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Euler's factorial series and global relations

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To the memory of Marc Huttner
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ABSTRACT

Using Padé approximations to the series $E(z) = \sum_{k=0}^{\infty} k! (-z)^k$, we address arithmetic and analytical questions related to its values in both p -adic and Archimedean valuations.

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1. Introduction

In his 1760 paper on divergent series [5], L. Euler introduced and studied the formal (hypergeometric) series

$$E(z) := \sum_{k=0}^{\infty} k! (-z)^k \quad (1)$$

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(see also [1,2] and, in particular, [8, Section 2.5]), which specialises at $z = 1$ to Wallis' series

$$W := \sum_{k=0}^{\infty} (-1)^k k!, \quad (2)$$

which has teased people's imagination since the times of Euler. Notice that the series (1) is a perfectly convergent p -adic series in the disc $|z|_p \leq 1$ for all primes p , so that one can discuss the arithmetic properties of its values, for example, at $z = 1$ (the Wallis case) and at $z = -1$. The irrationality of the latter specialisation,

$$K = K_p := \sum_{k=0}^{\infty} k!, \quad (3)$$

for any prime p is a folklore conjecture [12, p. 17] (see [11], also for a link of the problem to a combinatorial conjecture of H. Wilf). Already the expectation $|K_p|_p = 1$ (so that K_p is a p -adic unit) for all primes p , which is an equivalent form of Kurepa's conjecture [7] from 1971, remains open; see [4] for the latest achievements in this direction.

In what follows, we refer to the series in (1) as Euler's factorial series and label it $E_p(z)$ when treat it as the function in the p -adic domain for a given prime p .

Theorem 1. *Given $\xi \in \mathbb{Z} \setminus \{0\}$, let \mathcal{P} be a subset of prime numbers such that*

$$\limsup_{n \rightarrow \infty} c^n n! \prod_{p \in \mathcal{P}} |n!|_p^2 = 0, \quad \text{where } c = c(\xi; \mathcal{P}) := 4|\xi| \prod_{p \in \mathcal{P}} |\xi|_p^2. \quad (4)$$

Then either there exists a prime $p \in \mathcal{P}$ for which $E_p(\xi)$ is irrational, or there are two distinct primes $p, q \in \mathcal{P}$ such that $E_p(\xi) \neq E_q(\xi)$ (while $E_p(\xi), E_q(\xi) \in \mathbb{Q}$).

Because $\prod_p |n!|_p = 1/n!$ when the product is taken over all primes, condition (4) is clearly satisfied for any subset \mathcal{P} whose complement in the set of all primes is finite. This also suggests more exotic choices of \mathcal{P} . Furthermore, the conclusion of the theorem is contrasted with Euler's sum

$$\sum_{k=0}^{\infty} k \cdot k! = -1,$$

which is valid in any p -adic valuation (and follows from $\sum_{k=0}^{n-1} k \cdot k! = n! - 1$)—an example of what is called a *global* relation.

The idea behind the proof of Theorem 1 is to construct approximations p_n/q_n to the number $\omega_p = E_p(\xi)$ in question, which do not depend on p and approximate the number considerably well for each p -adic valuation: $q_n \omega_p - p_n = r_{n,p}$ for $n = 0, 1, 2, \dots$; $|r_{n,p}|_p \rightarrow 0$ as $n \rightarrow \infty$ and there are infinitely many indices n for which $r_{n,p} \neq 0$ for at

least one prime $p \in \mathcal{P}$. Assume that $\omega_p = a/b$, the same rational number, for all $p \in \mathcal{P}$. Then $q_n a - p_n b \in \mathbb{Z} \setminus \{0\}$, so that $0 < |q_n a - p_n b|_p \leq 1$, for infinitely many indices n and all primes p , hence

$$\begin{aligned} 1 &= |q_n a - p_n b| \prod_p |q_n a - p_n b|_p \leq |q_n a - p_n b| \prod_{p \in \mathcal{P}} |q_n a - p_n b|_p \\ &\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |br_{n,p}|_p \\ &\leq (|a| + |b|) \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p \end{aligned}$$

for those n . The condition

$$\limsup_{n \rightarrow \infty} \max\{|q_n|, |p_n|\} \prod_{p \in \mathcal{P}} |r_{n,p}|_p = 0 \quad (5)$$

contradicts the latter estimate, thus making the \mathcal{P} -global linear relation $\omega_p = a/b$ impossible.

