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Mean values of multivariable multiplicative functions and applications to the average number of cyclic subgroups and multivariable averages associated with the LCM function.

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Abstract

We use multiple zeta functions to prove, under suitable assumptions, precise asymptotic formulas for the averages of multivariable multiplicative functions. As applications, we prove some conjectures on the average number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ and multivariable averages associated with the LCM function.

Keywords: Mean values of multivariable arithmetic functions, multiplicative functions, Zeta functions, meromorphic continuation, tauberian theorems, subgroups averages, LCM multivariable averages.
2000 MSC: 11N37, 11M32, 11M41, 11M45

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1. Introduction

Our paper is motivated by the following recent results and conjectures. Let $n \in \mathbb{N}$ and for $m_1, \ldots, m_n \in \mathbb{N}$ let $c_n(m_1, \ldots, m_n)$ denote the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$. W. G. Nowak and L. Tóth [5] (2014) proved the asymptotic formula

$$\sum_{1 \le m_1, m_2 \le x} c_2(m_1, m_2) = x^2 \left(\frac{12}{\pi^4} (\ln x)^3 + a_2 (\ln x)^2 + a_1 (\ln x) + a_0 \right) + O(x^{\frac{1117}{701}+\varepsilon}) \quad \text{as} \quad x \to \infty,$$

where a_0, a_1 and a_2 are explicit constants. This error term was improved by L. Tóth and W. Zhai [9] (2018) into $O(x^{3/2} (\ln x)^{13/2})$. The case $n = 3$ was investigated by L.

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Tóth and W. Zhai [10] (2020) showing that

$$\sum_{1\le m_1,m_2,m_3\le x} c_3(m_1,m_2,m_3) = x^3 \sum_{j=0}^{7} c_j(\ln x)^j + O(x^{8/3+\varepsilon}),$$

where c_j $(0 \le j \le 7)$ are explicit constants. For the proof they used a multidimensional Perron formula and the complex integration method. It is natural to conjecture that such a result holds for $n \ge 4$.

T. Hilberdink, F. Luca, and L. Tóth [4] (2020) investigated the following three averages associated with the LCM function:

$$S_n(x) := \sum_{1\le m_1,\ldots,m_n\le x} \frac{1}{\mathrm{lcm}(m_1,\ldots,m_n)}, \tag{1}$$

$$U_n(x) := \sum_{\substack{1\le m_1,\ldots,m_n\le x\\ \gcd(m_1,\ldots,m_n)=1}} \frac{1}{\mathrm{lcm}(m_1,\ldots,m_n)}, \tag{2}$$

and

$$V_n(x) := \sum_{1\le m_1,\ldots,m_n\le x} \frac{m_1\ldots m_n}{\mathrm{lcm}(m_1,\ldots,m_n)}. \tag{3}$$

By using the convolution method, they obtained in their paper asymptotic formulas with error terms for $S_2(x)$, $U_2(x)$ and $V_2(x)$. For $n \ge 3$, they only obtained the estimates

$$(\ln x)^{2^n-1} \ll S_n(x) \ll (\ln x)^{2^n-1}, \quad (\ln x)^{2^n-2} \ll U_n(x) \ll (\ln x)^{2^n-2},$$
$$x^n \ll V_n(x) \ll x^n\ (\ln x)^{2^n-2} \quad \text{as} \quad x\to\infty,$$

and conjectured that asymptotic formulas with error terms also exist for these three averages for $n \ge 3$.

In order to prove these conjectures, we introduce a reasonably large class of mutivariable multiplicative functions (see Definition 2). For a function $f : \mathbb{N}^n \to \mathbb{R}_+$ in this class, we establish in Theorem 1 the existence of the meromorphic continuation of the associated multiple zeta function

$$\mathbf{s} = (s_1,\ldots,s_n) \to \mathcal{M}(f;\mathbf{s}) := \sum_{m_1\ge1,\ldots,m_n\ge1} \frac{f(m_1,\ldots,m_n)}{m_1^{s_1}\ldots m_n^{s_n}}$$

and derive several precise properties of this meromorphic continuation. By combining our Theorem 1 and La Bretèche's multivariable Tauberian Theorem (i.e., Theorems 1 and 2 of [1] (2001)) *we deduce in our Theorem 2 a precise asymptotic formula for the multivariable average*

$$\mathcal{N}_\infty(f;x) := \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n\\ \|\mathbf{m}\|_\infty=\max_i m_i\le x}} f(m_1,\ldots,m_n) \quad \textit{as } x\to\infty,$$

and derive from it four corollaries.

Our first application, namely Corollary 1, establishes the conjecture concerning the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$, in any dimension n. Our Corollaries 2, 3 and 4 prove the conjectures on the three sums above associated with the LCM function.

Variants of Theorem 2 with other norm choices can be obtained by combining our Theorem 1 and the first author's multivariable tauberian theorem (i.e., Corollary 2 of [2] (2012)). For example, for the class of Hölder's norms $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \cdots + |x_n|^d}$ $(d \geq 1)$, *we obtain in Theorem 3 an asymptotic for the multivariable average*

$$\mathcal{N}_d(f;x) := \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d+\cdots+m_n^d} \leq x}} f(m_1,\ldots,m_n) \quad \textit{as } x \to \infty.$$

As an application of Theorem 3, we derive in Corollaries 5 and 6 the analogues of Corollaries 1 and 4 for the Hölder's norms $\| \ \|_d$.

1.1. Notations

1. $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$; $\mathbb{R}_+ = [0, \infty)$.

2. The expression: $f(\lambda, \mathbf{y}, \mathbf{x}) \ll_{\mathbf{y}} g(\mathbf{x})$ uniformly in $\mathbf{x} \in X$ and $\lambda \in \Lambda$ means there exists $A = A(\mathbf{y}) > 0$, such that, $\forall \mathbf{x} \in X$ and $\forall \lambda \in \Lambda \quad |f(\lambda, \mathbf{y}, \mathbf{x})| \leq A g(\mathbf{x})$;

3. Let $d \in [1, +\infty[$, for any $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$, we set $\|\mathbf{x}\|_d = \sqrt[d]{|x_1|^d + ... + |x_n|^d}$, and $\|\mathbf{x}\|_\infty = \max_{i=1,\ldots,n} |x_i|$. We denote the canonical basis of $\mathbb{R}^n$ by $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ (i.e. $\mathbf{e}_{i,j} = 1$ if $i = j$ and $\mathbf{e}_{i,j} = 0$ if $i \neq j$). The standard inner product on $\mathbb{R}^n$ is denoted by $\langle ., . \rangle$. We set also $\mathbf{0} = (0, \ldots, 0)$ and $\mathbf{1} = (1, \ldots, 1)$;

4. We denote a vector in $\mathbb{C}^n$ by $\mathbf{s} = (s_1, \ldots, s_n)$, and write $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau}$, where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ are the real resp. imaginary components of $\mathbf{s}$ (i.e. $\sigma_i = \Re(s_i)$ and $\tau_i = \Im(s_i)$ for all i). We also write $\langle \mathbf{x}, \mathbf{s} \rangle$ for $\sum_i x_i s_i$ if $\mathbf{x} \in \mathbb{R}^n, \mathbf{s} \in \mathbb{C}^n$;

5. A function $f : \mathbb{N}^n \to \mathbb{C}$ is said to be multiplicative if for all $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$ and $\mathbf{m}' = (m'_1, \ldots, m'_n) \in \mathbb{N}^n$ satisfying $\gcd(\mathrm{lcm}(m_i), \mathrm{lcm}(m'_i)) = 1$ we have $f(m_1 m'_1, \ldots, m_n m'_n) = f(\mathbf{m}) \cdot f(\mathbf{m}')$;

6. Let F be a meromorphic function on a domain $\mathcal{D}$ of $\mathbb{C}^n$ and let $\mathcal{S}$ be the support of its polar divisor. F is said to be of moderate growth if $\exists a, b > 0$ such that $\forall \delta > 0$, $F(\mathbf{s}) \ll_{\boldsymbol{\sigma},\delta} 1 + \|\boldsymbol{\tau}\|_1^{a\|\boldsymbol{\sigma}\|_1 + b}$ uniformly in $\mathbf{s} = \boldsymbol{\sigma} + i\boldsymbol{\tau} \in \mathcal{D}$ verifying $d(\mathbf{s}, \mathcal{S}) \geq \delta$;

2. A class of multivariable multiplicative functions and statement of the main results

2.1. A class of multivariable multiplicative functions

To simplify the exposition, we introduce first the following three definitions.

Definition 1. *A quadruple $(g, \kappa, \mathbf{c}, \delta)$ is said to be a data if*

1. *$g : \mathbb{N}_0^n \to \mathbb{N}_0$ is a function of subexponential growth; that is g verifies for any $\varepsilon > 0$ $g(\boldsymbol{\nu}) \ll_\varepsilon e^{\varepsilon \|\boldsymbol{\nu}\|_1}$ uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$;*
2. *$\kappa : \mathbb{N}_0^n \to [1, \infty) \cup \{0\}$ is a function verifying $\kappa(\mathbf{0}) = 0$ and $\inf_{\boldsymbol{\nu} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}} \frac{\kappa(\boldsymbol{\nu})}{\|\boldsymbol{\nu}\|_1} > 0$*
3. *$\mathbf{c} = (c_1, \ldots, c_n) \in [0, \infty)^n$ and $\delta \in (0, \infty)$.*

We now introduce the class of multivariable multiplicative functions on which we will focus in this paper.

Definition 2. *Let $(g, \kappa, \mathbf{c}, \delta)$ be a data as in definition 1.*
A multivariable multiplicative function $f : \mathbb{N}^n \to \mathbb{R}$ is said to be in the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ if for any $\varepsilon > 0$,

$$f(p^{\nu_1}, \ldots, p^{\nu_n}) - g(\boldsymbol{\nu})\ p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle - \kappa(\boldsymbol{\nu})} \ll_\varepsilon e^{\varepsilon \|\boldsymbol{\nu}\|_1}\ p^{\langle \mathbf{c}, \boldsymbol{\nu} \rangle - \kappa(\boldsymbol{\nu}) - \delta}, \tag{4}$$

uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and p prime number.

We will need also the following integral definition.

Definition 3. *Let I be a finite subset of $\mathbb{N}_0^n \setminus \{\mathbf{0}\}$, $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ be a finite sequence of positive integers and $\mathbf{c} = (c_1, \ldots, c_n) \in [0, \infty)^n$. We denote by $\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^r$ the elements of I where $r = \#I$, and define the finite sequence q_k $(0 \leq k \leq r)$ by*

$$q_0 = 0 \quad \text{and} \quad q_k = \sum_{j=1}^{k} u(\boldsymbol{\nu}^j)\ \forall k = 1, \ldots, r.$$

We define then for $x > 0$ the integral

$$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) := \int_{\mathcal{A}(I, \mathbf{u}; x)} \frac{dy_1 \ldots dy_{q_r}}{\prod_{k=1}^{r} \prod_{\ell=q_{k-1}+1}^{q_k} y_\ell^{1 - \langle \boldsymbol{\nu}^k, \mathbf{c} \rangle}},$$

where $$\mathcal{A}(I, \mathbf{u}; x) := \left\{ \mathbf{y} \in [1, \infty)^{q_r};\ \prod_{k=1}^{r} \prod_{\ell=q_{k-1}+1}^{q_k} y_\ell^{\langle \boldsymbol{\nu}^k, \mathbf{e}_j \rangle} \leq x \quad \forall j = 1, \ldots, n \right\}.$$

2.2. Statement of the main results

Let $f : \mathbb{N}^n \to \mathbb{R}$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see definitions 1 and 2 above).
We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \quad \text{is nonempty.} \tag{5}$$

The following theorem is *the main analytic ingredient* of this paper:

Theorem 1. *1. the multiple zeta function*

$$\mathbf{s} = (s_1, \dots, s_n) \to \mathcal{M}(f; \mathbf{s}) := \sum_{m_1 \geq 1, \dots, m_n \geq 1} \frac{f(m_1, \dots, m_n)}{m_1^{s_1} \dots m_n^{s_n}}$$

converges absolutely in the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > c_i \ \forall i = 1, \dots, n\}$*;*

2. there exists $\varepsilon_0 > 0$ *such that the function*

$$\mathbf{s} = (s_1, \dots, s_n) \to \mathcal{H}(f, \mathbf{c}; \mathbf{s}) := \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{s} \rangle^{g(\boldsymbol{\nu})} \right) \mathcal{M}(f; \mathbf{c} + \mathbf{s})$$

has **holomorphic** *continuation to the domain* $\{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > -\varepsilon_0 \ \forall i = 1, \dots, n\}$ *and verifies in it the following estimate: for all* $\varepsilon > 0$,

$$\mathcal{H}(f, \mathbf{c}; \mathbf{s}) \ll_\varepsilon \prod_{\boldsymbol{\nu} \in I} (|\langle \boldsymbol{\nu}, \mathbf{s} \rangle| + 1)^{g(\boldsymbol{\nu})\left(1 - \frac{1}{2} \min(0, \Re(\langle \boldsymbol{\nu}, \mathbf{s} \rangle))\right) + \varepsilon};$$

3. $\mathcal{H}(f, \mathbf{c}; \mathbf{0})$ *is given by the following convergent Euler product:*

$$\mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})} \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \boldsymbol{\nu}, \mathbf{c} \rangle}} \right). \tag{6}$$

Combining our Theorem 1 and La Bretèche's multivariable Tauberian Theorem (i.e Theorems 1 and 2 of [1] (2001)) yields to the following multivariable mean value theorem:

Theorem 2. *Let* $f : \mathbb{N}^n \to \mathbb{R}_+$ *be a nonnegative multivariable multiplicative function satisfying assumptions of Theorem 1. Set* $J := \{\mathbf{e}_i \mid c_i = 0\}$ *where* $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ *is the canonical basis of* $\mathbb{R}^n$*. Set also* $\rho := \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) + \#J - \mathrm{Rank}\,(I \cup J)$*.*
Then, there exist a polynomial Q_∞ *of degree at most* ρ *and a positive constant* $\mu_\infty > 0$ *such that*

$$\mathcal{N}_\infty(f; x) := \sum_{\substack{\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \leq x}} f(m_1, \dots, m_n) = x^{\|\mathbf{c}\|_1} Q_\infty(\ln x) + O\left(x^{\|\mathbf{c}\|_1 - \mu_\infty}\right) \quad \text{as } x \to \infty.$$

Furthermore, if we assume in addition that the two following assumptions hold:

1. *Rank* $(I \cup J) = n$;

2. $\mathbf{1} = (1, \ldots, 1)$ *is in the interior of the cone generated by* $I \cup J$*; that is* $\mathbf{1} \in$ *con*$^* (I \cup J) := \{\sum_{\boldsymbol{\nu} \in I \cup J} \lambda_{\boldsymbol{\nu}} \boldsymbol{\nu} \mid \lambda_{\boldsymbol{\nu}} \in (0, \infty)\ \forall \boldsymbol{\nu} \in I \cup J\}$,

Then, the degree of the polynomial Q_∞ *is equal to* $\rho = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) + \#J - n$ *and the main term of* $\mathcal{N}_\infty(f; x)$ *is given by*

$$\mathcal{N}_\infty(f; x) = C_n(f) K_n(f, \|\|_\infty)\ x^{\|\mathbf{c}\|_1} (\ln x)^\rho + O\left((\ln x)^{\rho - 1}\right) \quad \text{as } x \to \infty,$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ *is defined by the Euler product (6) and*

$$K_n(f, \|\|_\infty) := \lim_{x \to \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\ x^{-\|\mathbf{c}\|_1} (\ln x)^{-\rho} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ *is the integral (see definition 3) associated to the finite set* I*, the finite sequence* $\mathbf{u} = (g(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ *and to the vector* $\mathbf{c}$.

