48, 60, 61].

**Conjecture 8.1.** *Let  $X$  be a Niemeier root system, and  $m = m^X$  the Coxeter number of any simple component of  $X$ . Then there is a bi-graded infinite-dimensional  $G^X$ -module*

$$\check{K}^X = \bigoplus_{r \in I^X \subseteq \mathbb{Z}/2m\mathbb{Z}} \bigoplus_{\substack{D \leq 0 \\ D \equiv r^2 \pmod{4m}}} \check{K}_{r,-D/4m}^X$$

such that the vector-valued mock modular form  $(H_{g,r}^X(\tau))_r$  is a McKay-Thompson series for  $\check{K}^X$  related to the graded trace of  $g$  on  $\check{K}^X$  by

$$H_{g,r}^X(\tau) - 2q^{-1/4m} \delta_{r,1} + \sum_{\substack{D \leq 0 \\ D \equiv r^2 \pmod{4m}}} \text{tr}(g \mid \check{K}_{r,-D/4m}^X) q^{-D/4m}.$$

As mentioned, Gannon [76] proved the conjecture in the case of  $X = A_1^{24}$ , and in [61], Duncan, Griffin, and Ono proved the remaining cases of the conjecture. Their proof adapts some of Gannon's methods, and also requires recent work of Imamoğlu-Raum-Richter [99] and Mertens [113] on the principle of *holomorphic projection*, which stems from original work on the topic by Gross and Zagier [91]. For the sake of brevity, we define this concept in Definition 8.2 below but refrain from explicitly stating the technical growth conditions which the functions  $g$  must satisfy, and also the definition of the constant  $c_0 = c_0^g$ . Both of these things may be found in [91, 99, 113]. Recall that  $\tau = u + iv \in \mathbb{H}$ .

**Definition 8.2.** Let  $g : \mathbb{H} \rightarrow \mathbb{C}$  be a (not necessarily holomorphic) modular form on some  $\Gamma_0(N)$  with integer weight  $k \geq 2$ , and Fourier expansion

$$g(\tau) = \sum_{n=-\infty}^{\infty} a_g(n, v) q^n.$$

If  $g$  additionally exhibits suitable growth conditions, the **holomorphic projection** of  $g$ ,  $\pi_{\text{hol}}g$ , is defined by

$$(\pi_{\text{hol}}g)(\tau) = (\pi_{\text{hol}}^k g)(\tau) := c_0 + \sum_{n=1}^{\infty} c(n) q^n,$$

where for  $n \in \mathbb{N}$ ,

$$c(n) = c^g(n) := \frac{(4\pi n)^{k-1}}{(k-2)!} \int_0^{\infty} a_g(n, v) e^{-4\pi n v} v^{k-2} dv.$$

The holomorphic projection operator is aptly named, as indicated by the following theorem (see [91, 99, 113]).

**Theorem 8.3.** *Let  $g$  be as in Definition 8.2. The following are true.*

- i) *If  $g$  is holomorphic, then  $\pi_{\text{hol}}g = g$ .*
- ii) *If  $k \geq 4$ , then the holomorphic projection  $\pi_{\text{hol}}g$  is a holomorphic modular form of weight  $k$ . If  $k = 2$ , then  $\pi_{\text{hol}}g$  is a quasimodular form of weight 2.*

(As briefly mentioned in Section 2, quasimodular forms were originally defined by Kaneko and Zagier [102], and the holomorphic Eisenstein series  $E_2$  is an example of such a form.)

A main ingredient to the proof of Umbral Moonshine from [61, 76] is to show that the mock modular forms  $H_{g,r}^X(\tau)$  are replicable. In this case, the authors of [61] establish replicability via the principle of holomorphic projection just discussed. To give a taste of what holomorphic projection can lead to in the mock modular setting, we describe a recent result of Imamoğlu, Raum, and Richter. For  $n \in \mathbb{N}$ , it was shown in [99] that the coefficients  $\alpha_f(n)$  of Ramanujan's mock theta function  $f(q)$  from (2.8) satisfy

$$(8.4) \quad \sum_{3m^2+m \leq 2n} (m + \frac{1}{6}) \alpha_f(n - \frac{3}{2}m^2 - \frac{1}{2}m) = \frac{4}{3}\sigma(n) - \frac{16}{3}\sigma(\frac{n}{2}) - 2 \sum_{2n=ab} d^*(N_{a,b}, \tilde{N}_{a,b}),$$

where  $\sigma(n)$  is equal to the sum of divisors of  $n$  or 0, depending on whether or not  $n$  is an integer,  $N_{a,b}, \tilde{N}_{a,b}$  are explicitly defined linear functions in  $a$  and  $b$ , and  $d^*$  is linearly defined in terms of  $N_{a,b}$  and  $\tilde{N}_{a,b}$ . The coefficients  $\sigma$  in (8.4) arise from the quasimodular form  $E_2$ , and the terms involving  $d^*(N, \tilde{N})$  arise from integrals coming from the non-holomorphic part of the completion of the mock modular form  $f$  (see Section 2, Section 4, and Section 5). Using the principle of holomorphic projection, similar recursions to (8.4) are found in [61] for the coefficients of the mock modular forms in the Umbral Conjecture 8.1.

There have since been further developments related to Umbral Moonshine, and, there is still much to explore on the algebraic side of the story. In addition to the works already mentioned in this section, see, for example, recent work by Cheng, Griffin, Harrison, Mertens,

Ono, Persson, Rolen, Trebat-Leder, and Volpato [49, 88, 121, 122].

*Closing remarks.* In closing, we urge the interested reader to survey the literature, and to consult the references below, for more developments beyond those mentioned in this article on the continuously expanding theory of mock modular forms, and their applications to number theory and other areas of mathematics. The last century, especially the last 15 years, has shown many advances in the subject, which remains an active area of research today.

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DEPARTMENT OF MATHEMATICS AND STATISTICS, AMHERST COLLEGE, AMHERST, MA 01002, USA  
*E-mail address:* afolsom@amherst.edu