The result in [Theorem 1](#) can be put in a general context of global relations for Euler-type series; the corresponding settings can be found in the paper [\[3\]](#). We do not pursue this route here as our principal motivation is a sufficiently elementary arithmetic treatment of an analytical function that have some historical value. The rational approximations to $E(z)$ we construct in [Section 2](#) are Padé approximations; in spite of being known for centuries, implicitly from the continued fraction for $E(z)$ given by Euler himself [\[5\]](#) and explicitly from the work of T.J. Stieltjes [\[13\]](#), these Padé approximations remain a useful source for arithmetic and analytical investigations. In [Section 3](#) we revisit Euler's summation of Wallis' series [\(2\)](#) using the approximations; we also complement the derivation by providing 'Archimedean analogue(s)' for the divergent series $K = E(-1)$ from [\(3\)](#)—the case when the classical strategy does not work.

Our way of constructing the Padé approximations is inspired by a related Padé construction of M. Hata and M. Huttner in [\[6\]](#). The construction was a particular favourite of Marc Huttner who, for the span of his mathematical life, remained a passionate advocate of interplay between Picard–Fuchs linear differential equations and Padé approximations. We dedicate this work to his memory.

2. Hypergeometric series and Padé approximations

Euler's factorial series [\(1\)](#), up to the sign change $z \mapsto -z$, is the particular $a = 1$ instance of the hypergeometric series

$${}_2F_0(a, 1 \mid z) = \sum_{k=0}^{\infty} (a)_k z^k. \quad (6)$$

Here and in what follows, we use the Pochhammer notation $(a)_k$ which is defined inductively by $(a)_0 = 1$ and $(a)_{k+1} = (a+k)(a)_k$ for $k \in \mathbb{Z}_{\geq 0}$. Our Padé approximations below are given more generally for the function (6).

Theorem 2. For $n, \lambda \in \mathbb{Z}_{\geq 0}$, take

$$B_{n,\lambda}(z) = \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i z^{n-i}}{(a)_{i+\lambda}}.$$

Then $\deg_z B_{n,\lambda} = n$ and for a polynomial $A_{n,\lambda}(z)$ of degree $\deg_z A_{n,\lambda} \leq n + \lambda - 1$ we have

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) - A_{n,\lambda}(z) = L_{n,\lambda}(z), \quad (7)$$

where $\text{ord}_{z=0} L_{n,\lambda}(z) = 2n + \lambda$. Explicitly,

$$L_{n,\lambda}(z) = (-1)^n n! z^{2n+\lambda} \sum_{k=0}^{\infty} k! \binom{n+k}{k} \binom{n+k+a+\lambda-1}{k} z^k. \quad (8)$$

Proof. Relation (7) means that there is a ‘gap’ of length n in the power series expansion

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) = A_{n,\lambda}(z) + L_{n,\lambda}(z).$$

Write $B_{n,\lambda}(z) = \sum_{h=0}^n b_h z^h$ and consider the series expansion of the product

$$B_{n,\lambda}(z) {}_2F_0(a, 1 \mid z) = \sum_{l=0}^{\infty} r_l z^l,$$

where $r_l = \sum_{h+k=l} b_h (a)_k$; in particular,

$$r_l = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(a)_{i+\lambda+m}}{(a)_{i+\lambda}} = \sum_{i=0}^n (-1)^i \binom{n}{i} (a+i+\lambda)_m$$

with $m = l - n - \lambda$ for $l > n + \lambda - 1$. To verify the desired ‘gap’ condition,