Remark 1. *The existence of the limit* $K_n(f, \|\|_\infty)$ *follows from the proof of Theorem2. If* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset I \cup J$*, then the two assumptions Rank* $(I \cup J) = n$ *and* $\mathbf{1} \in$ *con*$^* (I \cup J)$ *clearly hold.*

Combining our Theorem 1 and the first author's multivariable tauberian theorem (i.e corollary 2 of [2] (2012)) yields to the following multivariable mean value theorem for Hölder's norms $\|\mathbf{x}\|_d := \sqrt[d]{|x_1|^d + \cdots + |x_n|^d}$ $(d \geq 1)$:

Theorem 3. *Let* $f : \mathbb{N}^n \to \mathbb{R}_+$ *be a nonnegative multivariable multiplicative function satisfying assumptions of Theorem 1.*
Assume that $\mathbf{c} = (c_1, \ldots, c_n) \in (0, \infty)^n$*. Set*

1. $\rho := \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) -$ *Rank*(I);

2. $I_{\mathbf{c}} := \{\langle \mathbf{c}, \boldsymbol{\nu} \rangle^{-1} \boldsymbol{\nu} \mid \boldsymbol{\nu} \in I\}$ *and* $\mathbf{u} := (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in I_{\mathbf{c}}}$ *where* $u(\boldsymbol{\beta}) = \sum_{\boldsymbol{\nu} \in I, \langle \mathbf{c}, \boldsymbol{\nu} \rangle^{-1} \boldsymbol{\nu} = \boldsymbol{\beta}} g(\boldsymbol{\nu})$.

Then, there exist a polynomial Q *of degree at most* ρ *and a positive constant* $\mu > 0$ *such that*

$$\mathcal{N}_d(f; x) := \sum_{\substack{\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n \\ \|\mathbf{m}\|_d = \sqrt[d]{m_1^d + \cdots + m_n^d} \leq x}} f(m_1, \ldots, m_n) = x^{\|\mathbf{c}\|_1} Q(\ln x) + O\left(x^{\|\mathbf{c}\|_1 - \mu}\right) \quad \text{as } x \to \infty.$$

Furthermore, if we assume in addition that Rank$(I) = n$ *and* $\mathbf{1} \in$ *con*$^* (I)$*, then, the degree of the polynomial* Q *is equal to* $\rho = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) - n$ *and the main term of* $\mathcal{N}_d(f; x)$ *is given by*

$$\mathcal{N}_d(f; x) = C_n(f) K_n(f, \|\|_d)\ x^{\|\mathbf{c}\|_1} (\ln x)^\rho + O\left((\ln x)^{\rho - 1}\right) \quad \text{as } x \to \infty,$$

where $C_n(f) := \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ *is defined by the Euler product (6) above and*

$$K_n(f, \|\|_d) := \left(\prod_{\boldsymbol{\nu}\in I} \langle \boldsymbol{\nu}, \mathbf{c}\rangle^{-g(\boldsymbol{\nu})}\right) \frac{d^{\rho+1}\ A_0(\mathcal{T}_{\mathbf{c}}, P_d)}{\|\mathbf{c}\|_1\ \rho!} > 0.$$

where $A_0(\mathcal{T}_{\mathbf{c}}, P_d) > 0$ *is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the pair* $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$ *and the polynomial* $P_d = X_1^d + \cdots + X_n^d$.

2.3. Applications

We will now give the applications that motivated our general results of section §2.2.

2.3.1. On the average number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$

Let $n \in \mathbb{N}$. For $m_1, \ldots, m_n \in \mathbb{N}$ denote by $c_n(m_1, \ldots, m_n)$ the number of cyclic subgroups of the group $\mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$. Set

$$G_n(x) := \sum_{1\le m_1,\ldots,m_n\le x} c_n(m_1, \ldots, m_n).$$

As we mentioned in the introduction, precise asymptotic for $G_2(x)$ was obtained by W. G. Nowak and L. Tóth in [5] (2014) and improved by L. Tóth and W. Zhai in [9] (2018). The case $n = 3$ was also investigated by L. Tóth and W. Zhai in [10] (2020). It is natural to conjecture that such a result holds for $n \ge 4$. The following result establish this conjecture in any dimension n.

Corollary 1. *Let* $n \in \mathbb{N}$. *There exists a polynomial* Q_1 *of degree* $2^n - 1$ *and* $\mu_1 > 0$ *such that*

$$G_n(x) := \sum_{1\le m_1,\ldots,m_n\le x} c_n(m_1, \ldots, m_n) = x^n\ Q_1(\ln x) + O(x^{n-\mu_1}) \quad \textit{as} \quad x \to \infty.$$

In particular, we have

$$G_n(x) = C_n(c_n) K_n(c_n, \|\|_\infty)\ x^n (\ln x)^{2^n-1} + O\left(x^n (\ln x)^{2^n-2}\right) \quad \textit{as} \quad x \to \infty,$$

where

$$C_n(c_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n+n-1} \left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n} \frac{c_n(p^{\nu_1}, \ldots, p^{\nu_n})}{p^{\|\boldsymbol{\nu}\|_1}}\right) > 0 \tag{7}$$

and

$$K_n(c_n, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}; x)\ x^{-n} (\ln x)^{-2^n+1} > 0, \qquad \textit{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ *is the integral (see definition 3) associated to* $I = \{0,1\}^n \setminus \{\mathbf{0}\}$, *to the sequence* $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ *defined by* $u(\mathbf{e}_i) = 2\ \forall i = 1, \ldots, n$ *and* $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I \setminus \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ *and to the vector* $\mathbf{c} = \mathbf{1}$.

Remark 2. *We will compute more explicitly in §7 below the constants* $C_n(c_n)$ *and* $K_n(c_n, \|\|_\infty)$ *for* $n = 2$ *and* $n = 3$*. In particular, we will prove in §7.1 and §7.3 that* $C_2(c_2) = \frac{36}{\pi^4}$ *and* $K_2(c_2, \|\|_\infty) = \frac{1}{3}$*. Thus, our mains term in the asymptotic of* $G_2(x)$ *agree with the main term obtained by the convolution method in [5] (2014) by W. G. Nowak and L. Tóth.*

2.3.2. Some multivariable averages associated to the LCM function

As we mentioned in the introduction, T. Hilberdink, F. Luca, and L. Tóth introduced in [4] (2020) the three averages (1), (2) and (3) associated to the LCM function and obtained in this paper asymptotic formulas for $S_2(x)$, $U_2(x)$ and $V_2(x)$. For $n \geq 3$, they only obtained the following estimates

$$(\ln x)^{2^n-1} \ll S_n(x) \ll (\ln x)^{2^n-1}, \quad (\ln x)^{2^n-2} \ll U_n(x) \ll (\ln x)^{2^n-2},$$

$$x^n \ll V_n(x) \ll x^n \ (\ln x)^{2^n-2},$$

and conjectured that asymptotic formulas also exist for these three averages for $n \geq 3$. The following three corollaries prove these conjectures.

Corollary 2. *Let* $n \in \mathbb{N}$*. There exists a polynomial* Q_2 *of degree* $2^n - 1$ *and* $\mu_2 > 0$ *such that*

$$S_n(x) := \sum_{1 \leq m_1, \ldots, m_n \leq x} \frac{1}{\operatorname{lcm}(m_1, \ldots, m_n)} = Q_2(\ln x) + O(x^{-\mu_2}) \quad \text{as} \quad x \to \infty.$$

In particular, we have

$$S_n(x) = C_n(s_n) K_n(s_n, \|\|_\infty) \ (\ln x)^{2^n-1} + O\left((\ln x)^{2^n-2}\right) \quad \text{as} \quad x \to \infty,$$

where

$$C_n(s_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \tag{8}$$

and

$$K_n(s_n, \|\|_\infty) := \lim_{x \to \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) \ (\ln x)^{-2^n+1} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ *is the integral (see definition 3) associated to* $I = \{0,1\}^n \setminus \{\mathbf{0}\}$*, to the sequence* $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$ *defined by* $u(\boldsymbol{\nu}) = 1 \ \forall \boldsymbol{\nu} \in I$ *and to the vector* $\mathbf{c} = \mathbf{0}$*.*

Corollary 3. *Let* $n \in \mathbb{N} \setminus \{1\}$*. There exists a polynomial* Q_3 *of degree* $2^n - 2$ *and* $\mu_3 > 0$ *such that*

$$U_n(x) := \sum_{\substack{1 \leq m_1, \ldots, m_n \leq x \\ \gcd(m_1, \ldots, m_n) = 1}} \frac{1}{\operatorname{lcm}(m_1, \ldots, m_n)} = Q_3(\ln x) + O(x^{-\mu_3}) \quad \text{as} \quad x \to \infty.$$

In particular, we have

$$U_n(x) = C_n(u_n)K_n(u_n, \|\|\|_\infty)\ (\ln x)^{2^n-2} + O\left((\ln x)^{2^n-3}\right) \qquad as \qquad x \to \infty,$$

where

$$C_n(u_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \tag{9}$$

and

$$K_n(u_n, \|\|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-2^n+2} > 0, \qquad where$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see definition 3) associated to $I = \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$.

Corollary 4. *Let $n \in \mathbb{N}$. There exists a polynomial Q_4 of degree $2^n - n - 1$ and $\mu_4 > 0$ such that*

$$V_n(x) := \sum_{1\le m_1,\ldots,m_n\le x} \frac{m_1 \ldots m_n}{\mathrm{lcm}(m_1, \ldots, m_n)} = x^n\ Q_4(\ln x) + O(x^{n-\mu_4}) \qquad as \qquad x \to \infty.$$

In particular, we have

$$V_n(x) = C_n(v_n)K_n(v_n, \|\|\|_\infty)\ x^n\ (\ln x)^{2^n-n-1} + O\left(x^n\ (\ln x)^{2^n-n-2}\right) \qquad as \qquad x \to \infty,$$

where

$$C_n(v_n) := \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \tag{10}$$

and

$$K_n(v_n, \|\|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\ x^{-n}(\ln x)^{-2^n+n+1} > 0, \qquad where$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see definition 3) associated to $I = \{0,1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{1}$.

Remark 3. *The constants $C_n(s_n)$, $C_n(u_n)$ and $C_n(v_n)$ are equal. We will compute more explicitly in sections §7.2, §7.4, §7.5 and §7.6 below the constants $C_n(.)$ and $K_n(., \|\|\|_\infty)$ for $n = 2$ and $n = 3$. More precisely, we will prove that*

1. *$C_2(s_2) = C_2(u_2) = C_2(v_2) = \frac{6}{\pi^2}$ and*
   $C_3(s_3) = C_3(u_3) = C_3(v_3) = \prod_p \left(1 - \frac{9}{p^2} + \frac{16}{p^3} - \frac{9}{p^4} + \frac{1}{p^6}\right)$;
2. *$K_2(s_2, \|\|\|_\infty) = \frac{1}{3}$, $K_2(u_2, \|\|\|_\infty) = 1$ and $K_2(v_2, \|\|\|_\infty) = 1$;*
3. *$K_3(s_3, \|\|\|_\infty) = \frac{11}{3366}$, $K_3(u_3, \|\|\|_\infty) = \frac{11}{480}$ and $K_3(v_3, \|\|\|_\infty) = \frac{1}{16}$.*

In particular, our mains terms in the asymptotic of $S_2(x)$, $U_2(x)$ and $V_2(x)$ agree with those obtained by the convolution method in [4] (2020) by T. Hilberdink, F. Luca, and L. Tóth.

2.3.3. Multivariable averages with other norms

The following two results give analogues of corollaries 1 and 4 for some other choices of norms.

Corollary 5. *Let $n \in \mathbb{N}$ and $d \geq 1$. There exists a polynomial Q_5 of degree $2^n - 1$ and $\mu_5 > 0$ such that*

$$G_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n \\ \|\mathbf{m}\|_d=\sqrt[d]{m_1^d+\cdots+m_n^d}\leq x}} c_n(m_1,\dots,m_n) = x^n\, Q_5(\ln x) + O(x^{n-\mu_5}) \quad \text{as} \quad x \to \infty.$$

Moreover, if we set $\tilde{I} := \left\{\|\boldsymbol{\nu}\|_1^{-1}\boldsymbol{\nu};\ \boldsymbol{\nu} \in \{0,1\}^n \setminus \{\mathbf{0}\}\right\}$ and $\mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}$ where $u(\boldsymbol{\beta}) = 2$ if $\boldsymbol{\beta} \in \{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ and $u(\boldsymbol{\beta}) = 1$ otherwise, then

$$G_{n,d}(x) = C_n(c_n)\ K_n(c_n, \|\|_d)\ x^n(\ln x)^{2^n-1} + O\left(x^n(\ln x)^{2^n-2}\right) \quad \text{as} \quad x \to \infty,$$

where $C_n(c_n) > 0$ is given by (7) and

$$K_n(c_n, \|\|_d) = \left(\prod_{k=2}^{n} k^{-\binom{n}{k}}\right) \frac{d^{2^n}\ A_0(\mathcal{T}, P_d)}{n\ (2^n-1)!} > 0,$$

where $A_0(\mathcal{T}, P_d)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the pair $\mathcal{T} = (\tilde{I}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \cdots + X_n^d$.

Corollary 6. *Let $n \in \mathbb{N}$ and $d \geq 1$. There exists a polynomial Q_6 of degree $2^n - n - 1$ and $\mu_6 > 0$ such that*

$$V_{n,d}(x) := \sum_{\substack{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n \\ \|\mathbf{m}\|_d=\sqrt[d]{m_1^d+\cdots+m_n^d}\leq x}} \frac{m_1 \dots m_n}{\mathrm{lcm}(m_1,\dots,m_n)} = x^n\, Q_6(\ln x) + O(x^{n-\mu_6}) \quad \text{as} \quad x \to \infty.$$

Moreover, if we set $\tilde{I} := \left\{\|\boldsymbol{\nu}\|_1^{-1}\boldsymbol{\nu};\ \boldsymbol{\nu} \in \{0,1\}^n \setminus \{\mathbf{0}\}\right\}$ and $\mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}$ where $u(\boldsymbol{\beta}) = 1\ \forall \boldsymbol{\beta} \in \tilde{I}$, then

$$V_{n,d}(x) = C_n(v_n)\ K_n(v_n, \|\|_d)\ x^n(\ln x)^{2^n-n-1} + O\left(x^n(\ln x)^{2^n-n-2}\right) \quad \text{as} \quad x \to \infty,$$

where $C_n(v_n) > 0$ is given by (10) and

$$K_n(v_n, \|\|_d) = \left(\prod_{k=2}^{n} k^{-\binom{n}{k}}\right) \frac{d^{2^n-n}\ A_0(\mathcal{T}, P_d)}{n\ (2^n-n-1)!} > 0,$$

where $A_0(\mathcal{T}, P_d)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the pair $\mathcal{T} = (\tilde{I}, \mathbf{u})$ and the polynomial $P_d = X_1^d + \cdots + X_n^d$.

Remark 4. *The constants $C_n(.)$, are independent on the choice of the norm. The constants $K_n(.)$ depend on the choice of the norm. We will compute more explicitly in §7.7 below the constants $K_n(c_n, \|\|_d)$ and $K_n(v_n, \|\|_d)$ for $n = 2$ and $n = 3$. More precisely, we will prove that*

1. $K_2(c_2, \|\|_d) = \dfrac{1}{6d}\dfrac{\Gamma(1/d)^2}{\Gamma(2/d)}$ *and* $K_2(v_2, \|\|_d) = \dfrac{1}{2d}\dfrac{\Gamma(1/d)^2}{\Gamma(2/d)}$;
2. $K_3(c_3, \|\|_d) = \dfrac{31\ \Gamma(1/d)^3}{30240\ d^2\ \Gamma(3/d)}$ *and* $K_3(v_3, \|\|_d) = \dfrac{\Gamma(1/d)^3}{2\ d^2\ \Gamma(3/d)}$.

3. Proof of Theorem 1

Let $f : \mathbb{N}^n \to \mathbb{R}$ be a multiplicative function in the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$.
Define for $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and p prime $V(p, \boldsymbol{\nu})$ by the formula

$$f(p^{\nu_1}, \dots, p^{\nu_n}) = (g(\boldsymbol{\nu}) + V(p, \boldsymbol{\nu}))\ p^{\langle \mathbf{c}, \boldsymbol{\nu}\rangle - \kappa(\boldsymbol{\nu})} \tag{11}$$

Since $(g, \kappa, \mathbf{c}, \delta)$ is a data, point 1 of definition 1 and assumption (4) can then be written in the following more convenient equivalent form:

$$\forall \varepsilon > 0, \quad g(\boldsymbol{\nu}) \ll_\varepsilon e^{\varepsilon\|\boldsymbol{\nu}\|_1} \quad \text{and} \quad V(p, \boldsymbol{\nu}) \ll_\varepsilon e^{\varepsilon\|\boldsymbol{\nu}\|_1}\ p^{-\delta}, \tag{12}$$

uniformly in $\boldsymbol{\nu} \in \mathbb{N}_0^n$ and in p prime number.