$$r_{n+\lambda} = r_{n+\lambda+1} = \cdots = r_{n+\lambda+n-1} = 0, \quad (9)$$

we introduce the shift operators $N = N_a$ and $\Delta = \Delta_a = N - \text{id}$ defined on functions $f(a)$ by

$$Nf(a) = f(a+1) \quad \text{and} \quad \Delta f(a) = f(a+1) - f(a).$$

It follows from

$$\Delta^n(a + \lambda)_m = (-1)^n(-m)_n(a + \lambda + n)_{m-n} \quad \text{for } n \in \mathbb{Z}_{\geq 0}$$

that

$$\begin{aligned} r_l &= \sum_{i=0}^n (-1)^i \binom{n}{i} N^i(a + \lambda)_m = (\text{id} - N)^n(a + \lambda)_m \\ &= (-\Delta)^n(a + \lambda)_m = (-m)_n(a + \lambda + n)_{m-n}, \end{aligned}$$

which, in turn, implies (9) because of the vanishing of $(-m)_n$ for $m = 0, 1, \dots, n-1$. The explicit expression for r_l just found also gives

$$r_{2n+\lambda+k} = (-n-k)_n(a + \lambda + n)_k = (-1)^n n! \binom{n+k}{k} k! \binom{n+k+a+\lambda-1}{k},$$

hence the closed form (8). We also have

$$A_{n,\lambda}(z) = \sum_{l=0}^{n+\lambda-1} r_l z^l = \sum_{l=0}^{n+\lambda-1} z^l \sum_{\substack{i=0 \\ i \geq n-l}}^n (-1)^i \binom{n}{i} \frac{(a)_{i+l-n}}{(a)_{i+\lambda}}.$$

This concludes our proof of the theorem. \square

From now on we choose $a = 1$, $\lambda = 0$, change z to $-z$ and renormalise the corresponding Padé approximations produced by Theorem 2 by multiplying them by $(-1)^n n!$:

$$\begin{aligned} Q_n(z) &:= (-1)^n n! B_{n,0}(-z) = \sum_{i=0}^n \binom{n}{i} \frac{n!}{i!} z^{n-i} = \sum_{i=0}^n i! \binom{n}{i}^2 z^i, \\ P_n(z) &:= (-1)^n n! A_{n,0}(-z) = \sum_{l=0}^{n-1} (-z)^l \sum_{i=n-l}^n (-1)^{n-i} \binom{n}{i} \frac{n! (i+l-n)!}{i!} \\ &= \sum_{l=0}^{n-1} (-z)^l \sum_{i=0}^l (-1)^i i! (l-i)! \binom{n}{i}^2 \end{aligned}$$

and

$$R_n(z) := (-1)^n n! L_{n,0}(-z) = n!^2 z^{2n} \sum_{k=0}^{\infty} (-1)^k k! \binom{n+k}{k}^2 z^k.$$

In this notation, the Padé approximation formula (7) may be rewritten as

$$Q_n(z)E(z) - P_n(z) = R_n(z) \tag{10}$$

for $n \in \mathbb{Z}_{>0}$. Observe that

$$Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z) = n!^2 z^{2n}. \quad (11)$$

Indeed, the standard Padé walkabout proves the identity

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = Q_{n+1}(z)R_n(z) - Q_n(z)R_{n+1}(z), \quad (12)$$

in which the degree of the left-hand side is at most $2n$ while the order of the right-hand side is at least $2n$. Further, the lowest term of the right-hand side in (12) is the product of the constant term 1 of $Q_{n+1}(z)$ and the lowest term $n!^2 z^{2n}$ of $R_n(z)$, confirming (11).