Moreover, point 2 of definition 1 implies that there exists $\beta > 0$ such that

$$\kappa(\boldsymbol{\nu}) \geq \max(1, \beta\|\boldsymbol{\nu}\|_1) \quad \forall \boldsymbol{\nu} \in \mathbb{N}_0^n \setminus \{\mathbf{0}\}. \tag{13}$$

3.1. Proof of point 1 of Theorem 1

Let $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{C}^n$ be such that $\sigma_i = \Re(s_i) > c_i\ \forall i = 1, \dots, n$.
Set $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_n)$ and $\eta = \frac{1}{2}\min_{i=1,\dots,n}(\sigma_i - c_i) > 0$.
So, we have $\sigma_i \geq c_i + 2\eta\ \forall i$ and $\langle \boldsymbol{\sigma}, \boldsymbol{\nu}\rangle \geq \langle \mathbf{c}, \boldsymbol{\nu}\rangle + 2\eta\|\boldsymbol{\nu}\|_1\ \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$. Choose $\varepsilon > 0$ small enough such that $e^\varepsilon < 2^\eta$. It follows then from (11) and (12) that we have for any prime number p,

$$\begin{aligned}
\sum_p \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \left|\frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \mathbf{s}, \boldsymbol{\nu}\rangle}}\right| &= \sum_p \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \frac{|f(p^{\nu_1}, \dots, p^{\nu_n})|}{p^{\langle \boldsymbol{\sigma}, \boldsymbol{\nu}\rangle}} \ll_\varepsilon \sum_p \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1} p^{\langle \mathbf{c}, \boldsymbol{\nu}\rangle - \kappa(\boldsymbol{\nu})}}{p^{\langle \mathbf{c}, \boldsymbol{\nu}\rangle + 2\eta\|\boldsymbol{\nu}\|_1}} \\
&\ll_\varepsilon \sum_p \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1}}{p^{\kappa(\boldsymbol{\nu}) + 2\eta\|\boldsymbol{\nu}\|_1}} \ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1}}{p^{\eta\|\boldsymbol{\nu}\|_1}} \\
&\ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} \sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \left(\frac{e^\varepsilon}{2^\eta}\right)^{\|\boldsymbol{\nu}\|_1} \ll_\varepsilon \sum_p \frac{1}{p^{1+\eta}} < \infty.
\end{aligned}$$

The multiplicativity of f implies then that $\mathbf{s} \to \mathcal{M}(f;\mathbf{s})$ converges absolutely and that

$$\mathcal{M}(f;\mathbf{s}) = \sum_{\mathbf{m}\in\mathbb{N}^n} \frac{f(m_1,\ldots,m_n)}{m^{s_1}\ldots m_n^{s_n}} = \prod_p \left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n} \frac{f(p^{\nu_1},\ldots,p^{\nu_n})}{p^{\langle \mathbf{s},\boldsymbol{\nu}\rangle}} \right). \tag{14}$$

This ends the proof of point 1 of Theorem 1. $\square$

3.2. Two useful lemmas

Recall that $I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\}$ is an nonempty set.
For all $t \in \mathbb{R}$, set $U_t := \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i = \Re(s_i) > t \ \forall i = 1,\ldots,n\}$.
We need the following two lemmas:

Lemma 1. *There exists $\varepsilon_1, \eta_1 > 0$ such that for any prime number p, the function*

$$\mathbf{s} \mapsto R_p(\mathbf{s}) := \left(\sum_{\|\boldsymbol{\nu}\|_1 \geq 1} \frac{f(p^{\nu_1},\ldots,p^{\nu_n})}{p^{\langle \mathbf{c}+\mathbf{s},\boldsymbol{\nu}\rangle}} \right) - \left(\sum_{\boldsymbol{\nu}\in I} \frac{g(\boldsymbol{\nu})}{p^{1+\langle \boldsymbol{\nu},\mathbf{s}\rangle}} \right)$$

is holomorphic in the domain $U_{-\varepsilon_1}$ and verifies in it the estimate

$$R_p(\mathbf{s}) \ll p^{-1-\eta_1} \quad \textit{uniformly in } p.$$

Lemma 2. *Set $\varepsilon_2 = \inf_{\boldsymbol{\nu}\in I} \frac{1}{4\|\boldsymbol{\nu}\|_1}$ and $\eta_2 = \frac{1}{2}$. Then, for any prime number p, the function*

$$\mathbf{s} \mapsto L_p(\mathbf{s}) := \left(\prod_{\boldsymbol{\nu}\in I} \left(1 - \frac{1}{p^{1+\langle \boldsymbol{\nu},\mathbf{s}\rangle}}\right)^{g(\boldsymbol{\nu})} \right) - 1 + \left(\sum_{\boldsymbol{\nu}\in I} \frac{g(\boldsymbol{\nu})}{p^{1+\langle \boldsymbol{\nu},\mathbf{s}\rangle}} \right)$$

is holomorphic in the domain $U_{-\varepsilon_2}$ and verifies in it the estimate

$$L_p(\mathbf{s}) \ll p^{-1-\eta_2} \quad \textit{uniformly in } p.$$

3.2.1. Proof of Lemma 1

Fix $\beta > 0$ such that (13) holds. Fix also a positive integer N verifying $N \geq \max\left(4\beta^{-1}, \max_{\boldsymbol{\nu}\in I} \|\boldsymbol{\nu}\|_1\right)$.
Identity (11) implies that for p prime number and $\mathbf{s} \in U_0 = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > 0 \ \forall i\}$, we have

$$R_p(\mathbf{s}) = R_p^1(\mathbf{s}) + R_p^2(\mathbf{s}), \quad \text{where} \tag{15}$$

$$R_p^1(\mathbf{s}) = \sum_{1\leq \|\boldsymbol{\nu}\|_1 \leq N} \frac{V(p,\boldsymbol{\nu})}{p^{\kappa(\boldsymbol{\nu})+\langle \boldsymbol{\nu},\mathbf{s}\rangle}} + \sum_{\substack{\boldsymbol{\nu}\notin I\\ 1\leq \|\boldsymbol{\nu}\|_1 \leq N}} \frac{g(\boldsymbol{\nu})}{p^{\kappa(\boldsymbol{\nu})+\langle \boldsymbol{\nu},\mathbf{s}\rangle}} \quad \text{and} \quad R_p^2(\mathbf{s}) = \sum_{\|\boldsymbol{\nu}\|_1 > N} \frac{g(\boldsymbol{\nu}) + V(p,\boldsymbol{\nu})}{p^{\kappa(\boldsymbol{\nu})+\langle \boldsymbol{\nu},\mathbf{s}\rangle}}.$$

To prove Lemma 1 it suffices to verify that both $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ and $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ satisfy its conclusions.

CLAIM 1: $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ satisfies the conclusions of Lemma 1.
Proof of CLAIM 1: It's clear that $\mathbf{s} \mapsto R_p^1(\mathbf{s})$ is holomorphic in the whole space $\mathbb{C}^n$.
Let $\varepsilon > 0$. It follows from (12) and (13) that for p prime number and $\mathbf{s} \in U_{-\varepsilon} = \{\mathbf{s} \in \mathbb{C}^n \mid \sigma_i > -\varepsilon \ \forall i\}$, we have

$$|R_p^1(\mathbf{s})| \leq \sum_{1 \leq \|\boldsymbol{\nu}\|_1 \leq N} \frac{|V(p,\boldsymbol{\nu})|}{p^{1+\langle \boldsymbol{\nu}, \boldsymbol{\sigma}\rangle}} + \sum_{\substack{\boldsymbol{\nu} \notin I \\ 1 \leq \|\boldsymbol{\nu}\|_1 \leq N}} \frac{g(\boldsymbol{\nu})}{p^{\kappa(\boldsymbol{\nu})+\langle \boldsymbol{\nu}, \boldsymbol{\sigma}\rangle}} \ll \sum_{\boldsymbol{\nu} \in I} \frac{p^{-\delta}}{p^{1-\varepsilon\|\boldsymbol{\nu}\|_1}} + \sum_{\substack{\boldsymbol{\nu} \notin I, \ g(\boldsymbol{\nu}) \neq 0 \\ 1 \leq \|\boldsymbol{\nu}\|_1 \leq N}} \frac{1}{p^{\kappa(\boldsymbol{\nu})-\varepsilon\|\boldsymbol{\nu}\|_1}} \tag{16}$$

Since $\kappa(\boldsymbol{\nu}) > 1$ if $\boldsymbol{\nu} \notin I \cup \{\mathbf{0}\}$ and $g(\boldsymbol{\nu}) \neq 0$, it is clear that we can choose $\varepsilon > 0$ small enough such that

$$\mu_1 := \min_{\boldsymbol{\nu} \in I} (\delta - \varepsilon\|\boldsymbol{\nu}\|_1) > 0 \text{ and } \mu_2 := \min\{\kappa(\boldsymbol{\nu}) - \varepsilon\|\boldsymbol{\nu}\|_1 - 1 \mid 1 \leq \|\boldsymbol{\nu}\|_1 \leq N, \ \boldsymbol{\nu} \notin I \text{ and } g(\boldsymbol{\nu}) \neq 0\} > 0.$$

Set $\mu = \min(\mu_1, \mu_2) > 0$. It follows then from (16) that we have $R_p^1(\mathbf{s}) \ll p^{-1-\mu}$ uniformly in p prime number and in $\mathbf{s} \in U_{-\varepsilon}$. This ends the proof of CLAIM 1. □

CLAIM 2: $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ satisfies the conclusions of Lemma 1.
Proof of CLAIM 2: Fix $\varepsilon > 0$ such that $e^{\varepsilon} < 2^{\beta/4}$. Assumptions (12) and (13) imply that we have uniformly in p prime number and in $\mathbf{s} \in U_{-\beta/2}$,

$$\begin{aligned} \sum_{\|\boldsymbol{\nu}\|_1 > N} \left| \frac{g(\boldsymbol{\nu}) + V(p,\boldsymbol{\nu})}{p^{\kappa(\boldsymbol{\nu})+\langle \boldsymbol{\nu}, \mathbf{s}\rangle}} \right| &\ll_{\varepsilon} \sum_{\|\boldsymbol{\nu}\|_1 > N} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1}}{p^{\beta\|\boldsymbol{\nu}\|_1 + \langle \boldsymbol{\nu}, \boldsymbol{\sigma}\rangle}} \leq \sum_{\|\boldsymbol{\nu}\|_1 > N} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1}}{p^{\frac{\beta}{2}\|\boldsymbol{\nu}\|_1}} \ll_{\varepsilon} \frac{1}{p^{\frac{\beta}{4}N}} \sum_{\|\boldsymbol{\nu}\|_1 > N} \frac{e^{\varepsilon\|\boldsymbol{\nu}\|_1}}{2^{\frac{\beta}{4}\|\boldsymbol{\nu}\|_1}} \\ &\ll_{\varepsilon} \frac{1}{p^{\frac{\beta}{4}N}} \sum_{\|\boldsymbol{\nu}\|_1 > N} \left(\frac{e^{\varepsilon}}{2^{\frac{\beta}{4}}}\right)^{\|\boldsymbol{\nu}\|_1} \ll_{\varepsilon} \frac{1}{p^{\frac{\beta}{4}N}} \leq \frac{1}{p^2}. \end{aligned}$$

We deduce that $\mathbf{s} \mapsto R_p^2(\mathbf{s})$ is holomorphic in $U_{-\beta/2}$ and verifies the estimates $R_p^2(\mathbf{s}) \ll p^{-2}$ uniformly in p prime number and $\mathbf{s} \in U_{-\beta/2}$. This ends the proof of CLAIM 2 and also ends the proof of Lemma 1. □

3.3. Proof of Lemma 2

It is clear that $\mathbf{s} \mapsto L_p(\mathbf{s})$ is holomorphic in $\mathbb{C}^n$ for any p.
Set now $\varepsilon_2 = \inf_{\boldsymbol{\nu} \in I} \frac{1}{4\|\boldsymbol{\nu}\|_1}$. It follows that for $\mathbf{s} \in U_{-\varepsilon_2}$ and $\boldsymbol{\nu} \in I$, $1 + \langle \boldsymbol{\nu}, \boldsymbol{\sigma}\rangle \geq 1 - \varepsilon_2\|\boldsymbol{\nu}\|_1 \geq 3/4$.
Newton Binomial theorem implies then that we have uniformly in $\mathbf{s} \in U_{-\varepsilon_2}$ and in p

prime number,

$$\begin{aligned}
|L_p(\mathbf{s})| &= \left|\left(\prod_{\boldsymbol{\nu}\in I}\left(1-\frac{1}{p^{1+\langle\boldsymbol{\nu},\mathbf{s}\rangle}}\right)^{g(\boldsymbol{\nu})}\right)-1+\left(\sum_{\boldsymbol{\nu}\in I}\frac{g(\boldsymbol{\nu})}{p^{1+\langle\boldsymbol{\nu},\mathbf{s}\rangle}}\right)\right| \\
&= \left|\sum_{\substack{0\le k_{\boldsymbol{\nu}}\le g(\boldsymbol{\nu})\ \forall\boldsymbol{\nu}\in I,\\ \sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}\ge 2}}\frac{\prod_{\boldsymbol{\nu}\in I}(-1)^{k_{\boldsymbol{\nu}}}\binom{g(\boldsymbol{\nu})}{k_{\boldsymbol{\nu}}}}{p^{\sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}(1+\langle\boldsymbol{\nu},\mathbf{s}\rangle)}}\right| \ll \sum_{\substack{0\le k_{\boldsymbol{\nu}}\le g(\boldsymbol{\nu})\ \forall\boldsymbol{\nu}\in I,\\ \sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}\ge 2}}\frac{1}{p^{\sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}(1+\langle\boldsymbol{\nu},\boldsymbol{\sigma}\rangle)}} \\
&\ll \sum_{\substack{0\le k_{\boldsymbol{\nu}}\le g(\boldsymbol{\nu})\ \forall\boldsymbol{\nu}\in I,\\ \sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}\ge 2}}\frac{1}{p^{\frac{3}{4}\sum_{\boldsymbol{\nu}\in I}k_{\boldsymbol{\nu}}}} \ll \frac{1}{p^{3/2}}.
\end{aligned}$$

This ends the proof of Lemma 2.

3.3.1. Proof of parts 2 and 3 of Theorem 1

Define the function $\mathbf{s}=(s_1,\dots,s_n)\mapsto\mathcal{E}(f;\mathbf{s})$ by

$$\mathcal{E}(f;\mathbf{s}) := \left(\prod_{\boldsymbol{\nu}\in I}\zeta\left(1+\langle\boldsymbol{\nu},\mathbf{s}\rangle\right)^{-g(\boldsymbol{\nu})}\right)\mathcal{M}(f;\mathbf{c}+\mathbf{s}). \tag{17}$$

Part 1 of Theorem 1 implies then that $\mathbf{s}\mapsto\mathcal{E}(f;\mathbf{s})$ converges absolutely in the domain $U_0=\{\mathbf{s}\in\mathbb{C}^n \mid \sigma_i>0\ \forall i\}$. Moreover, The multiplicativity of f imply that for all $\mathbf{s}\in U_0$:

$$\mathcal{E}(f;\mathbf{s}) = \prod_p \mathcal{E}_p(f;\mathbf{s}), \quad \text{where} \tag{18}$$

$$\mathcal{E}_p(f;\mathbf{s}) := \prod_{\boldsymbol{\nu}\in I}\left(1-\frac{1}{p^{1+\langle\boldsymbol{\nu},\mathbf{s}\rangle}}\right)^{g(\boldsymbol{\nu})}\left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n}\frac{f(p^{\nu_1},\dots,p^{\nu_n})}{p^{\langle\boldsymbol{\nu},\mathbf{c}+\mathbf{s}\rangle}}\right).$$

We will now prove the following needed lemma:

Lemma 3. *There exists $\varepsilon_0>0$ such that the Euler product $\mathbf{s}\mapsto\mathcal{E}(f;\mathbf{s})=\prod_p\mathcal{E}_p(f;\mathbf{s})$ converges absolutely and defines a bounded holomorphic function in the domain $U_{-\varepsilon_0}=\{\mathbf{s}\in\mathbb{C}^n \mid \sigma_i>-\varepsilon_0\ \forall i=1,\dots,n\}$.*

Proof of Lemma 3:
We will use in the sequel of this proof notation of Lemmas 1 and 2. Lemmas 1 and 2 imply that for any prime p and any $\mathbf{s}\in U_0$,

$$\begin{aligned}
\mathcal{E}_p(f;\mathbf{s}) &= \left(1-\left(\sum_{\boldsymbol{\nu}\in I}\frac{g(\boldsymbol{\nu})}{p^{1+\langle\boldsymbol{\nu},\mathbf{s}\rangle}}\right)+L_p(\mathbf{s})\right)\left(1+\left(\sum_{\boldsymbol{\nu}\in I}\frac{g(\boldsymbol{\nu})}{p^{1+\langle\boldsymbol{\nu},\mathbf{s}\rangle}}\right)+R_p(\mathbf{s})\right) \\
&= 1-A_p(\mathbf{s})^2+B_p(\mathbf{s})+C_p(\mathbf{s}), \quad \text{where}
\end{aligned} \tag{19}$$

$$A_p(\mathbf{s}) := \sum_{\boldsymbol{\nu}\in I} \frac{g(\boldsymbol{\nu})}{p^{1+\langle \boldsymbol{\nu},\mathbf{s}\rangle}},\ B_p(\mathbf{s}) := (1 - A_p(\mathbf{s}))\, R_p(\mathbf{s}) \text{ and } C_p(\mathbf{s}) := L_p(\mathbf{s})\,(1 + A_p(\mathbf{s}) + R_p(\mathbf{s}))\,.$$

Let $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 > 0$ the positive constants defined in Lemmas 1 and 2.
Set $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2) > 0$ and $\eta_0 = \min(\eta_1, \eta_2) = \min(\eta_1, 1/2) > 0$. Lemmas 1 and 2 imply that the three function A_p, B_p and C_p are holomorphic in $U_{-\varepsilon_0}$ and that we have uniformly in p prime number and $\mathbf{s} \in U_{-\varepsilon_0}$ the following estimates:

1. $A_p(\mathbf{s}) \ll \sum_{\boldsymbol{\nu}\in I} \frac{1}{p^{1+\langle \boldsymbol{\nu},\boldsymbol{\sigma}\rangle}} \le \sum_{\boldsymbol{\nu}\in I} \frac{1}{p^{1-\varepsilon_0\|\boldsymbol{\nu}\|_1}} \ll \frac{1}{p^{3/4}}$;
2. $A_p(\mathbf{s})^2 \ll \frac{1}{p^{3/2}} \ll \frac{1}{p^{1+\eta_0}}$;
3. $B_p(\mathbf{s}) \ll \left(1 + \frac{1}{p^{3/4}}\right) \frac{1}{p^{1+\eta_1}} \ll \frac{1}{p^{1+\eta_0}}$;
4. $C_p(\mathbf{s}) \ll \frac{1}{p^{1+\eta_2}} \left(1 + \frac{1}{p^{3/4}} + \frac{1}{p^{1+\eta_1}}\right) \ll \frac{1}{p^{1+\eta_0}}$.

It follows that for any prime number p, the function $\mathbf{s} \mapsto \mathcal{E}_p(f;\mathbf{s}) - 1$ is holomorphic in $U_{-\varepsilon_0}$ and verifies $\mathcal{E}_p(f;\mathbf{s}) - 1 \ll \frac{1}{p^{1+\eta_0}}$ uniformly in $\mathbf{s} \in U_{-\varepsilon_0}$ and in the prime number p.
We deduce that the Euler product $\mathbf{s} \mapsto \mathcal{E}(f;\mathbf{s}) = \prod_p \mathcal{E}_p(f;\mathbf{s})$ converges absolutely and defines a bounded holomorphic function in $U_{-\varepsilon_0}$. This ends the proof of Lemma 3. □

We are now ready to prove points 2 and 3 of Theorem 1. Combining part 1 of Theorem 1, (17) and (18) implies that for $\mathbf{s} \in U_0$,

$$\mathcal{H}(f, \mathbf{c}; \mathbf{s}) := \left(\prod_{\boldsymbol{\nu}\in I} \langle \boldsymbol{\nu}, \mathbf{s}\rangle^{g(\boldsymbol{\nu})}\right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) = \left(\prod_{\boldsymbol{\nu}\in I} \left(\langle \boldsymbol{\nu}, \mathbf{s}\rangle \zeta\left(1 + \langle \boldsymbol{\nu}, \mathbf{s}\rangle\right)\right)^{g(\boldsymbol{\nu})}\right) \mathcal{E}(f; \mathbf{s}). \tag{20}$$

Part 2 of Theorem 1 follows then from Lemma 3 and the following two classical properties of Riemann zeta function: $s \mapsto s\zeta(1 + s)$ is holomorphic in $\mathbb{C}$ and verifies in the half-plane $\{\Re(s) > -1\}$ the estimate $s\ \zeta(1 + s) \ll_\varepsilon (1 + |s|)^{1-\frac{1}{2}\min(0,\Re(s))+\varepsilon}$, $\forall \varepsilon > 0$.
Moreover, since $s\zeta(1 + s)|_{s=0} = 1$, we deduce from (20) and (18) that

$$\mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \mathcal{E}(f; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})} \left(\sum_{\boldsymbol{\nu}\in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \dots, p^{\nu_n})}{p^{\langle \boldsymbol{\nu}, \mathbf{c}\rangle}}\right).$$

This ends the proof of point 3 and also the proof of Theorem 1. □

4. Proofs of Theorems 2 and 3

4.1. Proof of Theorem 2

We will now explain how the combination of our Theorem 1 and La Bretèche's multivariable Tauberian Theorem (i.e Theorems 1 and 2 of [1] (2001)) yields to our Theorem 2. Our notations are different from La Bretèche's notations. To simplify the exposition, we will first recall La Bretèche's Tauberian Theorem 1 and the part we use of his Tauberian Theorem 2 by using our notations:

Theorem A: (Theorem 1 of [1] (2001)):
Let $f:\mathbb{N}^n\to\mathbb{R}_+$ be a nonnegative function and F the associated Dirichlet's series defined by

$$F(\mathbf{s})=F(s_1,\dots,s_n)=\sum_{m_1,\dots,m_n\geq 1}\frac{f(m_1,\dots,m_n)}{m_1^{s_1}\dots m_n^{s_n}}.$$

Denote by $\mathcal{LR}_n^+(\mathbb{C})$ the set of $\mathbb{C}-$linear forms from $\mathbb{C}^n$ to $\mathbb{C}$ that are nonnegative on $(\mathbb{R}_+)^n$.
We assume that there exists $\mathbf{c}=(c_1,\dots,c_n)\in(\mathbb{R}_+)^n$ such that:

1. *$F(\mathbf{s})$ converges absolutely for $\mathbf{s}\in\mathbb{C}^n$ such that $\Re(s_i)>c_i$ $\forall i=1,\dots,n$;*
2. *There exist a finite family $\mathcal{L}=\left(\ell^{(i)}\right)_{1\leq i\leq q}$ of nonzero elements of $\mathcal{LR}_n^+(\mathbb{C})$, a finite family $\left(h^{(i)}\right)_{1\leq i\leq q'}$ of elements of $\mathcal{LR}_n^+(\mathbb{C})$ and $\delta_1,\delta_2,\delta_3>0$ such that the function H defined by*

$$H(\mathbf{s})=F(\mathbf{c}+\mathbf{s})\prod_{i=1}^{q}\ell^{(i)}(\mathbf{s})$$

has **holomorphic** *continuation to the domain*

$$\mathcal{D}(\delta_1,\delta_3):=\{\mathbf{s}\in\mathbb{C}^n\mid\Re\left(\ell^{(i)}(\mathbf{s})\right)>-\delta_1\ \forall i=1,\dots,q\ \text{and}\ \Re\left(h^{(i)}(\mathbf{s})\right)>-\delta_3\ \forall i=1,\dots,q'\}$$

and verifies the estimate: for $\varepsilon,\varepsilon'>0$ we have uniformly in $\mathbf{s}\in\mathcal{D}(\delta_1-\varepsilon',\delta_3-\varepsilon')$

$$H(\mathbf{s})\ll\prod_{i=1}^{q}\left(\left|\Im\left(\ell^{(i)}(\mathbf{s})\right)\right|+1\right)^{1-\delta_2\min\left(0,\Re\left(\ell^{(i)}(\mathbf{s})\right)\right)}\left(1+(|\Im(s_1)|+\dots+|\Im(s_n)|)^{\varepsilon}\right).$$

Set $J=J(\mathbf{c})=\{j\in\{1,\dots,n\}\mid c_j=0\}$. Denote by $w=\#J$ the cardinality of the set J and by $j_1<\dots<j_w$ its elements in increasing order. Define the w linear forms $\ell^{(q+i)}$ $(1\leq i\leq w)$ by $\ell^{q+i}(\mathbf{s})=\mathbf{e}_{j_i}^*(\mathbf{s})=s_{j_i}$.
Then, for any $\boldsymbol{\beta}=(\beta_1,\dots,\beta_n)\in(0,\infty)^n$, there exist a polynomial $Q_{\boldsymbol{\beta}}\in\mathbb{R}[X]$ of degree at most $q+w-Rank\left\{\ell^{(1)},\dots,\ell^{(q+w)}\right\}$ and $\theta>0$ such that

$$\sum_{1\leq m_1\leq x^{\beta_1}}\cdots\sum_{1\leq m_n\leq x^{\beta_n}}f(m_1,\dots,m_n)=x^{\langle\mathbf{c},\boldsymbol{\beta}\rangle}Q_{\boldsymbol{\beta}}(\log x)+O(x^{\langle\mathbf{c},\boldsymbol{\beta}\rangle-\theta})\ \text{as}\ x\to\infty.$$

Theorem B: (parts (ii) and (iv) from Theorem 2 of [1] (2001)):
Let $f : \mathbb{N}^n \to \mathbb{R}_+$ be a function satisfying assumptions of Theorem A.
Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in (0, \infty)^n$. Set $\mathcal{B} = \sum_{i=1}^{n} \beta_i \mathbf{e}_i^* \in \mathcal{LR}_n^+(\mathbb{C})$.

- **(ii)** *If the Dirichlet's series F satisfies the additional two assumptions:*
  (C1) There exists a function G such $H(\mathbf{s}) = G\left(\ell^{(1)}(\mathbf{s}), \ldots, \ell^{(q+w)}(\mathbf{s})\right)$.
  (C2) $\mathcal{B} \in Vect\left(\{\ell^{(k)} \mid k = 1, \ldots, q+w\}\right)$ and there is no subfamily $\mathcal{L}'$ of $\mathcal{L}_0 := \left(\ell^{(k)}\right)_{1 \le k \le q+w}$ such that $\mathcal{L}' \neq \mathcal{L}_0$, $\mathcal{B} \in Vect(\mathcal{L}')$ and $\#\mathcal{L}' - Rank(\mathcal{L}') = \#\mathcal{L}_0 - Rank(\mathcal{L}_0)$.

  Then, the polynomial $Q_{\boldsymbol{\beta}}$ satisfies the relation
  $$Q_{\boldsymbol{\beta}}(\log x) = H(\mathbf{0}) x^{-\langle \mathbf{c}, \boldsymbol{\beta} \rangle} \mathcal{I}_{\boldsymbol{\beta}}(x) + O\left((\log x)^{\rho - 1}\right),$$
  where $\rho := q + w - Rank\left\{\ell^{(1)}, \ldots, \ell^{(q+w)}\right\}$ and
  $$\mathcal{I}_{\boldsymbol{\beta}}(x) := \int_{\mathcal{A}_{\boldsymbol{\beta}}(x)} \frac{dy_1 \ldots dy_q}{\prod_{i=1}^{q} y_i^{1-\ell^{(i)}(\mathbf{c})}},$$
  with
  $$\mathcal{A}_{\boldsymbol{\beta}}(x) := \{\mathbf{y} \in [1, \infty)^q \mid \prod_{i=1}^{q} y_i^{\ell^{(i)}(\mathbf{e}_j)} \le x^{\beta_j} \quad \forall j = 1, \ldots, n\}.$$

- **(iv)** *If $Rank\left\{\ell^{(1)}, \ldots, \ell^{(q+w)}\right\} = n$, $H(\mathbf{0}) \neq 0$ and $\mathcal{B} \in con^*\left(\left\{\ell^{(1)}, \ldots, \ell^{(q+w)}\right\}\right)$, then $deg(Q_{\boldsymbol{\beta}}) = \rho = q + w - n$.*

Remark : If assumptions of point (iv) hold, then assumptions of the point (ii) also clearly hold.

Proof of Theorem 2:
Let $f : \mathbb{N}^n \to \mathbb{R}_+$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see definitions 1 and 2). We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \quad \text{is nonempty.}$$

We denote by $\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^r$ the elements of I where $r = \#I$,
and define the finite sequence q_k $(0 \le k \le r)$ by

$$q_0 = 0 \quad \text{and} \quad q_k = \sum_{j=1}^{k} g(\boldsymbol{\nu}^j) \ \forall k = 1, \ldots, r.$$

We define the linear forms $\ell^{(i)}$ $(1 \le i \le q_r)$ by

$$\ell^{(i)}(\mathbf{s}) = \langle \boldsymbol{\nu}^k, \mathbf{s} \rangle \quad \text{if } q_{k-1} < i \le q_k \text{ and } 1 \le k \le r.$$

We define also the set $J = J(\mathbf{c}) = \{j \in \{1,\dots,n\} \mid c_j = 0\}$. We denote by $w = \#J$ the cardinality of the set J and by $j_1 < \cdots < j_w$ its elements in increasing order. We define also the w linear forms $\ell^{(q+i)}$ $(1 \le i \le w)$ by

$$\ell^{q+i}(\mathbf{s}) = \mathbf{e}_{j_i}^*(\mathbf{s}) = s_{j_i} \quad (1 \le i \le w).$$

By using notation of our Theorem 1 it's easy to see that the Dirichlet's series associated to f is

$$F(\mathbf{s}) = \sum_{m_1,\dots,m_n \ge 1} \frac{f(m_1,\dots,m_n)}{m_1^{s_1}\dots m_n^{s_n}} = \mathcal{M}(f;\mathbf{s})$$

and

$$H(\mathbf{s}) = \left(\prod_{i=1}^{q_r} \ell^{(i)}(\mathbf{s})\right)\ F(\mathbf{c}+\mathbf{s}) = \left(\prod_{\boldsymbol{\nu}\in I} \langle \boldsymbol{\nu}, \mathbf{s}\rangle^{g(\boldsymbol{\nu})}\right)\ \mathcal{M}(f;\mathbf{c}+\mathbf{s}) = \mathcal{H}(f,\mathbf{c};\mathbf{s}).$$

Our Theorem 1 implies then that $F(\mathbf{s})$ converges absolutely if $\Re(s_i) > c_i\ \forall i = 1,\dots,n$ and that there exists $\varepsilon_0 > 0$ such that the function $\mathbf{s} \to H(\mathbf{s})$ has holomorphic continuation to the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > -\varepsilon_0\ \forall i = 1,\dots,n\}$ and verifies in it the following estimate: for all $\varepsilon > 0$,

$$\mathcal{H}(f,\mathbf{c};\mathbf{s}) \ll_\varepsilon \prod_{\boldsymbol{\nu}\in I} \left(|\langle \boldsymbol{\nu}, \mathbf{s}\rangle| + 1\right)^{g(\boldsymbol{\nu})\left(1-\frac{1}{2}\min(0,\Re(\langle \boldsymbol{\nu},\mathbf{s}\rangle))\right)+\varepsilon}.$$

For $i \in \{1,\dots,n\}$ set $h^{(i)}(\mathbf{s}) = s_i$ for all $\mathbf{s} = (s_1,\dots,s_n) \in \mathbb{C}^n$. Set also $\delta_1 = \delta_3 = \varepsilon_0$, $q = q_r$ and $q' = n$. It follows then that $\mathbf{s} \to H(\mathbf{s})$ has **holomorphic** continuation to the domain

$$\mathcal{D}(\delta_1,\delta_3) := \{\mathbf{s} \in \mathbb{C}^n \mid \Re\left(\ell^{(i)}(\mathbf{s})\right) > -\delta_1\ \forall i = 1,\dots,q \text{ and } \Re\left(h^{(i)}(\mathbf{s})\right) > -\delta_3\ \forall i = 1,\dots,q'\}$$

and verifies the estimate: for $\varepsilon, \varepsilon' > 0$ we have uniformly in $\mathbf{s} \in \mathcal{D}(\delta_1 - \varepsilon', \delta_3 - \varepsilon')$

$$H(\mathbf{s}) \ll \prod_{i=1}^{q} \left(|\Im\left(\ell^{(i)}(\mathbf{s})\right)| + 1\right)^{1-\delta_2 \min\left(0,\Re\left(\ell^{(i)}(\mathbf{s})\right)\right)} \left(1 + (|\Im(s_1)| + \cdots + |\Im(s_n)|)^\varepsilon\right),$$

where $\delta_2 = 1/2$. Thus, all the assumptions of Theorem A above hold. By applying Theorem A with $\boldsymbol{\beta} = \mathbf{1} = (1,\dots,1)$, we deduce that there exist a polynomial $Q_{\mathbf{1}}$ of degree at most

$$\rho = q_r + w - Rank\left\{\ell^{(1)},\dots,\ell^{(q+w)}\right\} = \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) + \#J - Rank\,(I \cup J)$$

and a positive constant $\eta > 0$ such that

$$\mathcal{N}_\infty(f;x) := \sum_{\substack{\mathbf{m}=(m_1,\dots,m_n)\in\mathbb{N}^n \\ \|\mathbf{m}\|_\infty = \max_i m_i \le x}} f(m_1,\dots,m_n) = x^{\|\mathbf{c}\|_1} Q_{\mathbf{1}}(\ln x) + O\left(x^{\|\mathbf{c}\|_1 - \eta}\right) \qquad \text{as } x \to \infty.$$

This ends the proof of the first part of our Theorem 2.