Proof of Theorem 1. We may use the formal series identity (10) to get appropriate numerical approximations for any $z = \xi \in \mathbb{Z} \setminus \{0\}$. For $n = 1, 2, \dots$, take $p_n = P_n(\xi) \in \mathbb{Z}$, $q_n = Q_n(\xi) \in \mathbb{Z}$ and define

$$r_{n,p} = R_n(\xi) = q_n E_p(\xi) - p_n$$

for each prime $p \in \mathcal{P}$. Using elementary summation formulas and trivial estimates for binomials we have

$$|q_n| \leq |\xi|^n n! \sum_{i=0}^n \binom{n}{i}^2 = |\xi|^n n! \binom{2n}{n} < 4^n |\xi|^n n!,$$

$$|p_n| \leq |\xi|^{n-1} n \sum_{i=0}^{n-1} i! (n-1-i)! \binom{n}{i}^2 \leq |\xi|^n n! \sum_{i=0}^n \binom{n}{i}^2 < 4^n |\xi|^n n!$$

and

$$|r_{n,p}|_p = |\xi|_p^{2n} |n!|_p^2 \left| \sum_{k=0}^{\infty} (-1)^k k! \binom{n+k}{k}^2 \xi^k \right|_p \leq |\xi|_p^{2n} |n!|_p^2. \quad (13)$$

Therefore, condition (5) follows from

$$\limsup_{n \rightarrow \infty} 4^n |\xi|^n n! \prod_{p \in \mathcal{P}} |\xi|_p^{2n} |n!|_p^2 = 0$$

and because for every $n > 0$ we have either $r_{n,p} \neq 0$ or $r_{n+1,p} \neq 0$ from (11), the theorem follows. \square

3. Summation of divergent series

Let again $\xi \in \mathbb{Z} \setminus \{0\}$ and fix a prime p . If we restrict, for simplicity, the sequence of indices n to the arithmetic progression $n \equiv 0 \pmod{p}$, then it is not hard to see from the calculation in Section 2 that the sequence of rational approximations $p_n/q_n = P_n(\xi)/Q_n(\xi)$ converges p -adically to $E_p(\xi)$. Indeed,

$$q_n = 1 + \sum_{i=1}^{p-1} i! \binom{n}{i}^2 \xi^i + \sum_{i=p}^n i! \binom{n}{i}^2 \xi^i \equiv 1 \pmod{p}$$

is a p -adic unit (all the binomials are divisible by p in the first sum, while $i!$ is divisible by p in the second one), and

$$\left| E_p(\xi) - \frac{p_n}{q_n} \right|_p = |q_n|_p^{-1} |q_n E_p(\xi) - p_n|_p = |r_{n,p}|_p \leq |n!|_p^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

by the estimate (13).

When $\xi > 0$, the same sequence of rational numbers $p_n/q_n = P_n(\xi)/Q_n(\xi)$ converges to the value at $z = \xi$ of the integral

$$\tilde{E}(z) = \int_0^\infty \frac{e^{-s}}{1+zs} \, ds \quad (14)$$

with respect to the Archimedean norm. The integral itself converges for any $z \in \mathbb{C} \setminus (-\infty, 0]$, though its formal Taylor expansion at $z = 0$ (obtained by expanding $1/(1+zs)$ into the power series under the integral sign) is precisely Euler's factorial series $E(z)$ as in (1). In particular, relation (10) remains valid with $E(z)$ replaced by $\tilde{E}(z)$ and $R_n(z)$ by

$$\tilde{R}_n(z) = n! z^{2n} \int_0^\infty \frac{s^n e^{-s}}{(1+zs)^{n+1}} \, ds \quad \text{for } n = 0, 1, 2, \dots,$$

so that

$$\left| \tilde{E}(\xi) - \frac{p_n}{q_n} \right| = \frac{\tilde{R}_n(\xi)}{Q_n(\xi)} \leq \frac{n^n}{(n+1)^{n+1}} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

where the estimates $Q_n(\xi) \geq n! \xi^n$ and

$$\tilde{R}_n(\xi) \leq n! \xi^{2n} \left(\max_{s>0} \frac{s^n}{(1+\xi s)^{n+1}} \right) \int_0^\infty e^{-s} \, ds = n! \xi^n \frac{n^n}{(n+1)^{n+1}}$$

were used. This computation reveals us that

$$\lim_{n \rightarrow \infty} \frac{p_n}{q_n} = \tilde{E}(\xi) = -x e^x \left(\gamma + \log x + \sum_{k=1}^\infty \frac{(-x)^k}{k \cdot k!} \right) \Big|_{x=1/\xi},$$

where $\gamma = 0.5772156649 \dots$ is Euler's constant. In particular,

$$W = \tilde{E}(1) = e \left(-\gamma + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \cdot k!} \right) = 0.5963473623 \dots$$

for Wallis' series (2). The resulting quantity is known as the Euler–Gompertz constant [8].

The strategy in the previous paragraph does not apply to $z = \xi < 0$, somewhat already observed by Stieltjes in [13]. The analytical continuation of the function $\tilde{E}(z)$ depends on whether we perform the integration along the upper or lower banks of the ray $[0, \infty)$ in (14); denote the corresponding values by $\tilde{E}_+(z)$ and $\tilde{E}_-(z)$, respectively. By considering the integration of $e^{-s}/(1+zs)$ along the curvilinear triangle that consists of the segment $[0, R]$ (along a particular bank), the arc $[R, R e^{\sqrt{-1}\theta}]$ followed by the segment $[R e^{\sqrt{-1}\theta}, 0]$, where $0 < \theta < \pi/2$ for the upper bank and $-\pi/2 < \theta < 0$ for the lower one, and then taking the limit as $R \rightarrow \infty$ (so that the integral along the arc tends to 0) we conclude that

$$\tilde{E}_{\pm}(z) = \int_0^{e^{\sqrt{-1}\theta}\infty} \frac{e^{-s}}{1+zs} ds = -xe^x \left(\gamma + \log|x| \pm \sqrt{-1}\pi + \sum_{k=1}^{\infty} \frac{(-x)^k}{k \cdot k!} \right) \Big|_{x=1/z}, \quad (15)$$

with the choice of θ arbitrary in the interval $0 < \theta < \pi/2$ for $\tilde{E}_+(z)$ and in the interval $-\pi/2 < \theta < 0$ for $\tilde{E}_-(z)$. In particular,

$$\begin{aligned} K = \tilde{E}_{\pm}(-1) &= \frac{1}{e} \left(\gamma + \sum_{k=1}^{\infty} \frac{1}{k \cdot k!} \pm \sqrt{-1}\pi \right) \\ &= 0.6971748832 \dots \pm \sqrt{-1} \cdot 1.1557273497 \dots \end{aligned}$$

for the series in (3).

4. Final remarks

In [14] we outline a different strategy of proving a result analogous to Theorem 1 on using the Hankel determinants generated by the tails of Euler's factorial series (1). As the condition on a subset of primes \mathcal{P} in that result is spiritually similar to (4), we do not detail the derivation here. However we stress that a potential combination of the two methods, namely, using the Hankel determinants generated by the Padé approximations of Euler's factorial series, may be a source of further novelties on the topic. A discussion on this type of construction in the Archimedean setting can be found in [15].

One consequence of the formula in (15), which uncovers a pair of *complex* conjugate values for $E(\xi)$ when $\xi < 0$, is that the *rational* approximations $p_n/q_n = P_n(\xi)/Q_n(\xi)$ do not converge at all in such cases. Interestingly enough, the Hankel determinants 'see' those complex values (15) as experimentally observed in [14].

Finally, we would like to note that nothing is known about the irrationality and transcendence of the Archimedean valuations of Euler's factorial series (1) at rational $z = \xi$ (see the discussion in [8, Sections 3.15, 3.16]). This is in contrast with its q -analogue

$$\sum_{k=0}^{\infty} z^k \prod_{i=1}^k (1 - q^i),$$

for which irrationality and linear independence results are known in Archimedean and non-Archimedean places alike—see [9]. Further details on a nice q -counterpart of the Padé approximation analysis can be found in [10].

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