Assume now in addition that the two following assumptions hold:

1. $Rank\,(I \cup J) = n$;

2. $\mathbf{1} = (1, \ldots, 1)$ is in the interior of the cone generated by $I \cup J$; that is $\mathbf{1} \in con^*\,(I \cup J) := \{\sum_{\boldsymbol{\nu} \in I \cup J} \lambda_{\boldsymbol{\nu}} \boldsymbol{\nu} \mid \lambda_{\boldsymbol{\nu}} \in (0, \infty)\ \forall \boldsymbol{\nu} \in I \cup J\}$,

By duality, we deduce that $Rank\left\{\ell^{(1)}, \ldots, \ell^{(q+w)}\right\} = n$ and $\mathbf{1}^* \in con^*\left(\left\{\ell^{(1)}, \ldots, \ell^{(q+w)}\right\}\right)$, Moreover since f is nonegative, our Theorem 1 implies that

$$H(\mathbf{0}) = \mathcal{H}(f, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})} \left(\sum_{\boldsymbol{\nu} \in \mathbb{N}_0^n} \frac{f(p^{\nu_1}, \ldots, p^{\nu_n})}{p^{\langle \boldsymbol{\nu}, \mathbf{c} \rangle}}\right) > 0.$$

It follows that assumptions of point (iv) (and therefore assumptions of point (ii)) of Theorem B above hold. Theorem B implies then that

$$deg(Q_{\mathbf{1}}) = \rho = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) + \#J - n$$

and

$$Q_{\mathbf{1}}(\log x) = H(\mathbf{0}) x^{-\|\mathbf{c}\|_1} \mathcal{I}_{\mathbf{1}}(x) + O\left((\log x)^{\rho - 1}\right), \tag{21}$$

where

$$\mathcal{I}_{\mathbf{1}}(x) = \int_{\mathcal{A}_{\mathbf{1}}(x)} \frac{dy_1 \ldots dy_{q_r}}{\prod_{i=1}^{q_r} y_i^{1 - \ell^{(i)}(\mathbf{c})}},$$

with

$$\mathcal{A}_{\mathbf{1}}(x) := \{\mathbf{y} \in [1, \infty)^{q_r} \mid \prod_{i=1}^{q_r} y_i^{\ell^{(i)}(\mathbf{e}_j)} \leq x \quad \forall j = 1, \ldots, n\}.$$

By using notations of Definition 3, it's easy to see that

$$\mathcal{I}_{\mathbf{1}}(x) = \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x) \quad \text{and} \quad \mathcal{A}_{\mathbf{1}}(x) = \mathcal{A}(I, \mathbf{u}; x),$$

where $\mathbf{u}$ is the sequence $\mathbf{u} = (g(\boldsymbol{\nu}))_{\boldsymbol{\nu} \in I}$.
Since the degree of the polynomial $Q_{\mathbf{1}}$ is equal to $\rho = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) + \#J - n$, there exists a positive constant $C > 0$ such that $Q_{\mathbf{1}}(x) = Cx^\rho + O\left(x^{\rho - 1}\right)$ as $x \to \infty$ and (21) implies that

$$H(\mathbf{0}) x^{-\|\mathbf{c}\|_1} \mathcal{I}_{\mathbf{1}}(x) = C(\log x)^\rho + O\left((\log x)^{\rho - 1}\right).$$

It follows that

$$C = H(\mathbf{0}) \lim_{x \to \infty} x^{-\|\mathbf{c}\|_1} (\log x)^{-\rho}\, \mathcal{I}_{\mathbf{1}}(x) = H(\mathbf{0}) \lim_{x \to \infty} x^{-\|\mathbf{c}\|_1} (\log x)^{-\rho}\, \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x).$$

We deduce that the main term of $\mathcal{N}_\infty(f; x)$ is given by

$$\mathcal{N}_\infty(f; x) = C_n(f) K_n(f, \|\|_\infty)\, x^{\|\mathbf{c}\|_1} (\ln x)^\rho + O\left((\ln x)^{\rho - 1}\right) \quad \text{as } x \to \infty,$$

where $C_n(f) := H(\mathbf{0}) = \mathcal{H}(f, \mathbf{c}; \mathbf{0}) > 0$ is defined by the Euler product (6) and

$$K_n(f, \|\|_\infty) := \lim_{x \to \infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\, x^{-\|\mathbf{c}\|_1} (\ln x)^{-\rho} > 0.$$

This ends the proof of Theorem 2. $\square$

4.2. Proof of Theorem 3

Let $f : \mathbb{N}^n \to \mathbb{R}$ be a multivariable multiplicative function. We assume that f belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ associated to the data $(g, \kappa, \mathbf{c}, \delta)$ (see definitions 1 and 2). We assume also that the finite set

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} \quad \text{is nonempty.}$$

We define the set $I_{\mathbf{c}} := \{\frac{1}{\langle \boldsymbol{\nu}, \mathbf{c}\rangle}\boldsymbol{\nu} \mid \boldsymbol{\nu} \in I\}$ and the sequence $\mathbf{u} := (u(\boldsymbol{\beta}))_{\boldsymbol{\beta} \in I_{\mathbf{c}}}$ where

$$u(\boldsymbol{\beta}) = \sum_{\boldsymbol{\nu} \in I;\ \frac{1}{\langle \boldsymbol{\nu}, \mathbf{c}\rangle}\boldsymbol{\nu} = \boldsymbol{\beta}} g(\boldsymbol{\nu}) \quad \text{for all} \quad \boldsymbol{\beta} \in I_{\mathbf{c}}.$$

We Define also the pair $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$.
Theorem 1 implies that

$$\mathbf{s} \to \mathcal{M}(f; \mathbf{s}) := \sum_{m_1 \geq 1, \ldots, m_n \geq 1} \frac{f(m_1, \ldots, m_n)}{m_1^{s_1} \ldots m_n^{s_n}}$$

converges absolutely in the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > c_i \ \forall i = 1, \ldots, n\}$; and that there exists $\varepsilon_0 > 0$ such that the function

$$\begin{aligned}
\mathbf{s} \to H(f; \mathcal{T}_{\mathbf{c}}; \mathbf{s}) \ &:= \left(\prod_{\boldsymbol{\beta} \in I_{\mathbf{c}}} \langle \boldsymbol{\beta}, \mathbf{s}\rangle^{u(\boldsymbol{\beta})}\right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\
&= \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{c}\rangle^{-g(\boldsymbol{\nu})}\right) \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{s}\rangle^{g(\boldsymbol{\nu})}\right) \mathcal{M}(f; \mathbf{c} + \mathbf{s}) \\
&= \left(\prod_{\boldsymbol{\nu} \in I} \langle \boldsymbol{\nu}, \mathbf{c}\rangle^{-g(\boldsymbol{\nu})}\right) \mathcal{H}(f, \mathbf{c}; \mathbf{s}) \qquad (22)
\end{aligned}$$

has *holomorphic* continuation to the domain $\{\mathbf{s} \in \mathbb{C}^n \mid \Re(s_i) > -\varepsilon_0 \ \forall i = 1, \ldots, n\}$ and verifies in it the following estimate: for all $\varepsilon > 0$,

$$H(f, \mathcal{T}_{\mathbf{c}}; \mathbf{s}) \ll_\varepsilon \prod_{\boldsymbol{\nu} \in I} (|\langle \boldsymbol{\nu}, \mathbf{s}\rangle| + 1)^{g(\boldsymbol{\nu})\left(1 - \frac{1}{2}\min(0, \Re(\langle \boldsymbol{\nu}, \mathbf{s}\rangle))\right) + \varepsilon};$$

We deduce that f is of *finite type* with $\mathcal{T}_{\mathbf{c}} := (I_{\mathbf{c}}, \mathbf{u})$ as a *regularizing pair* (see Definition 2 of [2] (2012)). It follows then from Corollary 2 of [2] (2012) that there exist a polynomial Q of degree at most

$$\rho := \left(\sum_{\boldsymbol{\beta} \in I_{\mathbf{c}}} u(\boldsymbol{\beta})\right) - Rank(I_{\mathbf{c}}) = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) - Rank(I)$$

and a positive constant $\mu > 0$ such that

$$\mathcal{N}_d(f;x) := \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_n)\in\mathbb{N}^n \\ \|\mathbf{m}\|_d=\sqrt[d]{m_1^d+\cdots+m_n^d}\leq x}} f(m_1,\ldots,m_n) = x^{\|\mathbf{c}\|_1}Q(\ln x)+O\left(x^{\|\mathbf{c}\|_1-\mu}\right) \quad \text{as } x\to\infty.$$

This ends the proof of part 1 of Theorem 3.

Assume now in addition that $Rank(I) = n$ and $\mathbf{1} \in con^*(I)$. It follows that

1. $Rank(I_\mathbf{c}) = n$ and it's clear then that there exists a function holomorphic in a tubular neighborhood of $\mathbf{0}$ such that $H(f,\mathcal{T}_\mathbf{c};\mathbf{s}) = K\left((\langle\boldsymbol{\beta},\mathbf{s}\rangle)_{\boldsymbol{\beta}\in I_\mathbf{c}}\right)$;
2. $\mathbf{1} \in con^*(I_\mathbf{c})$.

Therefore, the additional assumptions 1 and 2 of Theorem 3 of [2] (2012) are satisfied and the second part of Corollary 2 of [2] (2012) implies then that

$$\mathcal{N}_d(f;x) = C_0(f,P_d)\ x^{\|\mathbf{c}\|_1}(\ln x)^\rho + O\left((\ln x)^{\rho-1}\right) \quad \text{as } x\to\infty,$$

where $C_0(f,P_d) := \dfrac{H(f,\mathcal{T}_\mathbf{c};\mathbf{0})d^{\rho+1}\ A_0(\mathcal{T}_\mathbf{c},P_d)}{\|\mathbf{c}\|_1\ \rho!}$ and $A_0(\mathcal{T}_\mathbf{c},P_d) > 0$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the pair $\mathcal{T}_\mathbf{c} := (I_\mathbf{c},\mathbf{u})$ and the polynomial $P_d = X_1^d+\cdots+X_n^d$.
Combining (22) and the expression of $\mathcal{H}(f,\mathbf{c};\mathbf{0})$ given by theorem 1 implies that

$$H(f,\mathcal{T}_\mathbf{c};\mathbf{0}) = \left(\prod_{\boldsymbol{\nu}\in I}\langle\boldsymbol{\nu},\mathbf{c}\rangle^{-g(\boldsymbol{\nu})}\right)\ C_n(f),$$

where $C_n(f) := \mathcal{H}(f,\mathbf{c};\mathbf{0}) > 0$ is defined by the Euler product (6). Moreover, if we set

$$K_n(f,\|\|_d) := \left(\prod_{\boldsymbol{\nu}\in I}\langle\boldsymbol{\nu},\mathbf{c}\rangle^{-g(\boldsymbol{\nu})}\right)\ \frac{d^{\rho+1}\ A_0(\mathcal{T}_\mathbf{c},P_d)}{\|\mathbf{c}\|_1\ \rho!} > 0,$$

then the the constant $C_0(f,P_d)$ is positive and is given by

$$C_0(f,P_d) = C_n(f)\ K_n(f,\|\|_d) > 0.$$

In particular, the degree of the polynomial Q is equal to $\rho = \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) - n$. This ends the proof of Theorem 3. □

5. Proof of Corollary 1

Define the function $g_1 : \mathbb{N}_0^n \to \mathbb{N}_0$ by

$$g_1(\boldsymbol{\nu}) = \begin{cases} 1 & \text{if } \exists i\neq j\in\{1,\ldots,n\} \text{ such that } \nu_i=\nu_j=\|\boldsymbol{\nu}\|_\infty; \\ \|\boldsymbol{\nu}\|_\infty - \max\left(\{\nu_i \mid i=1,\ldots,n\}\setminus\{\|\boldsymbol{\nu}\|_\infty\}\right)+1 & \text{otherwise,}\end{cases}$$

where $\|\boldsymbol{\nu}\|_\infty = \max_{i=1,\dots,n} \nu_i$.
We will first prove the following needed lemma.

Lemma 4. *We have*

$$c_n(p^{\nu_1},\dots,p^{\nu_n}) = g_1(\boldsymbol{\nu})\ p^{\|\boldsymbol{\nu}\|_1-\|\boldsymbol{\nu}\|_\infty} + O\left((1+\|\boldsymbol{\nu}\|_1)p^{\|\boldsymbol{\nu}\|_1-\|\boldsymbol{\nu}\|_\infty-1}\right)$$

uniformly in $\boldsymbol{\nu}=(\nu_1,\dots,\nu_n)\in\mathbb{N}_0^n$ *and* p *prime number.*

Proof of Lemma 4:
In the proof of this lemma we will use the notations: $a\wedge b=\min(a,b)$ and $a\vee b=\max(a,b)$.
First we recall the following formula proved by Tóth in [8] (2012):

$$c_n(p^{\nu_1},\dots,p^{\nu_n}) = \sum_{\substack{0\le \ell_i\le v_i\\ i\in[\![1,n]\!]}} \frac{\varphi(p^{\ell_1})\cdots\varphi(p^{\ell_n})}{\varphi\left(p^{\max\{\ell_1,\dots,\ell_n\}}\right)}, \tag{23}$$

where φ is the Euler's totient function.
If $n=1$, then $c_1(p^{\nu_1})=1+\nu_1$ and the lemma holds.
Let $n\ge 2$. Set $k=n-1\in\mathbb{N}$.
Let p be a prime number and $\boldsymbol{\nu}=(\nu_1,\dots,\nu_n)\in\mathbb{N}_0^n$. Without loss of generality we can assume that

$$\nu_1\le\nu_2\le\cdots\le\nu_n.$$

It follows that

$$\begin{aligned}
c_n(p^{\nu_1},\dots,p^{\nu_n}) &= \sum_{\substack{0\le\ell_i\le\nu_i\\ i\in[\![1,k+1]\!]\\ \ell_1\vee\cdots\vee\ell_k\le\ell_{k+1}}} \varphi(p^{\ell_1})\cdots\varphi(p^{\ell_k}) + \sum_{\substack{0\le\ell_i\le\nu_i\\ i\in[\![1,k+1]\!]\\ \ell_1\vee\cdots\vee\ell_k>\ell_{k+1}}} \varphi(p^{\ell_{k+1}})\frac{\varphi(p^{\ell_1})\cdots\varphi(p^{\ell_k})}{\varphi\left(p^{\max\{\ell_1,\dots,\ell_k\}}\right)}\\
&= \sum_{\ell=0}^{\nu_{k+1}} p^{\nu_1\wedge\ell+\cdots+\nu_k\wedge\ell} + \sum_{\substack{0\le\ell_i\le\nu_i\\ i\in[\![1,k]\!]\\ \ell_1\vee\cdots\vee\ell_k\ge1}} \frac{\varphi(p^{\ell_1})\cdots\varphi(p^{\ell_k})}{\varphi\left(p^{\max\{\ell_1,\dots,\ell_k\}}\right)} \sum_{\ell_{k+1}=0}^{\ell_1\vee\cdots\vee\ell_k-1}\varphi(p^{\ell_{k+1}})\\
&= (\nu_{k+1}-\nu_k+1)p^{\nu_1+\cdots+\nu_k} + \sum_{\ell=0}^{\nu_k-1} p^{\nu_1\wedge\ell+\cdots+\nu_k\wedge\ell}\\
&\quad+ \sum_{\substack{0\le\ell_i\le\nu_i\\ i\in[\![1,k]\!]\\ \ell_1\vee\cdots\vee\ell_k\ge1}} \frac{\varphi(p^{\ell_1})\cdots\varphi(p^{\ell_k})}{\varphi\left(p^{\max\{\ell_1,\dots,\ell_k\}}\right)}p^{\ell_1\vee\cdots\vee\ell_k-1}\\
&= (\nu_{k+1}-\nu_k+1)p^{\nu_1+\cdots+\nu_k} + \sum_{\ell=0}^{\nu_k-1} p^{\nu_1\wedge\ell+\cdots+\nu_k\wedge\ell} + \sum_{\substack{0\le\ell_i\le\nu_i\\ i\in[\![1,k]\!]}} \frac{\varphi(p^{\ell_1})\cdots\varphi(p^{\ell_k})}{p-1} - \frac{1}{p-1}
\end{aligned}$$

$$= (\nu_{k+1} - \nu_k + 1)p^{\nu_1+\cdots+\nu_k} + \sum_{\ell=0}^{\nu_k-1} p^{\nu_1\wedge\ell+\cdots+\nu_k\wedge\ell} + \frac{p^{\nu_1+\cdots+\nu_k}-1}{p-1}$$

$$= (\nu_{k+1} - \nu_k + 1)p^{\nu_1+\cdots+\nu_k} + \sum_{\ell=0}^{\nu_k-1} p^{\nu_1\wedge\ell+\cdots+\nu_{k-1}\wedge\ell}p^\ell + \sum_{\ell=0}^{\nu_1+\cdots+\nu_k-1} p^\ell$$

Thus, for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$ such that $\nu_1 \le \nu_2 \le \cdots \le \nu_n$, we have

$$c_n(p^{\nu_1}, \ldots, p^{\nu_n}) = (\nu_n - \nu_{n-1} + 1)p^{\nu_1+\cdots+\nu_{n-1}} + \sum_{\ell=0}^{\nu_{n-1}-1} p^{\nu_1\wedge\ell+\cdots+\nu_{n-2}\wedge\ell}p^\ell + \sum_{\ell=0}^{\nu_1+\cdots+\nu_{n-1}-1} p^\ell \tag{24}$$

We deduce that

$$\begin{aligned} & 0 \le c_n(p^{\nu_1}, \ldots, p^{\nu_n}) - (\nu_n - \nu_{n-1} + 1)p^{\|\boldsymbol{\nu}\|_1 - \|\boldsymbol{\nu}\|_\infty} \\ \le \ & p^{\nu_1+\cdots+\nu_{k-1}} \sum_{\ell=0}^{\nu_k-1} p^\ell + (\nu_1 + \cdots + \nu_k)p^{\nu_1+\cdots+\nu_k-1} \\ \le \ & (\nu_1 + \cdots + \nu_{k-1} + 2\nu_k)p^{\nu_1+\cdots+\nu_k-1} \le 2\|\boldsymbol{\nu}\|_1 \ p^{\|\boldsymbol{\nu}\|_1 - \|\boldsymbol{\nu}\|_\infty - 1}. \end{aligned}$$

This ends the proof of Lemma 4. □

We will now use Lemma 4 to prove Corollary 1.
It's clear that $c_n : (m_1, \ldots, m_n) \mapsto c_n(m_1, \ldots, m_n)$ is a multiplicative function. Moreover, Lemma 4 implies that c_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see definition 2), where $g = g_1$, $\mathbf{c} = \mathbf{1} = (1, \ldots, 1)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i=1,\ldots,n} \nu_i$ $\forall \boldsymbol{\nu} \in \mathbb{N}_0^n$. Furthermore, if we denote by $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ the canonical basis of $\mathbb{R}^n$, then

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0,1\}^n \setminus \{\mathbf{0}\}.$$

Since $J = \{\mathbf{e}_i \mid c_i = 0\} = \emptyset$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n \in I = I \cup J$, it follows that the two assumptions $Rank(I \cup J) = n$ and $\mathbf{1} \in con^*(I \cup J)$ hold. Set

$$\rho := \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) + \#J - n = \left(\sum_{\boldsymbol{\nu} \in I} g(\boldsymbol{\nu})\right) - n.$$

Since $g(\mathbf{e}_i) = 2$ $\forall i = 1, \ldots, n$ and $g(\boldsymbol{\nu}) = 1$ $\forall \boldsymbol{\nu} \in I \setminus \{e_1, \ldots, e_n\}$, we have

$$\rho = 2n + (\#I - n) - n = \#I = 2^n - 1.$$

Theorem 2 implies then that there exist a polynomial Q_1 of degree ρ and a positive constant $\mu_1 > 0$ such that

$$\begin{aligned} G_n(x) \ &:= \sum_{1 \le m_1, \ldots, m_n \le x} c_n(m_1, \ldots, m_n) = x^n Q_1(\ln x) + O\left(x^{n-\mu_1}\right) \quad \text{as } x \to \infty, \\ &= C_n(c_n) K_n(c_n, \|\|_\infty) \ x^n (\ln x)^{2^n-1} + O\left(x^n (\ln x)^{2^n-2}\right) \quad \text{as} \quad x \to \infty, \end{aligned}$$

where

$$C_n(c_n) := \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n+n-1} \left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n} \frac{c_n(p^{\nu_1}, \ldots, p^{\nu_n})}{p^{\|\boldsymbol{\nu}\|_1}}\right) > 0$$

and

$$K_n(c_n, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}; x)\ x^{-n} (\ln x)^{-2^n+1} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see definition 3) associated to the set $I = \{0,1\}^n \setminus \{\mathbf{0}\}$ and to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\mathbf{e}_i) = 2\ \forall i = 1, \ldots, n$ and $u(\boldsymbol{\nu}) = 1$ $\forall \boldsymbol{\nu} \in I \setminus \{e_1, \ldots, e_n\}$ and to the vector $\mathbf{c} = \mathbf{1}$. This ends the proof of corollary 1. □

6. Proof of Corollaries 2, 3, 4, 5 and 6

6.1. Proof of Corollary 2

Let $s_n : \mathbb{N}^n \to \mathbb{R}_+$ be the function defined by

$$s_n(m_1, \ldots, m_n) = \frac{1}{\operatorname{lcm}(m_1, \ldots, m_n)} \quad \forall (m_1, \ldots, m_n) \in \mathbb{N}^n.$$

It clear that the function s_n is multiplicative and that for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$ and p prime number, we have $s_n(p^{\nu_1}, \ldots, p^{\nu_n}) = p^{-\max_{i=1,\ldots,n} \nu_i}$. Thus, s_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see definition 2), where $g \equiv 1$, $\mathbf{c} = \mathbf{0} = (0, \ldots, 0)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i=1,\ldots,n} \nu_i\ \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$.

Moreover, we have

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0,1\}^n \setminus \{\mathbf{0}\}$$

and $J = \{\mathbf{e}_i \mid c_i = 0\} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. It follows that the two assumptions $Rank(I \cup J) = n$ and $\mathbf{1} \in con^*(I \cup J)$ hold. Moreover, $\rho := \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) + \#J - n = \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) = 2^n - 1$.
Theorem 2 implies then that there exist a polynomial Q_2 of degree $\rho = 2^n - 1$ and a positive constant $\mu_2 > 0$ such that

$$\begin{aligned} S_n(x) &:= \sum_{1\le m_1,\ldots,m_n\le x} \frac{1}{\operatorname{lcm}(m_1, \ldots, m_n)} = Q_2(\ln x) + O(x^{-\mu_2}) \quad \text{as} \quad x \to \infty, \\ &= C_n(s_n) K_n(s_n, \|\|_\infty)\ (\ln x)^{2^n-1} + O\left((\ln x)^{2^n-2}\right) \quad \text{as} \quad x \to \infty, \end{aligned}$$

where

$$\begin{aligned} C_n(s_n) &:= \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n} \frac{1}{p^{\|\boldsymbol{\nu}\|_\infty}}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \end{aligned}$$

$$\text{and} \quad K_n(s_n, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I,\mathbf{u},\mathbf{c};x)\,(\ln x)^{-2^n+1} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I,\mathbf{u},\mathbf{c};x)$ is the integral (see definition 3) associated to $I = \{0,1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$. This ends the proof of corollary 2. □

6.2. Proof of Corollary 3

Let $n \in \mathbb{N} \setminus \{1\}$. Let $u_n : \mathbb{N}^n \to \mathbb{R}_+$ be the function defined by

$$u_n(m_1,\ldots,m_n) = \frac{1}{\operatorname{lcm}(m_1,\ldots,m_n)} \text{ if } \gcd(m_1,\ldots,m_n) = 1 \text{ and } u_n(m_1,\ldots,m_n) = 0 \text{ otherwise.}$$

It is clear that the function u_n is multiplicative and that for $\boldsymbol{\nu} = (\nu_1,\ldots,\nu_n) \in \mathbb{N}_0^n$ and p prime number, we have

$$u_n(p^{\nu_1},\ldots,p^{\nu_n}) = p^{-\max_{i=1,\ldots,n}\nu_i} \text{ if } \min_{i=1,\ldots,n} \nu_i = 0 \text{ and } u_n(p^{\nu_1},\ldots,p^{\nu_n}) = 0 \text{ otherwise.}$$

Thus, u_n belongs to the class $\mathcal{C}(g,\kappa,\mathbf{c},\delta)$ (see definition 2), where $\mathbf{c} = \mathbf{0} = (0,\ldots,0)$, $\delta = 1$, κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i=1,\ldots,n}\nu_i\ \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$ and g is the function defined by

$$g(\boldsymbol{\nu}) = 1 \text{ if } \min_{i=1,\ldots,n} \nu_i = 0 \text{ and } g(\boldsymbol{\nu}) = 0 \text{ otherwise.}$$

Thus, we have $I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0,1\}^n \setminus \{\mathbf{0},\mathbf{1}\}$ and $J = \{\mathbf{e}_i \mid c_i = 0\} = \{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$. It follows that the two assumptions $Rank(I \cup J) = n$ and $\mathbf{1} \in con^*(I \cup J)$ hold. Moreover, $\rho := \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) + \#J - n = \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) = 2^n - 2$.
Theorem 2 implies then that there exist a polynomial Q_3 of degree $2^n - 2$ ans $\mu_3 > 0$ such that

$$\begin{aligned} U_n(x) &:= \sum_{\substack{1\le m_1,\ldots,m_n\le x\\ \gcd(m_1,\ldots,m_n)=1}} \frac{1}{\operatorname{lcm}(m_1,\ldots,m_n)} = Q_3(\ln x) + O(x^{-\mu_3}) \quad \text{as} \quad x\to\infty \\ &= C_n(u_n)K_n(u_n,\|\|_\infty)\,(\ln x)^{2^n-2} + O\left((\ln x)^{2^n-3}\right) \quad \text{as} \quad x\to\infty, \end{aligned}$$

where

$$\begin{aligned} C_n(u_n) &:= \mathcal{H}(c_n,\mathbf{c};\mathbf{0}) = \prod_p \left(1-\frac{1}{p}\right)^{2^n-2} \left(\sum_{\substack{\boldsymbol{\nu}\in\mathbb{N}_0^n\\ \min_{i=1,\ldots,n}\nu_i=0}} \frac{1}{p^{\|\boldsymbol{\nu}\|_\infty}} \right) \\ &= \prod_p \left(1-\frac{1}{p}\right)^{2^n-2} \left(1 + \sum_{k=1}^{\infty} \frac{(k+1)^n + (k-1)^n - 2k^n}{p^k}\right) \\ &= \prod_p \left(1-\frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \end{aligned}$$

$$\text{and} \quad K_n(u_n, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-2^n+2} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see definition 3) associated to $I = \{0,1\}^n \setminus \{\mathbf{0}, \mathbf{1}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{0}$. This ends the proof of corollary 3. □

6.3. Proof of Corollary 4

Let $v_n : \mathbb{N}^n \to \mathbb{R}_+$ be the function defined by

$$v_n(m_1, \ldots, m_n) = \frac{m_1 \ldots m_n}{\operatorname{lcm}(m_1, \ldots, m_n)} \quad \forall (m_1, \ldots, m_n) \in \mathbb{N}^n.$$

It is clear that the function f is multiplicative and that for $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$ and p prime number, we have

$$v_n(p^{\nu_1}, \ldots, p^{\nu_n}) = p^{\|\boldsymbol{\nu}\|_1 - \max_{i=1,\ldots,n} \nu_i}.$$

Thus, v_n belongs to the class $\mathcal{C}(g, \kappa, \mathbf{c}, \delta)$ (see definition 2), where $g \equiv 1$, $\mathbf{c} = \mathbf{1} = (1, \ldots, 1)$, $\delta = 1$ and κ is the function defined by $\kappa(\boldsymbol{\nu}) = \max_{i=1,\ldots,n} \nu_i\ \forall \boldsymbol{\nu} \in \mathbb{N}_0^n$.

Moreover, we have

$$I = I(\kappa, g) := \{\boldsymbol{\nu} \in \mathbb{N}_0^n \mid \kappa(\boldsymbol{\nu}) = 1 \text{ and } g(\boldsymbol{\nu}) \neq 0\} = \{0,1\}^n \setminus \{\mathbf{0}\}$$

and $J = \{\mathbf{e}_i \mid c_i = 0\} = \emptyset$. It follows that the two assumptions $Rank(I \cup J) = n$ and $\mathbf{1} \in con^*(I \cup J)$ hold. Moreover, $\rho := \left(\sum_{\boldsymbol{\nu}\in I} g(\boldsymbol{\nu})\right) + \#J - n = 2^n - 1 - n$. Theorem 2 implies then that there exist a polynomial Q_4 of degree $\rho = 2^n - n - 1$ and a positive constant $\mu_4 > 0$ such that

$$\begin{aligned} V_n(x) &:= \sum_{1\le m_1,\ldots,m_n\le x} \frac{m_1 \ldots, m_n}{\operatorname{lcm}(m_1, \ldots, m_n)} = x^n Q_4(\ln x) + O(x^{n-\mu_4}) \quad \text{as} \quad x \to \infty \\ &= C_n(v_n) K_n(v_n, \|\|_\infty)\ x^n\ (\ln x)^{2^n-n-1} + O\left(x^n\ (\ln x)^{2^n-n-2}\right) \quad \text{as} \quad x \to \infty, \end{aligned}$$

where

$$\begin{aligned} C_n(v_n) &:= \mathcal{H}(c_n, \mathbf{c}; \mathbf{0}) = \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{\boldsymbol{\nu}\in\mathbb{N}_0^n} \frac{1}{p^{\|\boldsymbol{\nu}\|_\infty}}\right) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^{2^n-1} \left(\sum_{k=0}^{\infty} \frac{(k+1)^n - k^n}{p^k}\right) > 0, \end{aligned}$$

$$\text{and} \quad K_n(v_n, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)\ x^{-n} (\ln x)^{-2^n+n+1} > 0, \qquad \text{where}$$

$\mathcal{I}_n(I, \mathbf{u}, \mathbf{c}; x)$ is the integral (see definition 3) associated to $I = \{0,1\}^n \setminus \{\mathbf{0}\}$, to the sequence $\mathbf{u} = (u(\boldsymbol{\nu}))_{\boldsymbol{\nu}\in I}$ defined by $u(\boldsymbol{\nu}) = 1\ \forall \boldsymbol{\nu} \in I$ and to the vector $\mathbf{c} = \mathbf{1}$. This ends the proof of corollary 4. □

6.4. Proof of Corollaries 5 and 6

Proof of corollary 5 (resp. corollary 6) is similar to the proof of corollary 1 (resp. corollary 4) by using Theorem 3 instead of Theorem 2 and the identity $\prod_{\boldsymbol{\nu}\in\{0,1\}^n,\ \|\boldsymbol{\nu}\|_1\geq 2}\|\boldsymbol{\nu}\|_1 = \prod_{k=2}^{n} k^{\binom{n}{k}}$.

7. Explicit computations of the constants $C_n(.)$ and $K_n(.,.)$ in dimensions $n=2$ and $n=3$

We will use the software WX Maxima to compute some iterated integrals below.

7.1. Computation of $C_2(c_2)$ in corollaries 1 and 5

The identity (24) implies that for $\boldsymbol{\nu}\in\mathbb{N}_0^2$ such that $0\leq\nu_1\leq\nu_2$ we have

$$c_2(p^{\nu_1},p^{\nu_2}) = (\nu_2-\nu_1-1)p^{\nu_1} + 2\frac{p^{\nu_1+1}-1}{p-1}.$$

We deduce by symmetry that

$$\begin{aligned} C_2(c_2) &= \prod_p \left(1-\frac{1}{p}\right)^5 \left[2\sum_{\nu_2\geq 0}\sum_{\nu_1=0}^{\nu_2}\frac{c_2(p^{\nu_1},p^{\nu_2})}{p^{\nu_1+\nu_2}} - \sum_{\nu_1\geq 0}\frac{c_2(p^{\nu_1},p^{\nu_1})}{p^{2\nu_1}}\right] \\ &= \prod_p \left(\frac{p-1}{p}\right)^5 \left[2\frac{p^2(p^2+p+2)}{(p-1)^3(p+1)} - \frac{p(p^2+1)}{(p-1)^3}\right] = \prod_p\left(1-\frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)} = \frac{36}{\pi^4}. \end{aligned}$$

7.2. Computation of $C_n(.)$ $(n=2,3)$ in corollaries 2, 3, 4 and 6

Constants $C_n(.)$ in corollaries 2, 3, 4 and 6 are equal. We will denote them by C_n in this subsection.

• In dimension $n=2$, we have

$$C_2 = \prod_p\left(1-\frac{1}{p}\right)^3\left(\sum_{k\geq 0}\frac{2k+1}{p^k}\right) = \prod_p\frac{(p-1)^3}{p^3}\left(\frac{2p}{(p-1)^2}+\frac{p}{p-1}\right) = \prod_p\frac{p^2-1}{p^2} = \zeta(2)^{-1} = \frac{6}{\pi^2}.$$

• In dimension $n=3$, we have

$$\begin{aligned} C_3 &= \prod_p\left(1-\frac{1}{p}\right)^7\left(\sum_{k\geq 0}\frac{3k^2+3k+1}{p^k}\right) = \prod_p\frac{(p-1)^7}{p^7}\left(\frac{3p(p+1)}{(p-1)^3}+\frac{3p}{(p-1)^2}+\frac{p}{p-1}\right) \\ &= \prod_p\frac{(p-1)^4(p^2+4p+1)}{p^6} = \prod_p\left(1-\frac{9}{p^2}+\frac{16}{p^3}-\frac{9}{p^4}+\frac{1}{p^6}\right) \end{aligned}$$

7.3. Computation of $K_n(c_n, \|\|_\infty)$ $(n = 2, 3)$ in corollary 1

• In dimension $n = 2$, we have: $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (2, 2, 1)$ and $\mathbf{c} = \mathbf{1}$. It follows from definition 3 that

$$\begin{aligned}
\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) &= \int_{\substack{\boldsymbol{y} \in [1,+\infty[^5 \\ y_1 y_2 y_5 \leq x \\ y_3 y_4 y_5 \leq x}} y_5 \, d\boldsymbol{y} = \int_{y_5 \in [1,x[} y_5 \Big(\int_{\substack{y_1, y_2 \in [1,+\infty[\\ y_1 y_2 \leq x/y_5}} dy_1 \, dy_2 \Big)^2 dy_5 \\
&= \int_{y_5 \in [1,x[} y_5 \left(\frac{x}{y_5} \ln \left(\frac{x}{y_5} \right) - \frac{x}{y_5} + 1 \right)^2 dy_5 \\
&= \frac{1}{3} x^2 \ln^3(x) - x^2 \ln^2(x) + x^2 \ln(x) - 2x \ln(x) + \frac{x^2}{2} - \frac{1}{2},
\end{aligned}$$

which implies that

$$K_2(c_2, \|\|_\infty) := \lim_{x \to \infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) \; x^{-2} (\ln x)^{-3} = \frac{1}{3}.$$

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = (2, 2, 2, 1, 1, 1, 1)$ and $\mathbf{c} = \mathbf{1}$. It follows from definition 3 that

$$\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) = \int_{\substack{\boldsymbol{y} \in [1,+\infty[^{10} \\ y_1 y_2 y_7 y_8 y_{10} \leq x \\ y_3 y_4 y_7 y_9 y_{10} \leq x \\ y_5 y_6 y_8 y_9 y_{10} \leq x}} y_7 y_8 y_9 y_{10}^2 \, d\boldsymbol{y} = \int_{\substack{y_8, y_9, y_{10} \in [1,+\infty[\\ y_8 y_9 y_{10} \leq x}} y_8 y_9 y_{10}^2 \int_{\substack{y_1, \ldots, y_7 \in [1,+\infty[\\ y_1 y_2 y_7 \leq x/y_8 y_{10} \\ y_3 y_4 y_7 \leq x/y_9 y_{10} \\ y_5 y_6 \leq x/y_8 y_9 y_{10}}} y_7 \, d\boldsymbol{y}.$$

By using the software WX Maxima we obtain that

$$\begin{aligned}
\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) = x^3 \bigg(&\frac{47}{16128} \ln^7(x) - \frac{217}{11520} \ln^6(x) + \frac{11}{240} \ln^5(x) - \frac{1}{32} \ln^4(x) + \frac{4}{3} \ln^3(x) \\
&- \frac{1}{4} \ln(x) - \frac{973}{36} \bigg) + O\left(x^{2+\varepsilon} \right).
\end{aligned}$$

which implies that $K_3(c_3, \|\|_\infty) := \lim_{x \to \infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) \; x^{-3} (\ln x)^{-7} = \dfrac{47}{16128}$.

7.4. Computation of $K_n(s_n, \|\|_\infty)$ $(n = 2, 3)$ in corollary 2

• In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (1, 1, 1)$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int_{\substack{y_1, y_2, y_3 \in [1,+\infty[\\ y_1 y_3 \leq x \\ y_2 y_3 \leq x}} \frac{d\boldsymbol{y}}{y_1 y_2 y_3} = \frac{1}{3} \ln^3(x).$$

which implies that $K_2(s_2, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-3} = \frac{1}{3}$.

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = \mathbf{1}$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) = \int\limits_{\substack{y_1,\cdots,y_7\in[1,+\infty[\\ y_1y_4y_5y_7\le x\\ y_2y_4y_6y_7\le x\\ y_3y_5y_6y_7\le x}} \frac{d\boldsymbol{y}}{y_1y_2y_3y_4y_5y_6y_7} = \ln^7(x) \int\limits_{\substack{z_1,\ldots,z_7\in[0,+\infty[\\ z_1+z_4+z_5+z_7\le 1\\ z_2+z_4+z_6+z_7\le 1\\ z_3+z_5+z_6+z_7\le 1}} d\boldsymbol{z} = \frac{11}{3360}\ln^7(x).$$

We deduce that $K_3(s_3, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-7} = \frac{11}{3360}$.

7.5. Computation of $K_n(u_n, \|\|_\infty)$ $(n = 2, 3)$ in corollary 3

• In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathbf{u} = (1, 1)$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int\limits_{\substack{y_1,y_2\in[1,+\infty[\\ y_1\le x,\ y_2\le x}} \frac{d\boldsymbol{y}}{y_1y_2} = \ln^2(x)$$

and therefore that $K_2(u_2, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-2} = 1$.

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3\}$, $\mathbf{u} = \mathbf{1}$ and $\mathbf{c} = \mathbf{0}$. It follows that

$$\mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x) = \int\limits_{\substack{y_1,\cdots,y_6\in[1,+\infty[\\ y_1y_4y_5\le x\\ y_2y_4y_6\le x\\ y_3y_5y_6\le x}} \frac{d\boldsymbol{y}}{y_1y_2y_3y_4y_5y_6} = \ln^6(x) \int\limits_{\substack{z_1,\ldots,z_6\in[0,+\infty[\\ z_1+z_4+z_5\le 1\\ z_2+z_4+z_6\le 1\\ z_3+z_5+z_6\le 1}} d\boldsymbol{z} = \frac{11}{480}\ln^6(x)$$

and therefore that $K_3(u_3, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_3(I, \mathbf{u}, \mathbf{c}; x)\ (\ln x)^{-6} = \frac{11}{480}$.

7.6. Computation of $K_n(v_n, \|\|_\infty)$ $(n = 2, 3)$ in corollary 4

• In dimension $n = 2$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$, $\mathbf{u} = (1, 1, 1)$ and $\mathbf{c} = (1, 1)$. It follows that $\mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x) = \int\limits_{\substack{y_1,y_2,y_3\in[1,+\infty[\\ y_1y_3\le x,\ y_2y_3\le x}} y_3 d\boldsymbol{y} = x^2\ln(x) - \frac{3}{2}x^2 + 2x - \frac{1}{2}$ and therefore that $K_2(v_2, \|\|_\infty) := \lim_{x\to\infty} \mathcal{I}_2(I, \mathbf{u}, \mathbf{c}; x)\ x^{-2}\ (\ln x)^{-1} = 1$.

• In dimension $n = 3$: We have $I = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3\}$,

$\mathbf{u}=\mathbf{1}$ and $\mathbf{c}=(1,1,1)$. It follows that

$$
\begin{aligned}
\mathcal{I}_3(I,\mathbf{u},\mathbf{c};x) &= \int\limits_{\substack{y_1,\cdots,y_7\in[1,+\infty[\\ y_1y_4y_5y_7\le x\\ y_2y_4y_6y_7\le x\\ y_3y_5y_6y_7\le x}} y_4y_5y_6y_7^2\,d\boldsymbol{y} = \ln^7(x)\int\limits_{\substack{z_1,\dots,z_7\in[0,+\infty[\\ z_1+z_4+z_5+z_7\le 1\\ z_2+z_4+z_6+z_7\le 1\\ z_3+z_5+z_6+z_7\le 1}} x^{z_1+\cdots+z_3+2z_4+\cdots+2z_6+3z_7}\,d\boldsymbol{z}\\
&= \ln^7(x)\int\limits_{\substack{z_5,z_6,z_7\in[0,+\infty[\\ z_5+z_6+z_7\le 1}}\ \int\limits_{\substack{z_1,z_2,z_4\in[0,+\infty[\\ z_1+z_4\le 1-z_5-z_7\\ z_2+z_4\le 1-z_6-z_7}} x^{z_1+z_2+2z_4+\cdots+2z_6+3z_7}\,\frac{x^{1-z_5-z_6-z_7}-1}{\ln(x)}\,dz_{1,2,4}\,dz_{5,6,7}\\
&= \ln^4(x)\int\limits_{\substack{z_5,z_6,z_7\in[0,+\infty[\\ z_5+z_6+z_7\le 1}}\ \int\limits_{z_4=0}^{1-z_7-z_5\vee z_6} x^{2z_4+\cdots+2z_6+3z_7}(x^{1-z_5-z_6-z_7}-1)(x^{1-z_4-z_5-z_7}-1)\times\\
&\quad\times(x^{1-z_4-z_6-z_7}-1)\,dz_4\,dz_{5,6,7}
\end{aligned}
$$

By symmetry in z_5 and z_6 we get

$$
\begin{aligned}
\mathcal{I}_3(I,\mathbf{u},\mathbf{c};x) &= 2\ln^4(x)\left(\int\limits_{z_7=0}^{1}\ \int\limits_{z_6=0}^{(1-z_7)/2}\ \int\limits_{z_5=0}^{z_6}\ \int\limits_{z_4=0}^{1-z_6-z_7} + \int\limits_{z_7=0}^{1}\ \int\limits_{z_6=(1-z_7)/2}^{1-z_7}\ \int\limits_{z_5=0}^{1-z_6-z_7}\ \int\limits_{z_4=0}^{1-z_6-z_7}\right)\\
&= \frac{1}{16}x^3\ln^4(x) - \frac{1}{4}x^3\ln^3(x) + \frac{5}{2}x^3\ln(x) - \frac{67}{12}x^3 + O(x^{2+\varepsilon}).
\end{aligned}
$$

We deduce that $K_3(v_3,\|\|_\infty) := \lim_{x\to\infty}\mathcal{I}_3(I,\mathbf{u},\mathbf{c};x)\ x^{-3}\ (\ln x)^{-4} = \frac{1}{16}$.

7.7. Computation of $K_n(.,\|\|_d)$ $(n=2,3)$ in corollaries 5 and 6

7.7.1. Sargos's volume constant

First we will recall some notations from §2.3.1 of [2] (2012). Let $Q(\mathbf{X})=\sum_{\boldsymbol{\alpha}\in supp(Q)} a_{\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}$ be a generalized polynomial with positive coefficients that depends upon all the variables $X_1,\dots,X_n$. We apply the discussion in [7] (1987) (see also [6] (1988)) to define a "volume constant" for Q.

By definition, the Newton polyhedron of Q (at infinity) is the set $\mathcal{E}^\infty(Q) := \left(conv(supp(Q)) - \mathbb{R}_+^n\right)$.

Let G_0 be the smallest face of $\mathcal{E}^\infty(Q)$ which meets the diagonal $\Delta=\mathbb{R}_+\mathbf{1}$. We denote by σ_0 the unique positive real number t that satisfies $t^{-1}\mathbf{1}\in G_0$. Thus, there exists a unique vector subspace $\overrightarrow{G}_0$ of largest codimension ρ_0 such that $G_0\subset\sigma_0^{-1}\mathbf{1}+\overrightarrow{G}_0$. Both ρ_0,σ_0 evidently depend upon Q, but it is not necessary to indicate this in the notation. We also set $Q_{G_0}(X)=\sum_{\boldsymbol{\alpha}\in G_0} a_{\boldsymbol{\alpha}}\mathbf{X}^{\boldsymbol{\alpha}}$.

There exist finitely many facets of $\mathcal{E}^\infty(Q)$ that intersect in G_0. We denote their normalized polar vectors by $\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_N$.

By a permutation of the coordinates X_i one can suppose that $\oplus_{i=1}^{\rho_0}\mathbb{R}\mathbf{e}_i \oplus \overrightarrow{G_0} = \mathbb{R}^n$, and that $\{\mathbf{e}_{m+1},\dots,\mathbf{e}_n\}$ is the set of vectors to which G_0 is parallel (i.e. for which $G_0 = G_0 - \mathbb{R}_+\mathbf{e}_i$). If G_0 is compact then $m = n$.
Set $\Lambda = Conv\{\mathbf{0}, \boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_N, \mathbf{e}_{\rho_0+1},\dots,\mathbf{e}_n\}$. It follows that $dim\Lambda = n$.

Definition 4. *The volume constant associated to Q is:*

$$A_0(Q) := n!\ Vol(\Lambda)\ \int_{[1,+\infty)^{n-m}} \left(\int_{\mathbb{R}_+^{m-\rho_0}} P_{G_0}^{-\sigma_0}(\mathbf{1},\mathbf{x},\mathbf{y})\ d\mathbf{x} \right) d\mathbf{y}\,.$$

In ([7] (1987), chap 3, th. 1.6) (also see [6] (1988)), P. Sargos proved the following important result:

Theorem (P. Sargos): *Let Q be a generalized polynomial with positive coefficients. Then $s \mapsto Y(Q;s) := \int_{[1,+\infty)^n} Q(\mathbf{x})^{-s}\ d\mathbf{x}$ converges absolutely in $\{\Re s > \sigma_0\}$, and has a meromorphic continuation to $\mathbb{C}$ with largest pole at $s = \sigma_0$ of order ρ_0. In addition, the volume constant of Q is given by*

$$A_0(Q) = \lim_{s\to\sigma_0} (s-\sigma_0)^{\rho_0}\ Y(Q;s) > 0. \tag{25}$$

7.7.2. Mellin's Formula

We will also use the following classical Mellin's formula:
Let $w_1,\dots,w_r \in \mathbb{C}$ such that $\Re(w_i) > 0\ \forall i = 0,\dots,r$, Let $\rho_1,\dots,\rho_r > 0$. Then, for $s \in \mathbb{C}$ verifying $\Re(s) > \rho_1 + \cdots + \rho_r$, we have :

$$\frac{\Gamma(s)}{\left(\sum_{k=0}^r w_k\right)^s} = \frac{1}{(2\pi i)^r} \int_{(\rho_1)} \cdots \int_{(\rho_r)} \frac{\Gamma(s - z_1 - \cdots - z_r)\ \prod_{i=1}^r \Gamma(z_i)\ d\mathbf{z}}{w_0^{s-z_1-\cdots-z_r}\ \left(\prod_{k=1}^r w_k^{z_k}\right)}, \tag{26}$$

where the notation $\int_{(\rho)}$ denote the integral on the vertical line $\Re(s) = \rho$.

7.7.3. Computation of $K_n(c_n, \|\|_d)$ $(n = 2,3)$ in corollary 5

• In dimension $n = 2$: Corollary 5 implies that

$$K_2(c_2, \|\|_d) = \frac{d^4}{24}\ A_0(\mathcal{T}, P), \tag{27}$$

where $A_0(\mathcal{T}, P)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the polynomial $P = X_1^d + X_2^d$ and the pair $\mathcal{T} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}\right)$, where

$\tilde{I} = \{\frac{1}{2}(\boldsymbol{e_1} + \boldsymbol{e_2}), \boldsymbol{e_1}, \boldsymbol{e_2}\}$, and $u\left(\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right) = 1$ and $u(\mathbf{e}_1) = u(\mathbf{e}_2) = 2$.
It follows then from the construction given in §2.3.3 of [2] (2012) that

$$A_0(\mathcal{T}, P) = A_0(Q), \tag{28}$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(X_1, X_2, X_3, X_4, X_5) = X_1^{d/2} X_2^d X_3^d + X_1^{d/2} X_4^d X_5^d.$$

By using notations of §7.7.1, we have

$$\mathcal{E}^\infty(Q) := conv\left(supp(Q) - \mathbb{R}_+^5\right) = conv\left(\{(d/2, d, d, 0, 0), (d/2, 0, 0, d, d)\} - \mathbb{R}_+^5\right),$$

$$G_0 = \text{conv}\{(d/2, d, d, 0, 0), (d/2, 0, 0, d, d)\}, \quad \sigma_0 = 2/d \quad \text{and} \quad \rho_0 = 4.$$

Sargos's Theorem above implies then that

$$A_0(Q) = \lim_{s \to 2/d} \left(s - \frac{2}{d}\right)^4 Y(Q; s). \tag{29}$$

We will now compute the principal part of the integral $Y(Q; s)$.
First we remark that for $\Re(s) > 2/d$, we have

$$Y(Q; s) = \int_{[1,\infty)^5} (x_1^{d/2} x_2^d x_3^d + x_1^{d/2} x_4^d x_5^d)^{-s} dx_{1,2,3,4,5} = \frac{2}{d} \frac{1}{s - \frac{2}{d}} \int_{[1,\infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} \tag{30}$$

Mellin's formula (26) implies that for $\Re(s) > 2/d$,

$$\begin{aligned} \int_{[1,\infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} &= \frac{1}{2\pi i} \int_{[1,\infty)^4} \int_{(2/d)} \frac{\Gamma(s-z)\Gamma(z)}{\Gamma(s)} (x_2 x_3)^{-d(s-z)} (x_4 x_5)^{-dz} dx_{2,3,4,5} dz \\ &= \frac{1}{2\pi i} \int_{(2/d)} \frac{\Gamma(s-z)\Gamma(z)}{d^4 \Gamma(s)} \frac{1}{\left[(s-z) - \frac{1}{d}\right]^2} \frac{1}{\left[z - \frac{1}{d}\right]^2} dz \end{aligned}$$

Moving the integration line to left until $\frac{1}{2d}$ and using residues theorem imply that

$$\begin{aligned} \int_{[1,\infty)^4} (x_2^d x_3^d + x_4^d x_5^d)^{-s} dx_{2,3,4,5} &= \frac{1}{2\pi i} \int_{\left(\frac{1}{2d}\right)} \frac{\Gamma(s-z)\Gamma(z)}{d^4 \Gamma(s)} \frac{1}{\left[(s-z) - \frac{1}{d}\right]^2} \frac{1}{\left[z - \frac{1}{d}\right]^2} dz \\ &+ \frac{\Gamma^{(1)}\left(\frac{1}{d}\right)\Gamma\left(s - \frac{1}{d}\right)}{d^4 \Gamma(s) \left[s - \frac{2}{d}\right]^2} - \frac{\Gamma\left(\frac{1}{d}\right)\Gamma^{(1)}\left(s - \frac{1}{d}\right)}{d^4 \Gamma(s) \left[s - \frac{2}{d}\right]^2} + \frac{2\Gamma\left(\frac{1}{d}\right)\Gamma\left(s - \frac{1}{d}\right)}{d^4 \Gamma(s) \left[s - \frac{2}{d}\right]^3} \end{aligned} \tag{31}$$

Since the integral in the right side of (31) defines a holomorphic function in $\Re(s) > \frac{3}{2d}$, we deduce by using in addition (30) that

$$A_0(Q) = \lim_{s \to 2/d} \left(s - \frac{2}{d}\right)^4 Y(Q; s) = \frac{4\Gamma\left(\frac{1}{d}\right)^2}{d^5 \Gamma\left(\frac{2}{d}\right)}. \tag{32}$$

Combining (27), (40) and (32) implies that

$$K_2(c_2, \|\|_d) = \frac{1}{6d}\frac{\Gamma\left(\frac{1}{d}\right)^2}{\Gamma\left(\frac{2}{d}\right)}. \tag{33}$$

• In dimension $n = 3$: Corollary 5 implies that

$$K_3(c_3, \|\|_d) = \frac{d^8}{72 \times 7!}\ A_0(\mathcal{T}, P), \tag{34}$$

where $A_0(\mathcal{T}, P)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the polynomial $P = X_1^d + X_2^d + X_3^d$ and the pair $\mathcal{T} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}\right)$, where

$$\tilde{I} = \left\{\frac{1}{3}(\boldsymbol{e_1}+\boldsymbol{e_2}+\boldsymbol{e_3}), \frac{1}{2}(\boldsymbol{e_1}+\boldsymbol{e_2}), \frac{1}{2}(\boldsymbol{e_1}+\boldsymbol{e_3}), \frac{1}{2}(\boldsymbol{e_2}+\boldsymbol{e_3}), \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\right\}$$

and $\mathbf{u} = (1,1,1,1,2,2,2)$. It follows then from the construction given in §2.3.3 of [2] (2012) that

$$A_0(\mathcal{T}, P) = A_0(Q), \tag{35}$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(x_1,\ldots,x_{10}) = x_1^{d/3}x_2^{d/2}x_3^{d/2}x_5^dx_8^d + x_1^{d/3}x_2^{d/2}x_4^{d/2}x_6^dx_9^d + x_1^{d/3}x_3^{d/2}x_4^{d/2}x_7^dx_{10}^d.$$

By using notations of §7.7.1, we have
$G_0 = \mathrm{conv}\{(\frac{d}{3},\frac{d}{2},\frac{d}{2},0,d,d,0,0,0,0),(\frac{d}{3},\frac{d}{2},0,\frac{d}{2},0,0,d,d,0,0),(\frac{d}{3},0,\frac{d}{2},\frac{d}{2},0,0,0,0,d,d)\}$,

$$\sigma_0 = 3/d \quad \text{and} \quad \rho_0 = 8.$$

Sargos's Theorem implies then that

$$A_0(Q) = \lim_{s\to 3/d}\left(s-\frac{3}{d}\right)^8 Y(Q;s). \tag{36}$$

We will now compute the principal part of the integral $Y(Q;s)$ at $s = 3/d$.
Mellin's formula (26) implies that for $\Re(s) > 3/d$,

$$Y(Q;s) = \frac{24}{d^{10}\left[s-\frac{3}{d}\right]}\frac{1}{(2\pi i)^2}\int_{\substack{(\rho_1)=\frac{3}{2d}\\(\rho_2)=\frac{3}{2d}}}\frac{\Gamma(s-z_1-z_2)\Gamma(z_1)\Gamma(z_2)\Gamma(s)^{-1}\,dz_{1,2}}{\left(s-z_1-z_2-\frac{1}{d}\right)^2\left(z_1+z_2-\frac{2}{d}\right)\prod_{j=1}^{2}\left[\left(z_j-\frac{1}{d}\right)^2\left(s-z_j-\frac{2}{d}\right)\right]}.$$

By using the residue theorem, we obtain (the details of computation are left to the reader) that for $\Re(s) > 3/d$,

$$Y(Q;s) = \frac{372\Gamma\left(\frac{1}{d}\right)^2\Gamma\left(s-\frac{2}{d}\right)}{d^{10}\Gamma(s)\left[s-\frac{3}{d}\right]^8} + \frac{H(s)}{\left[s-\frac{3}{d}\right]^7}, \tag{37}$$

where H is a holomorphic function in the bigger domain $\Omega = \left\{\Re(s) > \frac{5}{2d}\right\}$.
Combining (34), (35), (36) and (37) implies that

$$K_3(c_3, \|\|_d) = \frac{31\ \Gamma\left(\frac{1}{d}\right)^3}{30240\ d^2\ \Gamma\left(\frac{3}{d}\right)}. \tag{38}$$

7.7.4. Computation of $K_n(v_n, \|\|_d)$ $(n = 2, 3)$ in corollary 6

• In dimension $n = 2$: Corollary 6 implies that

$$K_2(v_2, \|\|_d) = \frac{d^2}{4}\ A_0(\mathcal{T}, P), \tag{39}$$

where $A_0(\mathcal{T}, P)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated tothe polynomial $P = X_1^d + X_2^d$ and the pair $\mathcal{T} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}\right)$, where $\tilde{I} = \{\frac{1}{2}(\boldsymbol{e_1} + \boldsymbol{e_2}), \boldsymbol{e_1}, \boldsymbol{e_2}\}$, and $u\left(\frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right) = u(\mathbf{e}_1) = u(\mathbf{e}_2) = 1$.
It follows then from the construction given in §2.3.3 of [2] (2012) that

$$A_0(\mathcal{T}, P) = A_0(Q), \tag{40}$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(X_1, X_2, X_3) = X_1^{d/2} X_2^d + X_1^{d/2} X_3^d.$$

By using notations of §7.7.1, we have

$$G_0 = \mathrm{conv}\{(d/2, d, 0), (d/2, 0, d)\},\ \sigma_0 = 2/d,\ \rho_0 = 2 \text{ and } \Lambda = \mathrm{conv}\left\{\mathbf{0}, \frac{1}{d}(2, 0, 0), \frac{1}{d}(0, 1, 1), \boldsymbol{e_3}\right\}.$$

It follows then from Definition 4 above that

$$A_0(\mathcal{T}, P) = A_0(Q) = 3!\,\mathrm{Vol}(\Lambda) \int_{\mathbb{R}_+} Q(\mathbf{1}, x_3)^{-2/d}\, dx_3 = \frac{2}{d^3} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}.$$

By using in addition (39) we obtain that

$$K_2(v_2, \|\|_d) = \frac{1}{2d} \frac{\Gamma(1/d)^2}{\Gamma(2/d)}. \tag{41}$$

• In dimension $n = 3$: Corollary 6 implies that

$$K_3(v_3, \|\|_d) = \frac{d^5}{3 \times 4!}\ A_0(\mathcal{T}, P), \tag{42}$$

where $A_0(\mathcal{T}, P)$ is the mixed volume constant (see §2.3.3 of [2] (2012)) associated to the polynomial $P = X_1^d + X_2^d + X_3^d$ and the pair $\mathcal{T} = \left(\tilde{I}, \mathbf{u} = (u(\boldsymbol{\beta}))_{\boldsymbol{\beta}\in\tilde{I}}\right)$, where

$$\tilde{I} = \left\{\frac{1}{3}(\boldsymbol{e_1} + \boldsymbol{e_2} + \boldsymbol{e_3}), \frac{1}{2}(\boldsymbol{e_1} + \boldsymbol{e_2}), \frac{1}{2}(\boldsymbol{e_1} + \boldsymbol{e_3}), \frac{1}{2}(\boldsymbol{e_2} + \boldsymbol{e_3}), \boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\right\}$$

and $\mathbf{u} = (1, 1, 1, 1, 1, 1, 1)$. It follows then from the construction given in §2.3.3 of [2] (2012) that

$$A_0(\mathcal{T}, P) = A_0(Q), \tag{43}$$

where $A_0(Q)$ is the volume constant associated to the polynomial

$$Q(x_1, \ldots, x_7) = x_1^{d/3} x_2^{d/2} x_3^{d/2} x_5^d + x_1^{d/3} x_2^{d/2} x_4^{d/2} x_6^d + x_1^{d/3} x_3^{d/2} x_4^{d/2} x_7^d.$$

By using notations of §7.7.1, we have

$$G_0 = \mathrm{conv}\{(d/3, d/2, d/2, 0, d, 0, 0), (d/3, d/2, 0, d/2, 0, d, 0), (d/3, 0, d/2, d/2, 0, 0, d)\},$$

$$\sigma_0 = 3/d \quad \text{and} \quad \rho_0 = 5.$$

Sargos's Theorem above implies then that

$$A_0(Q) = \lim_{s \to 3/d} \left(s - \frac{3}{d}\right)^5 Y(Q; s). \tag{44}$$

Using Mellin's formula (26) as in the proof of (37) implies that for $\Re(s) > 3/d$, we have

$$\begin{aligned} Y(Q; s) &= \int_{[1,\infty)^7} (x_1^{d/3} x_2^{d/2} x_3^{d/2} x_5^d + x_1^{d/3} x_2^{d/2} x_4^{d/2} x_6^d + x_1^{d/3} x_3^{d/2} x_4^{d/2} x_7^d)^{-s} dx_{1,2,3,4,5,6,7} \\ &= \frac{36 \Gamma\left(s - \frac{2}{d}\right) \Gamma\left(\frac{1}{d}\right)^2}{d^7 \Gamma(s) \left[s - \frac{3}{d}\right]^5} + \frac{H(s)}{\left[s - \frac{3}{d}\right]^4}, \end{aligned}$$

where H is a holomorphic function in the domain $\Omega = \left\{\Re(s) > \frac{5}{2d}\right\}$.
We deduce that

$$A_0(\mathcal{T}, P) = A_0(Q) = \frac{36\ \Gamma\left(1/d\right)^3}{d^7\ \Gamma\left(3/d\right)}.$$

It follows then from (42) that

$$K_3(v_3, \|\|_d) = \frac{\Gamma\left(1/d\right)^3}{2\ d^2\ \Gamma\left(3/d\right)}. \tag{45}$$

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